AN UNDERGRADUATES GUIDE TO A RELATIVE OF THE INDEPENDENT TRANSVERSAL PROBLEM IN THE CONTEXT OF INFINITE K_n-FREE GRAPHS.

JOSHUA DOW

ABSTRACT. This project aims to review and explain the following problem such that an undergraduate mathematics student can comprehend it. Given a sparse graph with the vertices partitioned into equally large classes, can we find an independent set which meets a certain number of these classes in large sets? This is a slight variation on the classic Happy Dean problem. In this problem suppose that the dean at your university is looking to throw a dinner party such that each faculty is present at the party but, for the sake of avoiding unpleasant dinner conversation, no two guests of the party hold strictly opposing opinions on certain topics. This is modelled by forming a graph with vertices corresponding to faculty members and edges connecting colleagues who cannot be present at the same dinner table. In our version, we are looking for an independent set meeting each faculty. The original phrasing of the problem serves to facilitate the readers' comprehension of the problem, following which it's demonstrated how the problem relates to independent sets, producing a more powerful result.

1. Introduction

The goal of this project is to take advanced mathematical results and provide enough background information along with bridging gaps in order to allow a undergraduate mathematics student to understand the material. The main motivation for understanding this problem comes from [1]. This project will draw from other sources to illuminate the results and I will clarify the gaps in those results to facilitate understanding of the material. I first explain notation that is used throughout the paper, and follow this with basic definitions and theorems pertinent to comprehending the main result. This is then followed with an overview of "The Happy Dean" problem, highlighting some solutions and how it relates to other concepts. Lastly, I claim that we can find an independent set which satisfies the "Happy Dean" Problem, and I explore several concrete examples of graphs for which we can find these independent sets.

This problems roots stem from a branch of mathematics known as Ramsey Theory. "Ramsey Theory deals with finding order amongst apparent chaos" [5]. A more useful definition is: "A typical result in Ramsey theory starts with some mathematical structure that is then cut into pieces. How big must the original structure be in order to ensure that at least one of the pieces has a given interesting property?"[6]. In our context our interesting property is that we have a set which is independent.

Dedicated to all of the wonderful professors I have had the pleasure of meeting over my years at the UofC. Special mention to Helen MacLeod for inspiring my love of mathematics.

2. Notation

From here on out, I make the following assumptions unless specifically stated otherwise. G represents a graph with vertex set V and edge set E.

From [1] I take the following assignments: r will always denote a nonzero natural number which we also identify with the set $\{0, 1, ..., r-1\}$, while κ will always stand for an infinite cardinal.

I use $[X]^k$ to denote the set of k-element subsets of X. The expression $A \subseteq^* B$ means $A \backslash B$ is finite; similarly, $A =*B$ means that $(A\backslash B) \cup (B\backslash A)$ is finite. For a graph G, let $G[W]$ denote the subgraph of G induced by W where W is a subset of the vertex set of G.

3. Mathematical Background

This section will lay the foundation for the rest of the results and theorems to follow. I assume the reader has a basic understanding of set operations, basic proof techniques, and combinatorics. I advise the reader to re-read this section if there is any misunderstanding when reaching the next chapter.

Definition 3.1. A *graph* G is created by two finite sets, V and E. The set of vertices of G, denoted: $V = \{v \in V \mid \text{v is a vertex of G}\}$. The set of edges of G, denoted: $E = \{e \in E \mid \text{e is an edge of G}\}$. We will also see these presented as $V(G)$ and $E(G)$ respectively.

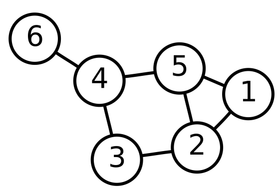

Figure 1. [4]

You can think of vertices as points or nodes that lie on a 2D plane. The ones I consider here do not have coordinates per say, they simply exist.

Here the elements of E are unordered pairs of vertices u, v such that u and v are connected. This connection is what we would call the edge e. From this definition when we refer to a particular edge we will denote it by $e = uv$, further, we say u and v are *adjacent*. For any set of edges E we say $v(E)$ to be the set of all vertices incident with edges of E. For a vertex v in a vertex-partitioned graph we denote by $V(x)$ the vertex partition containing x.

The *maximum degree* of some graph G is defined to be:

$$\Delta(G) = max\{deg(v) \mid v \in V(G)\}$$

and the minimum degree:

$$\delta(G) = min\{deg(v) \mid v \in V(G)\}$$

Consider a graph G. A sequence of vertices in a G, say, $v_1, v_2, ...v_k$ such that $v_i v_{i+1} \in E(G)$ for $1 \leq i < k$ is a *walk*. If the vertices in a walk are distinct, we call this a *path*. If the edges are distinct, we call is a *trail*. Keep in mind that every path is a trail. A *cycle* is a path $v_1, ..., v_k$ for $k \geq 3$ such that v_k is adjacent to v_1. Similarly, a trail which begins and ends at the same vertex is a *circuit*.

Definition 3.2. The *neighbourhood* of a vertex $v \in V$ for any graph G is all of the vertices such that: $N(v) = \{u \in V \mid uv \in E\}$[2]. In other words, it is the set of all vertices adjacent to v.

Consider a graph G whose vertices are marked with a sign or colour. These are referred to in many texts as signed graphs, however I will be calling them as coloured graphs. I denote a coloured graph by Σ. Like a regular or uncoloured graph, Σ has a set of edges and vertices. However, we have a new feature: a "colouring" function σ [3]. σ maps each vertex to some colour in what I will call our colour set: $\mho$.

We can define our colour set $\mho = \{$ red, blue, green $\}$ as an example.

Definition 3.3. A colouring of V, denoted V_i for $i < r$ is *balanced* if and only if:
(1) For all i $|V_i| = |V|$ when $|V| = \infty$.
(2) $||V_i| - |V_j|| \leq 1$ for all $i < j < r$ when $|V| < \infty$.

The above definition essentially says that when we have infinite vertices, we use each colour an infinite amount of times. When we have a countable number of vertices, the times we use each colour should be no more than once more than every other colour in $\mho$.

