EAS-230 Spring 2018

Programming Project # 1 (PP1) – Supplementary Materials

Background, [1]:

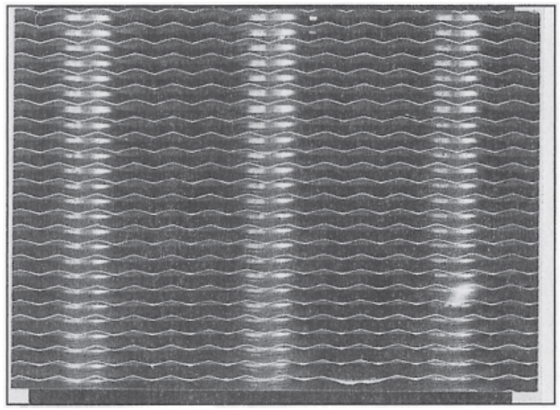


FIGURE 3–33
The thin plate fins of a car radiator greatly increase the rate of heat transfer to the air (photo by Yunus Çengel and James Kleiser).

3–6 ▪ HEAT TRANSFER FROM FINNED SURFACES

The rate of heat transfer from a surface at a temperature T_s to the surrounding medium at T_∞ is given by Newton's law of cooling as

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty)$$

where A_s is the heat transfer surface area and h is the convection heat transfer coefficient. When the temperatures T_s and T_∞ are fixed by design considerations, as is often the case, there are *two ways* to increase the rate of heat transfer: to increase the *convection heat transfer coefficient h* or to increase the *surface area A_s*. Increasing h may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate. The alternative is to increase the surface area by attaching to the surface *extended surfaces* called *fins* made of highly conductive materials such as aluminum. Finned surfaces are manufactured by extruding, welding, or wrapping a thin metal sheet on a surface. Fins enhance heat transfer from a surface by exposing a larger surface area to convection and radiation.

Finned surfaces are commonly used in practice to enhance heat transfer, and they often increase the rate of heat transfer from a surface severalfold. The car radiator shown in Fig. 3–33 is an example of a finned surface. The closely packed thin metal sheets attached to the hot water tubes increase the surface area for convection and thus the rate of convection heat transfer from the tubes to the air many times. There are a variety of innovative fin designs available in the market, and they seem to be limited only by imagination (Fig. 3–34).

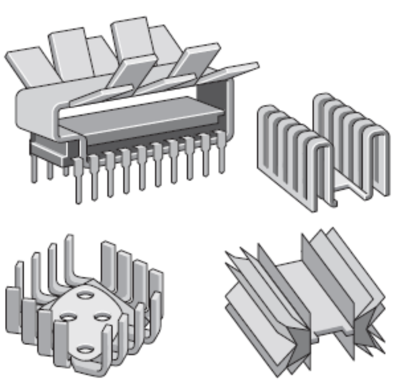


FIGURE 3–34
Some innovative fin designs.

In the analysis of fins, we consider *steady* operation with *no heat generation* in the fin, and we assume the thermal conductivity k of the material to remain constant. We also assume the convection heat transfer coefficient h to be *constant* and *uniform* over the entire surface of the fin for convenience in the analysis. We recognize that the convection heat transfer coefficient h, in general, varies along the fin as well as its circumference, and its value at a point is a strong function of the *fluid motion* at that point. The value of h is usually much lower at the *fin base* than it is at the *fin tip* because the fluid is surrounded by solid surfaces near the base, which seriously disrupt its motion to the point of "suffocating" it, while the fluid near the fin tip has little contact with a solid surface and thus encounters little resistance to flow. Therefore, adding too many fins on a surface may actually decrease the overall heat transfer when the decrease in h offsets any gain resulting from the increase in the surface area.

Fin Equation

Consider a volume element of a fin at location x having a length of Δx, cross-sectional area of A_c, and a perimeter of p, as shown in Fig. 3–35. Under steady conditions, the energy balance on this volume element can be expressed as

$$\left(\begin{array}{c}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ into} \\ \text{the element at } x\end{array}\right) = \left(\begin{array}{c}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ from the} \\ \text{element at } x + \Delta x\end{array}\right) + \left(\begin{array}{c}\text{Rate of } \textit{heat} \\ \textit{convection} \text{ from} \\ \text{the element}\end{array}\right)$$

or

$$\dot{Q}_{\text{cond}, x} = \dot{Q}_{\text{cond}, x + \Delta x} + \dot{Q}_{\text{conv}}$$

where

$$\dot{Q}_{\text{conv}} = h(p\,\Delta x)(T - T_\infty)$$

Substituting and dividing by Δx, we obtain

$$\frac{\dot{Q}_{\text{cond}, x+\Delta x} - \dot{Q}_{\text{cond}, x}}{\Delta x} + hp(T - T_\infty) = 0 \qquad \textbf{(3-52)}$$

Taking the limit as $\Delta x \to 0$ gives

$$\frac{d\dot{Q}_{\text{cond}}}{dx} + hp(T - T_\infty) = 0 \qquad \textbf{(3-53)}$$

