Zombie-Human Gas Dynamics in $\mathbb{R}^2$

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Introduction

Zombies pose a serious threat to our civilization. We wish to understand their large-scale population dynamics, specifically how they spread across a country or continent, so that we may better avoid obliteration.

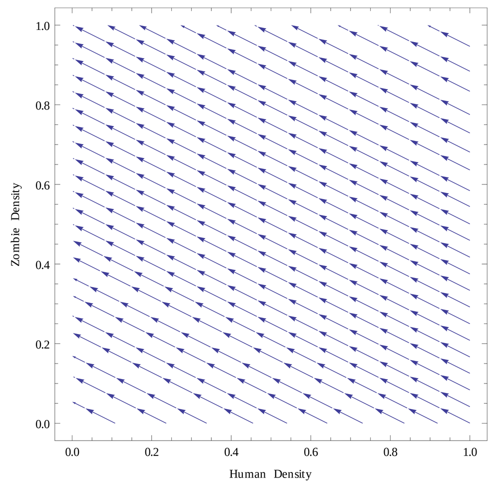
Basic Assumptions

1. That we are investigating a sufficiently large length scale that populations may be approximated by fluid densities
2. That humans and zombies act only according to their immediate locality
3. That zombies chase humans, and humans flee from zombies, along lines of steepest ascent/descent
4. That zombies can kill humans, converting them into zombies, and humans can permanently kill zombies
5. That humans and zombies slowly diffuse.

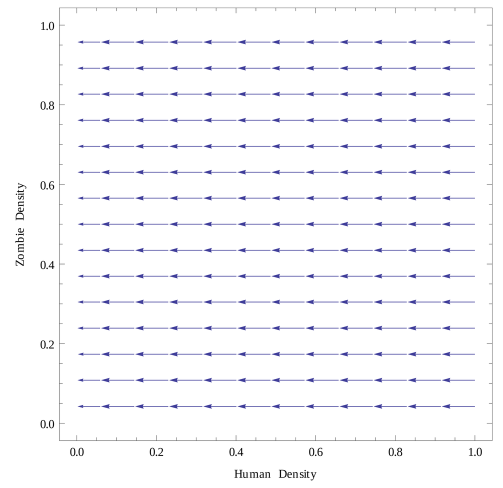
Spatial Homogeneity

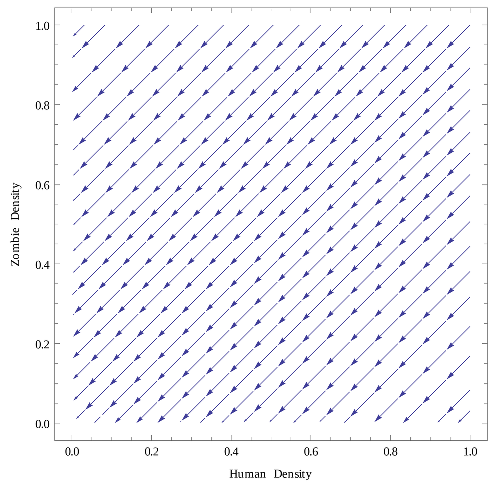
In the spatially homogeneous case, all spatial derivatives vanish. The system reduces to coupled ODEs, with phase space portraits as below:

$\alpha_Z > \alpha_H$



$\alpha_Z = \alpha_H$





$\alpha_H > \alpha_Z$

Visualization of $\alpha_H > \alpha_Z$

PDE Model Derivation

Let the zombie and human densities be ρ_Z and ρ_H respectively. Then equations of the form

$$\frac{\partial \rho_Z}{\partial t} + \nabla \cdot \mathbf{j_Z} = 0, \qquad \frac{\partial \rho_H}{\partial t} + \nabla \cdot \mathbf{j_H} = 0 \tag{1}$$

ensure that the total numbers of humans and zombies are separately conserved. In accordance with our earlier assumptions, pick $\mathbf{j_H}$ and $\mathbf{j_Z}$ to be

$$\mathbf{j_Z} = -\kappa_Z \nabla \rho_Z + \sigma_Z \rho_Z \nabla \rho_H, \qquad \mathbf{j_H} = -\kappa_H \nabla \rho_H - \sigma_H \rho_H \nabla \rho_Z. \tag{2}$$

these cause zombies/humans to have a mixture of diffusive behaviour and a form of "directed diffusion" in/against and in proportion to the gradient of the humans/zombies. The κ and σ parameters control the relative strengths of these effects.

Now by adding terms to the RHS, we can allow zombies to kill humans, converting them into zombies. Let these rates of conversion be proportional to the product of the densities (this is the simplest choice which makes any sense) and some parameter α_Z; then we arrive at

$$\frac{\partial \rho_Z}{\partial t} + \nabla \cdot \mathbf{j_Z} = +\alpha_Z \rho_Z \rho_H, \qquad \frac{\partial \rho_H}{\partial t} + \nabla \cdot \mathbf{j_H} = -\alpha_Z \rho_Z \rho_H. \tag{3}$$

Adding these equations together, we see that the quantity $\int_{\mathbb{R}^2} dxdy\, (\rho_Z + \rho_H)$ does not change with time, as it shouldn't. Finally, we allow humans to kill zombies, applying the same kind of reasoning, to get our full **Unified Human-Zombie Field Theory**:

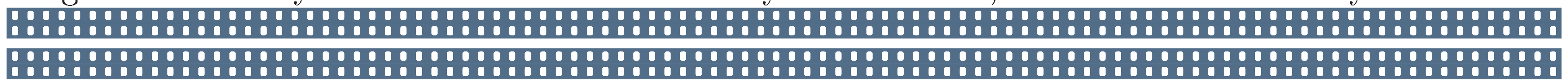
$$\frac{\partial \rho_Z}{\partial t} + \nabla \cdot \mathbf{j_Z} = (\alpha_Z - \alpha_H)\rho_Z \rho_H, \qquad \frac{\partial \rho_H}{\partial t} + \nabla \cdot \mathbf{j_H} = -\alpha_Z \rho_Z \rho_H. \tag{4}$$

Simulation Results: The Plight of the The Three Towns

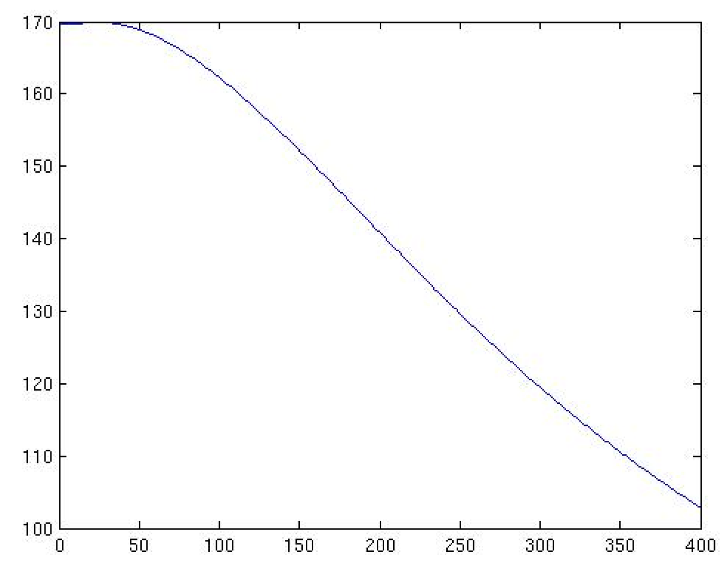
In order to demonstrate that our model makes sense, we have simulated a specific case study using an explicit finite difference scheme. Here, the action is confined to a square, where reflecting boundary conditions have been used to ensure that zombies and humans do not leave the domain. Zombie combat efficacy α_Z is held at 1, whilst their rage σ_Z is held at 2.5; both species' diffusion constants κ are pinned at 1 for the duration, whilst human combat efficacy α_H and fear σ_H vary with time t as

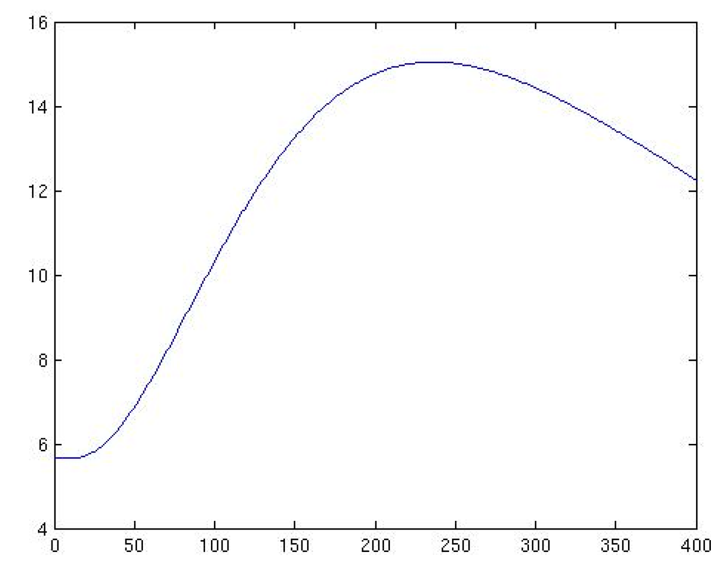
$$\alpha_H = 2(1 - e^{-\frac{t}{\tau_f}}) \qquad \sigma_H = 7.5(1 - e^{-\frac{t}{\tau_f}}) \tag{5}$$

where τ_f is the characteristic fightback time (=145 video frames; 400 in total) In the film clips below, 1 image is taken every 50 frames. The human density is shown first, with the zombie density below.



The left graph is the total human population over time; right is the total zombie population. Notice that the zombie population initially increases fairly rapidly, but then starts to decline as humans become more effective at killing zombies.





Conclusions, Further Work and Acknowledgements

The zombie-human field theory developed here seems to capture some of the phenomena commonly observed in zombie films. However, there are some issues which would need to be addressed if we were to carry this research further:

1. The finite-difference simulations do not appear to be stable unless sufficiently large diffusion parameters are used. Therefore we have not been able to explore the low-diffusion limit properly. This may be happening due to shock formation (which finite-difference would not be able to handle) or because the PDE system genuinely has inherent instability.
2. The RHS of the equations seems to be unrealistic, as a group of humans fighting two batches of zombies sustains just as many losses as a group fighting both batches combined. Thus it may be prudent to consider some different models.

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