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A General Approach to Obtain Series Solutions of Nonlinear Differential Equations

By S. Liao and Y. Tan

Based on homotopy, which is a basic concept in topology, a general analytic method (namely the homotopy analysis method) is proposed to obtain series solutions of nonlinear differential equations. Different from perturbation techniques, this approach is independent of small/large physical parameters. Besides, different from all previous analytic methods, it provides us with a simple way to adjust and control the convergence of solution series. Especially, it provides us with great freedom to replace a nonlinear differential equation of order n into an infinite number of linear differential equations of order k, where the order k is even unnecessary to be equal to the order n. In this paper, a nonlinear oscillation problem is used as example to describe the basic ideas of the homotopy analysis method. We illustrate that the second-order nonlinear oscillation equation can be replaced by an infinite number of (2κ)th-order linear differential equations, where $\kappa \geq 1$ can be any a positive integer. Then, the homotopy analysis method is further applied to solve a high-dimensional nonlinear differential equation with strong nonlinearity, i.e., the Gelfand equation. We illustrate that the second-order two or three-dimensional nonlinear Gelfand equation can be replaced by an infinite number of the fourth or sixth-order linear differential equations, respectively. In this way, it might be greatly simplified to solve some nonlinear problems, as illustrated in this paper. All of our series solutions agree well with numerical results. This paper illustrates that we might have much larger freedom and flexibility to solve

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nonlinear problems than we thought traditionally. It may keep us an open mind when solving nonlinear problems, and might bring forward some new and interesting mathematical problems to study.

1. Introduction

Generally speaking, it is easy to solve a linear differential equation. For example, let us consider the linear differential equation:

$$u''(t) + \lambda\, u(t) = 0, \quad u(0) = 1, \quad u'(0) = 0, \tag{1}$$

where $\lambda \in (-\infty, +\infty)$ is a constant. Obviously, for different values of λ, the linear equation $u''(t) + \lambda\, u(t) = 0$ has *different* types of solutions, i.e.,

$$u(t) = C_1 \exp(\sqrt{|\lambda|}t) + C_2 \exp(-\sqrt{|\lambda|}t), \quad \text{when } \lambda < 0, \tag{2}$$

$$u(t) = C_1 + C_2 t, \quad \text{when } \lambda = 0, \tag{3}$$

$$u(t) = C_1 \cos(\sqrt{\lambda}t) + C_2 \sin(\sqrt{\lambda}t), \quad \text{when } \lambda > 0, \tag{4}$$

where C_1 and C_2 are coefficients to be determined by the two initial conditions. So, for different value of λ, Equation (1) has different types of solutions:

$$u(t) = \begin{cases} \cosh(\sqrt{|\lambda|}t), & \text{when } \lambda < 0, \\ 1, & \text{when } \lambda = 0, \\ \cos(\sqrt{\lambda}t), & \text{when } \lambda > 0. \end{cases} \tag{5}$$

Here, we emphasize that, the character of the solution $u(t)$ (i.e., the kernel) of the *linear* differential equation (1) depends strongly upon the value of λ: it is unbounded when $\lambda < 0$, but is periodic when $\lambda > 0$.

However, it is much more difficult to analytically solve a nonlinear differential equation. Very few nonlinear problems have simple, closed-form solutions. In most cases, solutions of nonlinear differential equations can be expressed only by means of an infinite series. Up to now, there are some well-known analytic techniques for nonlinear problems, such as perturbation techniques [1–3], Adomian's decomposition method [4–9], the δ-expansion method [10, 11], Lyapunov's artificial small parameter method [12], and so on. All of these perturbation and nonperturbation techniques have the same characteristic: a *nonlinear* differential equation $\mathcal{A}u = 0$, where $\mathcal{A}$ is a nonlinear operator, is replaced by an *infinite* number of *linear* subproblems governed by the linear differential equation

$$\mathcal{L}u_n = f_n(u_0, u_1, u_2, \ldots, u_{n-1}), \tag{6}$$

where $\mathcal{L}$ is a linear operator that is dependent upon the original nonlinear equation and the used perturbation or nonperturbation method, and the right-hand side term f_n is dependent upon the known $u_0, u_1, u_2, \ldots, u_{n-1}$. And the solutions of the first M linear subproblems are used to give the Mth-order approximation

$$u \approx u_0 + \sum_{k=1}^{M} c_k u_k,$$

where c_k is a constant.[1] In general, a nonlinear equation $\mathcal{A}u = 0$ contains linear and nonlinear parts, i.e., $\mathcal{A}u = \mathcal{L}_0 u + \mathcal{N}_0 u$, where $\mathcal{L}_0$ and $\mathcal{N}_0$ are linear and nonlinear operators, respectively. Then, it is natural for us to ask the following questions:

1. Must the linear operator $\mathcal{L}$ in (6) for the linear subproblems have a close relationship with the linear operator $\mathcal{L}_0$ appeared in the original nonlinear equation $\mathcal{L}_0 u + \mathcal{N}_0 u = 0$?
2. Must the linear operator $\mathcal{L}$ in (6) for the linear subproblems have the same order as that of the original nonlinear operator $\mathcal{A}$?

It is hard for us to answer the above questions in general. In this paper, we investigate these interesting questions by some examples. Without loss of generality, let us first consider a nonlinear oscillation, governed by

$$u''(t) + \lambda u(t) + \epsilon u^3(t) = 0, \quad u(0) = 1, \quad u'(0) = 0, \tag{7}$$

where $\lambda \in (-\infty, +\infty)$ and $\epsilon \geq 0$ are physical parameters. Mathematically, the above equation contains the linear and nonlinear parts:

$$\mathcal{L}_0 u = u'' + \lambda u, \quad \mathcal{N}_0 u = \epsilon u^3.$$

Physically, the resilience of the oscillator is directly proportional to $-(\lambda u + \epsilon u^3)$. When $\lambda \geq 0$ and $\epsilon > 0$, there is one unique equilibrium point $u = 0$. However, when $\lambda < 0$ and $\epsilon > 0$, there are two equilibrium points $u = 0$ and $u = \sqrt{|\lambda/\epsilon|}$. From the physical points of view, the sum of the kinetic and potential energy of the oscillator keeps the same, therefore it is clear that the oscillation motion is *periodic*, no matter λ is positive or negative. Thus, from physical points of view, it is easy to know that $u(t)$ is periodic, even if we do not directly solve Equation (7).

Unfortunately, many traditional analytic methods can not give periodic solutions of Equation (7) for all possible values of λ. For example, let us use the straightforward perturbation technique to it. Regarding ϵ as a small parameter, one can expand $u(t)$ as follows

[1] If perturbation method is used, c_k is a function of perturbation quantity.

$$u(t) = u_0(t) + u_1(t)\epsilon + u_2(t)\epsilon^2 + \cdots.$$

Substituting it into Equation (7) and equating the coefficients of the like power of ϵ, one has an *infinite* number of *linear* differential equations:

$$u_0''(t) + \lambda u_0(t) = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0, \tag{8}$$

$$u_n''(t) + \lambda u_n(t) = -\sum_{i=0}^{n-1} u_i \sum_{j=0}^{n-1-i} u_j u_{n-1-i-j}, \quad u_n(0) = 0, \quad u_n'(0) = 0, \tag{9}$$

In this way, one replaces the second-order *nonlinear* differential equation (7) by an *infinite* number of the second-order *linear* differential equations (8) or (9). All of these linear equations have the *same* second-order linear operator

$$\mathcal{L}u = u'' + \lambda u = \mathcal{L}_0 u. \tag{10}$$

Obviously, for different value of λ, one has different types of the initial approximation $u_0(t)$ governed by (8), i.e.,

$$u_0(t) = \begin{cases} \cosh(\sqrt{|\lambda|}t), & \text{when } \lambda < 0, \\ 1, & \text{when } \lambda = 0, \\ \cos(\sqrt{\lambda}t), & \text{when } \lambda > 0. \end{cases} \tag{11}$$

Thus, the solutions of Equation (9) have different types of expressions for $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively. For example, when $\lambda = -9/4$ and $\epsilon = 1$, the second-order perturbation approximation reads

$$u(t) \approx \left(\frac{t^2}{64} + \frac{5t}{36} + \frac{5279}{10368}\right) e^{-3t/2} - \left(\frac{t}{192} + \frac{1}{108}\right) e^{-9t/2} + \frac{1}{10368} e^{-15t/2} + \left(\frac{t^2}{64} - \frac{5t}{36} + \frac{5279}{10368}\right) e^{3t/2} + \left(\frac{t}{192} - \frac{1}{108}\right) e^{9t/2} + \frac{1}{10368} e^{15t/2}.$$

When $\lambda = 0$ and $\epsilon = 1$, the second-order perturbation approximation is

$$u(t) \approx 1 - \frac{t^2}{2} + \frac{t^4}{8}.$$

When $\lambda = 9/4$ and $\epsilon = 1$, the second-order approximation reads

$$u(t) \approx \left(\frac{5135}{5184} - \frac{t^2}{32}\right) \cos\left(\frac{3t}{2}\right) + \frac{1}{108} \cos\left(\frac{9t}{2}\right) + \frac{1}{5184} \cos\left(\frac{15t}{2}\right) - \frac{2t}{9} \sin\left(\frac{3t}{2}\right) - \frac{t}{96} \sin\left(\frac{9t}{2}\right).$$

In general, when $\lambda < 0$, the straightforward perturbation approximations of Equation (7) have the form

$$u(t) = \sum_{m=0}^{M} a_m t^m \sum_{n=1}^{N} \left(b_n e^{-n\sqrt{|\lambda|}t} + c_n e^{n\sqrt{|\lambda|}t} \right), \tag{12}$$

with the property $|u(+\infty)| \to +\infty$. When $\lambda = 0$, the corresponding perturbation approximation is expressed by

$$u(t) = \sum_{n=0}^{N} a_n t^{2n}, \tag{13}$$

whose convergence radius is finite in most cases. When $\lambda > 0$, the corresponding perturbation approximation reads

$$u(t) = \sum_{m=0}^{M} a_m t^m \sum_{n=1}^{N} [b_n \cos(n\sqrt{\lambda}t) + c_n \sin(n\sqrt{\lambda}t)], \tag{14}$$

which contains the so-called secular terms such as $t \sin(\sqrt{\lambda}t)$ that tends to infinity as $t \to +\infty$. From above three types of the straightforward perturbation approximations, it is natural for us to guess that the solution of Equation (7) might be unbounded when $\lambda < 0$ or $\lambda > 0$. However, from the physical points of view, the solution of Equation (7) is *periodic* for *all* possible values of $\lambda \in (-\infty, +\infty)$ and $\epsilon > 0$, as shown in Figures 1 to 3 for $\epsilon = 1$ and $\lambda = -9/4$, 0 and 9/4, respectively. Note that, these perturbation approximations are valid only for small t, as shown in Figures 1 and 3. Note also that, by means of Adomian's decomposition method [4–9], one obtains exactly the same results as the perturbation ones. Obviously, for the considered example, $\mathcal{L} = \mathcal{L}_0$ can not give us good approximations, especially when $\lambda \leq 0$. So, this simple example clearly indicates that, for a nonlinear differential equation $\mathcal{L}_0 u + \mathcal{N}_0 u = 0$, where $\mathcal{L}_0$ and $\mathcal{N}_0$ are, respectively, linear and nonlinear operators, the linear operator $\mathcal{L}_0$ might completely *mislead* us: $\mathcal{L}_0$ might provide us wrong information, and therefore is *not* so important as we thought. This is mainly because the character of the solution of the linear equation $\mathcal{L}_0 u = 0$ might be completely different from the characters of the solution of the fully nonlinear ones $\mathcal{L}_0 u + \mathcal{N}_0 u = 0$. Therefore, it seems *unnecessary* that the linear operator $\mathcal{L}$ in the linear subproblems (6) should have a close relationship with $\mathcal{L}_0$ in the original nonlinear equation $\mathcal{L}_0 u + \mathcal{N}_0 u = 0$. In other words, we should have great *freedom* to choose the linear operator $\mathcal{L}$ of the linear subproblems (6).

Unfortunately, all of the current analytic techniques for nonlinear problems, such as perturbation techniques [1–3], Adomian's decomposition method [4–9], the δ-expansion method [10, 11], Lyapunov's artificial small parameter method [12], and so on, can *not* provide us with such kind of *freedom* to choose the linear operator $\mathcal{L}$ of linear subproblems.

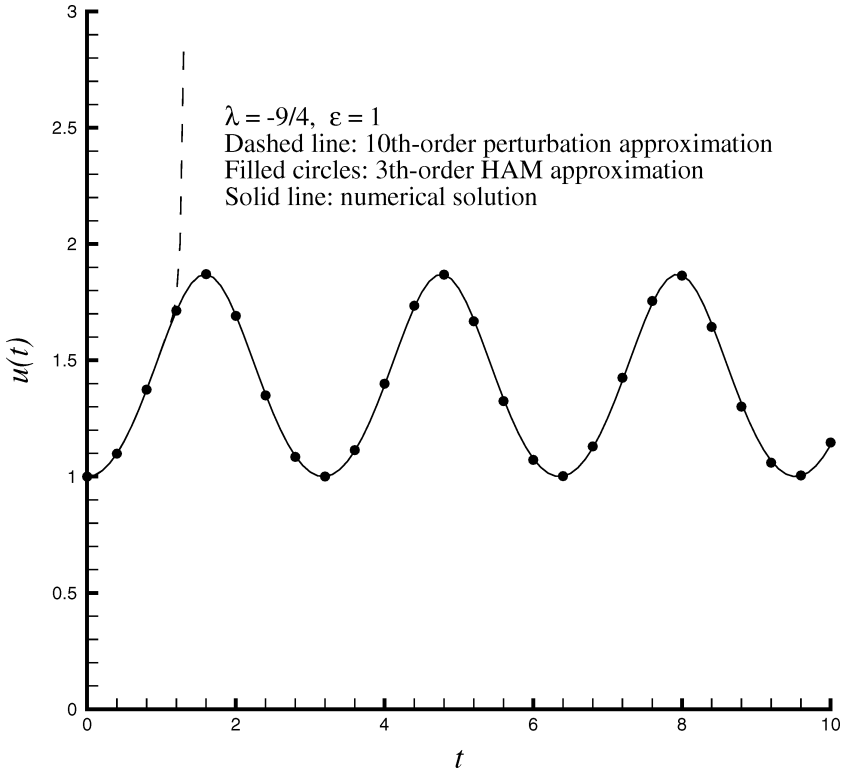

Figure 1. Comparison of the numerical solution with the 10th-order perturbation approximation and the third-order HAM approximation ($\hbar = -1/4$) of Equation (7) when $\lambda = -9/4$, $\epsilon = 1$.

2. The homotopy analysis method

It is well known that a real function can be expressed by some different basis functions. For example, let us consider the initial value problem

$$u'(t) + u^2(t) = 1, \quad u(0) = 0,$$

which has the closed-form solution

$$u(t) = \tanh(t).$$

Using polynomials as basis function, one has

$$\tanh(t) \sim t - \frac{t^3}{3} + \frac{2t^5}{5} - \frac{17t^7}{315} + \cdots,$$

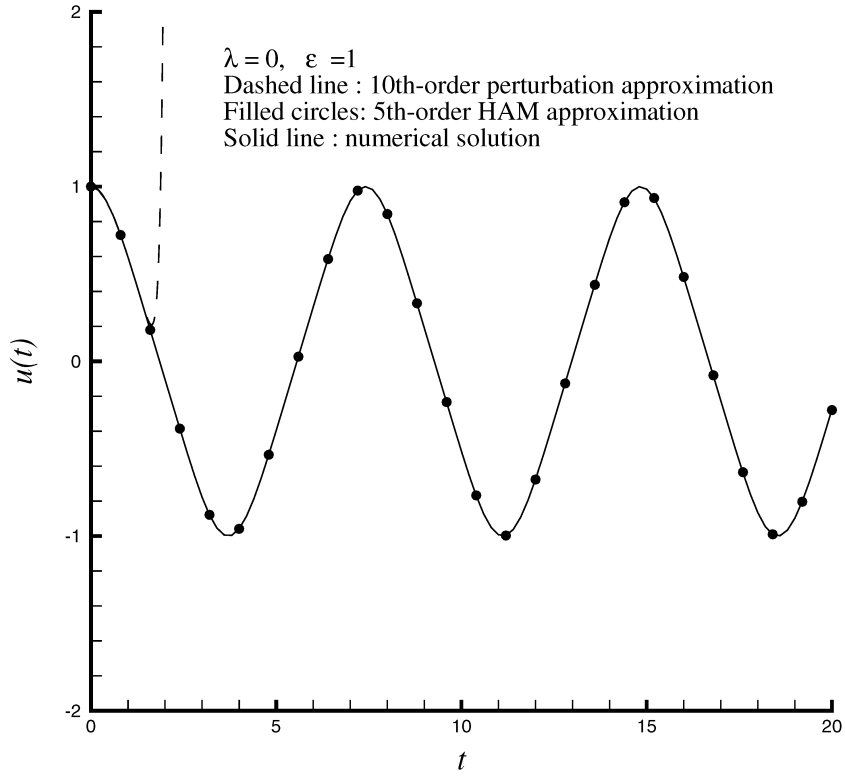

Figure 2. Comparison of the numerical solution with the 10th-order perturbation approximation and the fifth-order HAM approximation ($\hbar = -1$) of Equation (7) when $\lambda = 0$, $\epsilon = 1$.

which converges to the exact solution $u(t) = \tanh(t)$ only in a small region $0 \leq t < 3/2$. However, by means of exponential functions as basis function, one has

$$\tanh(t) \sim 1 + \lim_{m \to +\infty} \left[2 \sum_{n=1}^{m} (-1)^n e^{-2nt} + (-1)^{m+1} e^{-(2m+1)t} \right],$$

which converges to the exact solution $u(t) = \tanh(t)$ in the *whole* region $0 \leq t < +\infty$. Note that, even the first few terms of above expression give accurate approximation. For example, when $m = 2$, it gives

$$1 - 2e^{-2t} + 2e^{-4t} - e^{-5t},$$

which agrees well with the exact solution $\tanh(t)$ in the whole region $0 \leq t < +\infty$. For details, please refer to Liao ([13], Chapter 2). Therefore, one can get better approximations by means of better basis functions. In other words, solutions of nonlinear differential equations can be approximated more efficiently by means of better basis functions. For example, a periodic

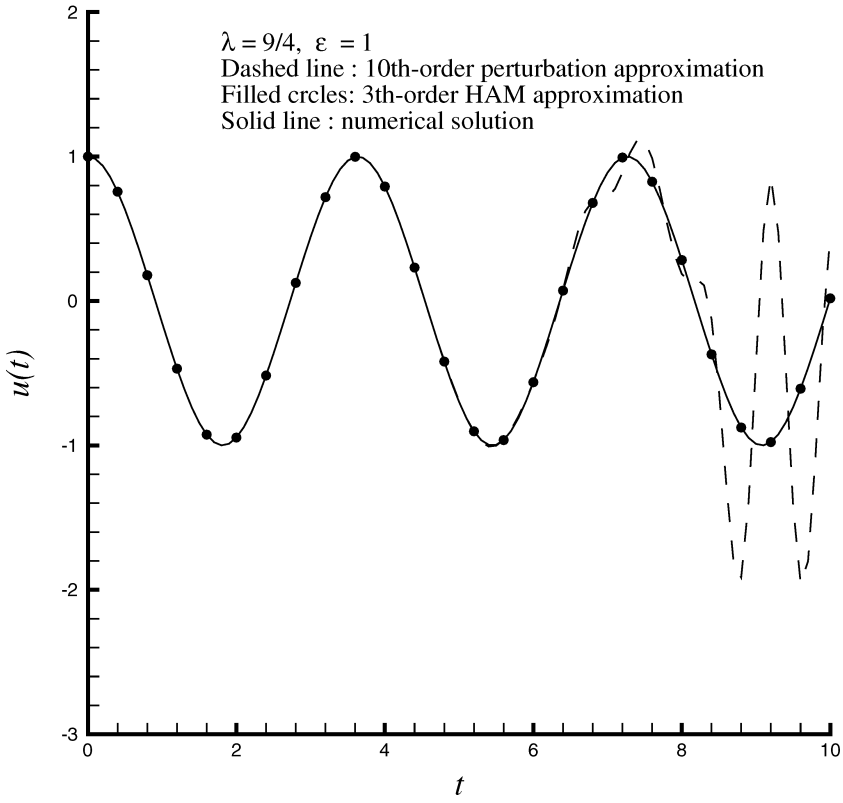

Figure 3. Comparison of the numerical solution with the 10th-order perturbation approximation and the third-order HAM approximation ($\hbar = -1/3$) when $\lambda = 9/4$, $\epsilon = 1$.

solution is expressed more efficiently by periodic basis functions than by polynomials.

