Decidability & Acceptability

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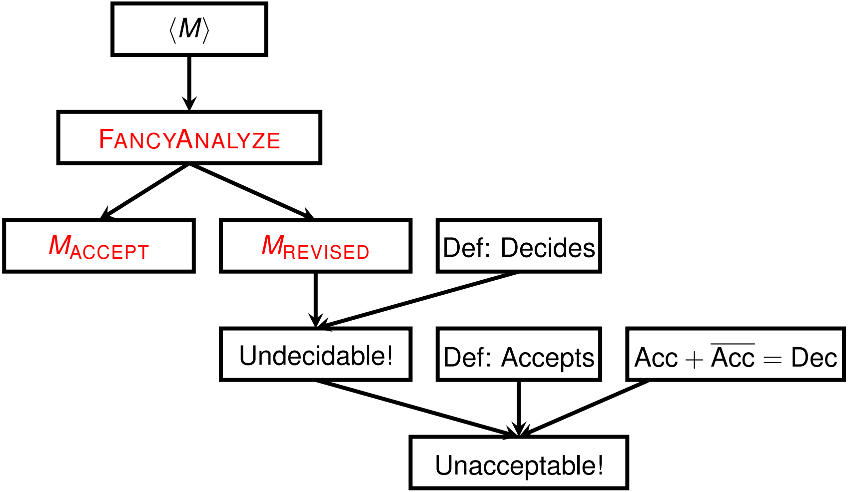
The Plan



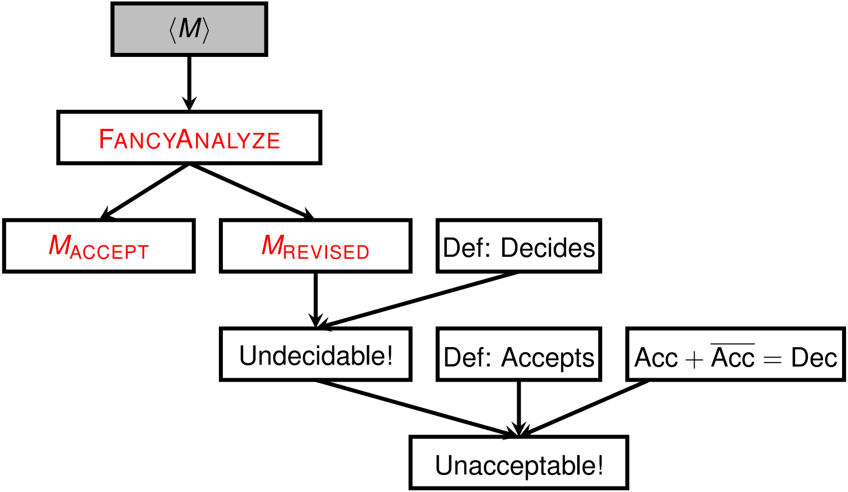
$\langle M \rangle$
FancyAnalyze
M_{ACCEPT}
M_{REVISED}
Def: Decides
Undecidable!
Def: Accepts
$\text{Acc} + \overline{\text{Acc}} = \text{Dec}$
Unacceptable!

String encodings of Turing Machines

First, we must allow a Turing Machine M to be encoded into a string $\langle M \rangle$.

Once encoded, it can be the *input* of a Turing Machine.

The Plan



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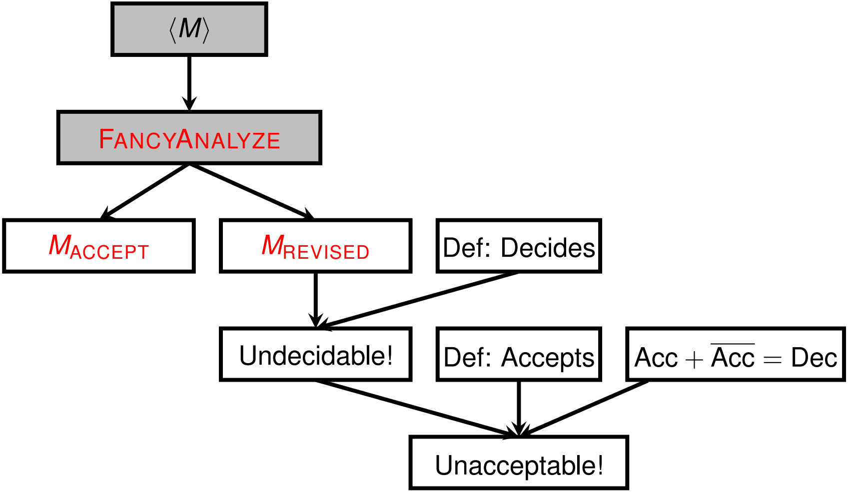
A fancy analyzer of Turing Machines