From Fourier's law of heat conduction we have

$$\dot{Q}_{\text{cond}} = -kA_c \frac{dT}{dx} \qquad \textbf{(3-54)}$$

where A_c is the cross-sectional area of the fin at location x. Substitution of this relation into Eq. 3–53 gives the differential equation governing heat transfer in fins,

$$\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right) - hp(T - T_\infty) = 0 \qquad \textbf{(3-55)}$$

In general, the cross-sectional area A_c and the perimeter p of a fin vary with x, which makes this differential equation difficult to solve. In the special case of *constant cross section* and *constant thermal conductivity,* the differential equation 3–55 reduces to

$$\frac{d^2\theta}{dx^2} - a^2\theta = 0 \qquad \textbf{(3-56)}$$

where

$$a^2 = \frac{hp}{kA_c} \qquad \textbf{(3-57)}$$

and $\theta = T - T_\infty$ is the *temperature excess.* At the fin base we have $\theta_b = T_b - T_\infty$.

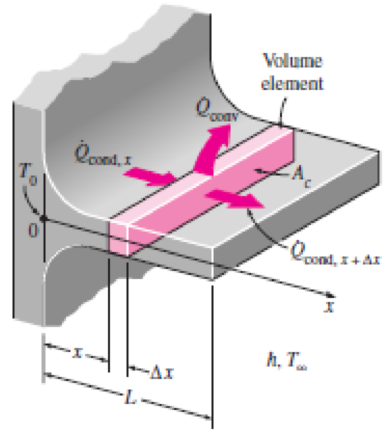


FIGURE 3–35
Volume element of a fin at location x having a length of Δx, cross-sectional area of A_c, and perimeter of p.

Fin Efficiency

Consider the surface of a *plane wall* at temperature T_b exposed to a medium at temperature T_∞. Heat is lost from the surface to the surrounding medium by convection with a heat transfer coefficient of h. Disregarding radiation or accounting for its contribution in the convection coefficient h, heat transfer from a surface area A_s is expressed as $\dot{Q} = hA_s(T_s - T_\infty)$.

Now let us consider a fin of constant cross-sectional area $A_c = A_b$ and length L that is attached to the surface with a perfect contact (Fig. 3–40). This time heat will flow from the surface to the fin *by conduction* and from the fin to the surrounding medium *by convection* with the same heat transfer coefficient h. The temperature of the fin will be T_b at the fin base and gradually decrease toward the fin tip. Convection from the fin surface causes the temperature at any cross section to drop somewhat from the midsection toward the outer surfaces. However, the cross-sectional area of the fins is usually very small, and thus the temperature at any cross section can be considered to be uniform. Also, the fin tip can be assumed for convenience and simplicity to be insulated by using the corrected length for the fin instead of the actual length.

In the limiting case of *zero thermal resistance* or *infinite thermal conductivity* ($k \to \infty$), the temperature of the fin will be uniform at the base value of T_b. The heat transfer from the fin will be *maximum* in this case and can be expressed as

$$\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}(T_b - T_\infty) \tag{3-67}$$

In reality, however, the temperature of the fin will drop along the fin, and thus the heat transfer from the fin will be less because of the decreasing temperature difference $T(x) - T_\infty$ toward the fin tip, as shown in Fig. 3–41. To account for the effect of this decrease in temperature on heat transfer, we define a **fin efficiency** as

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\begin{array}{c}\text{Ideal heat transfer rate from the fin}\\ \text{if the entire fin were at base temperature}\end{array}} \tag{3-68}$$

or

$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}}\dot{Q}_{\text{fin, max}} = \eta_{\text{fin}}hA_{\text{fin}}(T_b - T_\infty) \tag{3-69}$$

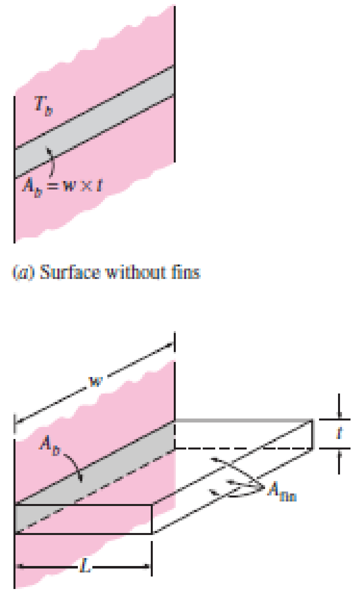
where A_{fin} is the total surface area of the fin. This relation enables us to determine the heat transfer from a fin when its efficiency is known. For the cases of constant cross section of *very long fins* and *fins with insulated tips,* the fin efficiency can be expressed as

$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{\text{fin}}(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{aL} \tag{3-70}$$

and

$$\eta_{\text{insulated tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{hA_{\text{fin}}(T_b - T_\infty)} = \frac{\tanh aL}{aL} \tag{3-71}$$

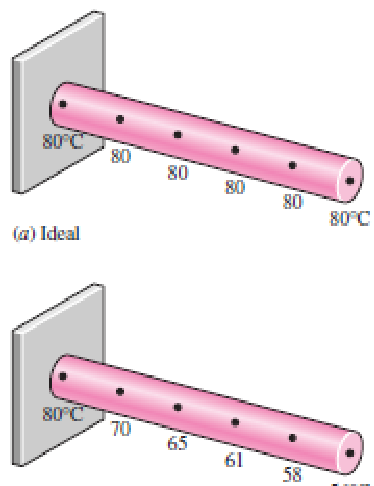
since $A_{\text{fin}} = pL$ for fins with constant cross section. Equation 3–71 can also be used for fins subjected to convection provided that the fin length L is replaced by the corrected length L_c.