It is well known that basis functions have close relationships with linear operators. Thus, it is very important to choose the linear operator $\mathcal{L}$ in the subproblems (6), because $\mathcal{L}$ determines the basis functions of the solution u_n. For example, the linear operator $\mathcal{L}u = u'' + \lambda u$ determines the basis functions of the above-mentioned approximations of Equation (7), given by the straightforward perturbation method [1–3] and Adomian's decomposition method [4–9]. When $\lambda < 0$, the straightforward perturbation approximations of Equation (7) are expressed by the set of basis functions

$$\left\{t^m e^{-n\sqrt{|\lambda|}t},\, t^m e^{n\sqrt{|\lambda|}t} \mid m > 0, n \geq 1\right\}. \tag{15}$$

When $\lambda = 0$, it is expressed by the set of basis functions

$$\{t^{2n} \mid n \geq 0\}. \tag{16}$$

When $\lambda > 0$, the straightforward perturbation approximations of Equation (7) are expressed by the set of basis functions

$$\{t^m \cos(n\sqrt{\lambda}t),\ t^m \sin(n\sqrt{\lambda}t) \mid m \geq 0, n \geq 1\}. \tag{17}$$

Unfortunately, all of above basis functions are *not* periodic, and thus do not match the principal character of Equation (7), whose solution is *periodic* for *all* possible values of $\lambda \in (-\infty, +\infty)$ and $\epsilon > 0$. Obviously, it is much better to express a periodic solution by the set of periodic basis functions

$$\{\cos(n\omega t),\ \sin(n\omega t) \mid n \geq 0\}, \tag{18}$$

where $\omega = 2\pi/T$ is the frequency, and T is the period of the solution. Certainly, for Equation (7), the periodic basis functions (18) are much better than those defined by (15)–(17). This example clearly verifies that, in principle, choosing the linear operator $\mathcal{L}$ in the linear subproblems (6) is to determine the basis functions of the solution of the original nonlinear equation $\mathcal{A}u = 0$. It is a pity that the traditional analytic techniques for nonlinear problems, such as perturbation techniques [1–3], Adomian's decomposition method [4–9], the δ-expansion method [10, 11], and so on, have *no* freedom to choose basis functions of approximate solutions, mainly because these methods have *no* freedom to choose the corresponding linear operator $\mathcal{L}$ of the related subproblems. Without this kind of freedom, approximations given by these analytic approaches are in general strongly dependent upon the value of physical parameters, as shown in Figures 1–3, for examples. Besides, all of these traditional methods can not provide a convenient way to *adjust* and *control* the convergence of approximation solutions.

2.1. Basic ideas

The homotopy is a basic concept in topology [14, 15], which is often applied to investigate the existence and uniqueness of solutions by pure mathematicians. Based on homotopy, some powerful numerical techniques, such as the homotopy continuation method [16–22] and the continuation method [23, 24] are developed, which are widely used to get numerical solutions of nonlinear problems. Based on the concept of homotopy, and with the rapid development of computer techniques and symbolic computation software, an analytical method for strongly nonlinear problems, namely the homotopy analysis method (HAM) [13, 25–29], has been developed since 1992. Different from perturbation techniques [1–3], the homotopy analysis method is *independent* of any small parameters at all. Besides, it provides us with a simple way to ensure the convergence of solution series, so that we can always get accurate enough approximations. Furthermore, as proved by Liao [13, 27], the homotopy analysis method logically contains the so-called nonperturbation methods such as Adomian's decomposition method [4–9], the

δ-expansion method [10, 11], Lyapunov's artificial small parameter method [12], and so on. Using the relationship between the homotopy analysis method and Adomian's decomposition method, Allan [30] investigated the accuracy of approximations given by the Adomian's decomposition method. Currently, Hayat et al. [31], Sajid et al. [32], and Abbasbandy [33–35] pointed out that the so-called "homotopy perturbation method" [36] proposed in 1999 is also a special case of the homotopy analysis method [13, 25] propounded in 1992. Thus, the homotopy analysis method is rather general. More importantly, it provides us with great freedom to choose better basis functions to approximate nonlinear problems. The homotopy analysis method has been successfully applied to many nonlinear problems in science and engineering, such as the similarity boundary-layer flows [27, 37–39], nonlinear heat transfer [40, 33, 35], nonlinear evaluation equations [41], nonlinear waves [34], viscous flows of non-Newtonian fluid [31, 32], Thomas–Fermi atom model [13], Volterra's population model [13], and so on. It has been applied in many fields of researches. For example, Zhu [42, 43] applied the HAM to give, for the first time, an explicit series solution of the famous Black–Scholes type equation in finance for American put option, which is a system of nonlinear PDEs with an unknown moving boundary. Besides, the HAM has been successfully applied to solve some PDEs in fluid mechanics and heat transfer, such as the unsteady boundary-layer viscous flows [28], the unsteady nonlinear heat transfer problem [44], and so on. Especially, the HAM has been successfully applied to find a few new solutions of some nonlinear problems [29, 40], and these new solutions have been never reported even by means of numerical techniques. All of these verify the great potential and validity of the HAM for strongly nonlinear problems in science and engineering.

In this section, we use the nonlinear oscillation problem (7) as an example to describe the basic ideas of the HAM and to illustrate its great freedom on the choice of basis functions. As mentioned before, from the physical points of view, the solution of Equation (7) is periodic for *all* possible values of $\lambda \in (-\infty, +\infty)$ and $\epsilon > 0$. So, it is better and more efficient to use a set of periodic basis functions to approximate it. Let ω and $T = 2\pi/\omega$ denote the frequency and the period of the solution $u(t)$, respectively. Using the transformation

$$\tau = \omega t,$$

Equation (7) becomes

$$\gamma u''(\tau) + \lambda u(\tau) + \epsilon u^3(\tau) = 0, \quad u(0) = 1, \quad u'(0) = 0, \tag{19}$$

where the frequency square $\gamma = \omega^2$ is unknown and is a function of both ϵ and λ. Note that, although the frequency square γ is unknown, it is now very clear that $u(\tau)$ is a function with the known period 2π, i.e., $u(\tau) = u(\tau + 2\pi)$ holds for any τ. Thus, considering the initial condition $u'(0) = 0, u(\tau)$ can be expressed by the periodic basis functions

$$\{\cos n\tau \mid n \geq 0\} \tag{20}$$

in the form

$$u(\tau) = \sum_{n=0}^{+\infty} a_n \cos(n\tau). \tag{21}$$

This provides us the so-called *solution expression* of $u(\tau)$. Our aim is to give a series solution of Equation (19), which *must* be expressed in the above expression.

Let $u_0(\tau)$ denote the initial approximation of $u(\tau)$. According to the solution expression (21) and considering the initial conditions in (19), it is natural to choose

$$u_0(\tau) = \delta + (1 - \delta)\cos\tau, \tag{22}$$

where the parameter δ is determined later. Note that $u_0(\tau)$ satisfies the initial conditions, i.e., $u_0(0) = 1$ and $u_0'(0) = 0$.

Let $\mathcal{L}$ denote an auxiliary linear operator, which will be chosen later. Obviously, $u = 0$ is a solution of the linear equation $\mathcal{L}u = 0$, and thus belongs to the kernel of the linear operator $\mathcal{L}$. Besides, let $\hbar$ denote a nonzero auxiliary parameter, $q \in [0, 1]$ an embedding parameter, respectively. For the sake of simplicity, we define the nonlinear operator

$$\mathcal{N}[\phi(\tau;q), \Gamma(q)] = \Gamma(q)\phi''(\tau;q) + \lambda\phi(\tau;q) + \epsilon\phi^3(\tau;q), \tag{23}$$

where the prime denotes the differentiation with respect to $\tau, \phi(\tau;\ q)$ is a mapping of $u(\tau), \Gamma(q)$ is a kind of mapping of γ, respectively. Then, we construct a homotopy

$$\mathcal{H}[\phi(\tau;q), q] := (1 - q)\mathcal{L}\left[\phi(\tau;q) - u_0(\tau)\right] - \hbar q\mathcal{N}[\phi(\tau;q), \Gamma(q)]. \tag{24}$$

When $q = 0$ and $q = 1$, we have, respectively,

$$\mathcal{H}[\phi(\tau;0), 0] := \mathcal{L}\left[\phi(\tau;0) - u_0(\tau)\right], \quad \text{when } q = 0, \tag{25}$$

$$\mathcal{H}[\phi(\tau;1), 1] := -\hbar\mathcal{N}[\phi(\tau;1), \Gamma(1)], \quad \text{when } q = 1. \tag{26}$$

Thus, as q increases from 0 to 1, the homotopy $\mathcal{H}[\phi(\tau;q), \Gamma(q)]$ *continuously* changes (or deforms) from $\mathcal{L}[\phi(\tau;0) - u_0(\tau)]$ to $-\hbar\mathcal{N}[\phi(\tau;1), \Gamma(1)]$. Thus, enforcing

$$\mathcal{H}[\phi(\tau;q), \Gamma(q)] = 0,$$

we have, from (24), a one-parameter *family* of differential equations in the embedding parameter $q \in [0, 1]$, i.e.,

$$(1 - q)\mathcal{L}\left[\phi(\tau;q) - u_0(\tau)\right] = \hbar q\mathcal{N}[\phi(\tau;q), \Gamma(q)], \tag{27}$$

subject to the initial conditions

$$\phi(0;q) = 1, \quad \phi'(0;q) = 0, \tag{28}$$

where the prime denotes the differentiation with respect to τ, and the nonlinear operator $\mathcal{N}$ is defined by (23).

Note that, the solutions $\phi(\tau;\, q)$ and $\Gamma(q)$ of Equations (27) and (28) are also dependent upon $q \in [0,\, 1]$. Especially, when $q = 0$, it holds

$$\mathcal{L}[\phi(\tau;0) - u_0(\tau)] = 0, \quad \phi(0;0) = 1, \quad \phi'(0;0) = 0. \tag{29}$$

According to the linear property of $\mathcal{L}$, i.e., $\mathcal{L}[0] = 0$, and using the fact that $u_0(\tau)$ satisfies the initial conditions, it is obvious that

$$\phi(\tau;0) = u_0(\tau) = \delta + (1-\delta)\cos\tau. \tag{30}$$

When $q = 1$, because $\hbar \neq 0$, Equations (27) and (28) are equivalent to

$$\mathcal{N}[\phi(\tau;1), \Gamma(1)] = 0, \quad \phi(0;1) = 1, \quad \phi'(0;1) = 0, \tag{31}$$

which are exactly the same as the original equation (19), provided

$$\phi(\tau;1) = u(\tau), \quad \Gamma(1) = \gamma. \tag{32}$$

So, as the embedding parameter $q \in [0,\, 1]$ increases from 0 to 1, $\phi(\tau;\, q)$ varies *continuously* from the initial guess $u_0(\tau)$ to the solution $u(\tau)$ of Equation (19), so does $\Gamma(q)$ from its initial guess γ_0 to the frequency square $\gamma = \omega^2$ of the solution $u(t)$ of Equation (7). Note that γ_0 is unknown right now and will be determined later. This kind of continuous variation is called deformation in topology. So, we call Equations (27) and (28) *the zeroth-order deformation equations*.

Note that $\phi(\tau;\, q)$ and $\Gamma(q)$ depend on the embedding parameter $q \in [0,\, 1]$, and thus can be expanded in a power series in q as follows

$$\phi(\tau;q) = u_0(\tau) + \sum_{n=1}^{+\infty} u_n(\tau) q^n, \tag{33}$$

$$\Gamma(q) = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n q^n, \tag{34}$$

where

$$u_n(\tau) = \left.\frac{1}{n!}\frac{\partial^n \phi(\tau;q)}{\partial q^n}\right|_{q=0}, \quad \gamma_n = \left.\frac{1}{n!}\frac{d^n \Gamma(q)}{dq^n}\right|_{q=0}. \tag{35}$$

Here, we use (30) and define $\gamma_0 = \Gamma(0)$. Note that, according to the fundamental theorems in calculus [45], $u_n(\tau)$ and γ_n are *unique*, and are completely determined by the zeroth-order deformation equations (27) and (28).

According to (32), we get the solution of Equation (19) just at $q = 1$. Therefore, it is important to ensure that the series (33) and (34) converge at

$q = 1$. However, in most cases, a power series has a bounded convergence radius. Fortunately, both $\phi(\tau; q)$ and $\Gamma(q)$ are dependent upon *not only* the embedding parameter $q \in [0, 1]$ *but also* the auxiliary parameter $\hbar$ and the auxiliary linear operator $\mathcal{L}$, because both of them also appear in the zeroth-order deformation equation (27). More importantly, we have great *freedom* to choose both $\hbar$ and $\mathcal{L}$. Thus, assuming that $\hbar$ and $\mathcal{L}$ are properly chosen so that the series (33) and (34) are convergent at $q = 1$, we have, using (32), the solution series

$$u(\tau) = u_0(\tau) + \sum_{n=1}^{+\infty} u_n(\tau), \tag{36}$$

$$\gamma = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n. \tag{37}$$

According to Liao's general proof ([13], Chapter 3), the above series must converge to the exact solution of the nonlinear oscillation equation (7), as long as they are not divergent.

For simplicity, define the vectors

$$\vec{u}_m = \{u_0(\tau), u_1(\tau), u_2(\tau), \ldots, u_m(\tau)\}, \quad \vec{\gamma}_m = \{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}.$$

According to the fundamental theorems in calculus [45], $u_n(\tau)$ is unique, thus the governing equation of $u_n(\tau)$ is unique, too, and therefore can be derived directly from the zeroth-order deformation equations (27) and (28). There are two different ways to do so. Each of them gives the same results, as shown below. First, differentiating Equations (27) and (28) n times with respect to q, then dividing by $n!$, and finally setting $q = 0$, we have, using the definitions (35) of $u_n(\tau)$ and γ_n, the so-called nth-order deformation equation

$$\mathcal{L}[u_n(\tau) - \chi_n u_{n-1}(\tau)] = \hbar R_n(\vec{u}_{n-1}, \vec{\gamma}_{n-1}), \tag{38}$$

subject to the initial conditions

$$u_n(0) = 0, \quad u_n'(0) = 0, \tag{39}$$

where

$$\begin{aligned} R_n(\vec{u}_{n-1}, \vec{\gamma}_{n-1}) &= \frac{1}{(n-1)!} \left\{ \frac{\partial^{n-1} \mathcal{N}[\phi(\tau; q); \Gamma(q)]}{\partial q^{n-1}} \right\} \Bigg|_{q=0} \\ &= \sum_{k=0}^{n-1} \gamma_k u''_{n-1-k}(\tau) + \lambda u_{n-1}(\tau) + \epsilon \sum_{k=0}^{n-1} u_{n-1-k}(\tau) \sum_{j=0}^{k} u_{k-j}(\tau) u_j(\tau), \end{aligned} \tag{40}$$

and

$$\chi_n = \begin{cases} 0, & n \le 1, \\ 1, & n > 1. \end{cases} \tag{41}$$

The detailed deduction of Equations (38) and (39) is given in Appendix A. Alternatively, as pointed out by Hayat et al. [31], directly substituting (33) and (34) into the zeroth-order deformation equations (27) and (28), and equating the coefficients of the like power of q, one can get exactly the *same* equations as (38) and (39). For details, please refer to Appendix B. This is mainly because, due to the fundamental theorems in calculus [45], $u_n(\tau)$ and γ_n in the series (33) and (34) are *unique*, and thus are certainly governed by the *same* equations.

Note that the high-order deformation equations (38) and (39) are linear. Thus, we transfer the nonlinear differential equation (19) of order 2 into infinite number of linear differential equations (38) and (39) with order k, the order of the auxiliary linear operator $\mathcal{L}$. It should be emphasized that, different from perturbation techniques, such kind of transformation does *not* need any small/large physical parameters. Besides, we have great *freedom* to choose the auxiliary linear operator $\mathcal{L}$ so that $\mathcal{L}$ might be quite different from $\mathcal{L}_0$ and the order k of $\mathcal{L}$ might be even *not* equal to 2, as illustrated in Section 2.2.