Suppose we wrote a function, FANCYANALYZE($\langle M \rangle$,w), that could do the following:

- Analyze how M (described by the string $\langle M \rangle$) behaves on input w. (Don't actually simulate M!)
- If M *accepts* w, then FANCYANALYZE returns *accept*.
- If M *rejects* w, then FANCYANALYZE returns *reject*.
- If M can be shown to get stuck in a loop on w, and therefore never accepts or rejects, then FANCYANALYZE returns *loops*.

Could a fancy programmer write FANCYANALYZE . . . ?

The Plan



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Using FANCYANALYZE to create a self-decider

M_{ACCEPT} identifies (decides) the class of "machines that accept their own description."

function $M_{\text{ACCEPT}}(\langle M \rangle)$
  Answer $\leftarrow$ FANCYANALYZE($\langle M \rangle$, $\langle M \rangle$)
  if Answer $=$ *accept* **then**
    return *accept*
  else if Answer $=$ *reject* or Answer $=$ *loops* **then**
    return *reject*
  end if
end function

A quick self-check

What does M_{ACCEPT} do on input $\langle M_{\text{ACCEPT}} \rangle$?

$M_{\text{ACCEPT}}(\langle M_{\text{ACCEPT}} \rangle) =$

- If $\textsc{FancyAnalyze}(\langle M_{\text{ACCEPT}} \rangle, \langle M_{\text{ACCEPT}} \rangle)$ returns *accept*, then $M_{\text{ACCEPT}}(\langle M_{\text{ACCEPT}} \rangle)$ returns *accept*.
- If $\textsc{FancyAnalyze}(\langle M_{\text{ACCEPT}} \rangle, \langle M_{\text{ACCEPT}} \rangle)$ returns *reject* or ~~*loop*~~, then $M_{\text{ACCEPT}}(\langle M_{\text{ACCEPT}} \rangle)$ returns *reject*.

Note: $\textsc{FancyAnalyze}(\langle M_{\text{ACCEPT}} \rangle, \langle M_{\text{ACCEPT}} \rangle)$ won't return *loop* because M_{ACCEPT} is a decider.

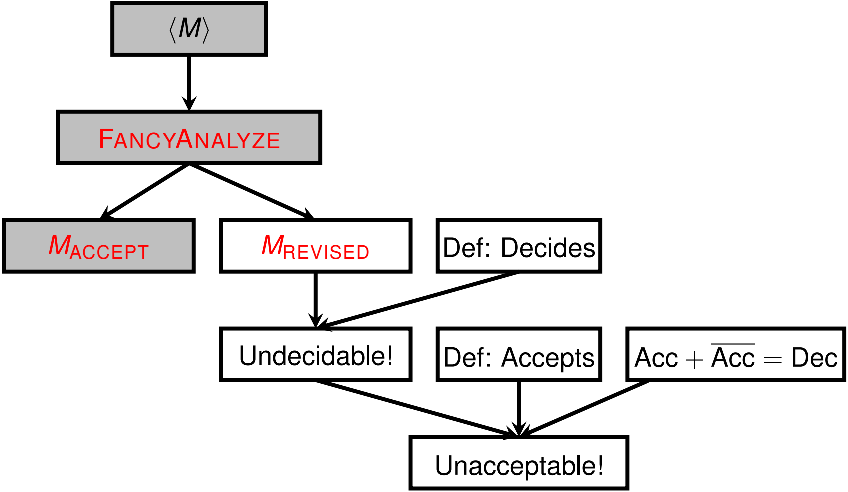
A quick self-check (in other words)

What does M_{ACCEPT} do on input $\langle M_{\text{ACCEPT}} \rangle$?

$M_{\text{ACCEPT}}(\langle M_{\text{ACCEPT}} \rangle) =$

- If M_{ACCEPT} accepts its own description, then it accepts its own description (duh).
- If M_{ACCEPT} rejects its own description, then it rejects its own description (duh).

