MIT 18.03 Problem Set 1B

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1 Problem 1

1.A Part a

This is a third-order, linear, inhomogenous differential equation.

1.B Part b

This is a first-order, linear, homogenous differential equation.

1.C Part c

This is a second-order, non-linear, homogenous differential equation.

1.D Part d

This is a first-order, linear, inhomogenous differential equation.

2 Problem 2

2.A Part a

$$\frac{dN(t)}{dt} = -k \cdot N(t)$$

where k is the positive proportionality constant and $\frac{dN(t)}{dt} \leq 0$ implies a decreasing concentration of ^{14}C.

2.B Part b

$$\frac{1}{N(t)} dN(t) = -kdt$$

$$\int \frac{1}{N(t)} dN(t) = -\int kdt$$

$$ln(N(t)) = -k \cdot t + C$$

$$N(t) = e^{-k \cdot t + C}$$

$$N(t) = Ce^{-k \cdot t}$$

We have the initial condition $N_0 = 1.5E - 12$. Therefore:

$$N(0) = Ce^{0}$$

$$N(0) = C$$

Therefore, we have:

$$N(t) = N(0)e^{-k \cdot t}$$

where k is the positive proportionality constant and $N_0 = 1.5E - 12$.

2.C Part c

If the half life of ^{14}C is 5730 years, then we have:

$$N(5730) = \frac{N(0)}{2}$$

This can be rewritten as:

$$N(0)e^{-k \cdot 5730} = \frac{1}{2}N(0)e^{-k \cdot 0}$$

Simplifying:

$$e^{-k \cdot 5730} = \frac{1}{2}$$

$$-k \cdot 5730 = -ln(2)$$

$$k = \frac{ln(2)}{5730}$$

2.D Part d

No. In the work for part c, we see that the proportionality constant is only dependent upon one factor in the case of half-life:

$$k = \frac{ln(2)}{T}$$

where T is the half-life. Therefore, my choice of a constant k would not change if the initial concentration of ^{14}C was greater. This makes sense as we are not actually changing the properties of the material itself.

2.E Part e

Given the ratio of concentration at t to the initial concentration, we have:

$$N(t) = 10^{-2} \cdot 1.5E - 12 = 1.5E - 14$$

Now, we can solve for t using $N(t) = N(0)e^{-\frac{\ln(2)}{5730} \cdot t}$:

$$N(0)e^{-\frac{\ln(2)}{5730} \cdot t} = 1.5E - 14$$

$$-\frac{\ln(2)}{5730} \cdot t = \ln(\frac{1.5E - 14}{1.5E - 12}) = ln(1.5E - 2)$$

$$t = -\ln(1.5E - 2)\frac{5730}{\ln(2)} = 34717.46$$

This means that the specimen is approximately 34717 years old.

3 Problem 3

3.A Part a

$$\frac{dy}{dt} = \frac{1}{1200}y^4$$

$$\int \frac{1}{y^4} = \int \frac{1}{1200}dt$$

$$-\frac{1}{3y^3} = \frac{t}{1200} + C$$

$$-3y^3 = \frac{1}{\frac{t}{1200} + C}$$

$$y^3 = \frac{1}{-\frac{t}{400} + C}$$

$$y(t) = \sqrt[3]{-\frac{1}{\frac{t}{400} + C}}$$

3.B Part b

Given that $y(0) = 2$:

$$y(0) = \sqrt[3]{\frac{1}{-\frac{0}{400} + C}}$$

$$2 = \sqrt[3]{\frac{1}{C}}$$

$$8 = \frac{1}{C}$$

$$C = \frac{1}{8}$$

Therefore, we have the specific-solution:

$$y(t) = \sqrt[3]{\frac{1}{-\frac{t}{400} + \frac{1}{8}}}$$

Now, we can solve for the inverse function, $t(y)$:

$$y = \sqrt[3]{\frac{1}{-\frac{t(y)}{400} + \frac{1}{8}}}$$

$$y^3 = \frac{1}{-\frac{t(y)}{400} + \frac{1}{8}}$$

$$\frac{1}{y^3} = -\frac{t(y)}{400} + \frac{1}{8}$$

$$\frac{1}{y^3} - \frac{1}{8} = -\frac{t(y)}{400}$$

$$\frac{400}{8} - \frac{400}{y^3} = t(y)$$

Taking the limit as $y \to \infty$:

$$\lim_{y\to\infty} \frac{400}{8} - \frac{400}{y^3} = \frac{400}{8}$$

It will take Dr.Nefario approximately 50 years to destroy the world.