(b) Surface with a fin

$A_{\text{fin}} = 2 \times w \times L + w \times t$
$\cong 2 \times w \times L$

FIGURE 3–40
Fins enhance heat transfer from a surface by enhancing surface area.



(b) Actual

FIGURE 3–41
Ideal and actual temperature distribution in a fin.

[1] Y. Cengel, "Heat Transfer – A Practical Approach," 2nd edition, McGraw-Hill.

Fin efficiency relations are developed for fins of various profiles and are plotted in Fig. 3–42 for fins on a *plain surface* and in Fig. 3–43 for *circular fins* of constant thickness. The fin surface area associated with each profile is also given on each figure. For most fins of constant thickness encountered in practice, the fin thickness t is too small relative to the fin length L, and thus the fin tip area is negligible.

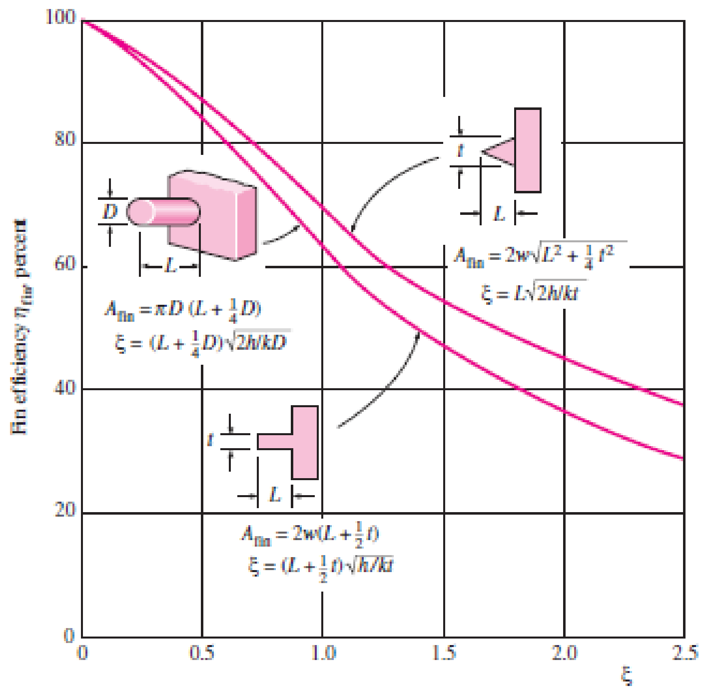


FIGURE 3–42
Efficiency of circular, rectangular, and triangular fins on a plain surface of width w (from Gardner, Ref. 6).

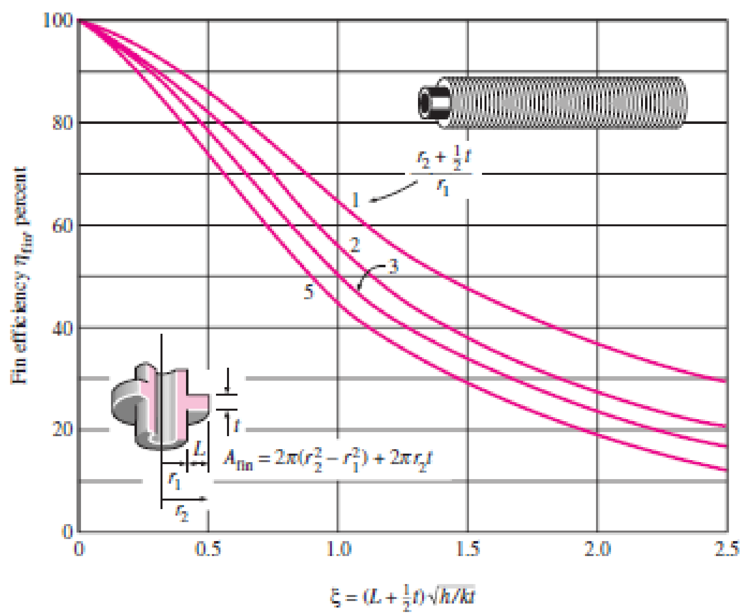


FIGURE 3–43
Efficiency of circular fins of length L and constant thickness t (from Gardner, Ref. 6).

Numerical Method in Heat Conduction:

5–3 ▪ ONE-DIMENSIONAL STEADY HEAT CONDUCTION

In this section we will develop the finite difference formulation of heat conduction in a plane wall using the energy balance approach and discuss how to solve the resulting equations. The **energy balance method** is based on *subdividing* the medium into a sufficient number of volume elements and then applying an *energy balance* on each element. This is done by first *selecting* the nodal points (or nodes) at which the temperatures are to be determined and then *forming elements* (or control volumes) over the nodes by drawing lines through the midpoints between the nodes. This way, the interior nodes remain at the middle of the elements, and the properties *at the* node such as the temperature and the rate of heat generation represent the *average* properties of the element. Sometimes it is convenient to think of temperature as varying *linearly* between the nodes, especially when expressing heat conduction between the elements using Fourier's law.