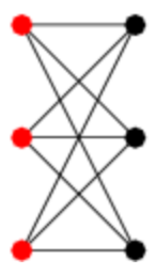

FIGURE 2. Finite Colouring [9]

Definition 3.4. A *Complete Graph* is a graph G such that every vertex is adjacent to every other vertex. We denote complete graphs by K_n.

For example, K_1 is a single vertex, K_2 is a line segment, K_3 is a triangle.

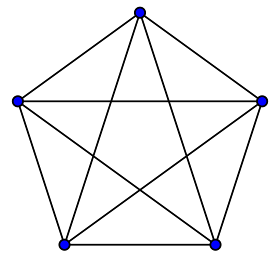

FIGURE 3. K_5 [10]

Definition 3.5. A *Clique* is a subset of vertices of a graph G such that each vertex in this subset is connected to every other vertex in the subset.

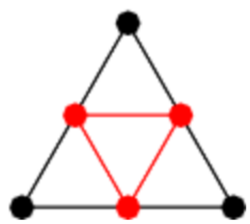

FIGURE 4. [11]

In the above figure the clique is the red section. In this particular graph, any subset consisting of three adjacent vertices will form a clique.

Definition 3.6. A *Bipartite Graph G* is a graph where V can be partitioned into two disjoint independent sets V_1 and V_2 such that every vertex in V_1 connects to at least vertex in V_2.

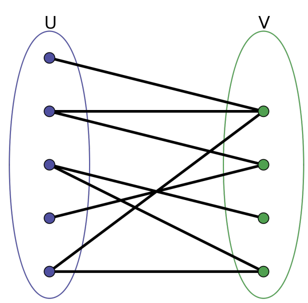

FIGURE 5. [11]

We can combine the above to definitions to form a complete bipartite graph which is denoted as $K_{\omega,\omega}$. Now each vertex in V_1 is connected to **every** vertex in V_2. We will denote the empty bipartite graph by $E_{\omega,\omega}$ on $2 \times \mathbb{N}$ from [1].

Definition 3.7. From [8] we define a *Half Graph* as a special bipartite graph where $|E| \approx |V| \times \frac{1}{2}$. We denote these as $H_{\omega,\omega}$.

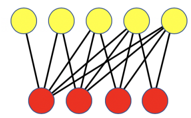

FIGURE 6

A half graph always has approximately half of the number of edges as a complete bipartite graph with the same number of vertices.

Definition 3.8. A *Subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. This is denoted by $H \subseteq G$.

Definition 3.9. Given some graph G and some set $S \subseteq V(G)$, the *subgraph of G induced by S* is the subgraph with vertex set S and edge set $E(G)$. We denote this by $\langle S \rangle$. Moreover, $\langle S \rangle$ contains all vertices of S and all edges of G whose end vertices are both in S.

Definition 3.10. A *Rado Graph* is a unique graph which has countably infinite vertices which is constructed by randomly assigning which vertices are connected by an edge with probability $\frac{1}{2}$ [7].

Definition 3.11. A *Henson Graph* is a Rado graph in which k-cliques (meaning a subset of order k) are forbidden [9]. I will describe these in more detail in later sections.

Suppose we have two vertex sets V_1, V_2, and a graph G. We say $G[V_1, V_2]$ is the graph with $E = \{uv \mid u \in E(V_1), v \in E(V_2), uv \in E(G)\}$. Notice that when $V_1 = V_2$ we have our definition of $G[W]$ as defined above.

Definition 3.12. A function or map is *Injective* when every element in the domain corresponds to **exactly** one element in the codomain.

Definition 3.13. A function or map is *Surjective* when every element in the codomain is mapped to by at least one element in the domain.

Definition 3.14. Consider a set of elements S. A *Partition* of S involves creating non-empty subsets such that each element of S is included in exactly one subset.

Definition 3.15. A function f which preserves structure such that $f(xy) = f(x)f(y)$ where x, y are elements of our domain is a *Homomorphism*.

Definition 3.16. A graph G is called *homogeneous* if there exists a subset $H \subseteq G$ with $V(H) < V(G)$ such that every embedding of H in G is equivalent to an isomorphism from G to itself.

Definition 3.17. For a subset W of $V(G)$, we say W *dominates* G if $N(W) = V(G)$.

Given two graphs G and H, assume there exists $V_{G1}, V_{H1} \subseteq V(G)$ and $V_{G2}, V_{H2} \subseteq V(H)$. From [13] we know there exists a graph homomorphism which we define as:

$$f : H[V_{G2}, V_{H2}] \to G[V_{G1}, V_{H1}]$$

$$f(u,v) \mapsto (a,b), a \in V_{G1}, b \in V_{H1}$$

When this homomorphism is injective, from [1] we write $H[V_{G2}, V_{H2}] \to G[V_{G1}, V_{H1}]$. When it is surjective we write $H[V_{G2}, V_{H2}] \twoheadrightarrow G[V_{G1}, V_{H1}]$.

Let G and H be two graphs. We define $G \bigotimes H$ on $V(G) \times V(H)$ and let $(u,v)(u',v') \in E(G \bigotimes H)$ if and only if $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ [1]. Each induced subgraph on a set of the form $u \times H$ is isomorphic to H in $G \bigotimes H$. For any $f : V(G) \to V(H)$ the subgraph induced on $(u, f(u)) : u \in V(G)$ is isomorphic to G.

Lastly, we say a graph H embeds into a graph G if there is some relabelling of edges and vertices of G such that we can be isomorphic to G.

4. The Happy Dean Problem

Suppose the dean of some university is looking to throw a dinner party such that each faculty is present but in order to avoid unpleasant dinner topics no two members present at this party may hold opposing opinions on certain topics. So the dean must choose his guests carefully.

Let G be a graph and let the vertex set of G represent the faculty members. I also assume that this set can be partitioned in such a way that each department is its own partition. Let the edge set of G be all of the conflicts between potential guests. Since only one member from each faculty is present we can additionally assume there exists no edge such that two members of the same faculty have an edge (conflict) between them.

Definition 4.1. An *independent transversal* of G is a set $T = \{v \mid v \in V(G)\}$ which contains exactly one vertex from each partition of $V(G)$.

