1) Prove that if n is any integer, then n^5 - $n = 5k$ for some integer k.

Claim: n^5 - $n = 5k$ for some integer k

Proof By Mathematical Induction on n: Base Case: Let $n = 0$, then $n^5 - n = 0^5 - 0 = 0$. Since $0 \in \mathbb{N}$ the case where $n = 0$ must be true.

Inductive Step: Suppose $k^5 - k = 5i$ for some integer i. We wish to show that $(k+1)^5 - (k+1) = 5j$ for some integer j.

$$\begin{aligned}
(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k+1) && (1)\\
&= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k - k && (2)\\
&= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 && (3)\\
&= 5i + 5k^4 + 10k^3 + 10k^2 && (4)\\
&= 5i + 5k^4 + 5k^3 * 2 + 5k^2 * 2 && (5)\\
&= 5j && (6)
\end{aligned}$$

I walk through the above steps as follows: (1) is an expansion of the polynomial, (2) is a grouping of like terms, (3) Notice that we achieve our Inductive Hypothesis, so we re-arrange terms, (4) We apply the Inductive Hypothesis, (5) We list each term as a multiple of 5, (6) Conclude that since $i, k \in \mathbb{Z}$ and j is the sum of integers then also $j \in \mathbb{Z}$.

Now we wish to extend n into the integers, so let us now denote n as $-n$ as to reach all negative integers as $\mathbb{Z}^- \cup \mathbb{N} = \mathbb{Z}$.

Base Case: Let $n = -1$, then $(-n)^5 - (-n) = (-1)^5 - (-1) = (-1) + (1) = 0$. Since $0 \in \mathbb{Z}$ the case where $n = -1$ must be true.

Inductive Step: Suppose $(-k)^5 - (-k) = 5q$ for some integer q. We wish to show that $(-k-1)^5 - (-k-1) = 5l$ for some integer l.

$$\begin{aligned}
(-k-1)^5 - (-k-1) &= -k^5 - 5k^4 - 10k^3 - 10k^2 - 5k - 1 - (-k-1) && (7)\\
&= -k^5 - -k - 5k^4 - 10k^3 - 10k^2 && (8)\\
&= 5q - 5k^4 - 10k^3 - 10k^2 && (9)\\
&= 5q - 5k^4 - 5k^3 * 2 - 5k^2 * 2 && (10)\\
&= 5l && (11)
\end{aligned}$$

As above, I will once again step through the procedure: (7) is an expansion of the polynomial, (8) is a grouping of similar terms, (9) Notice that we achieve our second Inductive Hypothesis, so we re-arrange terms, (10) We list each term as a multiple of 5, (6) Conclude that since $k, q \in \mathbb{Z}$ and l is the sum of integers then also $l \in \mathbb{Z}$.

We conclude that for $n \in \mathbb{Z}, n^5 - n = 5k, k \in \mathbb{Z}$. ∎

2) Prove that $(25)^{1/3}$ is not a rational number.

Claim: $(25)^{1/3} \notin \mathbb{Q}$

Proof: Suppose that $(25)^{1/3} \in \mathbb{Q}$. Then $(25)^{1/3} = \frac{m}{n}$, where $m, n \in \mathbb{Z}, n \neq 0$ and $\frac{m}{n}$ is in its most reduced form.

$$(25)^{1/3} = \frac{m}{n} \tag{12}$$
$$25 = \frac{m^3}{n^3} \tag{13}$$
$$25n^3 = m^3 \tag{14}$$
$$m = 25k, k \in \mathbb{Z} \tag{15}$$
$$25n^3 = (25k)^3 \tag{16}$$
$$25n^3 = 15625k^3 \tag{17}$$
$$n^3 = 625k^3 \tag{18}$$
$$625 = \frac{n^3}{k^3} \tag{19}$$
$$\sqrt[3]{625} = \frac{n}{k} \tag{20}$$

Since k < n < m, this contradicts our assumption that $\frac{m}{n}$ is in its most reduced form such that $\sqrt[3]{25} = \frac{m}{n}$. We can conclude that $\sqrt[3]{25} \notin \mathbb{Q}$. ∎

3) Prove that for some set X, if $A \cap X = B \cap X$ and $A \cup X = B \cup X$ then $A = B$.