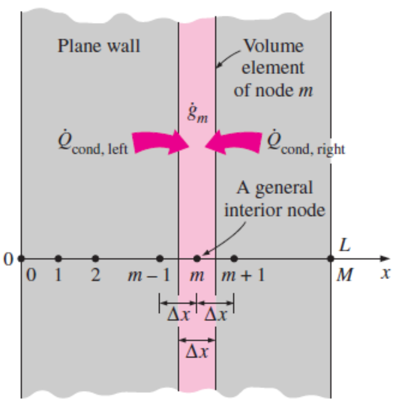


FIGURE 5–10
The nodal points and volume elements for the finite difference formulation of one-dimensional conduction in a plane wall.

To demonstrate the approach, again consider steady one-dimensional heat transfer in a plane wall of thickness L with heat generation $\dot{g}(x)$ and constant conductivity k. The wall is now subdivided into M equal regions of thickness $\Delta x = L/M$ in the x-direction, and the divisions between the regions are selected as the nodes. Therefore, we have $M + 1$ nodes labeled 0, 1, 2, . . . , $m - 1$, m, $m + 1$, . . . , M, as shown in Figure 5–10. The x-coordinate of any node m is simply $x_m = m\Delta x$, and the temperature at that point is $T(x_m) = T_m$. Elements are formed by drawing vertical lines through the midpoints between the nodes. Note that all interior elements represented by interior nodes are full-size elements (they have a thickness of Δx), whereas the two elements at the boundaries are half-sized.

To obtain a general difference equation for the interior nodes, consider the element represented by node m and the two neighboring nodes $m - 1$ and $m + 1$. Assuming the heat conduction to be *into* the element on all surfaces, an *energy balance* on the element can be expressed as

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at the left}\\ \text{surface}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at the right}\\ \text{surface}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

or

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, right}} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0 \tag{5-13}$$

since the energy content of a medium (or any part of it) does not change under *steady* conditions and thus $\Delta E_{\text{element}} = 0$. The rate of *heat generation* within the element can be expressed as

$$\dot{G}_{\text{element}} = \dot{g}_m V_{\text{element}} = \dot{g}_m A\Delta x \tag{5-14}$$

where $\dot{g}_m$ is the rate of heat generation per unit volume in W/m^3 evaluated at node m and treated as a constant for the entire element, and A is heat transfer area, which is simply the inner (or outer) surface area of the wall.

Recall that when temperature varies *linearly,* the steady rate of heat conduction across a plane wall of thickness L can be expressed as

$$\dot{Q}_{\text{cond}} = kA\frac{\Delta T}{L} \tag{5-15}$$

where ΔT is the temperature change across the wall and the direction of heat transfer is from the high temperature side to the low temperature. In the case of a plane wall with heat generation, the variation of temperature is not linear and thus the relation above is not applicable. However, the variation of temperature between the nodes can be *approximated* as being *linear* in the determination of heat conduction across a thin layer of thickness Δx between two nodes (Fig. 5–11). Obviously the smaller the distance Δx between two nodes, the more accurate is this approximation. (In fact, such approximations are the reason for classifying the numerical methods as approximate solution methods. In the limiting case of Δx approaching zero, the formulation becomes exact and we obtain a differential equation.) Noting that the direction of heat transfer on both surfaces of the element is assumed to be *toward* the node m, the rate of heat conduction at the left and right surfaces can be expressed as

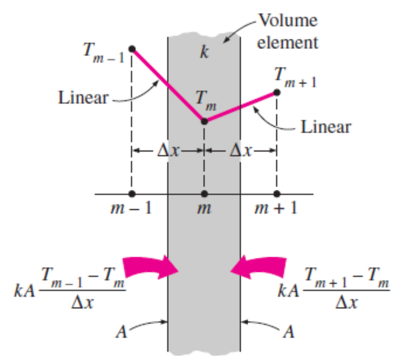


FIGURE 5–11
In finite difference formulation, the temperature is assumed to vary linearly between the nodes.

$$\dot{Q}_{\text{cond, left}} = kA\frac{T_{m-1} - T_m}{\Delta x} \quad \text{and} \quad \dot{Q}_{\text{cond, right}} = kA\frac{T_{m+1} - T_m}{\Delta x} \tag{5-16}$$

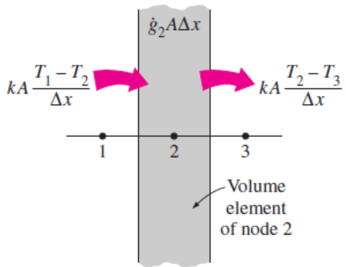
Substituting Eqs. 5–14 and 5–16 into Eq. 5–13 gives

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + \dot{g}_m A\Delta x = 0 \tag{5-17}$$

which simplifies to

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0, \qquad m = 1, 2, 3, \ldots, M - 1 \tag{5-18}$$

[1] Y. Cengel, "Heat Transfer – A Practical Approach," 2nd edition, McGraw-Hill.