This set T will be our solution to our happy dean problem. But does every graph G have a set like T? Well, the answer is we don't know. This is considered an NP-Complete problem. That means that there does not exist an algorithm which can find a solution in polynomial time with respect to the size of the input. I will not touch on NP-Completeness anymore, just know the problem is hard to solve.

The problem is hard to solve, so what? What else can we do? We can provide the dean with some rules or conditions necessary for an independent transversal to exist for some graph G. These conditions are easy to check.

The following two theorems from [14] show that we can use Theorem 4.3 to prove Theorem 4.2 because 4.3 is a more powerful result.

Theorem 4.2. *Let G be a graph with a vertex partition. Suppose each partition has size at least $2\Delta(G)$. Then G has an independent transversal.*

Theorem 4.3. *Let G be a graph, and suppose $V_1 \cup ... \cup V_m$ is a partition of $V(G)$ into m independent vertex partitions. Suppose G has no independent transversal, but for every edge e the graph $G - e$, formed by removing e from G, has an independent transversal. Let $e = xy \in E(G)$. Then there exists a subset $S \subseteq \{V_1, ..., V_m\}$ and a set of edges Z of $G_S = G[\bigcup_{V_i \in S} V_i]$ such that:*

(1) $V(x), V(Y) \in S$ *and* $e \in Z$.
(2) $V(Z)$ *dominates* G_S.
(3) $|Z| \leq |S| - 1$

Proof. Let G be a graph with $V_1 \cup ... \cup V_m$ as a partition of $V(G)$. Let $e = xy \in E(G)$. Assume G has no independent transversal. Suppose the number of independent vertex partitions is $m = 1$. Then we have no partition, just our original set $V(G)$ which fulfills our 3 conditions listed above. Assume $m \geq 2$ and that the above statement is true for all smaller values of m.

Pick some independent transversal T of $G - e$. Then $x, y \in T$, since otherwise T would be an independent transversal of G. Construct a graph H by removing $W = N(\{x, y\})$ from G and create a new vertex partitioning $Y = V(x) \cup V(y)$. We also remove all edges from inside Y. Each class V_i other than $V(x)$ and $V(y)$ just becomes $Y_i = V_i \ / \ W$ in H. The only partition class that has a possibility of being empty is Y since every other partition class contains an element of T.

If Y is empty, let $S = \{V(x), V(y)\}$ and $Z = \{e\}$ so $v(Z) = \{x, y\}$ dominates all of G_S.

So let Y be nonempty. Suppose that T' is an independent transversal for H. Let $v \in Y$ be the vertex of T'. Then either $v \in V(x)$ or $v \in V(y)$. In the first case, by definition of H the set $\{y\} \cup T$ is an independent transversal of G, and in the second case $\{x\} \cup T$ is an independent transversal of G. So H has no independent transversal.

Now we will remove edges one at a time from H until we obtain a new graph H' that has no independent transversal but the removal of any edge results in an independent transversal. Now Y contains no lone vertices because if it did $T/\{x, y\} \cup \{v\}$ would be an independent transversal of H'. Let q be any edge of H' that is adjacent to some vertex of Y. Since H' contains $m - 1$ partition classes, there exists a set S' and together with an edge set Z' in $H'_{S'}$ we satisfy our three conditions. Moreover, setting $S = S \setminus \{Y\} \cup \{V(x), V(y)\}$ and $Z = Z' \cup \{e\}$ then we are done. □

From the above proof we can see no two edges in Z share a vertex. If two edges did share a vertex then there would not be an independent transversal. Now how does 4.3 imply 4.2? If G is a graph as described by 4.2, and we assume that it has no independent transversal. We can remove edges one by one from G until the resulting graph satisfies the conditions of 4.3. Then let S be the subset of partition classes given by 4.3. There are then at most $2|S| - 2\Delta(G)$ vertices since $|v(Z)| = 2|S| - 2$. But G_S contains $2\Delta(G)|S|$ vertices. This violates the last two conditions of 4.3, so it must contain an independent transversal.

In 4.2 and 4.3 we form the result of a good committee, but more than that, we have formed a *complete matching* from faculties to departments. This gives us another way of checking if a complete matching exists. We will prove Hall's Theorem by use of Berge's Theorem which we will also prove. It should be clear that a matching in a graph is equivalent to a set of independent edges.

Definition 4.4. A *matching* in a graph G is a set of independent edges. That is a set of edges in which no two edges share a vertex.

Definition 4.5. Given a matching M in a graph G, the vertices in M are said to be *saturated* by M.

Definition 4.6. A *maximum* or *complete* matching in a graph is a matching that has the largest possible cardinality. A *maximal* matching is a matching that cannot be enlarged by adding any edges.

Definition 4.7. For a graph G and a matching M, an *M-alternating* path is a path in G where the edges alternate between edges in M and edges not in M. An *M-augmenting* path is an M-alternating path where the start and end vertices are not in M.

Theorem 4.8 (Berge's Theorem)**.** *Let M be a matching in a graph G. M is maximum if and only if G contains no M-augmenting paths.*

Proof. Suppose M is a matching in a graph G such that M is maximum. Next, suppose that P is an M-augmenting path. Say P is $v_1, v_2, ..., v_k$. By definition of an M-augmenting paths, k must be even. The edges of P are: $v_2v_3, v_4v_5, ..., v_{k-2}v_{k-1}$ are all edges of M while $v_1v_2, v_3v_4, ..., v_{k-1}v_k$ are not. We can define the set of edges $L = (M/\{v_2v_3, ..., v_{k-2}v_{k-1}\}) \cup \{v_1v_2, ..., v_{k-1}v_k\}$ then L is a matching that contains one more edge than M. This contradicts the maximality of M, so G contains no M-augmenting paths.

