MIT 18.03 Problem Set 1B

Joshua Pereira

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1 Problem 1

1.A Part a

We can rewrite the number i^{2024} as:

$$i^{2024} = [cos(\frac{\pi}{2}) + isin(\frac{\pi}{2})]^{2024}$$

$$= (e^{i\frac{\pi}{2}})^{2024} = e^{i1012\pi}$$

$$= \cos(1012\pi) + i\sin(1012\pi) \equiv cos(2\pi) + i\sin(2\pi) = 1$$

by the periodicity of sines and cosines. Therefore:

$$Re(i^{2024}) = 1$$
$$Im(i^{2024}) = 0$$

1.B Part b

We can rewrite the numbver $1 - e^{-\frac{i\pi}{4}}$:

$$1 - e^{-\frac{i\pi}{4}} = 1 - (\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})) = 1 - (\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})$$

$$(1 - \frac{\sqrt{2}}{2}) - i\frac{\sqrt{2}}{2}$$

Therefore:

$$Re(1 - e^{-\frac{i\pi}{4}}) = 1 - \frac{\sqrt{2}}{2}$$

$$Im(1 - e^{-\frac{i\pi}{4}}) = -\frac{\sqrt{2}}{2}$$

2 Problem 2

2.A Part a

$$z^3 = 27i$$

$$z^3 = 27e^{i\frac{\pi}{2}}$$

$$z = 3(e^{i\frac{\pi}{2}})^{\frac{1}{3}}$$

Therefore, we can take the 3rd-roots of unity as:

$$\theta' = \frac{\theta}{3}$$

$$\theta' = \frac{\frac{\pi}{2} + 2k\pi}{3} = \frac{\pi}{6} + k\frac{2\pi}{3}$$

Therefore:

$$k = 0, \theta' = \frac{\pi}{6}$$

$$k = 1, \theta' = \frac{5\pi}{6}$$

$$k = 2, \theta' = \frac{9\pi}{6} = \frac{3\pi}{2}$$

The 3-rd roots of unity are: $3e^{i\frac{\pi}{6}}, 3e^{i\frac{5\pi}{6}}, 3e^{i\frac{3\pi}{2}}$.

2.B Part b

$$e^z = 2$$

$$z = \ln(2)$$

There is only one root of unity: $\ln(2)e^{i\cdot 0}$

3 Problem 3

3.A Part a

Suppose $z_1 = a + bi$ and $z_2 = c + di$, where $z_2 > 0$. Therefore, the number $\bar{z_1/z_2}$:

$$\bar{z_1/z_2} = \frac{\bar{a+bi}}{c+di} = \frac{\bar{(a+bi)(c-di)}}{c^2+d^2}$$

$$\bar{\frac{(ac+(bc-ad)i+bd)}{c^2+d^2}} = \frac{ac+bd}{c^2+d^2} \bar{+} i\frac{bc-ad}{c^2+d^2}$$

Opening the conjugate notation:

$$z_1 \bar{/} z_2 = \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2}$$

Re-combining the conjugate:

$$\frac{ac+bd}{c^2+d^2} + i\frac{ad-bc}{c^2+d^2} = \frac{ac+adi+bd-bci}{(c^2+d^2)} = \frac{ac+adi-bdi^2-bci}{(c^2+d^2)}$$