The freedom on the choice of the auxiliary linear operator $\mathcal{L}$ is very important for us. Using this kind of freedom, we can express $\mathcal{L}$ in such a general form of the mth-order linear operator:

$$\mathcal{L}u = u^{(m)} + a_1(\tau)u^{(m-1)} + \cdots + a_{m-1}(\tau)u' + a_m(\tau)u,$$

where $u^{(k)}$ denotes the kth differentiation of $u(\tau)$ with respect to τ, $a_j(\tau)$ is a real function to be determined. Here, the order m and the coefficients $a_j(\tau)$ ($j = 1, 2, 3, \ldots, m$) of the linear operator $\mathcal{L}$ are unknown, and should be determined properly. Because the original nonlinear oscillation problem is governed by the second-order equation (19), it is *natural*[2] for us to choose $m = 2$, i.e.,

$$\mathcal{L}u = u'' + a_1(\tau)u' + a_2(\tau)u. \tag{42}$$

Let $w_1(\tau)$, $w_2(\tau)$ denote the non-zero solutions of $\mathcal{L}u = 0$, and $u_n^*(\tau)$ the special solution of (38). Then, the general solution of Equation (38) reads

$$u_n(\tau) = \chi_n u_{n-1}(\tau) + u_n^*(\tau) + C_1 w_1(\tau) + C_2 w_2(\tau), \tag{43}$$

where C_1, C_2 are constant coefficients. Note that our aim is to give series solutions with the known period 2π. So, $w_1(\tau)$ and $w_2(\tau)$ must be periodic functions such as $\sin(n\tau)$ and $\cos(n\tau)$, where n is a positive integer. Obviously, we should choose

[2]In fact, it is not so "natural" as one might think. We will show in the next subsection that one can use (2κ)th-order linear operator $\mathcal{L}$, defined by (57), to solve the same problem, where $\kappa = 1, 2, 3, \ldots$.

$$w_1(\tau) = \cos\tau, \quad w_2(\tau) = \sin\tau.$$

Because $w_1(\tau)$ and $w_2(\tau)$ are solutions (i.e., kernel) of $\mathcal{L}u = 0$, it holds for any non-zero constant coefficients C_1 and C_2 that

$$\mathcal{L}[C_1\cos\tau + C_2\sin\tau] = 0.$$

Then, using the above equation and the definition (42) of $\mathcal{L}$, we have

$$\{[a_2(\tau) - 1]\cos\tau - a_1(\tau)\sin\tau\}C_1 + \{[a_2(\tau) - 1]\sin\tau + a_1(\tau)\cos\tau\}C_2 = 0. \tag{44}$$

The above equation holds for *any* coefficients C_1, C_2 and *all* $\tau \in [0, +\infty)$, if and only if

$$a_1(\tau) = 0, \quad a_2(\tau) = 1. \tag{45}$$

Substituting them into (42), we obtain the auxiliary linear operator

$$\mathcal{L}u = u'' + u, \tag{46}$$

which has the property

$$\mathcal{L}[C_1\cos\tau + C_2\sin\tau] = 0 \tag{47}$$

for any constants C_1 and C_2. In other words, $\cos\tau$ and $\sin\tau$ belong to the kernel of the auxiliary linear operator $\mathcal{L}$. Then, the general solution of Equation (38) reads

$$u_n(\tau) = \chi_n u_{n-1}(\tau) + u_n^*(\tau) + C_1\cos\tau + C_2\sin\tau. \tag{48}$$

Using the initial condition (39), we have

$$C_1 = -\chi_n u_{n-1}(0) - u_n^*(0), \quad C_2 = 0.$$

Thus, the solution of the nth-order deformation Equations (38) and (39) reads

$$u_n(\tau) = \chi_n u_{n-1}(\tau) + u_n^*(\tau) - \left[\chi_n u_{n-1}(0) + u_n^*(0)\right]\cos\tau. \tag{49}$$

How to get a special solution $u_n^*(\tau)$ of Equations (38) and (39)? Note that both of $u_n^*(\tau)$ and γ_{n-1} are unknown, but we have only one governing equation for $u_n^*(\tau)$. So, one additional algebraic equation is needed to determine γ_{n-1}. To show how to get $u_n^*(\tau)$ and γ_{n-1}, let us consider the first-order equation

$$\mathcal{L}[u_1(\tau)] = \hbar R_1(u_0), \quad u_1(0) = 0, \quad u_1'(0) = 0,$$

where we have, using (22) and (40), that

$$\begin{aligned} R_1(\tau) &= \gamma_0 u_0''(\tau) + \lambda u_0(\tau) + \epsilon u_0^3(\tau) \\ &= A_0 + A_1\cos\tau + A_2\cos 2\tau + A_3\cos 3\tau, \end{aligned}$$

in which A_0, A_1, A_2 and A_3 are constant coefficients, and

$$A_0 = \left(\frac{\delta}{2}\right)[2\lambda + \epsilon(5\delta^2 - 6\delta + 3)],$$

$$A_1 = \left(\frac{\delta - 1}{4}\right)\left[4\gamma_0 - 4\lambda - \epsilon(15\delta^2 - 6\delta + 3)\right].$$

According to the property (47), if $A_1 \neq 0$, then $u_1(\tau)$ contains the so-called secular term $\tau\cos\tau$, which however is *not* periodic and thus disobeys the solution expression (21). To avoid this, we had to enforce $A_1 = 0$, which provides us with one additional algebraic equation of γ_0, i.e.,

$$4\gamma_0 - 4\lambda - \epsilon(15\delta^2 - 6\delta + 3) = 0,$$

whose solution is

$$\gamma_0 = \lambda + \frac{3\epsilon}{4} + \frac{\epsilon}{4}(15\delta^2 - 6\delta). \tag{50}$$

Thereafter, it is easy to get the special solution

$$u_1^*(\tau) = A_0 - \frac{A_2}{3}\cos 2\tau - \frac{A_3}{8}\cos 3\tau.$$

Then, substituting it into (49), we have

$$u_1(\tau) = A_0 - \left(A_0 - \frac{A_2}{3} - \frac{A_8}{8}\right)\cos\tau - \frac{A_2}{3}\cos 2\tau - \frac{A_3}{8}\cos 3\tau.$$

Similarly, we can solve γ_{n-1} and $u_n(\tau)$ successively in the order $n = 2, 3, 4, \ldots$.

Here, the key point is to ensure that $u_n(\tau)$ must obey *the solution expression* (21). In other words, the initial guess, the auxiliary linear operator, and so on, must be chosen in such a way that the solutions of all high-order deformation equations (38) and (39) can be expressed by the solution expression (21). This idea is so important in the frame of the homotopy analysis method that it is regarded as a fundamental rule, namely *the rule of solution expression*. For details, please refer to Liao ([13], chapters 2 and 3).

Note that there is a unknown parameter δ in the initial guess $u_0(\tau)$, which we also have freedom to choose. The first-order approximation reads

$$u(\tau) \approx u_0 + u_1$$

$$= (\delta + A_0) - \left(A_0 - \frac{A_2}{3} - \frac{A_8}{8} + \delta - 1\right)\cos\tau - \frac{A_2}{3}\cos 2\tau - \frac{A_3}{8}\cos 3\tau.$$

Obviously, at the first-order of approximation, we have the equilibrium point

$$\int_0^{2\pi} u(\tau)\,d\tau = A_0 + \delta.$$

So, physically, δ can be regarded as the initial guess of the equilibrium point. Assuming that δ is so properly chosen that no modification of equilibrium point is needed at the first-order of approximation, i.e., $A_0 = 0$, we have

$$\delta = \begin{cases} 0, & \text{when } \lambda \geq -3\epsilon/5, \\ \dfrac{1}{5}\left[3 + \sqrt{\left|6 + \dfrac{10\lambda}{\epsilon}\right|}\right], & \text{when } \lambda < -3\epsilon/5. \end{cases} \tag{51}$$

The mth-order approximations are given by

$$u(\tau) \approx u_0(\tau) + \sum_{k=1}^{m} u_k(\tau), \tag{52}$$

$$\gamma \approx \gamma_0 + \sum_{k=1}^{m} \gamma_k. \tag{53}$$

According to Liao's general proof ([13], chapter 3), the series of γ converges to the exact value of ω^2 as long as it is not divergent. It should be emphasized that, for given λ and ϵ, we still have freedom to choose the value of the auxiliary parameter $\hbar$. Because $\hbar$ appears in the high-order deformation equation (38), both of γ and $u(\tau)$ are *mathematically* dependent upon $\hbar$. However, *physically*, for given λ and ϵ, there exists a *unique* value of $\gamma = \omega^2$, and thus all convergent series of γ given by different values of $\hbar$ should give the same result. For example, let us consider the case of $\epsilon = 1$ with three different values of λ, i.e., $\lambda = -9/4,\ 0,\ 9/4$, respectively. In each case, the series of γ indeed converges to the *same* value in a finite region of $\hbar$, as shown in Figure 4. Note that, each curve has a horizontal part, which corresponds to the so-called *valid region* of $\hbar$: the corresponding series of γ converges to the exact value of ω^2 if one chooses a value of $\hbar$ in such a region. The so-called $\hbar$-curves of γ clearly indicate the valid regions of $\hbar$ for given λ and ϵ:

$$\begin{aligned} -0.4 < \hbar < 0, &\quad \text{when } \lambda = -9/4, \quad \epsilon = 1, \\ -2 < \hbar < 0, &\quad \text{when } \lambda = 0, \quad \epsilon = 1, \\ -0.6 < \hbar < 0, &\quad \text{when } \lambda = 9/4, \quad \epsilon = 1. \end{aligned}$$

By means of choosing a proper value of $\hbar$ in the corresponding valid region, we get the convergent result of γ, as shown in Table 1, and further more, the convergent series of the corresponding $u(\tau)$, as shown in Figures 1–3. Therefore, the auxiliary parameter $\hbar$ indeed provides us with a simple way to *control* and *adjust* the convergence of the solution series.

Here the auxiliary parameter $\hbar$ plays a very important role: it ensures the convergence of the solution series. As proved by Liao in general ([13], chapter 3), some other techniques such as Adomian's decomposition method [4], the

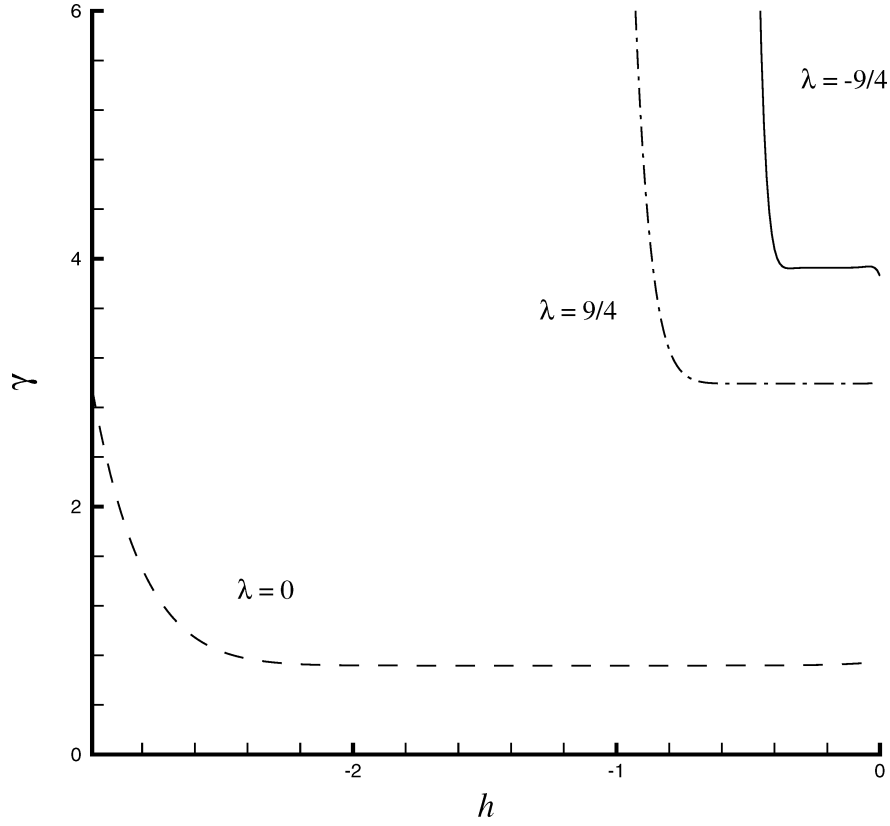

Figure 4. The 10th-order approximation of γ versus $\hbar$ when $\epsilon = 1$ for different values of λ. Solid line: $\lambda = -9/4$, Dashed line: $\lambda = 0$, Dash–dotted line: $\lambda = 9/4$.

Table 1
The HAM Approximation of $\gamma = \omega^2$ When $\epsilon = 1$ for Different λ in Equation (7)

Approx. order	$\lambda = -9/4, \hbar = -1/4$	$\lambda = 0, \hbar = -1$	$\lambda = 9/4, \hbar = -1/3$
2	4.00785	0.72656	2.99219
4	3.92638	0.71885	2.99217
6	3.92676	0.71794	2.99217
8	3.92803	0.71780	2.99217
10	3.92781	0.71778	2.99217
12	3.92780	0.71777	2.99217
14	3.92781	0.71777	2.99217
16	3.92781	0.71777	2.99217
18	3.92781	0.71777	2.99217
20	3.92781	0.71777	2.99217

δ-expansion method [10] and Lyapunov's artificial small parameter method [12] are only special cases of the homotopy analysis method when $\hbar = -1$. Currently, Hayat et al. [31], Sajid et al. [32], and Abbasbandy [33–35] pointed out that the so-called "homotopy perturbation method" [36] proposed in 1999 is also a special case of the homotopy analysis method (when $\hbar = -1$) [25, 13] propounded in 1992. However, using $\hbar = -1$, one can *not* get convergent value of $\gamma = \omega^2$ when $\epsilon = 1$ and $\lambda = \pm 9/4$, as shown in Figure 4. So, although choosing good basis functions is important, it is unfortunately *not* enough to ensure the convergence of solution series, and thus the auxiliary parameter $\hbar$ is *absolutely* necessary. Therefore, it is the auxiliary parameter $\hbar$ which provides us with a simple way to ensure the convergence of the solution series.

To show the importance of the auxiliary parameter $\hbar$, let us further consider the convergence of $\gamma = \omega^2$ in case of $\lambda = 0$ with all possible values of ϵ, i.e., $0 < \epsilon < +\infty$. The corresponding 10th-order HAM approximation is

$$\begin{aligned}\gamma = {} & 0.75\epsilon + 0.234375\epsilon^2\hbar + 0.791016\epsilon^3\hbar^2 + 1.628723\epsilon^4\hbar^3 + 2.260265\epsilon^5\hbar^4 \\ & + 2.202903\epsilon^6\hbar^5 + 1.522699\epsilon^7\hbar^6 + 0.735085\epsilon^8\hbar^7 + 0.236597\epsilon^9\hbar^8 \\ & + 0.0457473\epsilon^{10}\hbar^9 + 0.00402753\epsilon^{11}\hbar^{10}. \end{aligned} \tag{54}$$

The above expression of γ contains the auxiliary parameter $\hbar$. As a result, its convergence region is dependent upon $\hbar$: the convergence region of γ increases as $\hbar$ ($\hbar < 0$) increases and tends to zero, as shown in Figure 5. Note that the auxiliary parameter $\hbar$ is *unnecessary* to be a constant: $\hbar$ can be a function of physical parameters, such as ϵ and λ for this example. It is found that, when $\hbar = -(1+\epsilon)^{-1}$, even the second-order approximation of γ, i.e.,

$$\gamma = \frac{3\epsilon(128 + 248\epsilon + 123\epsilon^2)}{512(1+\epsilon)^2}, \tag{55}$$

agrees well with the numerical results in the *whole* region $0 < \epsilon < +\infty$, as shown in Figure 5. Thus, when $\lambda = 0$, we have a simple but rather accurate approximation of the period of oscillation

$$T = \frac{2\pi}{\sqrt{\gamma}} = \frac{32\sqrt{2}\pi(1+\epsilon)}{\sqrt{3\epsilon(128 + 248\epsilon + 123\epsilon^2)}}, \tag{56}$$

which is valid for *all* possible values of the physical parameter $0 < \epsilon < +\infty$, as shown in Figure 6. All of these indicate that, the auxiliary parameter $\hbar$ indeed provides us with a simple way to ensure the convergence of solution series. This is an obvious advantage of the homotopy analysis method over all other perturbation and nonperturbation techniques. It is the essential reason why the auxiliary parameter $\hbar$ is introduced in the zeroth-order deformation equation (27).

Note that even the second-order approximation (55) gives very good approximation of $\gamma = \omega^2$. In such kind of cases, like perturbation approximations,

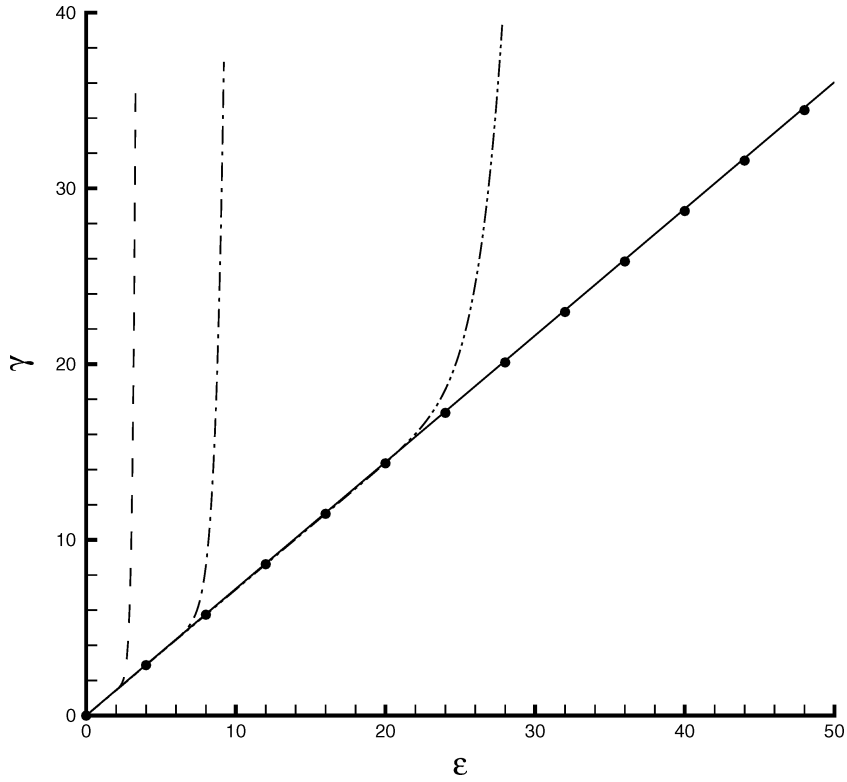

Figure 5. The curves of $\gamma \sim \epsilon$ at the 10th-order approximation when $\lambda = 0$. Dashed line: $\hbar = -1$; Dash–dotted line: $\hbar = -1/3$; Dash–dot–dotted line: $\hbar = -1/10$; Solid line: $\hbar = -1/(1 + \epsilon)$; Filled circles: numerical results.

the first few terms of the HAM solution series can give rather accurate results. However, different from perturbation approximations, the HAM result (55) is valid for *all* possible parameter $0 < \epsilon < +\infty$, and thus is independent of small physical parameters.

In this section, we illustrate that the homotopy analysis method have the following advantages:

1. Different from perturbation techniques, it is independent of small parameters.
2. Different from *all* other perturbation and non-perturbation methods, it provides us with great freedom to choose the auxiliary linear operator so that we can use better basis functions to approximate solutions of nonlinear problems more efficiently.
3. Different from *all* other perturbation and non-perturbation methods, it provides us with a simple way to ensure the convergence of solution series by means of the auxiliary parameter $\hbar$.