Claim: For sets A,B and for any arbitrary set X such that $A \cap X = B \cap X$ and $A \cup X = B \cup X$, that A = B.

Proof: Suppose sets A,B and for any arbitrary set X, $A \cap X = B \cap X$ and $A \cup X = B \cup X$

$$
\begin{aligned}
A &= A \cup (A \cap X) && (21)\\
&= (A \cup A) \cap (A \cup X) && (22)\\
&= A \cap (A \cup X) && (23)\\
&= A \cap (B \cup X) && (24)\\
&= (A \cap B) \cup (A \cap X) && (25)
\end{aligned}
$$

Now we will do B

$$
\begin{aligned}
B &= B \cup (B \cap X) && (26)\\
&= (B \cup B) \cap (B \cup X) && (27)\\
&= B \cap (B \cup X) && (28)\\
&= B \cap (A \cup X) && (29)\\
&= (B \cap A) \cup (B \cap X) && (30)\\
&= (A \cap B) \cup (A \cap X) && (31)
\end{aligned}
$$

I will now outline the transition in between steps. We go from (21) to (22) by distributive law. (23) to (24) by Assumption. (24) to (25) by distributive law. Now for the outline of B. We go from (26) to (27) by distributive law. (28) to (29) by Assumption. (29) to (30) by distributive law. Finally, (30) to (31) by commutative law.

We can conclude that for sets A,B and for any arbitrary set X such that $A \cap X = B \cap X$ and $A \cup X = B \cup X$, that A = B. ∎

4) Let A,B,C,D be sets with $C \subseteq A$ and $D \subseteq B$. Use the definition of set equality to prove $(A\times B)\setminus(C\times D) = ((A\times(B\setminus D))\cup((A\setminus C)\times B)$.

Claim: If A,B,C,D be sets such that $C \subseteq A$ and $D \subseteq B$ then $(A\times B)\setminus(C\times D) = ((A \times (B \setminus D)) \cup ((A \setminus C) \times B)$

Proof: We wish to show that $(A\times B)\setminus(C\times D) \subseteq ((A\times(B\setminus D))\cup((A\setminus C)\times B)$ and that $((A \times (B \setminus D)) \cup ((A \setminus C) \times B) \subseteq (A \times B) \setminus (C \times D)$.

Suppose $x \in ((A \times (B \setminus D)) \cup ((A \setminus C) \times B)$ then $x \in A \times (B \setminus D)$ OR $x \in ((A \setminus C) \times B)$. If $x \in A \times (B \setminus D)$ then x is an ordered pair (a, b) such that $a \in A, b \in B \setminus D$. Equivalently, $a \in A, b \in B$ and $b \notin D$. It must be the case that $x \in A \times B$ and $x \notin A \times D$. Then $x \notin C \times D$ as $b \notin D$. So $x \in A \times B$ and $x \notin C \times D$. Therefore we can conclude that $x \in (A \times B) \setminus (C \times D)$, this means that $(A \times B) \setminus (C \times D) \subseteq ((A \times (B \setminus D)) \cup ((A \setminus C) \times B)$.

Similarly, suppose $x \in (A \times B) \setminus (C \times D)$, then $x \in A \times B$ AND $x \notin C \times D$. X is then an ordered pair (a_1, b_1) such that $a_1 \in A, b_1 \in B$ and $a_1 \notin C, b_1 \notin D$. Then $x \in A \times (B \setminus D)$ as $a_1 \in A, b_1 \in B$ and $b_1 \notin D)$. We can also notice that $x \in (A\setminus C)\times B$ as $b_1 \in B, a_1 \in A$ and $a_1 \notin C$.This means that $x \in (A\times(B\setminus D))$ OR $x \in ((A \setminus C) \times B)$. This means that $((A \times (B \setminus D)) \cup ((A \setminus C) \times B) \subseteq (A \times B) \setminus (C \times D)$.

