LATEX BONUS 3

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Section 4.7

40. (a) Find the transition matrix from B to B',
(b) find the transition matrix from B' to B,
(c) verify that the two transition matrices are inverses of each other, and
(d) find the coordinate matrix $[\hat{x}]_B$, given the coordinate matrix $[\hat{x}]_{B'}$.

$$B = \{(1,1,1),(1,–1,1),(0,0,1)\}\ ,\ B' = \{(2,2,0),(0,1,1),(1,0,1)\}$$

$$[\hat{x}]_{B'} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Solution.

Form matrices B and B' using the basis vectors as columns.

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & –1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B' = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

P is the transition matrix from B' to B given by the expression $P[\hat{x}]_{B'} = [\hat{x}]_B$ and P^{-1} is the transition matrix from B to B' given by the expression $P^{-1}[\hat{x}]_B = [\hat{x}]_{B'}$.

If B' is invertible then it can be reduced to the identity matrix by a series of elementary transformations. This can be expressed as

$$E_nE_{n-1}...E_2E_1B' = I$$
$$E_nE_{n-1}...E_2E_1B'(B')^{-1} = I(B')^{-1}$$
$$E_nE_{n-1}...E_2E_1I = (B')^{-1}.$$

Thus, when we construct a matrix $[B' \quad I]$ and use elementary row operations to transform the left side into the identity matrix, we have completed the following operation

$$[B' \quad I] \to [I \quad (B')^{-1}].$$

You can think of this operation as *taking I into B'*. The result is an inverse matrix $(B')^{-1}$ satisfying $B'(B')^{-1} = I$.

Finding the transition matrix from B to B' and from B' to B is related to the method of finding the inverse of B'. When we construct a matrix $[B' \quad B]$ and use elementary row operations to transform the left side into the identity matrix, we have completed the following operation

$$[B' \quad B] \rightarrow [I \quad P^{-1}].$$

In this process, we *take* B *into* B' by applying to B the operations which reduce B' to I. The resulting matrix is defined by P^{-1}. Continuing on, if we take the inverse process

$$[B \quad B'] \rightarrow [I \quad P]$$

we *take* B' *into* B by applying to B' the operations which reduce B to I. The resulting matrix is defined by P, which happens to be the inverse of P^{-1}.

(a) P^{-1} is the transition matrix from B to B' denoted by the process $[B' \quad B] \rightarrow [I \quad P^{-1}]$.

$$\begin{bmatrix} 2 & 0 & 1 & & 1 & 1 & 0 \\ 2 & 1 & 0 & & 1 & -1 & 0 \\ 0 & 1 & 1 & & 1 & 1 & 1 \end{bmatrix} \longrightarrow \textit{GJE/mathematica} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & & 1/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & & 1/2 & 3/2 & 1/2 \end{bmatrix}$$

The transition matrix P^{-1} from B to B' is $\begin{bmatrix} 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 3/2 & 1/2 \end{bmatrix}$.

(b) P is the transition matrix from B' to B denoted by the process $[B \quad B'] \rightarrow [I \quad P]$.

$$\begin{bmatrix} 1 & 1 & 0 & & 2 & 0 & 1 \\ 1 & -1 & 0 & & 2 & 1 & 0 \\ 1 & 1 & 1 & & 0 & 1 & 1 \end{bmatrix} \longrightarrow \textit{GJE/mathematica} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & & 2 & 1/2 & 1/2 \\ 0 & 1 & 0 & & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & & -2 & 1 & 0 \end{bmatrix}$$

The transition matrix P from B' to B is $\begin{bmatrix} 2 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \\ -2 & 1 & 0 \end{bmatrix}$.

(c) $PP^{-1} = \begin{bmatrix} 2 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 3/2 & 1/2 \end{bmatrix}$

$$= \begin{bmatrix} (2)(\frac{1}{4}) + (\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2}) & (2)(-\frac{1}{4}) + (\frac{1}{2})(-\frac{1}{2}) + (\frac{1}{2})(\frac{3}{2}) & (2)(-\frac{1}{4}) + (\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2}) \\ 0 + (-\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2}) & 0 + (-\frac{1}{2})(-\frac{1}{2}) + (\frac{1}{2})(\frac{3}{2}) & 0 + (-\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2}) \\ (-2)(\frac{1}{4}) + (1)(\frac{1}{2}) + 0 & (-2)(-\frac{1}{4}) + (1)(-\frac{1}{2}) + 0 & (-2)(-\frac{1}{4}) + (1)(\frac{1}{2}) + 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

(d) $P[\hat{x}]_{B'} = [\hat{x}]_B$

$$\begin{bmatrix} 2 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + 3 \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -1 \end{bmatrix}$$

$$[\hat{x}]_B = \begin{bmatrix} 6 \\ -1 \\ -1 \end{bmatrix}$$

Section 5.2

66. Show that f and g are orthogonal in the inner product space C[a,b] with the inner product

$\langle f, g\rangle = \int_a^b f(x)g(x)dx$.