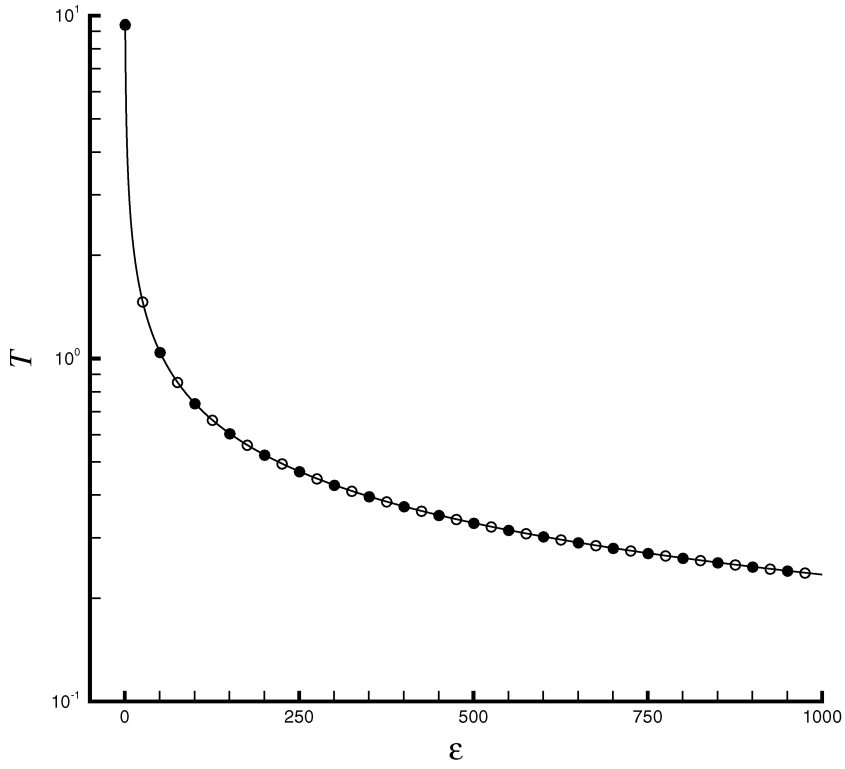

Figure 6. The period T versus ϵ when $\lambda = 0$. Solid line: numerical result; Open circles: HAM results given by (56); Filled circles: HAM results given by (64).

Here, we emphasize once again that, although choosing a set of good basis functions is very important, it is unfortunately *not* enough to ensure the convergence of solution series, and thus the auxiliary parameter $\hbar$ is *absolutely* necessary.

2.2. The order of the auxiliary linear operator $\mathcal{L}$

In Section 2.1, we choose a second-order auxiliary linear operator $\mathcal{L}u = u'' + u$, mainly because Equation (19) of nonlinear oscillation is second-order. In other words, the second-order nonlinear differential equation (19) is replaced by an infinite number of the second-order linear subproblems governed by (38) and (39): the order keeps the same.

The order of the auxiliary linear operator $\mathcal{L}$ of the linear subproblems *must* be equal to that of the original nonlinear ones? If one employs traditional methods, the answer seems to be out of question: yes. However, many of our traditional thoughts and concepts have been proved to be wrong: one famous example is Newton's concept of the absolute time, which was replaced by Einstein's concept of the time–space. In this subsection, using Equation (19)

as an example, we show that, against the traditional thoughts, the order of the auxiliary linear operator $\mathcal{L}$ of the linear subproblems is *unnecessary* to be the same as that of the original nonlinear ones.

Here, we use some auxiliary linear operators $\mathcal{L}$ of different orders, but keep others (such as u_0, γ_0 and δ, except $\hbar$) the same. Let us consider a more general expression of the auxiliary linear operator (46), i.e.,

$$\mathcal{L}u = \frac{\partial^{2\kappa} u}{\partial \tau^{2\kappa}} + (-1)^{\kappa+1} u, \quad \kappa = 1, 2, 3, \ldots, \tag{57}$$

where $\kappa \geq 1$ is *any* a positive integer. When $\kappa = 1$, it is exactly the same as the second-order linear operator (46). When $\kappa = 2$, 3, 4, it becomes the fourth, sixth, eighth-order linear operator, respectively. The linear operator (57) is (2κ)th-order, whose inverse operator $\mathcal{L}^{-1}$ has the property

$$\mathcal{L}^{-1}[C_1 \sin(m\tau) + C_2 \cos(m\tau)] = \frac{(-1)^{\kappa+1}[C_1 \sin(m\tau) + C_2 \cos(m\tau)]}{(1 - m^{2\kappa})} \tag{58}$$

and

$$\mathcal{L}^{-1}[C_1] = (-1)^{\kappa+1} C_1 \tag{59}$$

for any constant coefficients C_1 and C_2.

When $\kappa = 2$, the auxiliary linear operator (57) becomes $\mathcal{L}u = u^{(4)} - u$, where $u^{(4)}$ denotes the fourth-order differentiation with respect to τ. This fourth-order linear operator has the property

$$\mathcal{L}\left[C_1 \cos\tau + C_2 \sin\tau + C_3 e^{\tau} + C_4 e^{-\tau}\right] = 0 \tag{60}$$

for any constant coefficients C_1, C_2, C_3, and C_4. Using the definition of $\mathcal{L}u = u^{(4)} - u$, the general solution of Equation (38) reads

$$u_n(\tau) = \chi_n u_{n-1}(\tau) + u_n^*(\tau) + C_1 \cos\tau + C_2 \sin\tau + C_3 e^{\tau} + C_4 e^{-\tau}, \tag{61}$$

where $u_n^*(\tau)$ is a special solution of (38), and C_1, C_2, C_3, C_4, are integral coefficients. Note that there are *four* integral coefficients C_1, C_2, C_3, and C_4, but we have only *two* initial conditions (39). However, the terms e^{τ} and $e^{-\tau}$ are *not* periodic functions, and thus *disobey* the solution expression (21). Therefore, they can not appear in the expression of $u_n(\tau)$, according to the so-called *rule of solution expression*: all solutions of the high-order deformation equations (38) and (39) must be expressed in the form (21). So, to obey the solution expression (21), we must enforce $C_3 = C_4 = 0$. In other words, the solution expression (21) implies one additional period condition

$$u_n(\tau) = u_n(\tau + 2\pi),$$

which enforces $C_3 = C_4 = 0$. This condition comes from the periodic property of $u(\tau)$, i.e.,

$$u(\tau) = u(\tau + 2\pi).$$

Thereafter, C_1 and C_2 are uniquely determined by the two initial conditions (39).

Similarly, when $\kappa = 3$, the auxiliary linear operator (57) becomes $\mathcal{L}u = u^{(6)} + u$, and the general solution of Equation (38) reads

$$u_n(\tau) = \chi_n u_{n-1}(\tau) + u_n^*(\tau) + C_1 \cos\tau + C_2 \sin\tau + e^{\sqrt{3}\tau/2}(C_3 \cos\tau + C_4 \sin\tau) + e^{-\sqrt{3}\tau/2}(C_5 \cos\tau + C_6 \sin\tau)\big], \quad (62)$$

where C_i is a constant coefficient. Similarly, to obey the solution expression (21), we must enforce

$$C_3 = C_4 = C_5 = C_6 = 0.$$

Then, C_1 and C_2 are uniquely determined by the two initial conditions (39).

Similarly, for any positive integers $\kappa \geq 1$, we can always get a *periodic* solution $u_n(\tau)$ of Equations (38) and (39) by means of the (2κ)th-order auxiliary linear operator $\mathcal{L}$ defined by (57). Here, it should be emphasized that, although e^{τ}, $e^{\tau}\cos\tau$ and $e^{\tau}\sin\tau$, which tend to infinity as $\tau \to +\infty$, are so-called secular terms, the terms $e^{-\tau}$, $e^{-\tau}\cos\tau$, $e^{-\tau}\sin\tau$ tend to zero as $\tau \to +\infty$ and thus do not belong to the traditional definition of secular term. So, using the traditional ideas of "avoiding the secular terms," we can *not* avoid the appearance of the non-period terms $e^{-\tau}$, $e^{-\tau}\cos\tau$, $e^{-\tau}\sin\tau$ in $u_n(\tau)$. In this meaning, the so-called *rule of solution expression* in the frame of the HAM is more general than the traditional idea of "avoiding secular terms."

For given λ, ϵ, and κ, we can get convergent solution series of $u(\tau)$ and γ in the similar way as mentioned in Section 2.1. It should be emphasized that, for all auxiliary linear operator $\mathcal{L}$ defined by (57), we still have great freedom to choose a proper value of the auxiliary parameter $\hbar$. For example, let us consider again the cases of $\epsilon = 1$ and $\lambda = -9/4$, 0, $9/4$, respectively. When $\kappa = 2$, the curves of $\gamma \sim \hbar$ for three different cases are as shown in Figures 7 and 8. From the so-called $\hbar$-curves of $\gamma \sim \hbar$, it is easy to find out the valid regions of $\hbar$ for different cases:

$$\begin{aligned} &0 < \hbar < 0.4, \quad \text{when } \lambda = -9/4,\ \epsilon = 1,\ \kappa = 2,\\ &0 < \hbar < 8, \quad \text{when } \lambda = 0,\ \epsilon = 1,\ \kappa = 2,\\ &0 < \hbar < 20, \quad \text{when } \lambda = 9/4,\ \epsilon = 1,\ \kappa = 2. \end{aligned}$$

Choosing a proper value of $\hbar$ in the above valid region in each case, we obtain the convergent series solution for both γ and $u(t)$, which agree well with numerical results, as shown in Figure 9. Note that, the valid regions of $\hbar$ in case of $\kappa = 2$ are quite different from those in case of $\kappa = 1$: $\hbar$ is negative in case of $\kappa = 1$, but must be positive in case of $\kappa = 2$. This once again indicates that the auxiliary parameter $\hbar$ provides us with a simple way to ensure the convergence

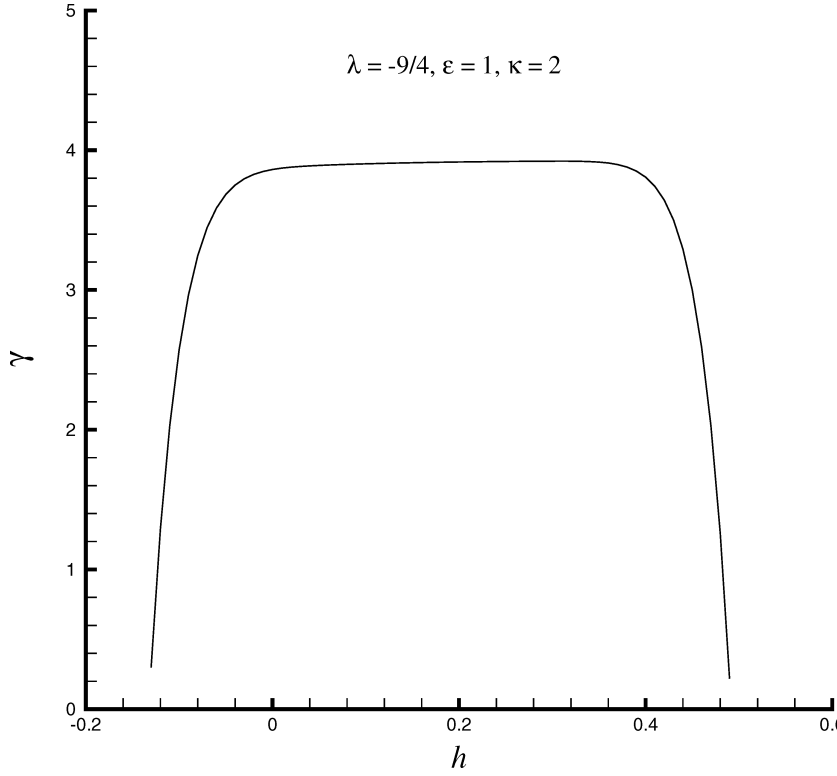

Figure 7. The curve $\gamma \sim \epsilon$ when $\lambda = -9/4$, $\epsilon = 1$ and $\kappa = 2$.

of the solution series. Besides, it verifies that, although choosing good basis functions is very important, it is however *not* enough to ensure the convergence of solution series. Thus, it is *absolutely* necessary to choose a proper value of the auxiliary parameter $\hbar$ so as to get accurate results of a nonlinear problem.

To further show the importance of the auxiliary parameter $\hbar$, let us consider Equation (19) in case of $\kappa = 2$, $\lambda = 0$ for all possible values of ϵ, i.e., $0 < \epsilon < +\infty$. In this case, the convergence region of the series of γ increases as $\hbar$ ($\hbar > 0$) decreases, as shown in Figure 10. Especially, when $\hbar = 1/(1 + \epsilon/15)$, even the second-order approximation of γ, i.e.,

$$\gamma = \frac{3\epsilon(1497600 + 190320\epsilon + 6401\epsilon^2)}{26624(15 + \epsilon)^2}, \tag{63}$$

agrees well with the numerical results for all *possible* physical parameter $0 < \epsilon < +\infty$, as shown in Figure 10. The above result gives a simple but accurate expression of the period of oscillation

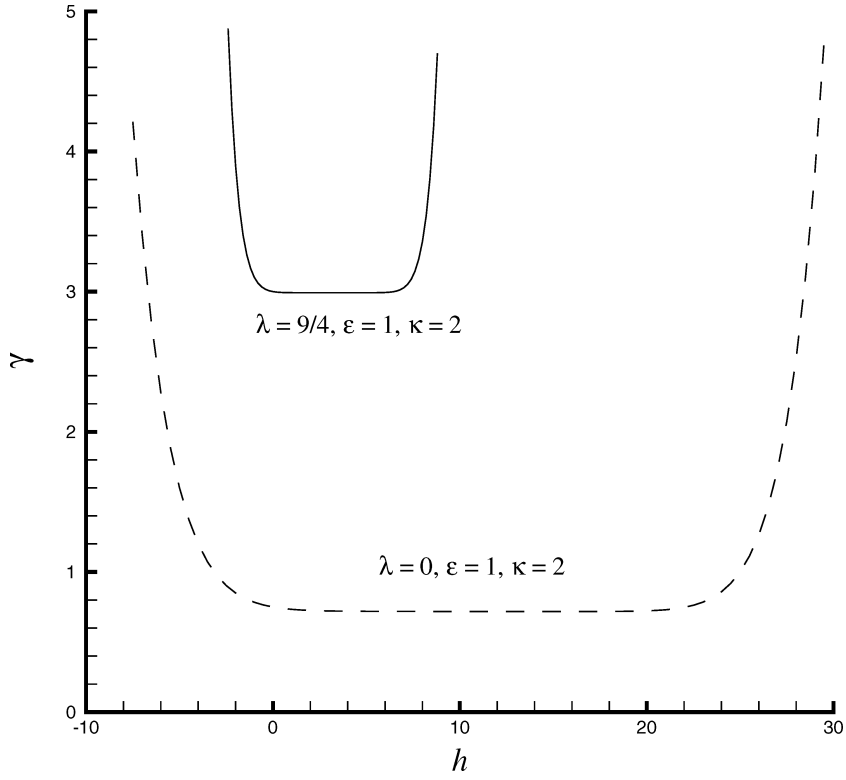

Figure 8. The curve $\gamma \sim \epsilon$ when $\kappa = 2$, $\epsilon = 1$, $\lambda = 9/4$ and $\lambda = 0$. Solid line: $\lambda = 9/4$; Dashed line: $\lambda = 0$.

$$T = \frac{64\sqrt{26}\pi(15+\epsilon)}{\sqrt{3\epsilon(1497600 + 190320\epsilon + 6401\epsilon^2)}}, \tag{64}$$

which agrees well with the numerical results for *all* $0 < \epsilon < +\infty$, as shown in Figure 6. Similarly, when $\kappa = 3$ and $\epsilon = 1$, by means of plotting the $\gamma \sim \hbar$ curves as shown in Figures 11 and 12, it is easy to find the corresponding valid region of $\hbar$ for different values of λ, i.e.,

$$\begin{aligned} -0.4 < \hbar < 0, &\quad \text{when } \lambda = -9/4,\ \epsilon = 1,\ \kappa = 3, \\ -60 < \hbar < 0, &\quad \text{when } \lambda = 0,\ \epsilon = 1,\ \kappa = 3, \\ -200 < \hbar < 0, &\quad \text{when } \lambda = 9/4,\ \epsilon = 1,\ \kappa = 3. \end{aligned}$$

For given λ, by means of choosing a proper value of $\hbar$ in the corresponding valid region of $\hbar$, we get convergent solution series for both γ and $u(t)$, which agree well with numerical results, as shown in Figure 13. Note that, different from the case of $\kappa = 2$, all valid values of $\hbar$ for $\kappa = 3$ are negative. Besides, in case of $\kappa = 3$, $\lambda = 0$ and $0 < \epsilon < +\infty$, the convergence region of the series of

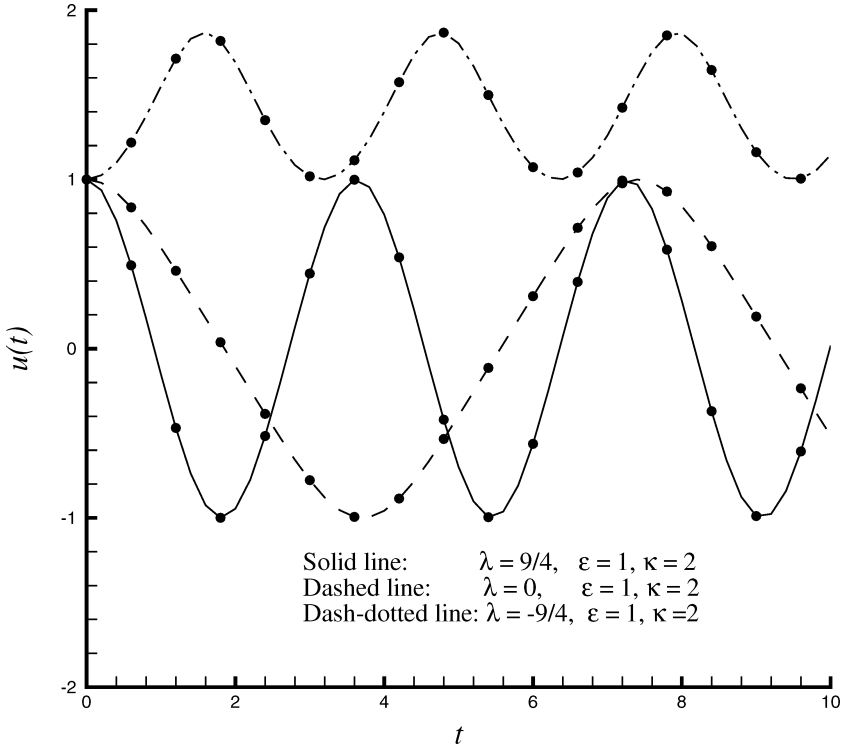

Figure 9. Comparison of the numerical solutions with the HAM approximations of $u(t)$ when $\kappa = 2$, $\epsilon = 1$. Dash–dotted line: 20th-order HAM solution when $\lambda = -9/4$ by means of $\hbar = 1/4$; Dashed line: 10th-order HAM solution when $\lambda = 0$ by means of $\hbar = 10$; Solid line: 10th-order HAM solution when $\lambda = 9/4$ by means of $\hbar = 5$.