We can conclude that by definition of set equality, if we have sets A,B,C,D such that $C \subseteq A$ and $D \subseteq B$ then $(A\times B)\setminus(C\times D) = ((A\times(B\setminus D))\cup((A\setminus C)\times B)$.
∎

5) Prove or disprove by a well documented example, injectivity and surjectivity for the following two maps:

- **$f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2 + 2z + 3$.**
- **$g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $g(n) = (n, (n + 1)^2)$.**

Claim: f is not injective nor surjective.

Proof: Consider the example $f(0 + i0)$ and $f(-2 + i0)$

$$\begin{aligned} f(0 + i0) &= 0^2 + 2 * 0 + 3 = 3 && (32) \\ f(-2 + i0) &= (-2 + i0)^2 + 2 * (-2 + i0) + 3 && (33) \\ &= (-2)^2 + 2 * (-2) + 3 = 3. && (34) \end{aligned}$$

Since we have two elements from the domain which map to the image of f, f cannot be injective.

For surjectivity, remember that $\mathbb{R} \subset \mathbb{C}$. Notice that over the real numbers, $f(z)$ has a minimum at $z = -1$ and also that f(z) is a parabola in the real plane. Then it is impossible for $f(z) < 2$ and as $f(z)$ does not reach every element in the image, f cannot be surjective.

We can conclude that f is neither injective or surjective.
∎

Claim: g is injective but not surjective.

Proof: Suppose $g(a) = g(b)$, where $a, b \in \mathbb{Z}$ then $(a, (a+1)^2) = (b, (b+1)^2)$, this immediately implies that $a = b$ by the first entries in the ordered pairs.Thus, g is injective.

g is not surjective however. Consider $g(n) = (0, 0)$. Notice that $(0, 0) \in \mathbb{Z} \times \mathbb{Z}$ but there is no such $n \in \mathbb{Z}$ such that $g(n) = (0, 0)$. Thus, g is not surjective. ∎

6) Show that the following conditions are equivalent for a mapping $f : A \to B$, where $A, B \neq \emptyset$

- **f is injective.**
- **There is a $g : B \to A$ such that $g \circ f = 1_A$**
- **If $h_1 : C \to A$ and $h_2 : C \to A$ satisfy if $f \circ h_1 = f \circ h_2$ then $h_1 = h_2$.**

Claim: f is injective $\equiv$ There is a $g : B \to A$ such that $g \circ f = 1_A \equiv$ If $h_1 : C \to A$ and $h_2 : C \to A$ satisfy if $f \circ h_1 = f \circ h_2$ then $h_1 = h_2$

Proof: Suppose f is injective, then if $f(a) = f(b)$ then $a = b$. This implies that there is a way to back from the image to the domain as every element from the image maps to only one element from the domain. Then there must exist *some* function $g : B \to A$ such that $g(f(a)) = a$. In other words, there is a function $g : B \to A$ such that $g \circ f = 1_A$. Lastly, notice that in tandem a and b implies that if $h_1 : C \to A$ and $h_2 : C \to A$ satisfy if $f \circ h_1 = f \circ h_2$ then $h_1 = h_2$. If $f \circ h_1 = f \circ h_2$, then $h_1 = g \circ f(h_1) = g \circ f(h_2) = h_2$. Therefore, $h_1 = h_2$.

We can conclude that our claim is true and that all three are equivalent. ∎

7) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 5n^2 + 1$. Define a binary relation $\approx$ on $\mathbb{Z} \times \mathbb{Z}$ by $m \approx n$ just in case $f(m) = f(n)$. Show that $\approx$ is an equivalence relation and determine the partition associated to it.

Claim: $\approx$ is an equivalence relation with a partition set $\mathbb{P} = \{\{\mathbb{O}, (\mathbb{Z} \times \mathbb{Z}) \setminus \mathbb{O}\}\}$, where $\mathbb{O} = \{(x, -x) | x \in \mathbb{Z}\}$

Proof: To show that $\approx$ is an equivalence relation we must show that $\approx$ is reflexive, symmetric, and transitive.