$C[-1, 1]$, $f(x) = x$, $g(x) = \frac{1}{2}(3x^2 - 1)$

Solution.

Orthogonality is defined by the property $\langle v_i, v_j\rangle = 0$ where $i \neq j$.

First, we should note that $f(x) \neq g(x)$ satisfies the condition that $i \neq j$.

$\langle f, g\rangle = \int_a^b f(x)g(x)dx$

$= \int_{-1}^1 x\frac{1}{2}(3x^2 - 1)dx$

$$= \frac{1}{2}\int_{-1}^{1} x(3x^2 - 1)dx$$
$$= \frac{1}{2}\int_{-1}^{1} (3x^3 - x)dx$$

$3x^3 - x$ is an odd function and the interval $[-1, 1]$ is symmetric about the origin so its integral must be 0.

$$= \frac{1}{2}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1}$$
$$= \frac{1}{2}\left[(\frac{3}{4} - \frac{1}{2}) - (\frac{3}{4} - \frac{1}{2})\right]$$
$$= \frac{1}{2}\left[\frac{1}{4} - \frac{1}{4}\right]$$
$$= 0.$$

Section 6.1

56. Let T be a linear transformation from $M_{2,2}$ into $M_{2,2}$ such that

$$T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}, \quad T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix},$$

$$T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}, \quad T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}3 & -1\\1 & 0\end{bmatrix}.$$

Find $T\left(\begin{bmatrix}1 & 3\\-1 & 4\end{bmatrix}\right)$.

Solution.

$$T\left(\begin{bmatrix}1 & 3\\-1 & 4\end{bmatrix}\right) = (1)T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) + (3)T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) + (-1)T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) + (4)T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)$$

$$= \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix} + \begin{bmatrix}0 & 6\\3 & 3\end{bmatrix} + \begin{bmatrix}-1 & -2\\0 & -1\end{bmatrix} + \begin{bmatrix}12 & -4\\4 & 0\end{bmatrix}$$

$$= \begin{bmatrix}12 & -1\\7 & 4\end{bmatrix}$$

Section 6.3

49. Let $B = \{1, x, x^2, x^3\}$ be a basis for P_3, and $T := P_3 \to P_4$ be the linear transformation represented by

$$T(x^k) = \int_0^x t^k dt.$$

(a) Find the matrix A for T with respect to B and the standard basis for P_4.

(b) Use A to integrate $p(x) = 8 - 4x + 3x^3$.

Solution.

The transformation $T(x)$ can be represented by a transformation matrix A applied to coordinate vector $\hat{x}$. This is given by the expression $T(\mathbf{x}) = A\hat{x}$.

Let B' be the standard basis for P_4 given by $B' = \{1, x, x^2, x^3, x^4\}$.

In the case of our problem, the transformation matrix A maps the coordinate matrix $[p(x)]_B$ to $[p(x)]_{B'}$. This is given by the expression $A[p(x)]_B = [p(x)]_{B'}$. To determine the transformation matrix A we first apply $T(x^k)$ to each basis vector in B. The result of these transformations relative to B' (denoted $[T(x^k)]_{B'}$) produces vectors which we take as the columns of A.

$$T(1) = \int_0^x t^0 dt = x \qquad [T(1)]_{B'} = [0 + x + 0x^2 + 0x^3 + 0x^4]_{B'} = \{0, 1, 0, 0, 0\}$$

$$T(x) = \int_0^x t^1 dt = \frac{1}{2}x^2 \qquad [T(x)]_{B'} = [0+0x+\frac{1}{2}x^2+0x^3+0x^4]_{B'} = \{0, 0, \frac{1}{2}, 0, 0\}$$

$$T(x^2) = \int_0^x t^2 dt = \frac{1}{3}x^3 \qquad [T(x^2)]_{B'} = [0+0x+0x^2+\frac{1}{3}x^3+0x^4]_{B'} = \{0, 0, 0, \frac{1}{3}, 0\}$$