Conversely, suppose G has no M-augmenting paths and that L is a matching larger than M. Let H be a subgraph of G defined by: $V(H) = V(G)$ and $E(H)$ is the set of edges of G which appear exclusively in M or L. Since each $v \in V(G)$ lies on at most one edge from M and one edge from L, it must be that the degree of each vertex of H is 2 at most. Then each connected component of H is either a lone vertex, cycle, or path. If this component is a cycle, it must be even since the edges alternate between edges of M and edges of L. Since L contains one more edge than M there must be at least one component of H that is a path that begins and ends with edges from L by the pigeonhole principle. But, this path is an M-augmenting path which forms a contradiction to our assumptions. So no edge set L can exist. □

Theorem 4.9 (Hall's Theorem)**.** *Let G be a bipartite graph with partite sets X and Y. X can be matched into Y if and only if $|N(S)| \geq |S|$ for all subsets S of X.*

Proof. Let G be a bipartite graph with partite sets X and Y. Suppose that X can be matched into Y. Let $S \subseteq X$. Since S can be matched into Y by assumption we get $|S| \leq |N(S)|$.

Suppose now that $|N(S)| \geq |S|$ for all subsets S of X. Then consider a maximum matching M. Let $u \in X$ be an element not saturated by M. Define a set Z as the set of vertices in G that can be joined to U via an M-alternating path. Additionally, we define $S = A \cap X$ and $T = A \cap Y$. By Berge's Theorem, every vertex of T is saturated by M and u is the only unsaturated vertex of S. Then $|T| = |S| - 1$. Also, $N(S) = T$. But then $|N(S)| = |S| - 1 < |S|$ which is impossible. Then such a vertex u cannot possibly exist in X nor that M saturates all of X. □

We can create a "checklist" of what criteria leads to a graph having an independent transversal. [15] showed that if the set of all departments is twice as large as the maximum degree d and no faculty conflicts with more than d others we have an independent transversal. From above we also get that when every subset S of departments contains representatives from at least $|S|$ faculties we also have an independent transversal. So what does this all mean?

This means we require that all partition classes have the same size and partition classes must have size at least $2d$. This is a requirement of all graphs. But in 2000, [16] proved the following additional theorem:

Theorem 4.10. *Let G be a graph with vertex partition classes $V_1, ..., V_m$. Suppose that for every $I \subset \{1, ..., m\}$ there exists an independent set S_I in $G_I = G[\bigcup_{i \in I} V_i]$ such that every independent set T in G_I of size at most $|I| - 1$ can be extended by a vertex of S_I. Then G has an independent transversal.*

Ok perfect! We can now reliably check whether there exists an independent transversal and find it. Would it not be nice for the dean to be able to host multiple parties, permuting the guests such that no one gets left out over the year? We can do just that! Recall that partitions and colourings are equivalent, which brings us to the notion of strong colouring.

Definition 4.11. Let G be a graph with n vertices such that for an integer r, $r \mid n$ (r divides n). We say that G is *strongly r-colourable* if for every vertex partition of G of size r, there exists r disjoint independent transversals.

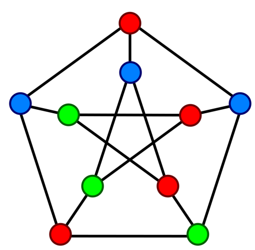

FIGURE 7. 3-colouring [20]

If r does not divide n, we can make G strongly r-colourable by adding lone vertices until $r \mid k$ where k is our new vertex count. If r is the smallest integer which makes G strongly r-colourable we define the *strong chromatic number* $s\chi(G) = r$.

It is not surprising that the strong chromatic number depends on the maximum degree. We can provide a linear bound for $s\chi(G)$ for a graph with maximum degree $\Delta(G)$. [17] gave us that $s\chi(G) \leq c\Delta(G)$. Can we do better?

Yes! We know for certain that our constant c is at least 2 and in fact $s\chi(G) \leq 3\Delta(G) - 1$ by [18]. So we can find multiple independent transversals given the conditions are satisfied! This is a very general, albeit powerful result, however we now turn our attention to what we really want: K_n-free graphs.

5. The First Rule Of Balanced Independent Sets...

Recall that a K_n-free graph G is a graph such that it contains no complete subgraphs with n vertices. Now we found the result above for some arbitrary graph G, but for some K_n-free graph G can we find $r = r(G, n)$ so that for every strong r-colouring of $V(G)$ there is an independent set $H = \{i < r \mid |H \cap V_i| = |V(G)|\}$ with at least n elements?

Whenever G is a K_n-free graph of infinite order, [21], shows that G always contains an independent set of size $|V(G)|$. Moreover, every graph G with κ many vertices either contains an independent set of size κ or an infinite clique.
I take the following definitions from [22]:

Definition 5.1. Let $R^*(n, m)$ be the smallest r so that any directed graph G with order r contains either an independent subset of size m or a set of n vertices in which the edge relation is transitive.

Definition 5.2. Let G be a graph and suppose $A, B \subseteq V(G)$ have cardinality $|V|$. We say A, B are a *rich pair* if and only if $G[A', B']$ is not empty whenever $A' \subseteq A$ and $B' \subseteq B$ have cardinality $|V|$.

Let us provide a few properties of rich pairs as proven by [1].

Lemma 5.3. *Suppose G is a graph such that $|V(G)| = \kappa$ and $A_0, A_1 \in [V]^\kappa$.*

(1) *Either A_0, A_1 is a rich pair or there exists a graph homomorphism from $K_{\kappa,\kappa} \to G[A_0, A_1]^c$.*
(2) *If A_0, A_1 is a rich pair then so are all subsets of A_0, A_1 which have size κ.*
(3) *If A_0, A_1 is a rich pair then there exists an $x < 2$ such that $|\{v \in A_x \mid |N(v) \cap A_{1-x}| < \kappa\}| < \kappa$. Here we call A_x essential in A_0, A_1.*
(4) *If A_0, A_1 is a rich pair then there exists $x < 2$ and $A'_i \subseteq A_i$ of size κ such that for all subsets $A''_0 \subseteq A'_0$ and $A''_1 \subseteq A'_1$ with cardinality κ, A''_x is essential in A''_0, A''_1. In this case we say A'_x is a strongly essential part of A'_0, A'_1.*
(5) *If A_0, A_1 is a rich pair, A_i is strongly essential, and $A'_i \subseteq A_i$ has size κ, then A'_i is strongly essential in A'_0, A'_1.*
(6) *If there exists a set of vertices $V* = \{A_i \mid i < n\}$ such that $|V*| = \kappa$ and A_i, A_j is a rich pair with A_i being strongly essential for all $i < j < n$, then K_n embeds into G.*

Using these above properties we can show that for infinite K_n-free graphs we can find homogeneous sets which are independent.