γ increases as $\hbar$ ($\hbar < 0$) increases and tends to zero, as shown in Figure 14. Similarly, when $\kappa = 3$, $\lambda = 0$ and $0 \leq \epsilon < +\infty$, even the second-order approximation of γ by means of $\hbar = -1/(1 + \epsilon/100)$, i.e.,

$$\gamma = \frac{\epsilon(4739280000 + 91530600\epsilon + 455753\epsilon^2)}{631904(100 + \epsilon)^2}, \tag{65}$$

agrees well with the numerical results for *all* possible physical parameter $0 < \epsilon < +\infty$, as shown in Figure 14. The above result gives a simple but accurate expression of the period of oscillation

$$T = \frac{56\sqrt{806}\pi(100 + \epsilon)}{\sqrt{\epsilon(4739280000 + 91530600\epsilon + 455753\epsilon^2)}}, \tag{66}$$

which agrees well with the numerical results for *all* possible values $0 < \epsilon < +\infty$. All of these once again indicate that the auxiliary parameter $\hbar$ indeed provides us with a simple way to ensure the convergence of solution series.

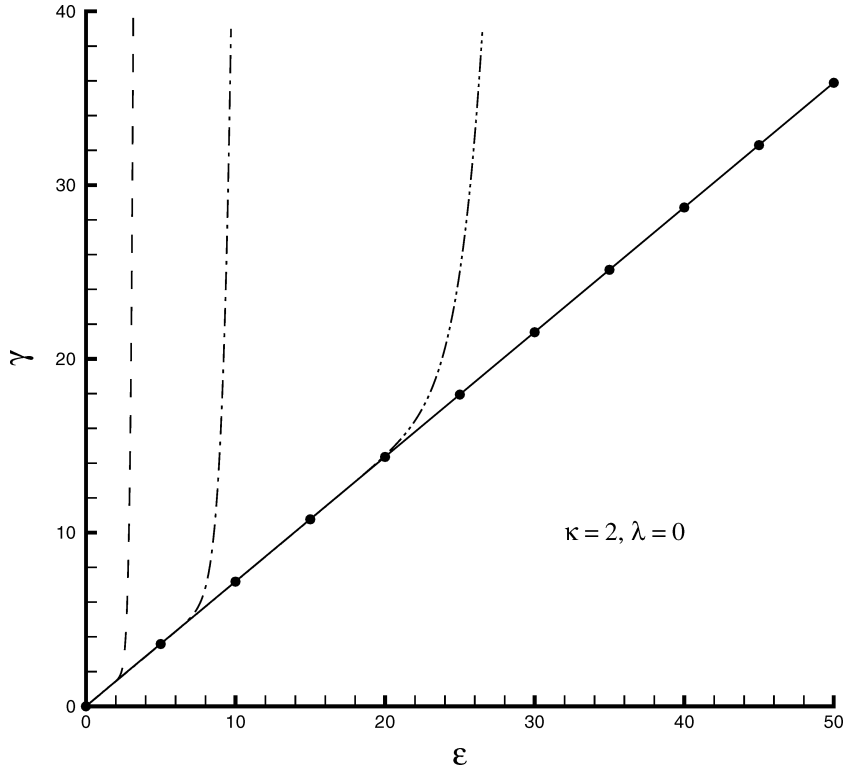

Figure 10. The curve $\gamma \sim \epsilon$ at the 10th-order approximation when $\kappa = 2$, $\lambda = 0$. Dashed line: $\hbar = 10$; Dash–dotted line: $\hbar = 3$; Dash–dot–dotted line: $\hbar = 1$; Solid line: formula (63); Filled circles: numerical results.

We investigate other cases of $\kappa \geq 3$ for given λ and $0 < \epsilon < +\infty$ in a similar way, and find out that we can always find the valid region of $\hbar$ by plotting the so-called $\gamma \sim \hbar$ curves, and then get convergent solutions series for both γ and $u(t)$ by choosing a proper value of $\hbar$. Our investigation indicate that the valid regions of $\hbar$ strongly depend upon the auxiliary linear operator $\mathcal{L}$: $\hbar$ must be negative for all odd integer κ but positive for all even integer κ. Thus, the auxiliary parameter $\hbar$ is absolutely necessary to ensure the convergence of solution series: without this kind of auxiliary parameter $\hbar$, the freedom on the choice of the auxiliary linear operator has no meanings at all, because one can not ensure the convergence of solution series. In other words, the freedom on the choice of the auxiliary linear operator $\mathcal{L}$ is based on the auxiliary parameter $\hbar$. As mentioned before, some methods such as Adomian's decomposition method [4–9], the δ-expansion method [10, 11], Lyapunov's artificial small parameter method [12] and the "homotopy perturbation method" [36] are only special cases of the homotopy analysis

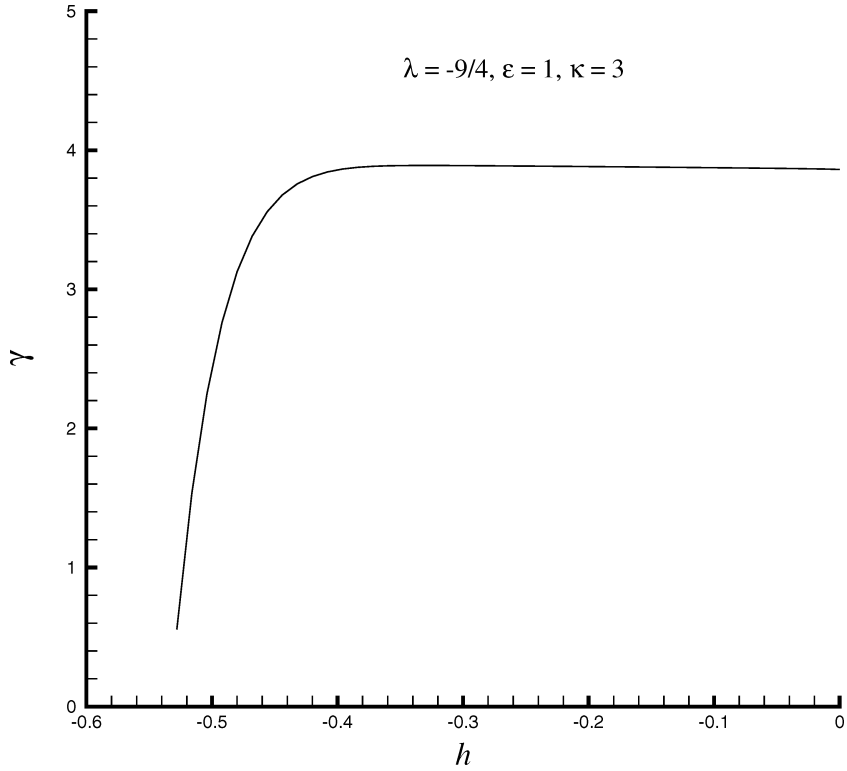

Figure 11. The curve $\gamma \sim \epsilon$ when $\lambda = -9/4$, $\epsilon = 1$ and $\kappa = 3$.

method when $\hbar = -1$. Thus, in essence, all of these methods have no freedom to choose better auxiliary linear operators to approximate nonlinear problems more efficiently, because they can not ensure the convergence of solution series.

Besides, one can use the so-called homotopy-Pàde technique [13] to accelerate the convergence of solution series. For details about the homotopy-Pàde technique, please refer to Liao ([13], Section 2.3.7). For example, when $\lambda = 0$ and $\epsilon = 1$, all of our homotopy-Pàde approximations of γ in case of $\kappa = 1$, 2, 3, 4, 5 converge to the *same* result $\gamma = 0.71777$, as shown in Table 2. Furthermore, it is found that, when $\lambda = 0$ and $0 < \epsilon < +\infty$, all of our homotopy-Pàde approximations of γ given by the auxiliary linear operators with *different* orders 2κ converge to the *same* result

$$\gamma = 0.71777\epsilon,$$

which agrees well with numerical ones. It is found that, for *all* possible values of $0 < \epsilon < +\infty$, the maximum errors of the second-order approximations (55), (63), and (65) of γ are only less than 0.5%.

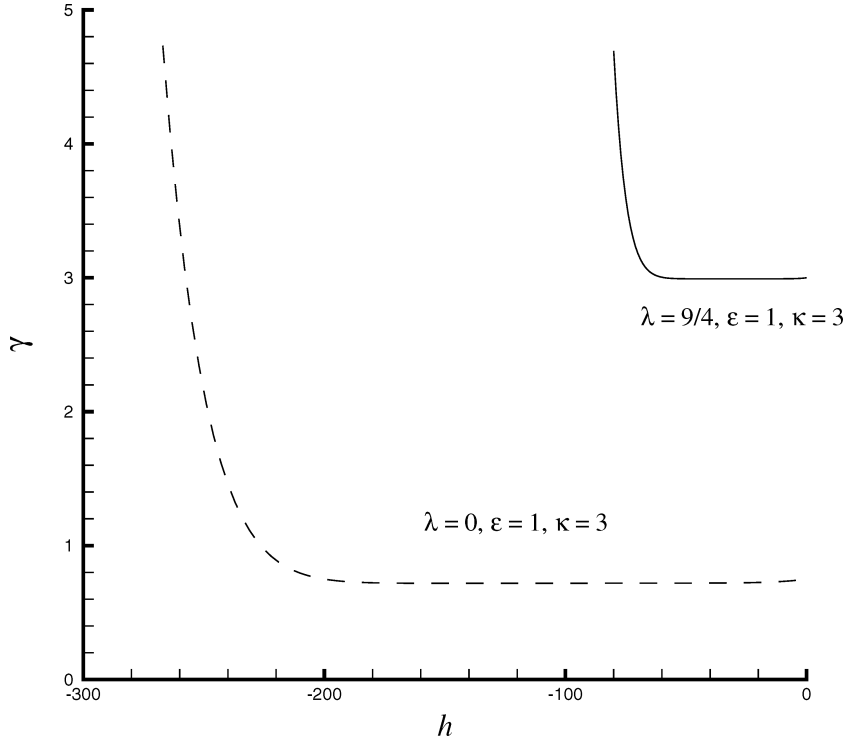

Figure 12. The curve $\gamma \sim \epsilon$ when $\kappa = 3$, $\epsilon = 1$, $\lambda = 9/4$ and $\lambda = 0$. Solid line: $\lambda = 9/4$; Dashed line: $\lambda = 0$.

Note that the order of the auxiliary linear operator $\mathcal{L}$ defined by (57) is 2κ, where $\kappa \geq 1$ is *any* a positive integer. However, the original governing equation (19) is only second-order. So, we illustrate here that, in the frame of the homotopy analysis method, the second-order *nonlinear* differential equation (19) can be replaced by an infinite number of fourth-order (when $\kappa = 2$), or sixth-order (when $\kappa = 3$), or eighth-order (when $\kappa = 4$) *linear* differential equations! In fact, the second-order *nonlinear* differential equation (19) can be replaced by an infinite number of the (2κ)th-order *linear* differential equations for *any* a positive integer $\kappa \geq 1$. *All* of these auxiliary linear operators with different orders give the *same* results that agree well with numerical ones. Therefore, the order of the auxiliary linear operator $\mathcal{L}$ in the linear subproblems (6) is *unnecessary* to be equal to that of the original nonlinear equation. This means that we have much greater freedom to solve nonlinear problems than we thought traditionally! Sometimes, this kind of freedom can greatly simplify solving nonlinear problems, as illustrated in Section 3.

It should be emphasized once again that the auxiliary parameter $\hbar$ is *absolutely* necessary to ensure the convergence of the solution series. For any

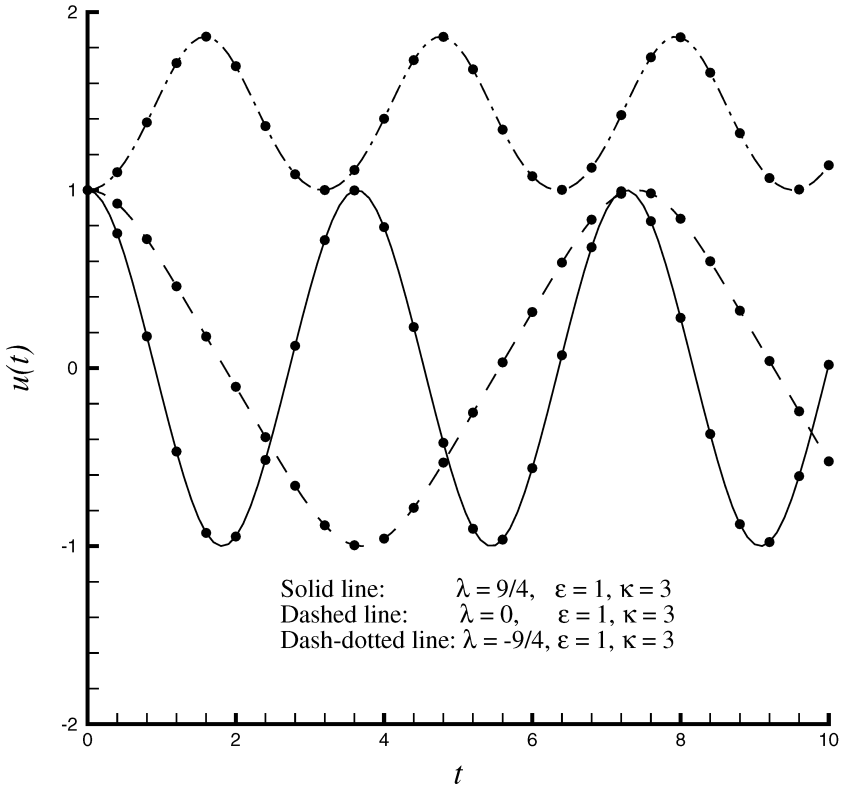

Figure 13. Comparison of the numerical solutions with the HAM approximations of $u(t)$ when $\kappa = 3$, $\epsilon = 1$. Dash–dotted line: 40th-order HAM solution when $\lambda = -9/4$ by means of $\hbar = -1/4$; Dashed line: fifth-order HAM solution when $\lambda = 0$ by means of $\hbar = -100$; Solid line: fifth-order HAM solution when $\lambda = 9/4$ by means of $\hbar = -40$.

a chosen linear operator (57) corresponding to *any* a positive integer of $\kappa \geq 1$, we can always find a valid region of $\hbar$ to ensure the convergence of the solutions series of γ and $u(t)$. Note that, for given λ and ϵ, the valid region of $\hbar$ depends strongly upon the auxiliary linear operator (57). For the considered nonlinear oscillation problem, the valid value of $\hbar$ is negative when κ is odd but positive when κ is even. Note that, when $\lambda = 0$ and $\lambda = 9/4$, the length of the valid region of $\hbar$ increases as the value of κ increases, as shown in Figures 4, 8, and 12. However, when $\lambda = -9/4$, it nearly keeps the same, as shown in Figures 4, 7, and 11. Therefore, the freedom on the choice of the auxiliary linear operator $\mathcal{L}$ and basis functions is based on such a kind of guarantee that the auxiliary parameter $\hbar$ can ensure the convergence of solution series. So, it is the auxiliary parameter $\hbar$ that provides us a solid base for the freedom on the choice of the auxiliary linear operator $\mathcal{L}$ and basis functions. Thus, the auxiliary parameter $\hbar$ is very important and absolutely necessary: without $\hbar$, such a kind of freedom has no meanings at all.

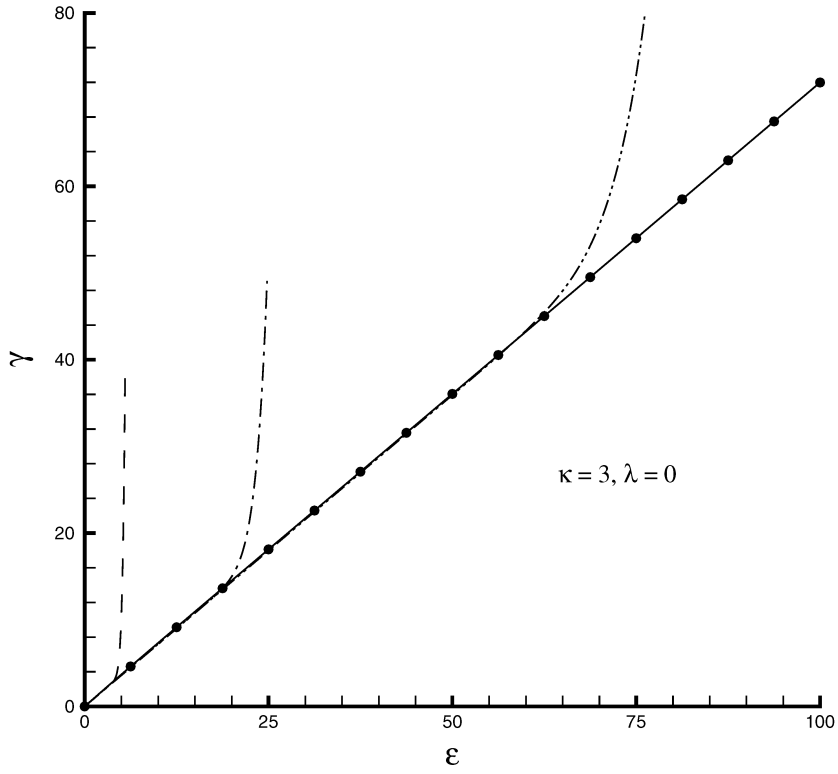

Figure 14. The curve $\gamma \sim \epsilon$ at the 10th-order approximation when $\kappa = 3$, $\lambda = 0$. Dashed line: $\hbar = -50$; Dash–dotted line: $\hbar = -10$; Dash–dot–dotted line: $\hbar = -3$; Solid line: formula (65); Filled circles: numerical results.

Table 2
$[m, m]$ Homotopy-Pàde Approximation of γ When $\lambda = 0$, $\epsilon = 1$ for Different κ

m	$\kappa = 1$	$\kappa = 2$	$\kappa = 3$	$\kappa = 4$	$\kappa = 5$
2	0.71780	0.71851	0.71966	0.72025	0.72047
4	0.71777	0.71779	0.71780	0.71731	0.72455
6	0.71777	0.71776	0.71784	0.71788	0.71790
8	0.71777	0.71777	0.71779	0.71785	0.71789
10	0.71777	0.71777	0.71777	0.71782	0.71786
12	0.71777	0.71777	0.71777	0.71776	0.71786
14	0.71777	0.71777	0.71777	0.71777	0.71780
16	0.71777	0.71777	0.71777	0.71777	0.71778
18	0.71777	0.71777	0.71777	0.71777	0.71777
20	0.71777	0.71777	0.71777	0.71777	0.71777

In summary, the auxiliary parameter $\hbar$ provides a simple way to ensure the convergence of solution series, and a solid base for the freedom of the choice of the auxiliary linear operators and basis functions. Here, we illustrate that, by means of the *rule of solution expression*, and using the freedom of the choice of the auxiliary linear operator and the auxiliary parameter $\hbar$, we can replace a nonlinear differential equation of order n into an infinite number of linear differential equations of order k, where k is *unnecessary* to be equal to n.