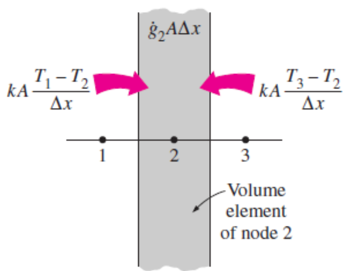


$$kA\frac{T_1 - T_2}{\Delta x} - kA\frac{T_2 - T_3}{\Delta x} + \dot{g}_2 A\Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 A\Delta x^2 / k = 0$$

(*a*) Assuming heat transfer to be out of the volume element at the right surface.



$$kA\frac{T_1 - T_2}{\Delta x} + kA\frac{T_3 - T_2}{\Delta x} + \dot{g}_2 A\Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 A\Delta x^2 / k = 0$$

(*b*) Assuming heat transfer to be into the volume element at all surfaces.

FIGURE 5–12

The assumed direction of heat transfer at surfaces of a volume element has no effect on the finite difference formulation.

which is *identical* to the difference equation (Eq. 5–11) obtained earlier. Again, this equation is applicable to each of the $M - 1$ interior nodes, and its application gives $M - 1$ equations for the determination of temperatures at $M + 1$ nodes. The two additional equations needed to solve for the $M + 1$ unknown nodal temperatures are obtained by applying the energy balance on the two elements at the boundaries (unless, of course, the boundary temperatures are specified).

You are probably thinking that if heat is conducted into the element from both sides, as assumed in the formulation, the temperature of the medium will have to rise and thus heat conduction cannot be steady. Perhaps a more realistic approach would be to assume the heat conduction to be *into* the element on the left side and *out of* the element on the right side. If you repeat the formulation using this assumption, you will again obtain the same result since the heat conduction term on the right side in this case will involve $T_m - T_{m+1}$ instead of $T_{m+1} - T_m$, which is subtracted instead of being added. Therefore, the assumed direction of heat conduction at the surfaces of the volume elements has no effect on the formulation, as shown in Figure 5–12. (Besides, the actual direction of heat transfer is usually not known.) However, it is convenient to assume heat conduction to be into the element at all surfaces and not worry about the sign of the conduction terms. Then all temperature differences in conduction relations are expressed as the temperature of the neighboring node minus the temperature of the node under consideration, and all conduction terms are added.

Boundary Conditions

Above we have developed a general relation for obtaining the finite difference equation for each interior node of a plane wall. This relation is not applicable to the nodes on the boundaries, however, since it requires the presence of nodes on both sides of the node under consideration, and a boundary node does not have a neighboring node on at least one side. Therefore, we need to obtain the finite difference equations of boundary nodes separately. This is best done by applying an *energy balance* on the volume elements of boundary nodes.

Boundary conditions most commonly encountered in practice are the *specified temperature, specified heat flux, convection,* and *radiation* boundary conditions, and here we develop the finite difference formulations for them for the case of steady one-dimensional heat conduction in a plane wall of thickness L as an example. The node number at the left surface at $x = 0$ is 0, and at the right surface at $x = L$ it is M. Note that the width of the volume element for either boundary node is $\Delta x/2$.

The **specified temperature** boundary condition is the simplest boundary condition to deal with. For one-dimensional heat transfer through a plane wall of thickness L, the *specified temperature boundary conditions* on both the left and right surfaces can be expressed as (Fig. 5–13)

$$\begin{aligned} T(0) &= T_0 = \text{Specified value} \\ T(L) &= T_M = \text{Specified value} \end{aligned} \qquad \textbf{(5-19)}$$

where T_0 and T_m are the specified temperatures at surfaces at $x = 0$ and $x = L$, respectively. Therefore, the specified temperature boundary conditions are incorporated by simply assigning the given surface temperatures to the boundary nodes. We do not need to write an energy balance in this case unless we decide to determine the rate of heat transfer into or out of the medium after the temperatures at the interior nodes are determined.

When other boundary conditions such as the *specified heat flux, convection, radiation,* or *combined convection and radiation* conditions are specified at a boundary, the finite difference equation for the node at that boundary is obtained by writing an *energy balance* on the volume element at that boundary. The energy balance is again expressed as

$$\sum_{\text{all sides}} \dot{Q} + \dot{G}_{\text{element}} = 0 \qquad \text{(5-20)}$$

for heat transfer under *steady* conditions. Again we assume all heat transfer to be *into* the volume element from all surfaces for convenience in formulation, except for specified heat flux since its direction is already specified. Specified heat flux is taken to be a *positive* quantity if into the medium and a *negative* quantity if out of the medium. Then the finite difference formulation at the node $m = 0$ (at the left boundary where $x = 0$) of a plane wall of thickness L during steady one-dimensional heat conduction can be expressed as (Fig. 5–14)

$$\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \qquad \text{(5-21)}$$

where $A\Delta x/2$ is the *volume* of the volume element (note that the boundary element has half thickness), $\dot{g}_0$ is the rate of heat generation per unit volume (in W/m^3) at $x = 0$, and A is the heat transfer area, which is constant for a plane wall. Note that we have Δx in the denominator of the second term instead of $\Delta x/2$. This is because the ratio in that term involves the temperature difference between nodes 0 and 1, and thus we must use the distance between those two nodes, which is Δx.