$$= \frac{(a-bi)(c+di)}{(c-di)(c+di)} = \frac{\bar{z}_1}{\bar{z}_2}\square$$

4 Problem 4

4.A Part a

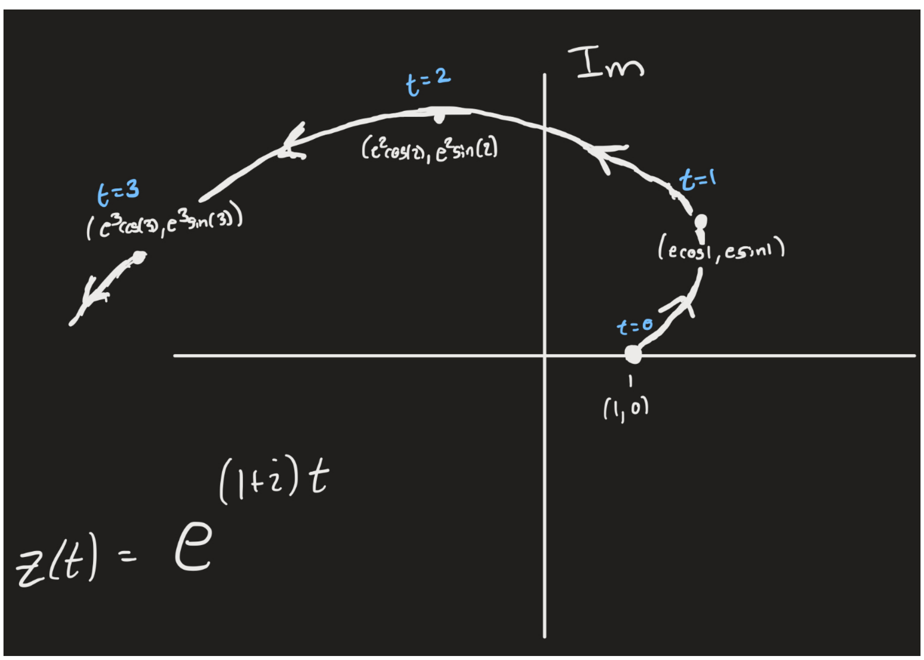

Figure 1: $z(t) = e^{(1+i)t} \forall t \in [0, \infty)$

4.B Part b

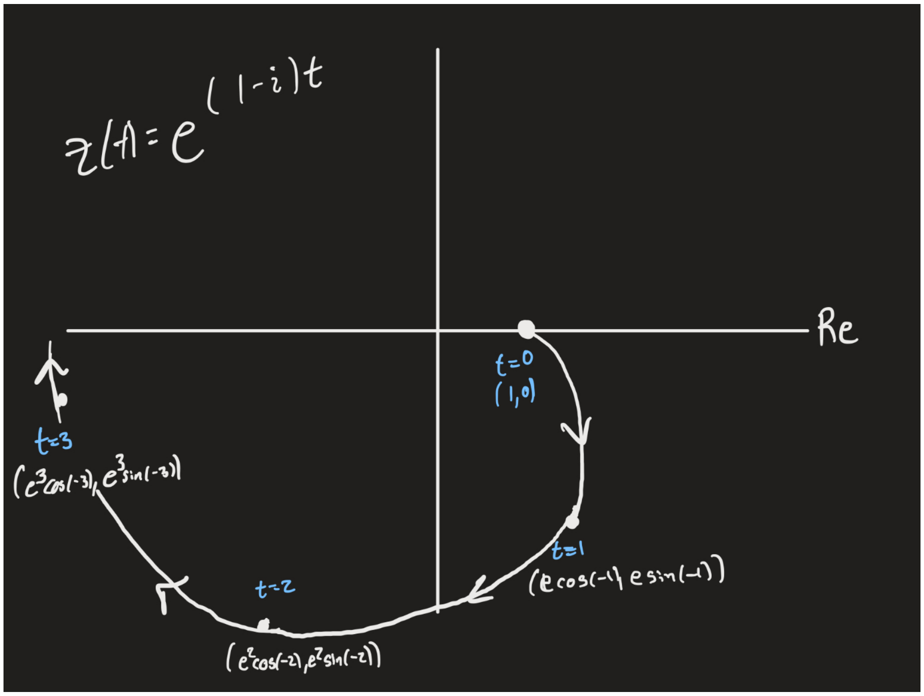

Figure 2: $z(t) = e^{(1-i)t} \forall t \in [0, \infty)$

5 Problem 5

5.A Part a

We can re-write the form $A\cos(\omega t - \phi)$:

$$A\cos(\omega t - \phi) = A(\cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi))$$

Therefore, given the form $-\frac{1}{2}\cos(2\pi t) + \frac{\sqrt{3}}{2}\sin(2\pi t)$, suppose $\omega = 2\pi$ and $\phi = \frac{4\pi}{3}$, and $A = 1$. Therefore, we have that:

$$\cos(2\pi t - \frac{4\pi}{3}) = -\frac{1}{2}\cos(2\pi t) + \frac{\sqrt{3}}{2}\sin(2\pi t)$$

5.B Part b

Given the form $Re(e^{i\frac{\pi}{5}}\cos(2t))$, we can exploit

$$e^{i\frac{\pi}{5}}\cos(2t) = (\cos(\frac{\pi}{5}) + i\sin(\frac{\pi}{5}))\cos(2t)$$

$$\cos(\frac{\pi}{5})\cos(2t) + i\sin(\frac{\pi}{5})\cos(2t)$$

Therefore, we can take the real part:

$$\cos(\frac{\pi}{5})\cos(2t)$$

Therefore, let $A = \cos(\frac{\pi}{5})$, $\phi = 0$, $\omega = 2$.

5.C Part c

$$2\sin(\frac{t-\pi}{2})$$

$$= 2\cos(\frac{t-\pi}{2} - \frac{\pi}{2}) = 2\cos(\frac{t}{2} - \frac{\pi}{2} - \frac{\pi}{2}) = 2\cos(\frac{t}{2} - \pi)$$

Therefore, let $A = 2$, $\phi = \pi$, and $\omega = \frac{1}{2}$.

5.D Part d

$$(i-1)e^{i\cdot 3t} = \frac{2}{\sqrt{2}}(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}))e^{i\cdot 3t} = \frac{2}{\sqrt{2}}e^{i\cdot\frac{3\pi}{4}}e^{i\cdot 3t} =$$

$$\frac{2}{\sqrt{2}}e^{i(\frac{3\pi}{4}+3t)}$$

Thus, we can take the imaginary part:

$$\frac{2}{\sqrt{2}}\sin(\frac{3\pi}{4} + 3t) =$$

$$\frac{2}{\sqrt{2}}\cos(\frac{3\pi}{4} + 3t - \frac{\pi}{2}) =$$

$$\frac{2}{\sqrt{2}}\cos(\frac{\pi}{4} + 3t)$$

where $A = \frac{2}{\sqrt{2}}$, $\phi = -\frac{\pi}{4}$, and $\omega = 3$.