3. Application to Gelfand equation

In Section 2, we describe the basic ideas of the homotopy analysis method, and illustrate that we have great freedom to choose the auxiliary linear operator. Using this kind of freedom, we can replace the second-order nonlinear differential equation (19) by an infinite number of linear 2κ-order linear differential equations, where $\kappa \geq 1$ is any a positive integer. In this section, we further illustrate that, sometimes, such kind of freedom makes it much easier to solve nonlinear problems.

For example, let us consider the so-called Gelfand equation [46, 47]

$$\begin{cases} \Delta u + \lambda e^u = 0, & \mathbf{x} \in \Omega \subset \mathbf{R}^N, \\ u = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{67}$$

where λ is an eigenvalue, u is the eigenfunction, the subscript denotes the differentiation to the spacial variables, $N = 1, 2, 3$ corresponds to the dimension, Ω is the domain, respectively. Mathematically, the difficulty comes from the exponential term $\exp(u)$, corresponding to a strong nonlinearity. Physically, Gelfand equation arises in several contexts, such as chemical reactor theory, the steady-state equation for a nonlinear heat conduction problem, questions on geometry and relativity about the expansion of universe, and so on.

Generally speaking, it is not easy to get eigenvalue and eigenfunction of a nonlinear differential equation, especially when the nonlinearity is strong and the dimension is high. The investigation on Gelfand problem [46, 47] has a long history. Liouville [46] gave a closed-form expression of eigenvalue for one-dimensional (1D) Gelfand equation. Based on Chebyshev functions, Boyd [47] provided an analytic approach and a numerical method for two-dimensional (2D) Gelfand equation. Boyd [47] gave a one-point analytic approximation

$$\lambda = 3.2Ae^{-0.64A}, \tag{68}$$

and a three-point analytic approximation

$$\lambda = (2.667A + 4.830B + 0.127C)\, e^{-0.381A-0.254B-0.018C}, \tag{69}$$

where $A = u(0, 0)$ and

$$B = A(0.829 - 0.566e^{0.463A} - 0.0787e^{-0.209A})/G,$$
$$C = A(-1.934 + 0.514e^{0.463A} + 1.975e^{-0.209A})/G,$$
$$G = 0.2763 + e^{0.463A} + 0.0483e^{-0.209A}.$$

This problem still attracts the attention of current researchers [48, 49]. In this section, the homotopy analysis method (HAM) [13] is employed to get series solution of eigenvalue and eigenfunction of 2D and 3D Gelfand equation.

3.1. The HAM approach for 2D Gelfand equation

First, let us consider the 2D case, i.e.,

$$\Delta u + \lambda e^u = 0 \tag{70}$$

on the square $[-1, 1] \times [-1, 1]$, subject to the boundary conditions

$$u = 0, \quad \text{on four walls.} \tag{71}$$

Write

$$A = u(0, 0) \tag{72}$$

and

$$u(x, y) = A + w(x, y), \tag{73}$$

where A is a given value. The original Gelfand equation becomes

$$\Delta w + \lambda e^A e^w = 0, \tag{74}$$

subject to the boundary conditions on four walls:

$$w(x, \pm 1) = -A, \quad w(\pm 1, y) = -A, \tag{75}$$

and the restriction condition:

$$w(0, 0) = 0. \tag{76}$$

From Equation (74), it is obvious that $w(x, y)$ is symmetric about x and y axis, i.e., $w(-x, y) = w(x, y) = w(x, -y)$. Thus, considering the above restriction condition, $w(x, y)$ can be expressed by the basis functions

$$\left\{x^{2m} y^{2n} \mid m = 1, 2, 3, \ldots, \quad n = 1, 2, 3, \ldots\right\} \tag{77}$$

in the form

$$w(x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} b_{m,n} x^{2m} y^{2n}, \tag{78}$$

where $b_{m,n}$ is a constant coefficient. This provides us the so-called *solution expression* of $w(x, y)$. Our aim is to find a series solution of $w(x, y)$ expressed by (78) for given A.

According to the restriction condition (76) and the solution expression (78), it is natural for us to choose such an initial guess

$$w_0(x, y) = 0. \tag{79}$$

According to Equation (74), we define such a nonlinear operator

$$\mathcal{N}[\Phi(x, y; q), \Lambda(q)] = \frac{\partial^2 \Phi(x, y; q)}{\partial x^2} + \frac{\partial^2 \Phi(x, y; q)}{\partial y^2} + e^A \Lambda(q) \exp[\Phi(x, y; q)], \tag{80}$$

where $q \in [0, 1]$ is an embedding parameter, $\Phi(x, y; q)$ and $\Lambda(q)$ relate to $w(x, y)$ and λ, respectively. Let $\mathcal{L}$ denote an auxiliary linear operator, which we will choose later, and $\hbar$ be a nonzero auxiliary parameter, respectively. We construct the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\Phi(x, y; q) - w_0(x, y)] = \hbar q \mathcal{N}[\Phi(x, y; q), \Lambda(q)] \tag{81}$$

on the square $[-1, 1] \times [-1, 1]$, subject to the boundary condition on four walls:

$$(1 - q)[\Phi(\pm 1, y; q) - w_0(\pm 1, y)] = \hbar q[\Phi(\pm 1, y; q) + A], \tag{82}$$

$$(1 - q)[\Phi(x, \pm 1; q) - w_0(x, \pm 1)] = \hbar q[\Phi(x, \pm 1; q) + A], \tag{83}$$

and the restriction condition at the origin:

$$(1 - q)[\Phi(0, 0; q) - w_0(0, 0)] = \hbar q \Phi(0, 0; q), \tag{84}$$

where $q \in [0, 1]$ is an embedding parameter.

Obviously, when $q = 0$, the above equations have the solution

$$\Phi(x, y; 0) = w_0(x, y). \tag{85}$$

When $q = 1$, because $\hbar \neq 0$, Equations (81)–(84) are equivalent to the original Equations (74)–(76), provided

$$\Phi(x, y; 1) = w(x, y), \quad \Lambda(1) = \lambda. \tag{86}$$

Thus, as q increases from 0 to 1, $\Phi(x, y; q)$ varies from the initial guess $w_0(x, y)$ to the eigenfunction $w(x, y)$, so does $\Lambda(q)$ from the initial guess $\lambda_0 = \Lambda(0)$ to the eigenvalue λ. By Taylor series and using (85), we have the Taylor series in the embedding parameter q as follows:

$$\Phi(x, y; q) = w_0(x, y) + \sum_{n=1}^{+\infty} w_n(x, y)\, q^n, \tag{87}$$

$$\Lambda(q) = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n \, q^n, \tag{88}$$

where

$$w_n(x, y) = \frac{1}{n!} \left. \frac{\partial^n \Phi(x, y; q)}{\partial q^n} \right|_{q=0}, \quad \lambda_n = \frac{1}{n!} \left. \frac{d^n \Lambda(q)}{dq^n} \right|_{q=0}. \tag{89}$$

Note that the series (87) and (88) contain the auxiliary parameter $\hbar$, which influences the convergence of (87) and (88). Assume that $\hbar$ is chosen properly so that the series (87) and (88) are convergent at $q = 1$. Then, using (86), we have

$$w(x, y) = w_0(x, y) + \sum_{n=1}^{+\infty} w_n(x, y), \tag{90}$$

$$\lambda = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n. \tag{91}$$

The unknown $w_n(x, y)$ and λ_{n-1}, where $n = 1, 2, 3, \ldots$, are obtained in the following way. For simplicity, define the vectors

$$\vec{w}_k = \{w_0(x, y), w_1(x, y), w_2(x, y), \ldots, w_k(x, y)\},$$
$$\vec{\lambda}_k = \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k\}.$$

Differentiating the zeroth-order deformation equations (81)–(84) n times with respect to q, then dividing by $n!$, and finally setting $q = 0$, we have the nth-order deformation equation[3]

$$\mathcal{L}[w_n(x, y) - \chi_n \, w_{n-1}(x, y)] = \hbar R_n(\vec{w}_{n-1}, \vec{\lambda}_{n-1}), \tag{92}$$

subject to the boundary conditions on four walls:

$$w_n(x, \pm 1) = \delta_n(x, \pm 1), \tag{93}$$

$$w_n(\pm 1, y) = \delta_n(\pm 1, y), \tag{94}$$

and the restriction condition at the origin:

$$w_n(0, 0) = (\chi_n + \hbar) w_{n-1}(0, 0), \tag{95}$$

where

$$\delta_n(x, y) = (\chi_n + \hbar) w_{n-1}(x, y) + \hbar(1 - \chi_n) A$$

[3] As mentioned in Section 2.1, there are two different ways to get the high-order deformation equations. Both of them give the *same* results. For details, please read Appendix A and B, or refer to Hayat et al. [31].

and

$$R_n(\vec{w}_{n-1}, \vec{\lambda}_{n-1}) = \Delta w_{n-1}(x, y) + e^A \sum_{k=0}^{n-1} \lambda_{n-1-k} S_k(x, y), \tag{96}$$

with the definitions

$$S_k(x, y) = \frac{1}{k!} \frac{\partial^k e^{\Phi(x,y;q)}}{\partial q^k}\bigg|_{q=0}. \tag{97}$$

According to (79) and (97), it holds,

$$S_0(x, y) = e^{w_0(x,y)} = 1.$$

Besides, we have

$$\frac{\partial e^{\Phi(x,y;q)}}{\partial q} = e^{\Phi(x,y;q)} \frac{\partial \Phi(x, y; q)}{\partial q}.$$

So, for $n \geq 1$, it holds

$$\begin{aligned}
\frac{1}{n!} \frac{\partial^n e^{\Phi(x,y;q)}}{\partial q^n} &= \frac{1}{n!} \frac{\partial^{n-1}}{\partial q^{n-1}} \left[e^{\Phi(x,y;q)} \frac{\partial \Phi(x, y; q)}{\partial q} \right] \\
&= \frac{1}{n!} \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} \frac{\partial^j e^{\Phi(x,y;q)}}{\partial q^j} \frac{\partial^{n-j} \Phi(x, y; q)}{\partial q^{k-j}} \\
&= \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) \left[\frac{1}{j!} \frac{\partial^j e^{\Phi(x,y;q)}}{\partial q^j} \right] \left[\frac{1}{(n-j)!} \frac{\partial^{n-j} \Phi(x, y; q)}{\partial q^{n-j}} \right].
\end{aligned} \tag{98}$$

Thus, using the definitions (89) and (97), we have the recurrence formula

$$S_0 = 1, \quad S_n = \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) S_j w_{n-j}. \tag{99}$$

In this way, it is easy to calculate the term $R_n(\vec{w}_{n-1}, \vec{\lambda}_{n-1})$ of the high-order deformation equation (92).

The corresponding auxiliary linear operator $\mathcal{L}$ should be chosen properly. First, to obey the solution expression (78) of $w(x, y)$, it should hold

$$\mathcal{L}[C_1] = 0 \tag{100}$$

for any a nonzero constant coefficient C_1. Second, because $w_0(x, y) = 0$, it holds $R_1 = \lambda_0 e^A$, thus, $R_m(\vec{w}_{m-1}, \vec{\lambda}_{m-1})$ may contain a nonzero constant. So, to obey the solution expression (78), the inverse operator $\mathcal{L}^{-1}$ of the auxiliary linear operator $\mathcal{L}$ should have the property:

$$\mathcal{L}^{-1}[1] = Cx^2y^2, \tag{101}$$

where C is a nonzero constant. More importantly, the linear auxiliary operator should be chosen so properly that it is convenient to satisfy the boundary conditions (93) and (94) at four walls. Let $w_m^*(x, y)$ denote a special solution of Equation (92). Obviously,

$$w_n^*(x, y) - w_n^*(x; \pm 1) - w_n^*(\pm 1, y) + w_n^*(\pm 1, \pm 1)$$

vanishes on the four walls, and besides

$$\delta_n(x, \pm 1) + \delta_n(\pm 1, y) - \delta_n(\pm 1, \pm 1)$$

satisfies the boundary conditions (93) and (94) on the four walls. Therefore,

$$\begin{aligned} w_n(x, y) = {} & w_n^*(x, y) - w_n^*(x, \pm 1) - w_n^*(\pm 1, y) + w_n^*(\pm 1, \pm 1) \\ & + \delta_n(x, \pm 1) + \delta_n(\pm 1, y) - \delta_n(\pm 1, \pm 1) \end{aligned} \tag{102}$$

is the solution of Equations (92)–(94), if the auxiliary operator $\mathcal{L}$ has the property

$$\mathcal{L}[f(x)] = \mathcal{L}[g(y)] = 0, \tag{103}$$

where $f(x)$ and $g(y)$ are *any* functions of x and y, respectively. There are an infinite number of linear operators $\mathcal{L}$ satisfying the properties (100), (101), and (103). For example,

$$\mathcal{L}u = \alpha_2 \left(\frac{1}{xy}\right) \frac{\partial^2 u}{\partial x \partial y}, \tag{104}$$

and

$$\mathcal{L}u = \alpha_4 \frac{\partial^4 u}{\partial x^2 \partial y^2}, \tag{105}$$

where α_2 and α_4 are constant coefficients. They are special cases of the operator:

$$\mathcal{L}u = \left(\frac{\alpha_2}{xy}\right) \frac{\partial^2 u}{\partial x \partial y} + \alpha_4 \frac{\partial^4 u}{\partial x^2 \partial y^2}, \tag{106}$$

whose inverse operator is

$$\mathcal{L}^{-1}[x^k y^n] = \frac{x^{k+2}\, y^{n+2}}{(k+2)(n+2)[\alpha_2 + \alpha_4(k+1)(n+1)]}. \tag{107}$$

Using the above inverse operator $\mathcal{L}^{-1}$, it is easy to get a special solution of Equation (92), i.e.,

$$w_n^*(x, y) = \hbar \mathcal{L}^{-1}[R_n(\vec{w}_{n-1}, \vec{\lambda}_{n-1})] + \chi_n w_{n-1}(x, y). \tag{108}$$

Thereafter, the solution $w_n(x, y)$ of the high-order deformation equations (92)–(94) is given by (102). Then, λ_{n-1} is determined by the linear algebraic

equation (95). The above approach needs only algebraic calculations. Thus, it is easy for us to obtain results at rather high order of approximations, especially by means of symbolic computation software such as Mathematica, Maple, MathLab and so on. For example, the corresponding Mathematica code for 2D Gelfand equation is given in Appendix C.[4]

As shown in Section 2, the convergence of the series (90) and (91) are determined by the auxiliary parameter $\hbar$. It is found that the nth-order approximation of $u(x, y)$ on the four walls is

$$A(1+\hbar)^n,$$

which vanishes as $n \to +\infty$, only if

$$|1+\hbar| < 1. \tag{109}$$

This restricts the choice of the value of $\hbar$. Especially, when $\hbar = -1$, the boundary conditions on the four walls are exactly satisfied at every order of approximation. However, when $|1 + \hbar| < 1$, the series of eigenvalue λ and eigenfunction $u(x, y)$ are divergent by means of the auxiliary linear operator (104) for any values of $\alpha_2 \neq 0$. It is interesting that, when $|1 + \hbar| < 1$, the series of eigenvalue and eigenfunction are convergent for any values of A by means of the auxiliary operator (105) if $\alpha_4 < -1/2$. For example, when $A = 3$ and $\hbar = -1$, the series of eigenvalue is convergent for *any* values of $\alpha_4 < -1/2$, as shown in Figure 15. When $\hbar = -1$ and $\alpha_4 = -1$, the 25th-order HAM approximation of the eigenvalue

$$\begin{aligned}\lambda \approx e^{-A}(&3.39463856A + 0.852428251A^2 + 0.14535358A^3\\ &+ 1.83839458 \times 10^{-2}A^4 + 1.59846843 \times 10^{-3}A^5\\ &+ 8.92565457 \times 10^{-5}A^6 + 2.83370030 \times 10^{-6}A^7\\ &- 8.12484778 \times 10^{-7}A^8 - 1.86368503 \times 10^{-7}A^9\\ &- 9.57976887 \times 10^{-9}A^{10} + 1.65428886 \times 10^{-9}A^{11} + \cdots),\end{aligned} \tag{110}$$

agrees well with Boyd's numerical results [47], as shown in Figure 16. Note that even the 10th-order HAM approximation of eigenvalue is accurate enough. As shown in Table 3, the eigenvalue of the 2D Gelfand equation has the maximum value 1.70205 at $A = 1.391$, which agrees with Boyd's numerical result $\lambda_{\max} = 1.702$ at $A = 1.39$.

Note that, we use here the auxiliary linear operator

$$\mathcal{L}u = -\frac{\partial^4 u}{\partial x^2 \partial y^2}.$$

[4]To get free electronic version of this code, please send email to the corresponding author.

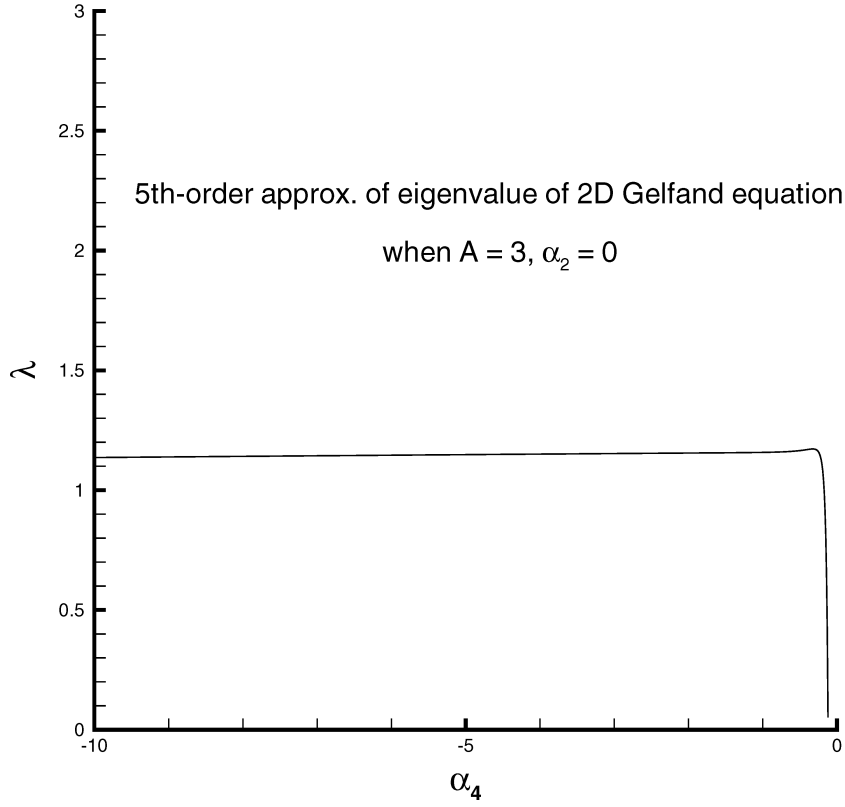

Figure 15. The fifth-order HAM approximation of the eigenvalue of 2D Gelfand equation when $A = 3$ and $\hbar = -1$ by means of (105).