The Plan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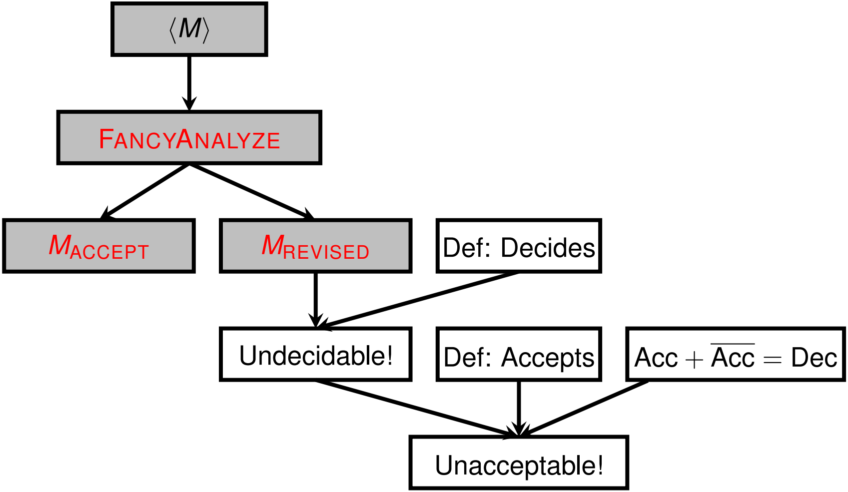
A backwards self-decider (why not?)

function $M_{\text{REVISED}}(\langle M \rangle)$
    Answer $\leftarrow$ FANCYANALYZE$(\langle M \rangle, \langle M \rangle)$
    if Answer $=$ *accept* **then**
        return *reject* $\triangleright$ opposite!
    else if Answer $=$ *reject* or Answer $=$ *loops* **then**
        return *accept* $\triangleright$ opposite!
    end if
end function

If you believe that we could implement M_{ACCEPT}, then you have to allow us to implement M_{REVISED}.

The Plan



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The sucker punch

What does M_{REVISED} do on input $\langle M_{\text{REVISED}} \rangle$?

$M_{\text{REVISED}}(\langle M_{\text{REVISED}} \rangle) =$

- If FANCYANALYZE($\langle M_{\text{REVISED}} \rangle$, $\langle M_{\text{REVISED}} \rangle$) returns *accept*, then $M_{\text{REVISED}}(\langle M_{\text{REVISED}} \rangle)$ returns *reject*!
- If FANCYANALYZE($\langle M_{\text{REVISED}} \rangle$, $\langle M_{\text{REVISED}} \rangle$) returns *reject*, then $M_{\text{REVISED}}(\langle M_{\text{REVISED}} \rangle)$ returns *accept*!

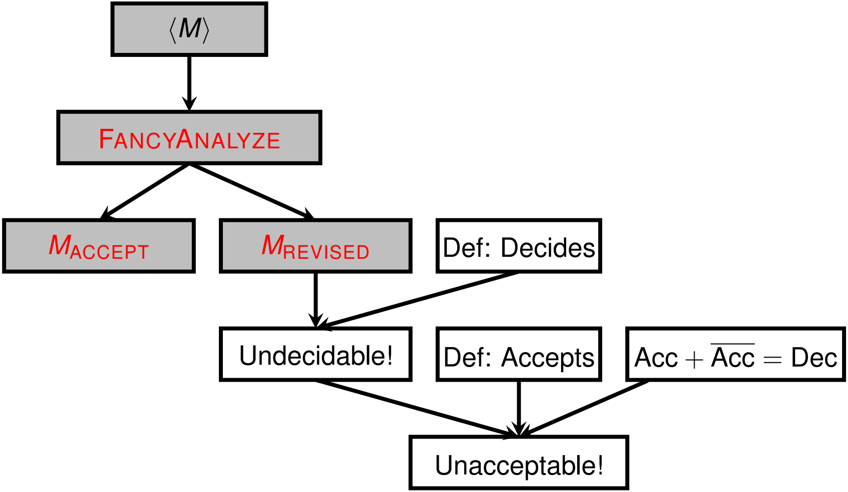
The sucker punch (in other words)

What does M_{REVISED} do on input $\langle M_{\text{REVISED}} \rangle$?

$M_{\text{REVISED}}(\langle M_{\text{REVISED}} \rangle) =$

- If M_{REVISED} accepts its own description, then it rejects its own description!
- If M_{REVISED} rejects its own description, then it accepts its own description!