The finite difference form of various boundary conditions can be obtained from Eq. 5–21 by replacing $\dot{Q}_{\text{left surface}}$ by a suitable expression. Next this is done for various boundary conditions at the left boundary.

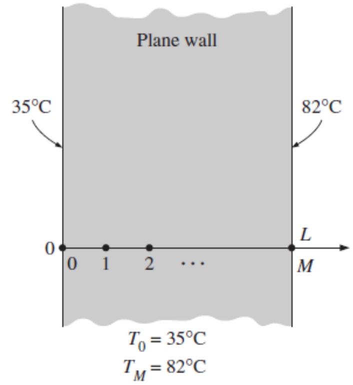
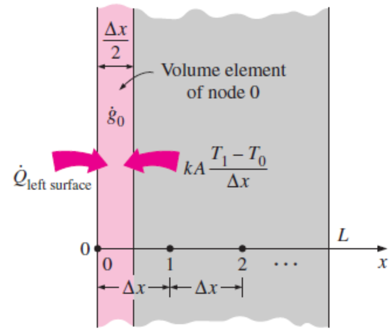


FIGURE 5–13
Finite difference formulation of specified temperature boundary conditions on both surfaces of a plane wall.



$$\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0 A\frac{\Delta x}{2} = 0$$

FIGURE 5–14
Schematic for the finite difference formulation of the left boundary node of a plane wall.

1. Specified Heat Flux Boundary Condition

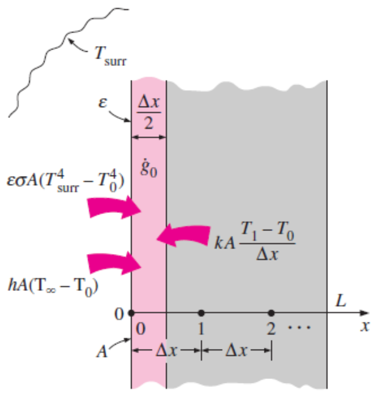
$$\dot{q}_0 A + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \qquad \text{(5-22)}$$

Special case: **Insulated Boundary** ($\dot{q}_0 = 0$)

$$kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \qquad \text{(5-23)}$$

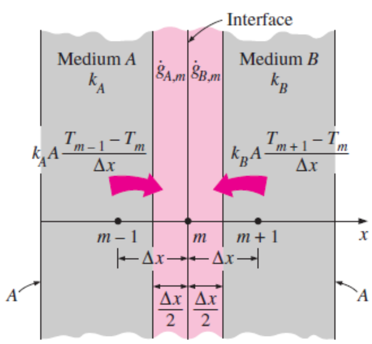
2. Convection Boundary Condition

$$hA(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \qquad \text{(5-24)}$$



$$hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0 A\frac{\Delta x}{2} = 0$$

FIGURE 5–15
Schematic for the finite difference formulation of combined convection and radiation on the left boundary of a plane wall.



$$k_A A\frac{T_{m-1} - T_m}{\Delta x} + k_B A\frac{T_{m+1} - T_m}{\Delta x} + \dot{g}_{A,m} A\frac{\Delta x}{2} + \dot{g}_{B,m} A\frac{\Delta x}{2} = 0$$

FIGURE 5–16
Schematic for the finite difference formulation of the interface boundary condition for two mediums A and B

3. Radiation Boundary Condition

$$\varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \quad \textbf{(5-25)}$$

4. Combined Convection and Radiation Boundary Condition (Fig. 5–15)

$$hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \quad \textbf{(5-26)}$$

or

$$h_{combined}\, A(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \quad \textbf{(5-27)}$$

5. Combined Convection, Radiation, and Heat Flux Boundary Condition

$$\dot{q}_0 A + hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \quad \textbf{(5-28)}$$

6. Interface Boundary Condition Two different solid media A and B are assumed to be in perfect contact, and thus at the same temperature at the interface at node m (Fig. 5–16). Subscripts A and B indicate properties of media *A and B, respectively.*

$$k_A A\frac{T_{m-1} - T_m}{\Delta x} + k_B A\frac{T_{m+1} - T_m}{\Delta x} + \dot{g}_{A,\,m}(A\Delta x/2) + \dot{g}_{B,\,m}(A\Delta x/2) = 0 \quad \textbf{(5-29)}$$

In these relations, $\dot{q}_0$ is the specified heat flux in W/m^2, h is the convection coefficient, $h_{combined}$ is the combined convection and radiation coefficient, T_∞ is the temperature of the surrounding medium, T_{surr} is the temperature of the surrounding surfaces, ε is the emissivity of the surface, and σ is the Stefan–Boltzman constant. The relations above can also be used for node M on the right boundary by replacing the subscript "0" by "M" and the subscript "1" by "$M - 1$".

Note that *absolute temperatures* must be used in radiation heat transfer calculations, and all temperatures should be expressed in K or R when a boundary condition involves radiation to avoid mistakes. We usually try to avoid the *radiation boundary condition* even in numerical solutions since it causes the finite difference equations to be *nonlinear,* which are more difficult to solve.