$$T(x^3) = \int_0^x t^3 dt = \frac{1}{4}x^4 \qquad [T(x^3)]_{B'} = [0+0x+0x^2+0x^3+\frac{1}{4}x^4]_{B'} = \{0, 0, 0, 0, \frac{1}{4}\}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}, \qquad [p(x)]_B = \begin{bmatrix} 8 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \qquad A[p(x)]_B = [p(x)]_{B'}$$

$$[p(x)]_{B'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 8 \\ -4 \\ 0 \\ 3 \end{bmatrix} = 8 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -2 \\ 0 \\ 3/4 \end{bmatrix}$$

$$\int_0^x p(x)dx = \int_0^x (8 - 4x + 3x^3)dx = 8x - 2x^2 + \frac{3}{4}x^4$$

Section 6.4

12. Find the matrix A' for T relative to the basis B'.

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \qquad T(x, y, z) = (x, x + 2y, x + y + 3z), \qquad B' = \{1, -1, 0), (0, 0, 1), (0, 1, -1)\}$$

Solution.

A' is a matrix *similar* to the standard transformation matrix A satisfying the expression $A' = P^{-1}AP$. To find the standard matrix for T we apply the transformation to the standard basis vectors in $\mathbb{R}^3$.

$$T(e_1) = (1, 1, 1), \qquad T(e_2) = (0, 2, 1) \qquad T(e_3) = (0, 0, 3)$$

We take these transformation vectors as the columns of the standard matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

P is the transformation matrix which maps coordinate vectors (relative to B') into B given by the expression $P[\hat{x}]_{B'} = [\hat{x}]_B$. We find P by *taking* B' *into* B using the process $[B \quad B'] \to [I \quad P]$.

$$\begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & & -1 & 0 & 1 \\ 0 & 0 & 1 & & 0 & 1 & -1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

We can find P^{-1} using the inverse process $[B' \quad B] \to [I \quad P^{-1}]$.

$$\begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ -1 & 0 & 1 & & 0 & 1 & 0 \\ 0 & 1 & -1 & & 0 & 0 & 1 \end{bmatrix} \to GJE/mathematica \to \begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & & 1 & 1 & 1 \\ 0 & 0 & 1 & & 1 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A' = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Therefore, $A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Section 7.1

46. Consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ whose matrix A relative to the standard basis is given. Find (a) the eigenvalues of A, (b) a basis for each of the corresponding eigenspaces, and (c) the matrix A' for T relative to the basis B', where B' is made up of the basis vectors found in part (b).

$$A = \begin{bmatrix} -8 & 16 \\ 1 & -2 \end{bmatrix}$$

Solution.

(a) An *eigenvalue* denoted by λ is a scalar value which, when multiplied against a **nonzero** *eigenvector* (denoted by $\hat{x}$), is equivalent to applying a transformation matrix A to the same eigenvector. These relationships are given by the expression $A\hat{x} = \lambda\hat{x}$.

From Theorem 7.2, an eigenvalue of A is a scalar λ such that

$$det(\lambda I - A) = 0 \quad \Rightarrow \quad \lambda I_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad \Rightarrow \quad det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -8 & 16 \\ 1 & -2 \end{bmatrix}\right) = 0 \quad \Rightarrow$$

$$\begin{vmatrix} \lambda+8 & -16 \\ -1 & \lambda+2 \end{vmatrix} = 0 \quad \Rightarrow \quad (\lambda+8)(\lambda+2) - (-16)(-1) = 0 \quad \Rightarrow$$

$$\lambda^2 + 10\lambda = 0 \quad \Rightarrow \quad (\lambda+0)(\lambda+10) = 0$$

Therefore, $\lambda_1 = 0$ and $\lambda_2 = -10$.

(b) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\hat{x} = \hat{0}$.

$$\lambda_1 I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -8 & 16 \\ 1 & -2 \end{bmatrix}\right)\hat{x} = \hat{0} \quad \Rightarrow \quad \begin{bmatrix} 8 & -16 \\ -1 & 2 \end{bmatrix}\hat{x} = \hat{0}$$

$$\begin{aligned} 8x_1 - 16x_2 &= 0 \\ -x_1 + 2x_2 &= 0 \end{aligned}$$

$$\left[\begin{array}{cc|c} 8 & -16 & 0 \\ -1 & 2 & 0 \end{array}\right] \to GJE/mathematica \to \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 2x_2 = 0 \quad \Rightarrow \quad x_1 = 2t\ , \ \ x_2 = t$$