There exist an infinite number of functions, such as

$$\sin x, \cos y, x\cos y, y\cos x, xg(y), yf(x)$$

and so on, satisfying

$$\mathcal{L}[\sin x] = \mathcal{L}[\cos y] = \mathcal{L}[x\cos y] = \mathcal{L}[y\sin x] = \mathcal{L}[xf(y)] = \mathcal{L}[yg(x)] = 0,$$

where $f(x)$, $g(y)$ are any real functions. However, all of them are *not* allowed to appear in the solution of Equation (92), because they disobey *the solution expression* (78). In other words, if $w_n^*(x, y)$ is a special solution of Equation (92), then

$$w_n^*(x, y) + B_1\sin x + B_2\cos y + B_3 x\cos y + B_4 y\sin x + B_5 xf(y) + B_6 yg(x) + \cdots$$

also satisfies Equation (92), where B_k is a coefficient. However, to obey the solution expression (78), we had to enforce

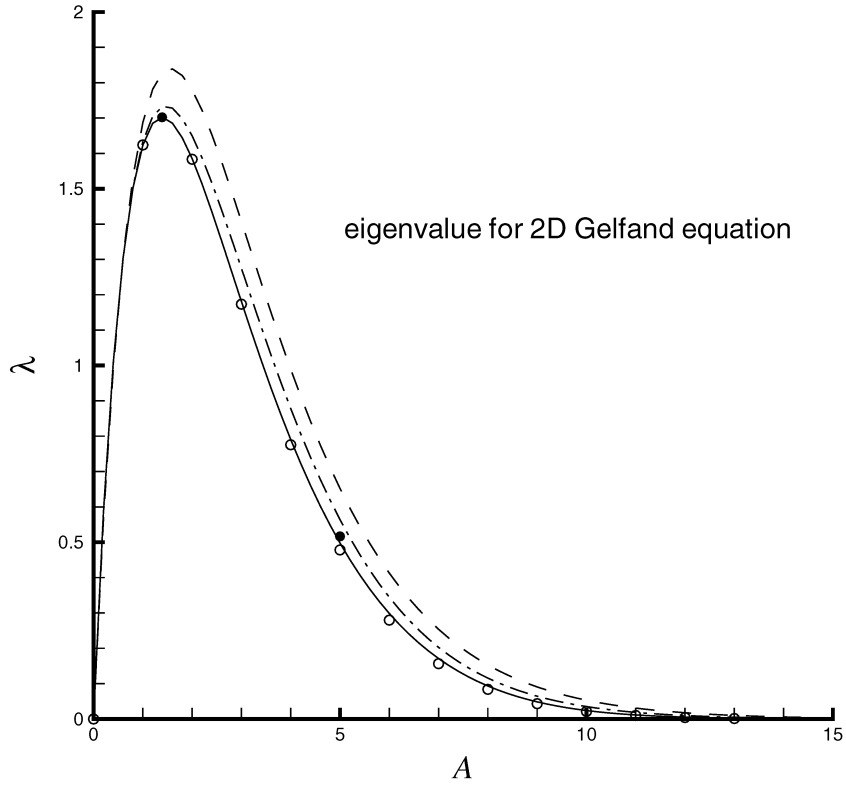

Figure 16. Eigenvalue of 2D Gelfand equation. Solid line: 25th-order HAM approximation (110); Open circles: fifth-order HAM approximation; Filled circles: Numerical results given by Boyd [47]; Dashed line: Boyd's one-point approximation (68); Dash–dotted line: Boyd's three-points approximation (69).

Table 3
Comparison of the Maximum Eigenvalue of 2D Gelfand Equation with Boyd's Analytic and Numerical Results [47]

	$\lambda_{\max}$	A_{limit}
5th-order HAM approx.	1.70115	1.383
10th-order HAM approx.	1.70200	1.389
15th-order HAM approx.	1.70205	1.391
20th-order HAM approx.	1.70205	1.391
25th-order HAM approx.	1.70205	1.391
One-point formula (68)	1.84	1.56
Three-point formula (69)	1.735	1.465
Numerical result	1.702	1.39

$$B_1 = B_2 = B_3 = B_4 = B_5 = B_6 = \cdots = 0.$$

As pointed out by Liao ([13], chapters 2 and 3) and mentioned in Section 2, the initial guess and the auxiliary linear operator must be chosen in such a way that the solutions of the high-order deformation equations exist and do not disobey the solution expression: this is so important in the frame of the homotopy analysis method that it is regarded as a rule, namely the *rule of solution expression*, as mentioned in Section 2. This illustrates once again that the so-called *rule of solution expression* has general meanings, and can greatly simplify finding solutions of some nonlinear problems.

In this subsection, we illustrate that the second-order 2D Gelfand equation can be replaced by an infinite number of fourth-order linear differential equations. More importantly, this approach greatly simplifies finding eigenvalue and eigenfunction of the original 2D Gelfand equation with strong nonlinearity.

3.2. The HAM approach for 3D Gelfand equation

Let us further consider the 3D Gelfand equation

$$\Delta u + \lambda e^u = 0, \tag{111}$$

on the cube $[-1, 1] \times [-1, 1] \times [-1, 1]$, subject to the boundary conditions

$$u = 0, \quad \text{on the six walls.} \tag{112}$$

Write

$$A = u(0, 0, 0) \tag{113}$$

and

$$u(x, y, z) = A + w(x, y, z). \tag{114}$$

The original equations become

$$\Delta w + \lambda e^A e^w = 0, \tag{115}$$

subject to the boundary conditions on the six walls:

$$w(\pm 1, y, z) = w(x, \pm 1, z) = w(x, y, \pm 1) = -A, \tag{116}$$

and the restriction condition:

$$w(0, 0, 0) = 0. \tag{117}$$

In a similar way as mentioned in Section 3.1, we have the solution series

$$w(x, y, z) = w_0(x, y, z) + \sum_{n=1}^{+\infty} w_n(x, y, z), \tag{118}$$

$$\lambda = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n. \tag{119}$$

Similarly, $w_0(x, y, z) = 0$, and $w_n(x, y, z)$ is given by

$$\begin{aligned} w_n(x, y, z) = {} & w_n^*(x, y, z) - w_n^*(\pm 1, y, z) - w_n^*(x, \pm 1, z) - w_n^*(x, y, \pm 1) \\ & + w_n^*(x, \pm 1, \pm 1) + w_n^*(\pm 1, y, \pm 1) + w_n^*(\pm 1, \pm 1, z) \\ & - w_n^*(\pm 1, \pm 1, \pm 1) \\ & + \delta_n(\pm 1, y, z) + \delta_n(x, \pm 1, z) + \delta_n(x, y, \pm 1) \\ & - \delta_n(x, \pm 1, \pm 1) - \delta_n(\pm 1, y, \pm 1) - \delta_n(\pm 1, \pm 1, z) \\ & + \delta_n(\pm 1, \pm 1, \pm 1), \end{aligned} \tag{120}$$

where

$$w_n^*(x, y, z) = \hbar \mathcal{L}^{-1}[R_n(\vec{w}_{n-1}, \vec{\lambda}_{n-1})] + \chi_n w_{n-1}(x, y, z), \tag{121}$$

$$\delta_n(x, y, z) = (\chi_n + \hbar) w_{n-1}(x, y, z) + \hbar\,(1 - \chi_n) A, \tag{122}$$

with the definition

$$R_n(\vec{w}_{n-1}, \vec{\lambda}_{n-1}) = \Delta w_{n-1}(x, y, z) + e^A \sum_{k=0}^{n-1} \lambda_{n-1-k} S_k(x, y, z), \tag{123}$$

in which $S_k(x, y, z)$ is given by the recurrence formula (99). Similarly, λ_{n-1} is determined by the linear algebraic equation:

$$w_n(0, 0, 0) = (\chi_n + \hbar) w_{n-1}(0, 0, 0). \tag{124}$$

In this way, using

$$w_0(x, y, z) = 0, \quad S_0(x, y, z) = 1,$$

it is easy for us to get $w_n(x, y, z)$ and λ_{n-1} one by one in the order $n = 1, 2, 3, \ldots$.

Similarly, we choose the auxiliary linear operator $\mathcal{L}$ in (121) as follows:

$$\mathcal{L} w = \left(\frac{\alpha_3}{xyz} \right) \frac{\partial^3 w}{\partial x \partial y \partial z} + \alpha_6 \frac{\partial^6 w}{\partial x^2 \partial y^2 \partial z^2}, \tag{125}$$

where α_3 and α_6 are constants. Its inverse operator is

$$\mathcal{L}^{-1}[x^l y^n z^k] = \frac{x^{l+2}\, y^{n+2}\, z^{k+2}}{(l+2)(n+2)(k+2)[\alpha_3 + \alpha_6 (l+1)(n+1)(k+1)]}. \tag{126}$$

It is found that the nth-order approximation of $u(x, y, z)$ on the six walls is

$$A(1 + \hbar)^n,$$

which vanishes as $n \to +\infty$ when $|1+\hbar| < 1$, i.e., $-2 < \hbar < 0$. Similarly, it is found that convergent series of eigenvalue and eigenfunction can not be obtained when $\alpha_3 \neq 0$ and $-2 < \hbar < 0$. However, the series of eigenvalue and eigenfunction are convergent when $-2 < \hbar < 0, \alpha_3 = 0$ and $\alpha_6 > 1/2$. For example, when $\hbar = -1$, $\alpha_3 = 0$ and $\alpha_6 = 1$, we have the 16th-order approximation of the eigenvalue:

$$\begin{aligned}\lambda = e^{-A}(&4.4938711591A + 1.2945236107A^2 \\ &+ 3.2051405745 \times 10^{-1}A^3 + 5.4513875743 \times 10^{-2}A^4 \\ &+ 7.4728993234 \times 10^{-3}A^5 + 7.3205808924 \times 10^{-4}A^6 \\ &+ 4.4124633612 \times 10^{-5}A^7 - 7.8131103398 \times 10^{-7}A^8 \\ &- 5.2875679521 \times 10^{-7}A^9 - 5.1192218589 \times 10^{-8}A^{10} \\ &+ 3.1141409410 \times 10^{-10}A^{11} + 5.6668899319 \times 10^{-10}A^{12} \\ &+ 4.0913909123 \times 10^{-11}A^{13} - 2.9126197658 \times 10^{-12}A^{14} \\ &- 4.5871791205 \times 10^{-13}A^{15} + 1.8121316154 \times 10^{-14}A^{16} \\ &+ 8.538605916 \times 10^{-16}A^{17}).\end{aligned} \tag{127}$$

It has the maximum value 1.60 at $A = 2.476$, as shown in Table 4. The curve of the eigenvalue of the 3D Gelfand equation is as shown in Figure 17.

Here, we illustrate that the second-order three-dimensional Gelfand equation can be replaced by an infinite number of the sixth-order linear differential equations. Note that our approach needs only algebraic calculations so that we can get results at rather high order of approximations. It shows that, sometimes, the freedom on the choice of the auxiliary linear operator might greatly simplify finding solutions of some nonlinear problems.

Table 4
The Maximum Eigenvalue of 3D Gelfand Equation

	$\lambda_{\max}$	A_{limit}
Second-order HAM approx.	1.75	2.716
Fourth-order HAM approx.	1.60	2.477
Sixth-order HAM approx.	1.60	2.475
Eighth-order HAM approx.	1.60	2.476
10th-order HAM approx.	1.60	2.476
12th-order HAM approx.	1.60	2.476
14th-order HAM approx.	1.60	2.476
16th-order HAM approx.	1.60	2.476

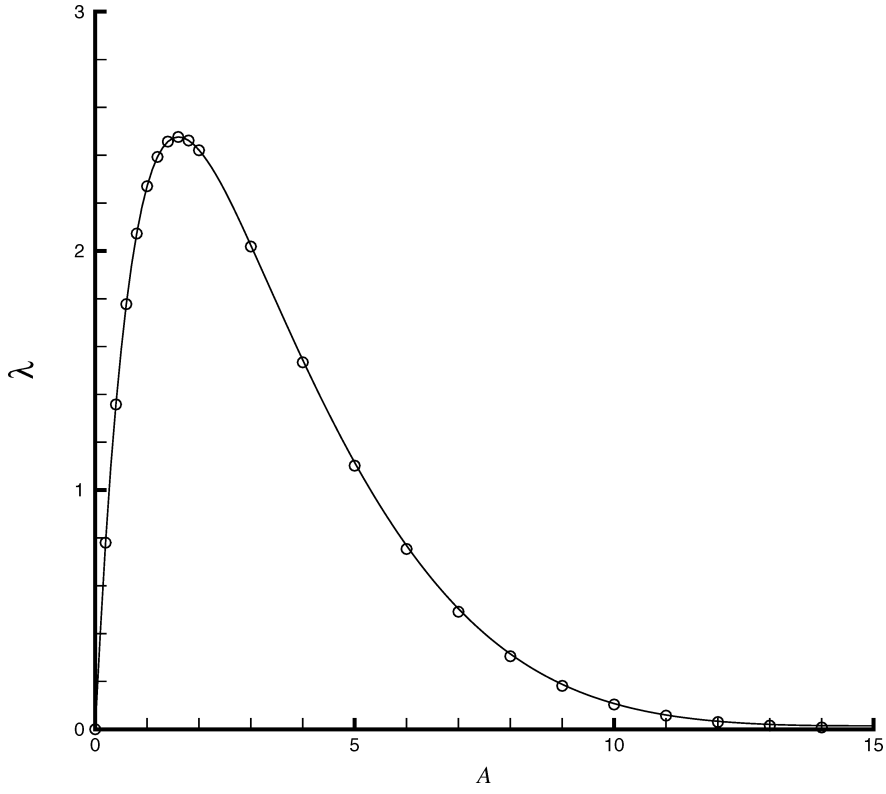

Figure 17. Eigenvalue of 3D Gelfand equation. Solid line: 16th-order HAM approximation (127); Open circles: 14th-order HAM approximation.

Finally, we point out that, it is much more complicated if one uses $\mathcal{L}u = \Delta u$ as the auxiliary linear operator. For 2D problem, the general solution of $\Delta u = 0$ is

$$u(x, y) = \left(C_1 e^{-\mu_1 x} + C_2 e^{\mu_1 x}\right)(C_3 \sin \mu_1 y + C_4 \cos \mu_1 y) + \left(C_5 e^{-\mu_2 y} + C_6 e^{\mu_2 y}\right)(C_7 \sin \mu_2 x + C_8 \cos \mu_2 x),$$

where C_k, μ_1, μ_2 are constant coefficients, and it is rather difficult to satisfy the boundary conditions at the four walls. In case of three dimension, the problem becomes even more complicated. Therefore, the linear operator $\mathcal{L}u = \Delta u$, which appears as the linear term in Gelfand equation $\Delta u + \lambda e^u = 0$, is *not* a good choice for us. Fortunately, we have great freedom to choose the auxiliary linear operator $\mathcal{L}$. And by means of this kind of freedom, it becomes much easier to solve the eigenvalue and eigenfunction of the high-dimensional Gelfand equation $\Delta u + \lambda e^u = 0$.

4. Discussion and conclusion

In this paper, the basic ideas of a general analytic method for nonlinear problems, namely the homotopy analysis method (HAM), are described first by means of a nonlinear oscillation problem as an example. As pointed out in Section 2.1, the homotopy analysis method has many advantages. Different from perturbation techniques [1–3], the homotopy analysis method is *independent* of any small physical parameters at all. Besides, it provides us with a simple way to ensure the convergence of the solution series, so that we can always get accurate enough approximations. Furthermore, as proved by Liao [13, 27], Hayat et al. [31] and Sajid et al. [32], and pointed out by Abbasbandy [33–35], the homotopy analysis method logically contains the so-called non-perturbation methods such as Adomian's decomposition method [4–9], the δ-expansion method [10, 11], Lyapunov's artificial small parameter method [12], and "homotopy perturbation method" [36], and thus is more general. In Section 2.2, we further illustrate that the second-order nonlinear differential equation (19) can be replaced by an infinite number of the (2κ)th-order linear differential equations, where $\kappa = 1, 2, 3, \ldots$. In Section 3, we verify that the second-order two- or three-dimensional nonlinear Gelfand equation can be replaced by an infinite number of the fourth or sixth-order linear differential equations, respectively. All of these illustrate that a nth-order nonlinear differential equation can be replaced by an infinite number of the kth-order linear differential equations, where the order k is *unnecessary* to be equal to the order n. Thus, we have much larger freedom to solve nonlinear problems than we traditionally thought. It is very interesting that, by means of this kind of freedom, it might become much easier to find solutions of some nonlinear problems, as illustrated in Section 3.

All of these base on the flexibility and the advantages of the homotopy analysis method: different from *all* other approximation techniques, the homotopy analysis method provides us great freedom to choose the basis functions of solutions and the corresponding auxiliary linear operator. More importantly, different from *all* other approximation techniques, the homotopy analysis method provides us with a simple way to ensure the convergence of solution series by means of introducing the nonzero auxiliary parameter $\hbar$. As shown in Sections 2 and 3, although choosing a set of good basis functions is very important, it is unfortunately *not* enough to ensure the convergence of solution series, and thus the auxiliary parameter $\hbar$ is *absolutely* necessary.

The freedom of the homotopy analysis method on the choice of basis functions and the corresponding auxiliary linear operators might greatly simplify finding solutions of some nonlinear problems, as illustrated in Section 3. This kind of freedom has general meanings and thus is very useful. For example, Liao, Su, and Chwang [44] successfully solved a *unsteady* nonlinear heat transfer problem with *combined* initial/boundary conditions by transferring it into an

infinite number of *steady-state* linear boundary-value problems. Currently, Liao [28] replaced a *unsteady* boundary-layer flow problem by means of an infinite number of *steady-state* linear boundary-value problems. Besides, this kind of freedom can be used to develop some new numerical techniques for nonlinear problems, such as the so-called "generalized boundary element method" [50–54]. All of these indicate that we indeed have much larger freedom and flexibility to solve nonlinear problems than we thought traditionally. And by means of this kind of freedom, it might become much easier to find solutions of some nonlinear problems, as illustrated in this paper and mentioned above.

The examples given in this paper might be helpful to keep us an open mind for nonlinear problems: it is our traditional thoughts that might be the largest restriction to our mind. It is a pity that a lot of things are still unclear now. For example, how can we find the *best* auxiliary linear operator among an *infinite* number of possible ones? Can we give some rigorous mathematical *proofs* in general? So, there are many interesting works to do in future. The freedom on the choice of the auxiliary linear operator might bring forward some new and interesting problems in applied and pure mathematics, and might, I wish, finally give us the "whole" freedom of finding solutions of nonlinear differential equations, if such kind of freedom really exists.