EXAMPLE 5–2 Heat Transfer from Triangular Fins

Consider an aluminum alloy fin (k = 180 W/m · °C) of triangular cross section with length L = 5 cm, base thickness b = 1 cm, and very large width w in the direction normal to the plane of paper, as shown in Figure 5–20. The base of the fin is maintained at a temperature of T_0 = 200°C. The fin is losing heat to the surrounding medium at T_∞ = 25°C with a heat transfer coefficient of h = 15 W/m² · °C. Using the finite difference method with six equally spaced nodes along the fin in the x-direction, determine (*a*) the temperatures at the nodes, (*b*) the rate of heat transfer from the fin for w = 1 m, and (*c*) the fin efficiency.

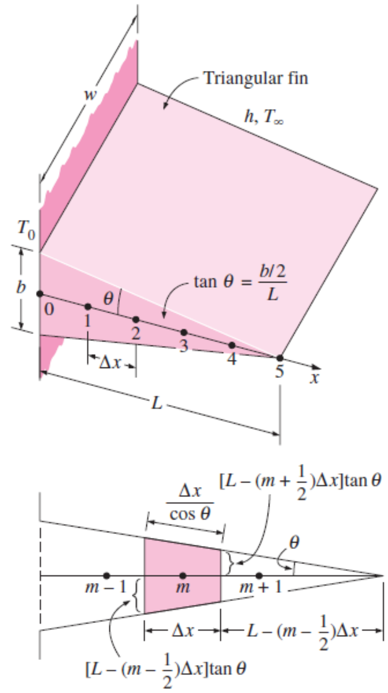


FIGURE 5–20
Schematic for Example 5–2 and the volume element of a general interior node of the fin.

SOLUTION A long triangular fin attached to a surface is considered. The nodal temperatures, the rate of heat transfer, and the fin efficiency are to be determined numerically using six equally spaced nodes.

Assumptions **1** Heat transfer is steady since there is no indication of any change with time. **2** The temperature along the fin varies in the x direction only. **3** Thermal conductivity is constant. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be k = 180 W/m · °C.

Analysis (*a*) The number of nodes in the fin is specified to be M = 6, and their location is as shown in the figure. Then the nodal spacing Δx becomes

$$\Delta x = \frac{L}{M-1} = \frac{0.05 \text{ m}}{6-1} = 0.01 \text{ m}$$

The temperature at node 0 is given to be T_0 = 200°C, and the temperatures at the remaining five nodes are to be determined. Therefore, we need to have five equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and the finite difference formulation for a general interior node m is obtained by applying an energy balance on the volume element of this node. Noting that heat transfer is steady and there is no heat generation in the fin and assuming heat transfer to be into the medium at all sides, the energy balance can be expressed as

$$\sum_{\text{all sides}} \dot{Q} = 0 \quad \rightarrow \quad kA_{\text{left}}\frac{T_{m-1}-T_m}{\Delta x} + kA_{\text{right}}\frac{T_{m+1}-T_m}{\Delta x} + hA_{\text{conv}}(T_\infty - T_m) = 0$$

Note that heat transfer areas are different for each node in this case, and using geometrical relations, they can be expressed as

$$A_{\text{left}} = (\text{Height} \times \text{Width})_{@m-\frac{1}{2}} = 2w[L-(m-1/2)\Delta x]\tan\theta$$
$$A_{\text{right}} = (\text{Height} \times \text{Width})_{@m+\frac{1}{2}} = 2w[L-(m+1/2)\Delta x]\tan\theta$$
$$A_{\text{conv}} = 2 \times \text{Length} \times \text{Width} = 2w(\Delta x/\cos\theta)$$

Substituting,

$$2kw[L-(m-\tfrac{1}{2})\Delta x]\tan\theta\,\frac{T_{m-1}-T_m}{\Delta x}$$
$$+\ 2kw[L-(m+\tfrac{1}{2})\Delta x]\tan\theta\,\frac{T_{m+1}-T_m}{\Delta x} + h\frac{2w\Delta x}{\cos\theta}(T_\infty - T_m) = 0$$

Dividing each term by $2kwL \tan\theta/\Delta x$ gives

$$\left[1-(m-\tfrac{1}{2})\frac{\Delta x}{L}\right](T_{m-1}-T_m)+\left[1-(m+\tfrac{1}{2})\frac{\Delta x}{L}\right](T_{m+1}-T_m)$$
$$+\frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty-T_m)=0$$

Note that

$$\tan\theta=\frac{b/2}{L}=\frac{0.5\text{ cm}}{5\text{ cm}}=0.1 \quad\rightarrow\quad \theta=\tan^{-1}0.1=5.71^\circ$$

Also, $\sin 5.71^\circ = 0.0995$. Then the substitution of known quantities gives

$$(5.5-m)T_{m-1}-(10.00838-2m)T_m+(4.5-m)T_{m+1}=-0.209$$

Now substituting 1, 2, 3, and 4 for m results in these finite difference equations for the interior nodes:

$$m=1:\quad -8.00838T_1+3.5T_2=-900.209 \qquad (1)$$
$$m=2:\quad 3.5T_1-6.00838T_2+2.5T_3=-0.209 \qquad (2)$$
$$m=3:\quad 2.5T_2-4.00838T_3+1.5T_4=-0.209 \qquad (3)$$
$$m=4:\quad 1.5T_3-2.00838T_4+0.5T_5=-0.209 \qquad (4)$$