Every eigenvector of λ_1 is of the form $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad t \neq 0.$

Therefore, the basis of the eigenspace of λ_1 is $B_{\lambda_1} = \{2, 1\}$.

$$\lambda_2 I_2 = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} \quad \Rightarrow \quad \left(\begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} - \begin{bmatrix} -8 & 16 \\ 1 & -2 \end{bmatrix} \right) \hat{x} = \hat{0} \quad \Rightarrow \quad \begin{bmatrix} -2 & -16 \\ -1 & -8 \end{bmatrix} \hat{x} = \hat{0}$$

$$\begin{aligned} -2x_1 - 16x_2 &= 0 \\ -x_1 - \ 8x_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} -2 & -16 & | & 0 \\ -1 & -8 & | & 0 \end{bmatrix} \to GJE/mathematica \to \begin{bmatrix} 1 & 8 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 + 8x_2 = 0 \quad \Rightarrow \quad x_1 = -8t \ , \ \ x_2 = t$$

Every eigenvector of λ_2 is of the form $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8t \\ t \end{bmatrix} = t \begin{bmatrix} -8 \\ 1 \end{bmatrix}, \quad t \neq 0.$

Therefore, the basis of the eigenspace of λ_2 is $B_{\lambda_2} = \{-8, 1\}$.

(c) Form basis B' using B_{λ_1} and B_{λ_2} as the column vectors.

$$B' = \begin{bmatrix} 2 & -8 \\ 1 & 1 \end{bmatrix}$$

$$[B \quad B'] \to [I \quad P] \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 2 & -8 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P = \begin{bmatrix} 2 & -8 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{det(P)} \begin{bmatrix} 1 & 8 \\ -1 & 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & 8 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/10 & 4/5 \\ -1/10 & 1/5 \end{bmatrix}$$

$$A' = P^{-1}AP = \begin{bmatrix} 1/10 & 4/5 \\ -1/10 & 1/5 \end{bmatrix} \begin{bmatrix} -8 & 16 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -8 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix}$$

Therefore, $A' = \begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix}$ is a diagonal matrix where $a_{1,1} = \lambda_1$ and $a_{2,2} = \lambda_2$.

Section 7.2

28. Find a basis B for the domain of T such that the matrix for T relative to B is diagonal.

$T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (-2x + 2y - 3z, 2x + y - 6z, -x - 2y)$

Solution.

We apply the transformation T to the standard basis vectors in $\mathbb{R}^3$ to produce the column vectors of the standard matrix A.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The eigenvalues of A are scalar values λ which satisfy $det(\lambda I - A) = 0$.

$$\begin{vmatrix} \lambda+2 & -2 & 3 \\ -2 & \lambda-1 & 6 \\ 1 & 2 & \lambda \end{vmatrix} = 0$$

$$(\lambda+2)(-1)^{1+1}\begin{vmatrix} \lambda-1 & 6 \\ 2 & \lambda \end{vmatrix} + (-2)(-1)^{1+2}\begin{vmatrix} -2 & 6 \\ 1 & \lambda \end{vmatrix} + (3)(-1)^{1+3}\begin{vmatrix} -2 & \lambda-1 \\ 1 & 2 \end{vmatrix} = 0$$

$$(\lambda+2)[(\lambda-1)(\lambda) - 12] + (2)(-2\lambda - 6) + (3)[-4 - (\lambda - 1)] = 0$$

$$(\lambda+2)(\lambda^2 - \lambda - 12) + (-4\lambda - 12) + (-3\lambda - 9) = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

Grouping will not help us factor this cubic polynomial. We can factor using the Rational Root Theorem because all of the coefficients are rational numbers. The Rational Root Theorem tells us that the possible roots for a given polynomial are the factors of the last term divided by the factors of the first term (denoted by $\pm\frac{factors\ of\ a_0}{factors\ of\ a_3}$).