Theorem 5.4. *Let $n, m \geq 2$ and G be an infinite K_n-free graph. Then $r(G, m) \leq R^*(n, m)$.*

Proof. Suppose a balanced partition $\{V_i \mid i < r\}$ of a K_n-free graph G and $|V(G)| = \kappa$. Let $[r]^2 = \{\{i_k, j_k\} \mid k < N\}$ such that $i_k < j_k$. We now define a sequence on the infinite independent subsets of V for all $i < r$:

$$W_i^{N-1} \subseteq ... \subseteq W_i^0 \subseteq W_i^{-1}$$

We know that such a subset exists by [21]. If $i \neq i_k$ and $ineqj_k$ we let $W_i^k = W_i^{k-1}$. Given a pair $W_{i_k}^{k-1}, W_{j_k}^{k-1}$ that is not a rich pair, we take subsets $W_{i_k}^k, W_{j_k}^k$ respectively, both of which have cardinality κ such that $\mid G[W_{i_k}^k, W_{j_k}^k] \mid= 0$. These subsets exists by Lemma 5.3 (1). Otherwise, we can find subsets where either $W_{i_k}^k$ or $W_{j_k}^k$ is strongly essential in $W_{i_k}^k, W_{j_k}^k$. Now we define a function f as follows:

(1) If $W_{i_k}^k, W_{j_k}^k$ is a rich pair and $W_{j_k}^k$ is strongly essential in $W_{i_k}^k, W_{j_k}^k$, then $f(\{i_k, j_k\}) = 0$.
(2) If $W_{i_k}^k, W_{j_k}^k$ is a rich pair and $W_{i_k}^k$ is strongly essential in $W_{i_k}^k, W_{j_k}^k$, then $f(\{i_k, j_k\}) = 1$.
(3) If $W_{i_k}^k, W_{j_k}^k$ is not rich, then $f(\{i_k, j_k\}) = 2$.

Now, construct a directed graph H with respect to r vertices by including $ij \in E(H)$ when $i < j$ and $f(i,j) = 1$. Likewise, we include $ji \in E(H)$ when $j < i$ and $f(i,j) = 0$. Otherwise, and without loss of generality, $ij \notin E(H)$. Then we enumerate over some arbitrary transitive set, call it T. T has size n, so when we apply Lemma 5.3 (6) to $\{W_{i_k} \mid k \in \{1, 2, ..., n\}\}$ which brings us always to an induced copy of K_n in G. This would contradict our assumption of G being K_n-free so such a transitive set cannot possibly exist. Lastly, if $r(G,m) = R^*(n,m)$ then there must be some independent set $\{i_k \mid k \in \{1, 2, ..., m\}\} \in H$. Thus $\bigcup\{W_{i_k} \mid k \in \{1, 2, ..., m\}\}$ is our desired independent set. □

Now that we have shown it is possible to find such an independent set for an infinite K_n-free graph G, it should be no surprise for the reader that it then should also be possible to find an independent set for when G is finite.

Theorem 5.5. *If $n, m \geq 2$ then there is a finite K_n-free graph G such that $r(G,m) = R^*(n,m)$.*

Proof. Let H be a directed graph with labeled vertices $\{0, 1, ..., R^*(n,m) - 2\}$ such that H has no transitive sets or independent sets of sizes n or m respectively. Let G be some $R^*(n,m) - 1$-partite graph; meaning G has $R^*(n,m) - 1$ independent vertex partitions. These partitions are defined by $V(G) = \bigcup V_i = \bigcup(\{i\} \times \mid V_{i_j} \cap A^* \mid)$ where V_{i_j} is a singleton vertex partition of V_i and some adjacent vertex j and A^* is some independent set that exists in G. We know such a set exists by Theorem 5.4. We define all V_i for all $i < r$. We construct $E(G) = \{\{(i,s),(j,t)\} \mid (i,j) \in E(H), s < t\}$.

Suppose $\{v_i \mid i \in \{0, 1, ..., r-1\}\}$ induces a copy of K_n in G. If $k_i < k_j$ and $\{(i,k_i),(j,k_j)\} \in E(G)$ then $ij \in E(H)$ which contradicts our assumption that D contains no transitive sets of cardinality n. So $r(G,m) \leq R^*(n,m)$.

Lastly, suppose an independent set A that meets two vertex partitions of G in infinitely many places. Then $G[V_i, V_j] = \emptyset$ and $ij, ji \notin E(D)$. But we assumed that D contains no independent sets of size m, so the set A cannot exist. So $r(G,m) \geq R^*(n,m)$.

Thus, $r(G,m) = R^*(n,m)$. □

Given any graph G, we can show whether or not it is K_n-free. If so, we can always find an independent set that satisfies a complete matching as stated in our motivation for this problem. We will now look at at $r(G,m)$ for arbitrary graphs G and prove some unique properties.

6. What is it about $r(G, \cdot)$?

I begin this section by noting all graphs G covered in this section are considered infinite and that given that $r(G, m)$ is defined for some G, there is no guarantee that G is K_n-free.

In this section I will be considering what [23] calls **flat graphs**.

Definition 6.1. A graph G is called *flat* if no subgraph H of G is isomorphic to K_n^m where K_n^m is the complete graph K_n with m new vertices added to each existing edge of K_n.

A useful property of graphs that we will need later from [23]:

Theorem 6.2. *Let G be a graph and $F \subseteq E(G)$ be an infinite subset. If $m \in \mathbb{N}$ there exists a finite $T \subseteq E(G)$ and infinite $F' \subseteq F$ such that for all unique edges $a, b \in F'$ the smallest length of a path from a to b is greater than m.*

We can consider flat graphs to be trees, planar graphs (edges only intersect at vertices), or graphs such that each vertex has finitely many neighbours.