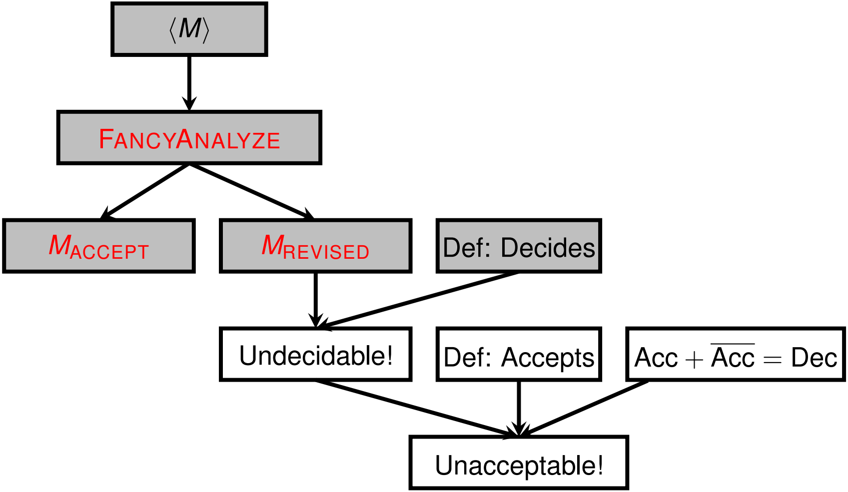
What went wrong?



Definition: Decides

foobar

The Plan



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Undecidable!

So FANCYANALYZE cannot exist (as a decider). The *language* it would have decided was,

$$\text{ACCEPT}_{\text{TM}} = \{(\langle M\rangle, w) : M \text{ accepts } w\}.$$

However, $\text{ACCEPT}_{\text{TM}}$ is easy to *accept*. Just literally simulate M on w (if M accepts w, so does the simulator).

The Halting Problem

FANCYANALYZE($\langle M \rangle$, w) could have been written differently:

- If M halts on w (accepts or rejects w), then FANCYANALYZE halts.
- Otherwise, loop forever.

The Halting Problem

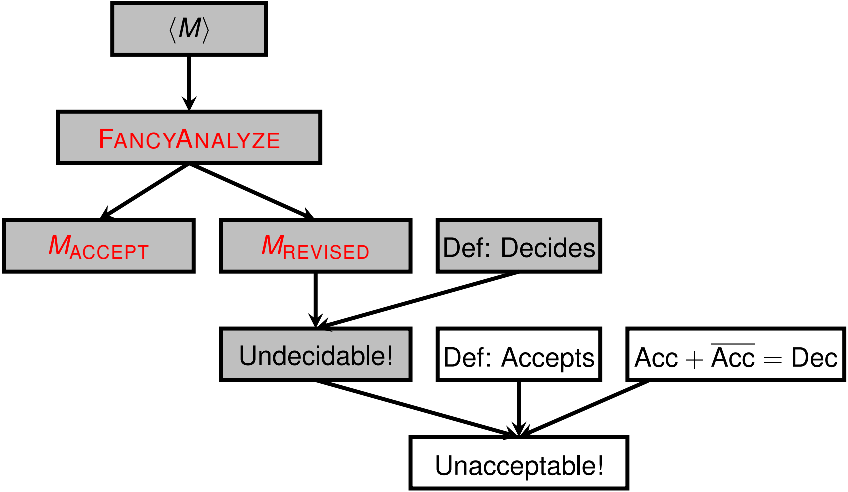
Then we make a variant, WACKOANALYZE, that does the opposite (loops forever if M halts, and halts if M loops forever).

Now, WACKOANALYZE, when executed on its own description, halts when it loops forever and loops forever when it halts (!).

Undecidable!

Either way, we have an undecidable problem (i.e., undecidable language).

The Plan



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Definition: Accepts

aka recognizes

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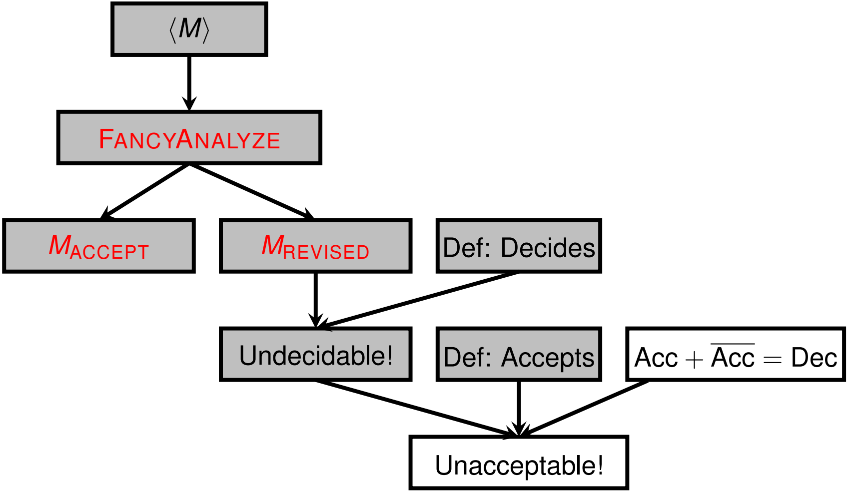