The finite difference equation for the boundary node 5 is obtained by writing an energy balance on the volume element of length $\Delta x/2$ at that boundary, again by assuming heat transfer to be into the medium at all sides (Fig. 5–21):

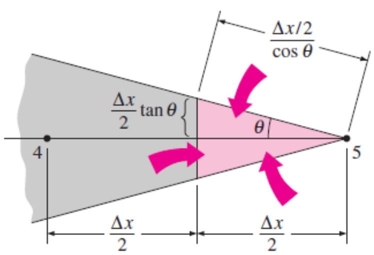


FIGURE 5–21
Schematic of the volume element of node 5 at the tip of a triangular fin.

$$kA_{\text{left}}\frac{T_4-T_5}{\Delta x}+hA_{\text{conv}}(T_\infty-T_5)=0$$

where

$$A_{\text{left}}=2w\frac{\Delta x}{2}\tan\theta \qquad \text{and} \qquad A_{\text{conv}}=2w\frac{\Delta x/2}{\cos\theta}$$

Canceling w in all terms and substituting the known quantities gives

$$T_4-1.00838T_5=-0.209 \qquad (5)$$

Equations (1) through (5) form a linear system of five algebraic equations in five unknowns. Solving them simultaneously using an equation solver gives

$$T_1=\mathbf{198.6^\circ C},\quad T_2=\mathbf{197.1^\circ C},\quad T_3=\mathbf{195.7^\circ C},$$
$$T_4=\mathbf{194.3^\circ C},\quad T_5=\mathbf{192.9^\circ C}$$

which is the desired solution for the nodal temperatures.

(*b*) The total rate of heat transfer from the fin is simply the sum of the heat transfer from each volume element to the ambient, and for $w = 1$ m it is determined from

$$\dot{Q}_{\text{fin}} = \sum_{m=0}^{5} \dot{Q}_{\text{element, }m} = \sum_{m=0}^{5} hA_{\text{conv, }m}(T_m - T_\infty)$$

Noting that the heat transfer surface area is $w\Delta x/\cos\theta$ for the boundary nodes 0 and 5, and twice as large for the interior nodes 1, 2, 3, and 4, we have

$$\begin{aligned}
\dot{Q}_{\text{fin}} &= h\frac{w\Delta x}{\cos\theta}[(T_0 - T_\infty) + 2(T_1 - T_\infty) + 2(T_2 - T_\infty) + 2(T_3 - T_\infty) \\
&\quad + 2(T_4 - T_\infty) + (T_5 - T_\infty)] \\
&= h\frac{w\Delta x}{\cos\theta}[T_0 + 2(T_1 + T_2 + T_3 + T_4) + T_5 - 10T_\infty] \\
&= (15\ \text{W/m}^2\cdot{}^\circ\text{C})\frac{(1\ \text{m})(0.01\ \text{m})}{\cos 5.71^\circ}[200 + 2\times 785.7 + 192.9 - 10\times 25] \\
&= \mathbf{258.4\ W}
\end{aligned}$$

(*c*) If the entire fin were at the base temperature of $T_0 = 200°C$, the total rate of heat transfer from the fin for $w = 1$ m would be

$$\begin{aligned}
\dot{Q}_{\max} &= hA_{\text{fin, total}}(T_0 - T_\infty) = h(2wL/\cos\theta)(T_0 - T_\infty) \\
&= (15\ \text{W/m}^2\cdot{}^\circ\text{C})[2(1\ \text{m})(0.05\ \text{m})/\cos 5.71^\circ](200 - 25)^\circ\text{C} \\
&= 263.8\ \text{W}
\end{aligned}$$

Then the fin efficiency is determined from

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\max}} = \frac{258.4\ \text{W}}{263.8\ \text{W}} = \mathbf{0.98}$$

which is less than 1, as expected. We could also determine the fin efficiency in this case from the proper fin efficiency curve in Chapter 3, which is based on the analytical solution. We would read 0.98 for the fin efficiency, which is identical to the value determined above numerically.

The finite difference formulation of steady heat conduction problems usually results in a system of N algebraic equations in N unknown nodal temperatures that need to be solved simultaneously. When N is small (such as 2 or 3), we can use the elementary *elimination method* to eliminate all unknowns except one and then solve for that unknown (see Example 5–1). The other unknowns are then determined by back substitution. When N is large, which is usually the case, the elimination method is not practical and we need to use a more systematic approach that can be adapted to computers.

There are numerous systematic approaches available in the literature, and they are broadly classified as **direct** and **iterative** methods. The direct methods are based on a fixed number of well-defined steps that result in the solution in a systematic manner. The iterative methods, on the other hand, are based on an initial guess for the solution that is refined by iteration until a specified convergence criterion is satisfied (Fig. 5–22). The direct methods usually require a large amount of computer memory and computation time,

Direct methods:
Solve in a systematic manner following a series of well-defined steps.
Iterative methods:
Start with an initial guess for the solution, and iterate until solution converges.

FIGURE 5–22
Two general categories of solution methods for solving systems of algebraic equations.