Theorem 6.3. *Suppose G is a flat graph. Then $r(G, m) = m$ for all $m \geq 2$.*

Proof. Suppose G is a flat graph. Let us partition $V(G)$ such that $V(G) = V_0 \cup V_1 \cup ... \cup V_{m-1}$ where $m \geq 2$. Let $n = 3$, by definition of flatness we can find $V_i' \subseteq V_i$ where $|V_i'| = \infty$ and a finite S_i such that each path with length two between the vertices of V_i' go through S_i. Suppose that for every i, $V_i' \cup S_i = \emptyset$. Next we remove all edges connected to the vertices of V_i'.

Take any vertex $v \in V_i'$. Now $\mid N(v) \cap V_j' \mid= 1$ for any $i \neq j < m$. If $\mid N(v) \cap V_j' \mid> 1$ then v would be present in a path of length two between vertices of V_j' which contradictions our assumption of disjointedness. So each V_i', V_j' is not a rich pair. By repeated application of Lemma 5.3 (1) we can find an independent set by removing edges that meets up with all partition classes in infinite sets. $\square$

Whenever G is one of our special case flat graphs, we know that $r(G, m)$ is always defined for all $m \geq 2$ but what does this really mean? Given that $r(G, 2)$ exists, every set of $|V|$ vertices must contain an independent set. Moreover, G is K_n-free where n $= |V|$ and $r(G, 2)$ exists.

A few more interested consequences: $r(G, m) \leq r(G, m+1)$ which comes straight from our definition. If we have two isomorphic graphs G_1 an G_2, $r(G_1, m) = r(G_2, m)$ which comes from the definition of isomorphic graphs. The only difference would be the labelling of the vertices between graphs, but this has no effect on the underlying independent set.

Above we showed that when G is a flat graph, $r(G, m) = m$ for all $m \geq 2$ however we can go further with a more general and powerful result. We will first show a case for when G contains no rich pairs.

Theorem 6.4. *Let G be a graph with no rich pairs and that $r(G, 2)$ is well defined. Then $r(G, m) = m$ for all $m \geq 2$.*

Proof. Suppose G is a graph which contains no rich pairs. Suppose that $V(G) = \bigcup V_i, i < m$ is a balanced partition of V. Now we can find an independent $V_i' \in [V_i]^{|V|}$ for each $i < m$ by Theorem 6.3. Since G contains no rich pairs find $W_i \subseteq V_i'$ such that no edge exists between W_i and W_j for $i < j < m$. Then the union of all such W_i is our independent set. □

This leads us to the main theorem of this section:

Theorem 6.5. *If $r(G, m) = m$ for all $m \geq 3$ then $r(G, m) = m$ for all $m \geq 2$.*

Proof. Let G be a graph. Suppose $r(G, m) = m$ for all $m \geq 3$. If G contains no rich pairs we are done by the above theorem. So suppose that G does contain at least one rich pair, say R_1, R_2. Again, by Theorem 6.3 we have that $r(G, m) = m \geq r(G[R_1, R_2], m-1) = 2(m-2)+1 = 2m-3$. So, $m = 3$. But if $| V(G) \backslash (R_1 \cup R_2) | = \kappa$ then we have a balanced partitioning responsible for $r(G, 3) > 3$ which contradicts our initial assumption. Also if $| V(G) \setminus (R_1 \cup R_2) | < \kappa$ then $3 = r(G[R_1, R_2], 3) = 5$. So G cannot contain any rich pairs. □

When a graph G contains a rich pair, the value of $r(G, m)$ for $m \geq 2$ is always greater than m. Consider an arbitrary tree T of size n. From above it should be trivial that $\Delta(T) <= n - 1$ and $r(T, m) = m$ when $m \geq 2$ since T is finite.

7. Dancing With The Stars

To ease our descent into more complex graphs, let us first start with **star graphs**. A graph G is a star graph when G contains at most one vertex with degree greater than 1. Moreover, we denote star graphs as S_n and is analogous to the complete bipartite graph $K_{1,n}$. That is, a tree with one internal node and n number of leaves. Additionally, from [24], we can say a graph G is a star if no path of length 3 exists and it cannot be coloured with two colours. To get us started with star graphs we

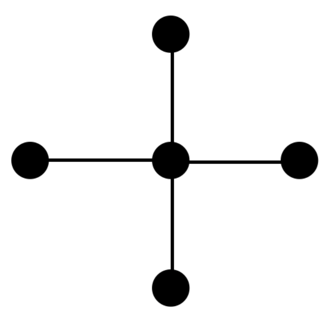

Figure 8. $K_{1,4}$

need a few preliminary results. From [27] we have:

Theorem 7.1. *If G is a bipartite graph then $s\chi(G) = \Delta(G)$.*

This tells us that for any vertex partitioning, where each partition class has size κ we can assign these to the edges of K_n such that there is no independent subset that is induced on G.

Suppose there exists graphs $G_1, ..., G_r$ such that $E(G_1) \cup ... \cup E(G_r) = E(K_n)$. If for all $i \leq k, s\chi(G_i) \leq k$ and each vertex belongs to at most $t-1$ of the $G_i's$ then $r(S_t, k) > n$ [26].

To conclude this section, I finish with a definite lower bound for star graphs.

Theorem 7.2. *Suppose $r \in \mathbb{N}$. If $r \mod 2 = 0$ then $r(S_r, 2) = 2r - 1$. Likewise, if $r \mod 2 = 1$ then $r(S_r, 2) = 2r$.*

Proof.

(1) r is even. Suppose $n = 2r - 2$. From [28] we can partition K_n into a complete matching of G_1 and $G_2, ..., G_{r-1}$ which are Hamiltonian. We know that $s\chi(G) \leq 2$ from [27] and each vertex belongs to $r-1$ other $G_i's$. From [26], we are done.
(2) r is odd. Suppose $n = 2r - 1$. If $r = 1$, then $n = 1$ and we are done. So let $r = 3$ and say we partition K_5 into two distinct cycles A and B. We know partitioning is possible from [28]. The only way we can find a complete matching between these two cycles is if the possible colours used are all the same. Say we colour A using just a single colour, and B with any colour which is not the one used to colour A. Again from [28], there exists no induced $K_{1,3}$ which can be coloured using the same colour as A.