Factors of -45: $\pm 1, 3, 5, 9, 15, 45$
Factors of 1: ± 1

Possible zeroes of $p(\lambda)$: $\pm\dfrac{1, 3, 5, 9, 15, 45}{\pm 1} = \pm 1, 3, 5, 9, 15, 45$

Now we plug in each of the possible zeroes into $p(\lambda)$ until we find a value that satisfies the equation. It turns out that $p(5) = 0$ and $p(-3) = 0$. Performing polynomial long division we determine the following

$$\frac{\lambda^3 + \lambda^2 - 21\lambda - 45}{\lambda - 5} = \lambda^2 + 6\lambda + 9 \qquad \Rightarrow \qquad (\lambda-5)(\lambda+3)(\lambda+3) = 0$$

$\lambda_1 = 5$, $\quad \lambda_2 = -3$ (*repeated*)

We find the eigenvectors by solving for $\hat{x}$ in $(\lambda I - A)\hat{x} = \hat{0}$.

$$\left(\begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix} - \begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}\right)\hat{x} = \hat{0} \quad \Rightarrow \quad \begin{bmatrix}7 & -2 & 3\\-2 & 4 & 6\\1 & 2 & 5\end{bmatrix}\hat{x} = 0$$

$$\begin{bmatrix}7 & -2 & 3 & | & 0\\-2 & 4 & 6 & | & 0\\1 & 2 & 5 & | & 0\end{bmatrix} \rightarrow GJE/mathematica \rightarrow \begin{bmatrix}1 & 0 & 1 & | & 0\\0 & 1 & 2 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix}$$

Let $x_3 = t$. Then, $x_1 = -t$, $x_2 = -2t$.

An eigenvector relative to λ_1 is always of the form $\hat{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-t\\-2t\\t\end{bmatrix} = t\begin{bmatrix}-1\\-2\\1\end{bmatrix}$, $t \neq 0$.

The basis of the eigenspace of λ_1 is $B_{\lambda_1} = \{(-1, -2, 1)\}$.

$$\left(\begin{bmatrix}-3 & 0 & 0\\0 & -3 & 0\\0 & 0 & -3\end{bmatrix} - \begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}\right)\hat{x} = \hat{0} \quad \Rightarrow \quad \begin{bmatrix}-1 & -2 & 3\\-2 & -4 & 6\\1 & 2 & -3\end{bmatrix}\hat{x} = 0$$

$$\begin{bmatrix}-1 & -2 & 3 & | & 0\\-2 & -4 & 6 & | & 0\\1 & 2 & -3 & | & 0\end{bmatrix} \rightarrow GJE/mathematica \rightarrow \begin{bmatrix}1 & 2 & -3 & | & 0\\0 & 0 & 0 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix}$$

Let $x_2 = s$ and $x_3 = t$. Then, $x_1 = -2s + 3t$.

An eigenvector relative to λ_2 is always of the form $\hat{x} = \begin{bmatrix}-2s+3t\\s\\t\end{bmatrix} = s\begin{bmatrix}-2\\1\\0\end{bmatrix} + t\begin{bmatrix}3\\0\\1\end{bmatrix}$; $s, t \neq 0$.

The basis of the eigenspace of λ_2 is $B_{\lambda_2} = \{(-2, 1, 0), (3, 0, 1)\}$.

Assuming the standard basis is denoted B', we form the basis B and construct a matrix of the same name using the basis vectors of the eigenspaces as columns.

$$B = \begin{bmatrix}-1 & -2 & 3\\-2 & 1 & 0\\1 & 0 & 1\end{bmatrix}$$

The matrix of T relative to this basis B is given by the expression $A' = P^{-1}AP$.

$[B' \quad B] \rightarrow [I \quad P^{-1}]$ Since $B' = I$ then $P^{-1} = B$.

$[B \quad B'] \rightarrow [I \quad P]$

$$\begin{bmatrix} -1 & -2 & 3 & & 1 & 0 & 0 \\ -2 & 1 & 0 & & 0 & 1 & 0 \\ 1 & 0 & 1 & & 0 & 0 & 1 \end{bmatrix} \to GJE/mathematica \to \begin{bmatrix} 1 & 0 & 0 & & -1/8 & -1/4 & 3/8 \\ 0 & 1 & 0 & & -1/4 & 1/2 & 3/4 \\ 0 & 0 & 1 & & 1/8 & 1/4 & 5/8 \end{bmatrix}$$

$$P = \begin{bmatrix} -1/8 & -1/4 & 3/8 \\ -1/4 & 1/2 & 3/4 \\ 1/8 & 1/4 & 5/8 \end{bmatrix}$$

$$A' = P^{-1}AP = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1/8 & -1/4 & 3/8 \\ -1/4 & 1/2 & 3/4 \\ 1/8 & 1/4 & 5/8 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Therefore, $A' = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a diagonal matrix where $a_{1,1} = \lambda_1$ and $a_{2,2} = a_{3,3} = \lambda_2$.