Reflexive: Suppose $n \in \mathbb{Z}$

$$f(n) = 5n^2 + 1 \tag{35}$$
$$5n^2 + 1 = 5n^2 + 1 \tag{36}$$
$$5n^2 = 5n^2 \tag{37}$$
$$n = n \tag{38}$$
$$f(n) = f(n) \tag{39}$$

So we can conclude that $\approx$ is reflexive.
Symmetric: Suppose $m, n \in \mathbb{Z}$ such that $m \approx n$

$$f(m) = f(n) \tag{40}$$
$$5m^2 + 1 = 5n^2 + 1 \tag{41}$$
$$5m^2 = 5n^2 \tag{42}$$
$$m^2 = n^2 \tag{43}$$
$$m = n \tag{44}$$

Similarly:

$$n = m \tag{45}$$
$$n^2 = m^2 \tag{46}$$
$$5n^2 = 5m^2 \tag{47}$$
$$5n^2 + 1 = 5m^2 + 1 \tag{48}$$
$$f(n) = f(m) \tag{49}$$
$$n \approx m \tag{50}$$

So we can conclude that $\approx$ is symmetric.
Transitive: Suppose $l, m, n \in \mathbb{Z}$ and that $l \approx m, m \approx n$

$$l \approx m \tag{51}$$
$$m \approx n \tag{52}$$
$$5l^2 + 1 = 5m^2 + 1, 5m^2 + 1 = 5n^2 + 1 \tag{53}$$
$$5l^2 + 1 = 5n^2 + 1 \tag{54}$$
$$5l^2 = 5n^2 \tag{55}$$
$$f(l) = f(n) \tag{56}$$
$$l \approx n \tag{57}$$

So we can conclude that $\approx$ is transitive.

With the proof of $\approx$ being reflexive, symmetric, and transitive then $\approx$ is an equivalence relation by definition. The last remark is on the partition $\mathbb{P}$. We have that each positive integer is related to its negative counter part in unordered pairs (x,-x). Due to the property of symmetry $\rightarrow$ (x, -x) $\equiv$ (-x,x). So it must be partitioned in such a way that $\mathbb{O}$ contains all the unordered pairs (x,-x) and $(\mathbb{Z} \times \mathbb{Z}) \setminus \mathbb{O}$ contains everything else. ∎

8) Use mathematical induction to prove that $1^3 + 2^3 + \ldots + n^3 = (\frac{n(n+1)}{2})^2$ **for all** $n \geq 1$**.**

Claim: $1^3 + 2^3 + \ldots + n^3 = (\frac{n(n+1)}{2})^2$, $\forall n \geq 1$.

Proof By Mathematical Induction on n: Base Case: Let $n = 1$, then $1 = 1^3 = (\frac{n(n+1)}{2})^2 = (\frac{1(1+1)}{2})^2 = (\frac{2}{2})^2 = 1^2 = 1$.

Inductive Step: Suppose $1^3 + \ldots + k^3 = (\frac{k(k+1)}{2})^2$ holds for $k \geq 1$. We wish to show that $1^3 + \ldots + (k+1)^3 = (\frac{(k+1)(k+2)}{2})^2$.

$$1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = (\frac{k(k+1)}{2})^2 + (k+1)^3 \tag{58}$$

$$= (\frac{k^2+k}{2})^2 + (k+1)^3 \tag{59}$$

$$= \frac{k^4+2k^3+k^2}{4} + (k+1)^3 \tag{60}$$

$$= \frac{k^4+2k^3+k^2}{4} + (k^3+3k^2+3k+1) \tag{61}$$

$$= \frac{k^4+2k^3+k^2}{4} + \frac{4}{4} * (k^3+3k^2+3k+1) \tag{62}$$

$$= \frac{k^4+6k^3+13k^2+12k+4}{4} \tag{63}$$

$$= \frac{(k+1)(k^3+5k^2+8k+4}{4} \tag{64}$$

$$= \frac{(k+1)(k+1)(k^2+4k+4}{4} \tag{65}$$

$$= \frac{(k+1)^2(k+2)^2}{4} \tag{66}$$

$$= (\frac{(k+1)(k+2)}{2})^2 \tag{67}$$

We can conclude that $1^3 + 2^3 + \ldots + n^3 = (\frac{n(n+1)}{2})^2$ for all $n \geq 1$. ∎

9) Prove that $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} < 2$. (Hint: try it first then try proving the stronger statement: $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} \leq 2 - \frac{1}{2^n}$

Claim: $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} < 2$.