$\square$

8. Henson's Graphs

I now move on to a bit more complicated type of graph, but much more useful and interesting.

From [29], Henson created a homogeneous K_n-free graph which is universal for all K_n-free graphs on a finite number of vertices. These graphs are denoted by H_n. I remind the reader here that *universal* in the context of graphs means it contains every finite graph and countably infinite graphs as an induced subgraph.

Additionally, from [30] we have Henson's Criterion: A countable graph G is universal for countable K_n-free graphs if and only if:

(1) G has no induced K_n subgraphs.
(2) If A, B are disjoint vertex subsets of $V(G)$ and $V(G) \setminus A$ has no induced K_{n-1} subgraphs, there must be another vertex which is connected in G to every vertex in A but to no vertex in B.

Theorem 8.1. *Suppose G is an infinite K_n-free graph, then $r(G, m) \leq r(H_n, m + 1) - 1$.*

Proof. Suppose G is an infinite K_n-free graph such that $V(G)$ can be expressed as a balanced partition $\{V_i \mid i < r(H_n, m+1) - 1\}$. Since H_n is universal, we can embed G into H_n. We define a function $f : V(H_{n+1}) \to r$. Now let $W_i = f(V_i)$ where $i < r$. Then $\{W_i | i < r\}$ is a balanced independent partition of H_n with cardinality $r(H_n, m + 1)$. Then there is an independent set L such that $\{i \leq r \mid |W_i \cap A|\}$

contains $m+1$ elements at a minimum. So $\{i < r \mid |W_i \cap A|\} \geq m$. Lastly, define $J = \bigcup\{f^{-1}(A \cap W_i) \mid i < r\}$. Then J is our desired independent set which is connected to at least m of our original $V_i's$ of G. □

As in the previous section I will now attempt to find bounds for the value of $r(H_n, m)$. Before I proceed with the main theorem for this section, I will be using Lemma 4.5 from [1].

Theorem 8.2. $R^*(n, m-1) + 1 = r(H_n, m)$ *for all* $n, m \geq 2$.

Proof. Suppose D is a directed graph with $R^*(n, m-1) - 1$ vertices. Additionally, assume that D contains no transitive sets of cardinality n or independent sets of size $m-1$. From [1] we can find $R^*(n, m-1)$ vertex partition classes such that every independent set is contained in at most $\alpha(D) + 1 \leq m - 1$. Here $\alpha(D)$ is the cardinality of the largest subset of $V(D)$ which is independent. Then $r(H_n, m) \geq R^*(n, m-1) + 1$.

Now suppose that we can partition $V(H_n)$ such that it is a balanced partition. Say this partition is: $V(H_n) = \{V_i \mid i \leq R^*(n, m-1)$. By definition of H_n we can find $W \subset V_{R^*(n,m-1)}$ which makes $H_n[W] \cong H_n$. Now from Lemma 1 in [1] and definition of a Henson graph, we can find $W_{R^*(n,m-1)} \subseteq W$ and $W_i \subseteq V_i$ where no edges exist between these two subsets.

We can remove edges from W_i until $H[W_i]$ is empty. It must be then that for all $i < j < R^*(n, m-1)$:

(1) $H[W_i, W_j] = \emptyset$ **or**
(2) There is a surjective homomorphism from $H_{k,k}$ to $H_n[W_i, W_j]$ **or**
(3) There is a surjective homomorphism from $H_{k,k}$ to $H_n[W_j, W_i]$

Construct a directed graph D where each edge in D only exists if and only if there is a surjective homomorphism from $H_{k,k}$ to the induced subgraph on H_n by each respective W_a. Then $|V(D)| = R^*(n, m-1)$ so we can find a transitive set of size n or independent set of size $m-1$. From Lemma 5.3 (6), there is no way for a transitive set of size n to exist. So, we can then find $R' \subseteq R^*(n, m-1)$ where $|R'| = m-1$ such that $H_n[W_i, W_j] = \emptyset$. Our independent set is then: $\bigcup\{W_i \mid i \in \{R^*(n, m-1) \cup R'\}$. Thus, $r(H_n, m) \leq R^*(n, m-1)$. □

9. Bounding $r(G, m)$

I conclude this paper with some more general bounds for $R^*(n, m)$. In addition, I will introduce new versions of $R^*(n, m)$ for a definite number of colourings for a graph G.

So far I have been using $R^*(n, m)$ to denote an arbitrary number of subsets used to create independent sets for a graph G. When I say $R_k^*(n, m)$ we are deliberate in our choice of exactly k subsets.

Theorem 9.1. *If* $p, q \in \mathbb{Z}^+$[31]*:*

$$R^*(n, m) \leq \frac{(n+m-2)!}{(n-1)!(m-1)!}$$

Theorem 9.2. *Let $k, m, n \in \mathbb{Z}^+$ where the smallest of m or n is greater than or equal to k. Then $R_k^*(n,m)$ exists. Moreover, for any such k, m, n:*

(1) $R_1^*(n,m) = m + n - 1$
(2) $R_k^*(k,m) = m$
(3) $R_k^*(n,k) = n$

Proof. Suppose $k = 1$, then we have one single subset. In this subset, if there is a transitive set with size less than m and independent set less than n, then certainly $R_1^*(n,m) \leq n + m - 1$.
Suppose $k = m$ and that each subset exclusively has either an independent set of size n or a transitive set of size m. Without loss of generality, suppose its an independent set of size n. If we did have any transitive sets, we would have a qualifying set of size m and we would be done. Then when $R_k^*(n,m) \geq n$ we know we have an independent set as described above. So $R_k^*(k,m) = m$ and likewise, $R_k^*(n,k) = n$.