4 Problem 4

4.A Part a

If $T_S(t) = T_S$ is a constant:

$$\frac{dT}{dt} = k(T_S - T)$$

By inspection, we can see that this is a differential equation where the change in temperature is proportional to the difference in the object's temperature at t and the ambient temperature, T_S. Therefore, we can have the general solution:

$$T(t) = Ce^{-kt} + T_S$$

Suppose at $t = 0$ the object has temperature $T(0) = T_0$:

$$T_0 = C + T_S$$

Where, $C = T_0 - T_S$ and:

$$T(t) = (T_0 - T_S)e^{-kt} + T_S$$

4.B Part b

We have the inhomogeneous equation:

$$\frac{dT}{dt} = k(\sin(at) - T)$$

$$\frac{dT}{dt} + Tk = k\sin(at)$$

We can solve the inhomogenous equation first:

$$\frac{dT}{dt} + Tk = 0$$

$$\frac{dT}{dt} = -Tk$$

$$\frac{1}{T}dT = -kdt$$

Integrating both sides:

$$ln(T) = -kt + C$$

$$T(t) = Ce^{-kt}$$

Where, $C = T_0$ and:

$$T_h(t) = T_0 e^{-kt}$$

Now, to find the inhomogenous equation, we solve for:

$$T(t) = u(t)T_h(t) = u(t) \cdot T_0 e^{-kt}$$

Therefore, we can substitute this back into the original equation to solve for u:

$$\frac{d}{dt}(u(t) \cdot T_0 e^{-kt}) + (u(t) \cdot T_0 e^{-kt}) \cdot k = k \sin(at)$$

$$(\frac{du}{dt}(T_0 e^{-kt})) - k(u(t)T_0)e^{-kt} + (ku(t) \cdot T_0 e^{-kt}) = k \sin(at)$$

$$\frac{du}{dt}(T_0 e^{-kt}) = k \sin(at)$$

$$\frac{du}{dt} T_0 = k \sin(at) e^{kt}$$

$$\int T_0 du = \int k \sin(at) e^{kt} dt$$

$$u(t) = \frac{e^k t}{T_0(k^2 + a^2)}(ksin(at) - acos(at)) + C$$

Plugging back in:

$$T(t) = (\frac{e^k t}{T_0(k^2 + a^2)}(ksin(at) - acos(at)) + C) \cdot T_0 e^{-kt}$$

Suppose $T(0) = T_0$:

$$T_0 = (\frac{e^k (0)}{T_0(k^2 + a^2)}(ksin(a(0)) - acos(a(0))) + C) \cdot T_0 e^{-k(0)}$$

$$T_0 = (\frac{-a}{T_0(k^2 + a^2)} + C) \cdot T_0$$

$$1 = \frac{-a}{T_0(k^2 + a^2)} + C$$

$$1 + \frac{a}{T_0(k^2 + a^2)} = C$$

Or,

$$\frac{T_0(k^2 + a^2) + a}{T_0(k^2 + a^2)} = C$$

Therefore:

$$T(t) = \frac{e^k t}{T_0(k^2+a^2)}(ksin(at) - acos(at)) + \frac{T_0(k^2+a^2)+a}{T_0(k^2+a^2)} \cdot T_0 e^{-kt}$$

4.C Part c

There are three components, the homogenous equation and two other scaled inhomogenous solutions. Therefore, by the superposition principle, we can plug in different values for a and multiply by various scalars to yield the components and add them to the homogenous equation. Therefore, using:

$$\frac{dT}{dt} + Tk = (20)\sin(\frac{1}{365}t)$$

and,

$$\frac{dT}{dt} + Tk = (10)\sin((1)t)$$

and our work from part a, we can superimpose the three ODEs to yield:

So,

$$T(t) = \frac{10e^k t}{T_0(k^2+1)}(ksin(t) - cos(t)) + \frac{10T_0(k^2+1)+1}{T_0(k^2+1)} \cdot T_0 e^{-kt} +$$

$$\frac{20e^k t}{T_0(k^2+\frac{1}{133225})}(ksin(\frac{1}{365}t) - \frac{1}{365}cos(\frac{1}{365}t)) + \frac{20T_0(k^2+\frac{1}{133225})+\frac{1}{365}}{T_0(k^2+\frac{1}{133225})} \cdot T_0 e^{-kt} +$$

$$+(T_0 - T_S)e^{-kt} + T_S$$

4.D Part d

Given that $T_s(t) = e^{-kt}$, we have:

$$\frac{dT}{dt} - kT = ke^{-kt}$$

From previous work, we have the homogeneous solution:

$$T_h(t) = T_0 e^{-kt}$$

By variation of parameters:

$$T = u(t) \cdot T_h(t)$$

Plugging back into the system:

$$\frac{d}{dt}(u(t) \cdot T_h(t)) + k(u(t) \cdot T_h(t)) = ke^{-kt}$$

Expanding on $T_h(t)$:

$$\frac{d}{dt}(u(t) \cdot T_0 e^{-kt}) + k(u(t) \cdot T_0 e^{-kt}) = ke^{-kt}$$

$$\frac{du}{dt} \cdot T_0 e^{-kt} - ku(t) \cdot T_0 e^{-kt} + ku(t) \cdot T_0 e^{-kt} = ke^{-kt}$$

This simplifies to:

$$\frac{du}{dt} \cdot T_0 e^{-kt} = ke^{-kt}$$

$$\frac{du}{dt} = \frac{k}{T_0}$$

By separation of variables:

$$\int du = \int \frac{k}{T_0} dt$$

$$u(t) = \frac{k}{T_0} t + C$$

Plugging back into the variation of parameters law:

$$T(t) = (\frac{k}{T_0} t + C) \cdot T_0 e^{-kt}$$

Where $T(0) = T_0$:

$$T_0 = C \cdot T_0$$

$$C = 1$$

Therefore:

$$T(t) = (\frac{k}{T_0} t + 1) \cdot T_0 e^{-kt}$$