MIT 6.3200 Problem Set 2

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February 28, 2024

1 Problem 1

1.A Part a

From our fifth lecture on DeGroot Social Learning, we are given that "a simple sufficient condition for a matrix to be aperiodic is that there exists some agent i such that $A_{ii} > 0$". More formally, A is aperiodic if "the greatest common divisor of all directed cycle lengths is 1". Furthermore, from the same lecture, we are given that if A is aperiodic, then $x(t)$ as $t \to \infty$ converges to a value.

Therefore, we can make the sufficient claim that if any agent places a positive weight on their own opinion, then the limit belief is well-defined due to the aperiodicity of the belief matrix, A. As we can see in the given example, all weights in A along the diagonal are greater than 0. So, this matrix has a specific limit belief $x^* = \lim_{t\to\infty} x(t)$ that must be well-defined.

We can write:

$$x^* = \lim_{t\to\infty} A^t \cdot x(0)$$

$$x^* = \lim_{t\to\infty} \begin{bmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^t \cdot x(0)$$

We can use a computer to determine the long-term $\lim_{t\to\infty} A^t$:

$$\lim_{t\to\infty} A^t = \begin{bmatrix} 0.22727273 & 0.36363636 & 0.40909091 \\ 0.22727273 & 0.36363636 & 0.40909091 \\ 0.22727273 & 0.36363636 & 0.40909091 \end{bmatrix}$$

Thus:

$$x^* = \begin{bmatrix} 0.22727273 & 0.36363636 & 0.40909091 \\ 0.22727273 & 0.36363636 & 0.40909091 \\ 0.22727273 & 0.36363636 & 0.40909091 \end{bmatrix} \cdot x(0)$$

1.B Part b

We can draw the network:

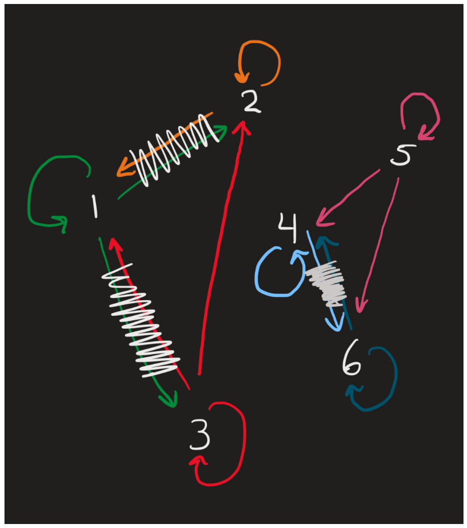

Figure 1: $N = 6$ graph for 1b.

where links between two nodes that can be virtually considered undirected as a result of bidirectionality is indicated by white accents. We know from lecture that although $\lim_{t\to\infty} A^t$ due to the aperiodicity of A, according to lecture, nodes that are in components that are strongly connected will share consensus long-run belief. Therefore, because A is aperiodic due to having at least one nonzero, positive element along its diagonal for reasons presented in part a, thee elements in the vector will not match because the graph is not strongly connected.

Therefore, $\lim_{t\to\infty} x(t)$ is well-defined due to the aperiodicity of A. But, because A is not strongly-connected, the components consisting of nodes $\{1, 2, 3\}$, $\{4, 6\}$, and $\{5\}$ will have different long-term beliefs compared to one another, but identical beliefs among themselves.

1.C Part c

Let $\Delta(t) = \max\limits_{j\in N}|x_i(t) - x_j(t)|$ and let $\underline{T} = \min\limits_{j\neq i} T_{ji}$. As well, suppose that $T_{ii} = 1$ and $T_{ji} > 0$. Suppose we have the expression:

$$\Delta(t+1) \leq (1 - \underline{T})\Delta(t)$$

We can rewrite this as:

$$\max_{j\in N}|x_i(t+1) - x_j(t+1)| \leq (1 - \underline{T})\max_{j\in N}|x_i(t) - x_j(t)|$$

It would suffice to show:

$$|x_i(t+1) - x_j(t+1)| \leq (1 - \underline{T}) \max_{j \in N} |x_i(t) - x_j(t)|$$

for some arbitrary j on the left. Re-writing the left:

$$|T_i \cdot x(t) - T_j \cdot x(t)|$$

We recall that $T_i \cdot x(t) = x_i(t)$ as $T_{ii} = 1$:

$$|x_i(t) - T_j \cdot x(t)|$$

Opening the dot product:

$$|x_i(t) + \sum^{k} -T_{jk} \cdot x_k(t)| = |x_i(t) + (-T_{j1} \cdot x_1(t) - ... - T_{ji} \cdot x_i(t)) - ... - T_{jN} \cdot x_N(t))|$$

$$= |(1 - T_{ji}) x_i(t) + (-T_{j1} \cdot x_1(t) - ... - T_{jN} \cdot x_N(t))| = |(1 - T_{ji}) x_i(t) + \sum_{k \neq i} -T_{jk} x_k(t)|$$

Recall that $\sum_k T_j k = 1$. Therefore:

$$= |(\sum_k T_{jk} - T_{ji}) x_i(t) + \sum_{k \neq i} -T_{jk} x_k(t))| = |\sum_{k \neq i} T_{jk} x_i(t) + \sum_{k \neq i} -T_{jk} x_k(t))| =$$

$$= |\sum_{k \neq i} T_{jk} (x_i(t) - x_k(t))|$$

By the triangle inequality:

$$|\sum_{k \neq i} T_{jk} (x_i(t) - x_k(t))| \leq \sum_{k \neq i} |T_{jk} (x_i(t) - x_k(t))| = \sum_{k \neq i} T_{jk} |(x_i(t) - x_k(t))|$$

Suppose we choose the maximum $|x_i(t) - x_k(t)|$ over all nodes k, then:

$$|\sum_{k \neq i} T_{jk} (x_i(t) - x_k(t))| \leq \max_{j \in N} |x_i(t) - x_j(t)| \sum_{k \neq i} T_{jk}$$

Furthermore, we know that:

$$\sum_{k \neq i} T_{jk} \leq 1 - \min_{j \neq i} T_{ji} = 1 - \underline{T}$$

So,

$$|\sum_{k \neq i} T_{jk} (x_i(t) - x_k(t))| \leq \max_{j \in N} |x_i(t) - x_j(t)| (1 - \min_{j \neq i} T_{ji})$$

Therefore, the inequality holds for any arbitrary j on the left-hand side. So, for the maximum of all j:

$$\max_{j \in N} |x_i(t) - x_j(t)| \leq \max_{j \in N} |x_i(t) - x_j(t)|(1 - \min_{j \neq i} T_{ji})$$

$$\Delta(t) \leq (1 - \underline{T})\Delta(t+1)\square$$

2 Problem 2

We are given that $\exists i$ and positive integer t such that:

$$(T^t)_{ji} = 0 \ \forall j$$

This means that there is a column in the matrix indexed by (j, i) $\forall j$ that is all zeroes. This means that as we take the t-th power of t, we will create a row and column in T^t that is also all zeroes. In fact, such a row and column correspond to agent i. We are also given that:

$$x(0) = \hat{x}(0) \ j \neq i$$

Therefore, because row and column i in T^t is composed of all zeroes, and because the ith element is the only element that can be different between $x(0)$ and $\hat{x}(0)$, the values of:

$$T^t x(0) = T^t \hat{x}(0)$$

Thus, because we can take a multiple of the constant t to infinity, and because we are given that:

$$\exists \lim_{t \to \infty} x(t), \lim_{t \to \infty} \hat{x}(t)$$

it follows that it must be the case that:

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \hat{x}(t)$$