□

So $R_k^*(n,m)$ always exists. From [32]:

Theorem 9.3. *If $n \geq 3$ then $R^*(n,n) > \lfloor 2^{\frac{k}{2}} \rfloor$.*

Finding the exact values for $R^*(n,m)$ is extremely difficult when viewed as a general problem. In fact we only know a handful of exact numbers. The problem is much easier to solve abstractly as shown throughout this paper. For now, the best we can do is bound these values. Erdös, who worked on these problems extensively once made a comment that it would be easier to fight aliens than find $R^*(6,6)$, even with the use of a computer.

10. Remarks

At this point, there might be some confusion as to how the later sections relate to our motivation for these problems: The Happy Dean Problem. Recall that we aimed to find an independent set for any arbitrary graph G that would satisfy our constraints on The Happy Dean. In Section 4 we did this explicitly for finite graphs. Sections 5 through 9, together give us that for infinite K_n-free graphs, we can absolutely find either an independent or transitive set. In the case that set is transitive, it is possible to transform it into an independent set which satisfies our criterion. The graph must be K_n free because remember that in K_n, each vertex has $n - 1$ neighbours, and so the only independent subsets which can find are lone vertices which is not useful in our context since a committee of one person is not very exciting.

There remains much to explore in this subfield of mathematics, and its something I look forward to learning more about. I'm hopeful this paper will spark similar interest in other undergraduate mathematics students.

References

1. Claude Laflamme,Andres A. Lopez, Daniel T. Soukup, Robert Woodrow, “Balanced independent sets in graphs omitting large cliques” arXiv:1611.06142 (2016)
2. Harris, John M., Jeffry L. Hirst, and Michael J. Mossinghoff. *Combinatorics and Graph Theory*. New York: Springer, 2008. Print. Undergraduate Texts in Mathematics.
3. Thomas Zaslavsky, Signed graph coloring, *Discrete Mathematics*, Volume 39, Issue 2, 1982, Pages 215-228, ISSN 0012-365X, https://doi.org/10.1016/0012-365X(82)90144-3.
4. AzaToth, https://upload.wikimedia.org/wikipedia/commons/5/5b/6n-graf.svg, [Public Domain]
5. Dickson, James O. *An Introduction to Ramsey Theory on Graphs*, Virginia Tech University(2011): 85. Web. 26 Jan.
6. Wikipedia contributors. ”Ramsey theory.” Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 21 Jan. 2020. Web. 23 Feb. 2020.
7. Erdös, P.; Rényi, A. (1963), ”Asymmetric graphs”, Acta Mathematica Academiae Scientiarum Hungaricae, 14: 295315, doi:10.1007/BF01895716, MR 0156334.
8. Erdös, Paul (1984), ”Some combinatorial, geometric and set theoretic problems in measure theory”, in Klzow, D.; Maharam-Stone, D. (eds.), Measure Theory Oberwolfach 1983, Lecture Notes in Mathematics, 1089, Springer
9. Weisstein, Eric W. ”Independent Set.” From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/IndependentSet.html
10. David Benbennick, https://upload.wikimedia.org/wikipedia/commons/c/cf/Complete_graph_K5.svg, [Public domain]
11. Weisstein, Eric W. ”Clique.” From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Clique.html
12. MistWiz, https://upload.wikimedia.org/wikipedia/commons/e/e8/Simple-bipartite-graph.svg, [Public Domain]
13. Hell, Pavol; Neetil, Jaroslav (2004), Graphs and Homomorphisms, Oxford Lecture Series in Mathematics and Its Applications, 28, Oxford University Press, ISBN 0-19-852817-5
14. P. Haxell, ”On Forming Committees.” The American Mathematical Monthly 118.9 (2011): 777-88. Web.
15. Haxell, Penny, Tibor Szab, and Gbor Tardos. ”Bounded Size Components partitions and Transversals.” Journal of Combinatorial Theory, Series B 88.2 (2003): 281-97. Web.
16. Aharoni, Ron, and Penny Haxell. ”Hall’s Theorem for Hypergraphs.” Journal of Graph Theory 35.2 (2000): 83-88. Web.
17. Alon, Noga. ”The Strong Chromatic Number of a Graph.” Random Structures & Algorithms 3.1 (1992): 1-7. Web.
18. Haxell, P. E. ”On the Strong Chromatic Number.” Combinatorics, Probability and Computing 13.6 (2004): 857-65. Web.
19. Kunen, Kenneth. Set Theory : An Introduction to Independence Proofs. Amsterdam: North-Holland, 1980. Print. Studies in Logic and the Foundations of Mathematics ; v. 102.
20. No Author Given, https://commons.wikimedia.org/wiki/File:Petersen_graph_3-coloring.svg, [Public Domain]
21. B. Dushnik and E. W. Miller (1941) Partially ordered sets, Amer, J. Math. 63, 600-610.
22. A. Gyárfás. Nonsymmetric party problems. Journal of Graph Theory 28.1 (1998): 43-47.
23. K. P. Podewski, M. Ziegler (1978) Stable Graphs. http://matwbn.icm.edu.pl/ksiazki/fm/fm100/fm100113.pdf
24. Guillaume Fertin, Andr Raspaud, Bruce Reed. Star Coloring of Graphs. Journal of Graph Theory,Wiley, 2004, 47 (3), pp.163-182.
25. S. A. Burr and J. A. Roberts. On Ramsey numbers for stars.Utilitas Math.4:217220, 1973.
26. Alon, Noga, M. Buci, Tom Kalvari, Eden Kuperwasser, and Tibor Szab. ”List Ramsey numbers.” (2019).
27. F. Galvin. The list chromatic index of a bipartite multigraph. J. Combin. Theory Ser. B63(1):153158, 1995.
28. B. Alspach and H. Gavlas. Cycle decompositions of K_n and $K_n - I$ .J. Combin. Theory Ser. B81(1):7799, 2001.
29. C. W. Henson. A family of countable homogeneous graphs. Pacific journal of mathematics 38.1 (1971):69-83.

30. Dobrinen, Natasha. "The Ramsey theory of Henson graphs." (2019).
31. Erdös, P.; Szekeres, G. A combinatorial problem in geometry. Compositio Mathematica, Volume 2 (1935)
32. Erdös, P. Some remarks on the theory of graphs. Bull. Amer. Math. Soc. 53 (1947), no. 4