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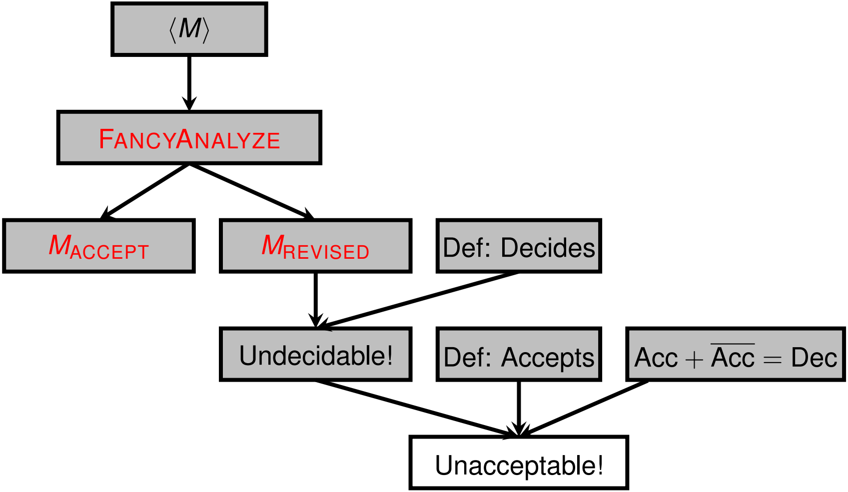

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Unacceptable!

We can also prove there exists a language that cannot be accepted (recognized).

It's simply,

$$\overline{\text{ACCEPT}_{\text{TM}}} = \{(\langle M \rangle, w) : M \text{ does not accept } w\}.$$

Unacceptable!

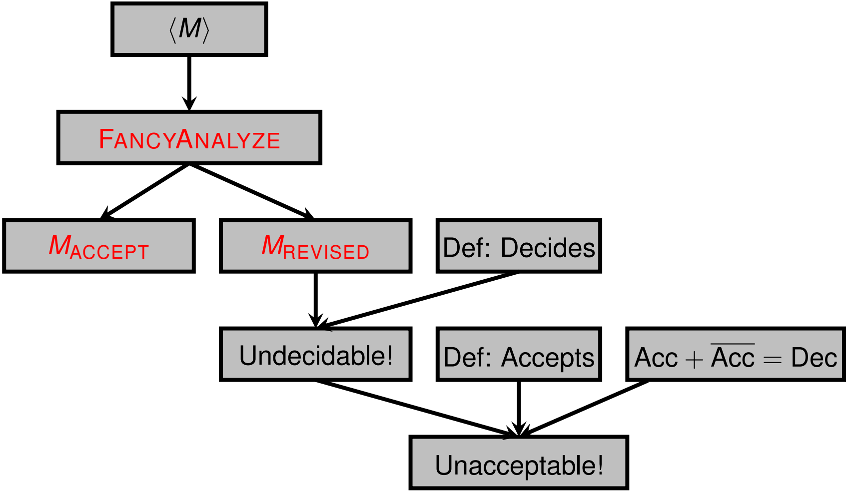
$\overline{\text{ACCEPT}_{\text{TM}}}$ is unacceptable.

Proof: Suppose it is acceptable. Then,

- $\overline{\text{ACCEPT}_{\text{TM}}}$ would be acceptable (by supposition).
- $\text{ACCEPT}_{\text{TM}}$ is acceptable (proved earlier).
- So, $\text{ACCEPT}_{\text{TM}}$ is decidable! Impossible (proved earlier).

So $\overline{\text{ACCEPT}_{\text{TM}}}$ is not acceptable.

The Plan



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Summary

So we have a language that is acceptable but not decidable:

$$\mathsf{ACCEPT}_{\mathsf{TM}} = \{(\langle M \rangle, w) : M \text{ accepts } w\}.$$

And we have a language that is not even acceptable:

$$\overline{\mathsf{ACCEPT}_{\mathsf{TM}}} = \{(\langle M \rangle, w) : M \text{ does not accept } w\}.$$

Practical implications

- You cannot write a *single* program that can check *every* program (including itself) and determine whether it is bug-free.
- Rice's theorem: There is no mechanical procedure to decide the answer to *every* non-trivial question about machines.
- More importantly: There is no mechanical procedure to decide the answer to every mathematical question about the natural numbers.