Proof: Notice that:

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} \tag{68}$$

can be rewritten as the following geometric series:

$$1 + \sum_{n=1}^{\infty} \frac{1}{2^n} \tag{69}$$

We can see that the limit of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is 1 from:

$$\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1. \tag{70}$$

So then if $\sum_{n=1}^{\infty} \frac{1}{2^n}$ comes infinitely close to 1, but never actually getting there. It makes sense that adding a definite 1 to a series which comes arbitrarily close to 1 will then arbitrarily come close to 2. We can verify this by taking the limit of $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2 \tag{71}$$

We can then see that:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} < 2 \tag{72}$$

$$1 + \sum_{n=1}^{\infty} \frac{1}{2^n} < 2 \tag{73}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} < 2 \tag{74}$$

We can conclude that $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} < 2$. ∎
Now we will attempt to prove the stronger claim listed above.

Claim: $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} \leq 2 - \frac{1}{2^n}$

Proof By Mathematical Induction on n: We already have shown that: $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} < 2$. So now we will show $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} \leq 2 - \frac{1}{2^n}$.
Base Case: Let n = 0, then $\frac{1}{2^0} \leq 2 - \frac{1}{2^0} = 1 \leq 2 - 1 = 1 \leq 1$.
Inductive Step: Suppose that $1 + \frac{1}{2} + \ldots + \frac{1}{2^k} < 2 - \frac{1}{2^k}$ for $k \geq 0$.

$$1 + \frac{1}{2} + \ldots + \frac{1}{2^k} \leq 2 - \frac{1}{2^k} \tag{75}$$

$$1 + \frac{1}{2} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \leq 2 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \tag{76}$$

$$\leq 2 - \frac{1}{2^k} * (1 + \frac{1}{2}) \tag{77}$$

$$\leq 2 - \frac{1}{2^k} * \frac{3}{2} \tag{78}$$

$$\leq 2 - \frac{3}{2^{k+1}} \tag{79}$$

Then we can see that $1 + \frac{1}{2} + \ldots + \frac{1}{2^{k+1}} \leq 2 - \frac{3}{2^{k+1}}$. And since $2 - \frac{1}{2^{k+1}} < 2 - \frac{3}{2^{k+1}}$ we can conclude that $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{k+1}} < 2 - \frac{1}{2^{k+1}}$. A walk-through of the steps is required; (52) is a statement of the Inductive Hypothesis. (53) we have added a similar term to both sides of the inequality. (54) Factoring of a common term. (55) Combining terms together.

We can conclude that $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} \leq 2 - \frac{1}{2^n}$. ∎

10) Define a sequence by the recurrence relation $a_0 = 2, an + 1 = 2an + 1$. Write down the first 7 terms of the sequence, and try to guess a closed form for the sequence. Use mathematical induction to prove that your answer is correct.

- $a_0 = 2$
- $a_1 = 5$
- $a_2 = 11$
- $a_3 = 23$
- $a_4 = 47$
- $a_5 = 95$
- $a_6 = 191$

Claim: $a_n = 3 * 2^n - 1, n \geq 0, n \in \mathbb{Z}$

Proof By Mathematical Induction on n: Base Case: Let n = 0, then $a_0 = 3 * 2^0 - 1 = 3 * 1 - 1 = 2$.
Inductive Step: Suppose that $a_k = 3 * 2^k - 1$ for some $k \in \mathbb{Z}, k \geq 0$. We wish to show that $a_{k+1} = 3 * 2^{k+1} - 1$.

$$a_{k+1} = 2a_k + 1 \tag{80}$$
$$= 2(3 * 2^k - 1) + 1 \tag{81}$$
$$= 3 * 2^{k+1} - 2 + 1 \tag{82}$$
$$= 3 * 2^{k+1} - 1 \tag{83}$$

We can conclude that for $n \geq 0, a_n = 3 * 2^n - 1$. ∎