Acknowledgments

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Appendix A: Detailed deduction of Equations (38) and (39) by the first approach

Differentiating Equation (27) n times with respect to q, then dividing by $n!$, we have for $n \geq 1$ that

$$\frac{1}{n!}\frac{d^n\{(1-q)\mathcal{L}[\phi(\tau;q)-u_0(\tau)]\}}{dq^n} = \frac{1}{n!}\frac{d^n\{\hbar q\mathcal{N}[\phi(\tau;q),\Gamma(q)]\}}{dq^n}. \tag{A.1}$$

Using the formula

$$(fg)^{(n)} = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}f^{(k)}g^{(n-k)},$$

where f, g are any real functions of q and $f^{(k)}$ denotes the kth-order differentiation with respect to q, we have

$$\begin{aligned}
&\frac{1}{n!}\frac{d^n\{(1-q)\mathcal{L}[\phi(\tau;q)-u_0(\tau)]\}}{dq^n}\\
&\quad=\frac{(1-q)}{n!}\frac{d^n}{dq^n}\mathcal{L}[\phi(\tau;q)-u_0(\tau)]-\frac{1}{(n-1)!}\frac{d^{n-1}}{dq^{n-1}}\mathcal{L}[\phi(\tau;q)-u_0(\tau)]\\
&\quad=(1-q)\mathcal{L}\left[\frac{1}{n!}\frac{d^n\phi(\tau;q)}{dq^n}\right]-\mathcal{L}\left[\frac{1}{(n-1)!}\frac{d^{n-1}\phi(\tau;q)}{dq^{n-1}}\right]\\
&\qquad+\frac{1}{(n-1)!}\frac{d^{n-1}\mathcal{L}[u_0(\tau)]}{dq^{n-1}}.
\end{aligned}$$

Setting $q=0$ in above expression and using the definition (35) of $u_n(\tau)$, we have

$$\begin{aligned}
&\frac{1}{n!}\frac{d^n\{(1-q)\mathcal{L}[\phi(\tau;q)-u_0(\tau)]\}}{dq^n}\bigg|_{q=0}\\
&\qquad=\mathcal{L}[u_n(\tau)]-\mathcal{L}[u_{n-1}(\tau)]+\frac{1}{(n-1)!}\frac{d^{n-1}\mathcal{L}[u_0(\tau)]}{dq^{n-1}},
\end{aligned}$$

which gives

$$\frac{1}{n!}\frac{d^n\{(1-q)\mathcal{L}[\phi(\tau;q)-u_0(\tau)]\}}{dq^n}\bigg|_{q=0}=\mathcal{L}[u_1(\tau)],\quad\text{when } n=1,$$

and

$$\frac{1}{n!}\frac{d^n\{(1-q)\mathcal{L}[\phi(\tau;q)-u_0(\tau)]\}}{dq^n}\bigg|_{q=0}=\mathcal{L}[u_n(\tau)-u_{n-1}(\tau)],\quad\text{when } n\geq 2,$$

respectively. Thus, using the definition (41), we have

$$\frac{1}{n!}\frac{d^n\{(1-q)\mathcal{L}[\phi(\tau;q)-u_0(\tau)]\}}{dq^n}\bigg|_{q=0}=\mathcal{L}[u_n(\tau)-\chi_n u_{n-1}(\tau)]. \tag{A.2}$$

Similarly, it holds

$$\begin{aligned}
&\frac{1}{n!}\frac{d^n\{\hbar q\mathcal{N}[\phi(\tau;q),\Gamma(q)]\}}{dq^n}\bigg|_{q=0}\\
&\quad=\hbar\left\{\frac{q}{n!}\frac{d^n\mathcal{N}[\phi(\tau;q),\Gamma(q)]}{dq^n}+\frac{1}{(n-1)!}\frac{d^{n-1}\mathcal{N}[\phi(\tau;q),\Gamma(q)]}{dq^{n-1}}\right\}\bigg|_{q=0}\\
&\quad=\frac{\hbar}{(n-1)!}\frac{d^{n-1}\mathcal{N}[\phi(\tau;q),\Gamma(q)]}{dq^{n-1}}\bigg|_{q=0}\\
&\quad=\hbar R_n,
\end{aligned} \tag{A.3}$$

where R_n is defined by (40). So, from (A.1), (A.2), and (A.3), we have the nth-order deformation equation

$$\mathcal{L}[u_n(\tau) - \chi_n u_{n-1}(\tau)] = \hbar R_n.$$

Then, according to the definition (40) of R_n and the definition (23) of $\mathcal{N}$, we have

$$R_n = \frac{1}{(n-1)!}\left\{\frac{\partial^{n-1}}{\partial q^{n-1}}[\Gamma(q)\phi''(\tau;q) + \lambda\phi(\tau;q) + \epsilon\phi^3(\tau;q)]\right\}\Big|_{q=0}.$$

It holds due to the definition (35) that

$$\begin{aligned}
&\frac{1}{(n-1)!}\left\{\frac{\partial^{n-1}}{\partial q^{n-1}}[\Gamma(q)\phi''(\tau;q)]\right\}\Big|_{q=0} \\
&= \frac{1}{(n-1)!}\left\{\sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-1-k)!}\frac{d^k\Gamma(q)}{dq^k}\frac{\partial^{n-1-k}\phi''(\tau;q)}{\partial q^{n-1-k}}\right\}\Big|_{q=0} \\
&= \left\{\sum_{k=0}^{n-1}\left[\frac{1}{k!}\frac{d^k\Gamma(q)}{dq^k}\right]\left[\frac{1}{(n-1-k)!}\frac{\partial^{n-1-k}\phi''(\tau;q)}{\partial q^{n-1-k}}\right]\right\}\Big|_{q=0} \\
&= \sum_{k=0}^{n-1}\left[\frac{1}{k!}\frac{d^k\Gamma(q)}{dq^k}\right]\Big|_{q=0}\frac{\partial^2}{\partial\tau^2}\left\{\left[\frac{1}{(n-1-k)!}\frac{\partial^{n-1-k}\phi(\tau;q)}{\partial q^{n-1-k}}\right]\Big|_{q=0}\right\} \\
&= \sum_{k=0}^{n-1}\gamma_k u''_{n-1-k}(\tau).
\end{aligned}$$

Similarly, we have

$$\frac{1}{(n-1)!}\left\{\frac{\partial^{n-1}}{\partial q^{n-1}}[\lambda\phi(\tau;q)]\right\}\Big|_{q=0} = \lambda u_{n-1}(\tau)$$

and

$$\begin{aligned}
&\frac{1}{(n-1)!}\left\{\frac{\partial^{n-1}}{\partial q^{n-1}}[\epsilon\phi^3(\tau;q)]\right\}\Big|_{q=0} \\
&= \frac{\epsilon}{(n-1)!}\left\{\sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-1-k)!}\frac{\partial^{n-1-k}\phi(\tau;q)}{\partial q^{n-1-k}}\frac{\partial^k[\phi^2(\tau;q)]}{\partial q^k}\right\}\Big|_{q=0} \\
&= \epsilon\left\{\sum_{k=0}^{n-1}\frac{1}{k!(n-1-k)!}\frac{\partial^{n-1-k}\phi(\tau;q)}{\partial q^{n-1-k}}\right. \\
&\quad\left.\times\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}\frac{\partial^{k-j}\phi(\tau;q)}{\partial q^{k-j}}\frac{\partial^j\phi(\tau;q)}{\partial q^j}\right\}\Big|_{q=0}
\end{aligned}$$

$$= \epsilon \left\{ \sum_{k=0}^{n-1} \left[\frac{1}{(n-1-k)!} \frac{\partial^{n-1-k} \phi(\tau;q)}{\partial q^{n-1-k}} \right] \right.$$
$$\left. \times \sum_{j=0}^{k} \left[\frac{1}{(k-j)!} \frac{\partial^{k-j} \phi(\tau;q)}{\partial q^{k-j}} \right] \left[\frac{1}{j!} \frac{\partial^{j} \phi(\tau;q)}{\partial q^{j}} \right] \right\} \Bigg|_{q=0}$$
$$= \epsilon \sum_{k=0}^{n-1} u_{n-1-k}(\tau) \sum_{j=0}^{k} u_{k-j}(\tau) u_j(\tau).$$

So, we have

$$R_n(\vec{u}_{n-1}, \vec{\gamma}_{n-1})$$
$$= \sum_{k=0}^{n-1} \gamma_k u''_{n-1-k}(\tau) + \lambda u_{n-1}(\tau) + \epsilon \sum_{k=0}^{n-1} u_{n-1-k}(\tau) \sum_{j=0}^{k} u_{k-j}(\tau) u_j(\tau). \quad \text{(A.4)}$$

Besides, differentiating (28) n times with respect to q, then dividing by $n!$, and finally setting $q = 0$, we have, using the definition (35) of $u_n(\tau)$, the corresponding initial conditions

$$u_n(0) = 0, \quad u'_n(0) = 0.$$

Note that, (A.2) and (A.3) are valid for *any* definitions of a linear operator $\mathcal{L}$ and a nonlinear operator $\mathcal{N}$. So, as long as the zero-order deformation equation is expressed by (27), the corresponding high-order deformation equations can be expressed in the *same* form

$$\mathcal{L}[u_n - \chi_n u_{n-1}] = \hbar R_n,$$

where only

$$R_n = \frac{1}{(n-1)!} \left. \frac{d^{n-1} \mathcal{N}[\phi]}{dq^{n-1}} \right|_{q=0}$$

is different for different definitions of nonlinear operator $\mathcal{N}$ related to different problems. For example, the high-order deformation equation (92) of Gelfand problem is in the same form. In this way, it is easy to give high-order deformation equations. Therefore, by means of the HAM, it is convenient to get high-order approximation by means of symbolic software such as Mathematica, Maple, MathLab and so on.

Appendix B: Detailed deduction of Equations (38) and (39) by the second approach

As pointed out by Hayat et al. [31] and Sajid et al. [32], directly substituting (33) and (34) into the zeroth-order deformation equations (27) and (28), and

equating the coefficients of the like power of q, one can get exactly the *same* equations as (38) and (39).

Using the series expression (33), we have

$$\begin{aligned}
&(1-q)\mathcal{L}\left[\phi(\tau;q)-u_0(\tau)\right] \\
&\quad=(1-q)\mathcal{L}\left[\sum_{k=0}^{+\infty}u_k(\tau)q^k-u_0(\tau)\right] \\
&\quad=(1-q)\mathcal{L}\left[\sum_{k=1}^{+\infty}u_k(\tau)q^k\right] \\
&\quad=\sum_{k=1}^{+\infty}\mathcal{L}\left[u_k(\tau)\right]q^k-\sum_{k=1}^{+\infty}\mathcal{L}\left[u_k(\tau)\right]q^{k+1} \\
&\quad=\sum_{k=1}^{+\infty}\mathcal{L}\left[u_k(\tau)\right]q^k-\sum_{k=2}^{+\infty}\mathcal{L}\left[u_{k-1}(\tau)\right]q^k \\
&\quad=\mathcal{L}[u_1(\tau)]q+\sum_{k=2}^{+\infty}\mathcal{L}\left[u_k(\tau)-u_{k-1}(\tau)\right]q^k \\
&\quad=\sum_{n=1}^{+\infty}\mathcal{L}\left[u_n(\tau)-\chi_n u_{n-1}(\tau)\right]q^n, \qquad \text{(B.1)}
\end{aligned}$$

where χ_n is defined by (41).

Substituting the series (33) and (34) into the definition (23), we have

$$\begin{aligned}
&\mathcal{N}[\phi(\tau;q),\Gamma(q)] \\
&\quad=\left(\sum_{i=0}^{+\infty}\gamma_i q^i\right)\left(\sum_{j=0}^{+\infty}u_j''(\tau)q^j\right)+\lambda\sum_{n=0}^{+\infty}u_n(\tau)q^n \\
&\qquad+\epsilon\left(\sum_{i=0}^{+\infty}u_i(\tau)q^i\right)\left(\sum_{j=0}^{+\infty}u_j(\tau)q^j\right)\left(\sum_{l=0}^{+\infty}u_l(\tau)q^l\right) \\
&\quad=\sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n}\gamma_k u_{n-k}''(\tau)\right]q^n+\lambda\sum_{n=0}^{+\infty}u_n(\tau)q^n \\
&\qquad+\epsilon\sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n}u_{n-k}(\tau)\sum_{j=0}^{k}u_j(\tau)u_{k-j}(\tau)\right]q^n \\
&\quad=\sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n}\gamma_k u_{n-k}''(\tau)+\lambda u_n(\tau)+\epsilon\sum_{k=0}^{n}u_{n-k}(\tau)\sum_{j=0}^{k}u_j(\tau)u_{k-j}(\tau)\right]q^n.
\end{aligned}$$

According to (40) or (A.4), the above expression reads

$$\mathcal{N}[\phi(\tau;q),\Gamma(q)] = \sum_{n=0}^{+\infty} R_{n+1}(\vec{u}_n, \vec{\gamma}_n) q^n. \tag{B.2}$$

Substituting (B.1) and (B.2) into the zeroth-order deformation equations (27), one has

$$\sum_{n=1}^{+\infty} \mathcal{L}[u_n(\tau) - \chi_n u_{n-1}(\tau)]\, q^n = \hbar \sum_{n=0}^{+\infty} R_{n+1}(\vec{u}_n, \vec{\gamma}_n) q^{n+1}.$$

Equating the coefficient of like-power of q in above equation, we have

$$\mathcal{L}[u_n(\tau) - \chi_n u_{n-1}(\tau)] = \hbar R_n(\vec{u}_{n-1}, \vec{\gamma}_{n-1}),$$

which is exactly the *same* as the high-order deformation equation (38).

Besides, substituting (33) into the initial conditions $\phi(0;\, q) = 1$ and $\phi'(0;\, q) = 0$, equating the coefficient of the like power of q, we have

$$u_k(0) = 0, \quad u_k'(0) = 0, \quad k = 1, 2, 3, \ldots,$$

which is exactly the *same* as the initial conditions (39).

In the approach mentioned above, q is *unnecessary* to be a small parameter at all, which is assumed in [36]. Therefore, no matter whether one regards the embedding parameter q as a small parameter or not, one should always get the *same* high-order deformation equations from the same zeroth-order deformation equations, as proved by Hayat et al. [31] and Sajid et al. [32], and pointed out by Abbasbandy [33–35]. This is mainly because the Taylor series of a real function is *unique*, according to the fundamental theorem in calculus [45].

So, according to Appendix A and B, we confirm the conclusion given by Hayat et al. [31], Sajid et al. [32] and Abbasbandy [33, 34] : the so-called "homotopy perturbation method" [36] (proposed in 1999) is indeed only a special case of the homotopy analysis method [13, 25–27] (propounded in 1992) when $\hbar = -1$, and thus cannot give any new things.

Appendix C: Mathematica code for 2D Gelfand equation[5]

```
(*************************************************************************************
For given A, we find such an eigenvalue λ and an eigenfunction w(x, y) that:

                    Δu + λ exp(u) = 0
```

[5]To get free electronic version of this code, please send email to the corresponding author.

subject to the boundary conditions:

$$u(\pm 1, y) = u(x, \pm 1) = 0, \quad u(0, 0) = A$$

```
**************************************************************************************)
≪Calculus‘Pade‘;
≪Graphics‘Graphics‘;

(**************************************************************************************
Define initial guess of w(x,y)
**************************************************************************************)
w[0] = beta;
U[0] = A + w[0];

(**************************************************************************************
Define the function chi[k]
**************************************************************************************)
chi[k_ ] := If[k <= 1, 0, 1];

(**************************************************************************************
Define the the auxiliary linear operator L
**************************************************************************************)
L[f_ ] := Module[{ },
Expand[alpha[2] * D[f, { x, 1 }, { y, 1 }]/x/y + alpha[4]*D[f, {x, 2}, {y, 2}]]
];

(**************************************************************************************
Define the inverse operator of the auxiliary linear operator L
**************************************************************************************)
Linv[x ∧ m_ * y ∧ n_ ] := x ∧ (m+2) * y ∧ (n+2)/(m+2)/(n+2)
/(alpha[2] + alpha[4] * (n+1) * (m+1) );
Linv[x ∧ m_ ] := x ∧ (m+2) * y ∧ 2/(m+2)/2/(alpha[2] + alpha[4]*(m+1));
Linv[y ∧ n_ ] := x ∧ 2 * y ∧ (n+2)/(n+2)/2/(alpha[2] + alpha[4]*(n+1));
Linv[c_ ] := c*x ∧ 2 * y ∧ 2/4/(alpha[2]+alpha[4]) /; FreeQ[c,x] && FreeQ[c,y];

(**************************************************************************************
The linear property of the inverse operator of L
Linv[f+g] = Linv[f]+Linv[g];
**************************************************************************************)
Linv[p_Plus] := Map[Linv, p];
Linv[c_ * f_ ] := c * Linv[f] /; FreeQ[c, x] && FreeQ[c, y];

(**************************************************************************************
Define GetR[k]
**************************************************************************************)
```

```
GetR[k_ ] := Module[{ temp, n },
temp[1] = D[w[k - 1], { x, 2 }] + D[w[k - 1], { y, 2 }];
temp[2] = Sum[delta[n] * GAMMA[k - 1 - n], { n, 0, k - 1 }];
R[k] = Expand[temp[1] + temp[2]];
];

(*****************************************************************************
Define GetRHS
*****************************************************************************)
GetRHS[k_ ] := Module[{ },
GetR[k];
RHS[k] = Expand[hbar * R[k]];
];

(*****************************************************************************
Define GetGAMMA[n]
*****************************************************************************)
GetGAMMA[n_ ] := Module[{ },
If[n == 0, GAMMA[0] = Exp[beta]];
If[n > 0, GAMMA[n] = Sum[(1-j/n)*GAMMA[j]*w[n-j],{j,0,n-1}]//Expand];
];

(*****************************************************************************
Define Getdelta: this module gets delta[k-1]
*****************************************************************************)
Getdelta[k_ ] := Module[{ eq, temp },
eq = w[k] - (hbar + chi[k] ) * w[k - 1] /. {x -> 0,y->0};
temp = Solve[eq == 0, delta[k - 1]];
delta[k - 1] = temp[[1, 1, 2]]//Expand;
];

(*****************************************************************************
Define GetwSpecial
*****************************************************************************)
GetwSpecial[k_ ] := Module[{ temp },
temp[0] = Expand[RHS[k]];
temp[1] = Linv[temp[0]];
temp[2] = temp[1] + chi[k] * w[k - 1]//Simplify;
wSpecial = temp[2]//Expand;
];

(*****************************************************************************
Define Getw
*****************************************************************************)
```

```
Getw[k_ ] := Module[{ temp, alpha },
alpha = (hbar + chi[k] ) * w[k - 1] + hbar * (1 - chi[k] ) * A;
temp[1] = wSpecial /. y -> 1;
temp[2] = wSpecial /. x -> 1;
temp[3] = wSpecial /. {x -> 1, y -> 1};
temp[4] = alpha /. x -> 1;
temp[5] = alpha /. y -> 1;
temp[6] = alpha /. {x -> 1, y -> 1};
temp[7] = wSpecial - temp[1]-temp[2]+temp[3]+temp[4]+temp[5]-temp[6];
w[k] = Simplify[temp[7]]//Expand;
];

(*****************************************************************************
This module gives the [m,n] homotopy-Pàde approximation of a series
For details, please refer to Section 2.3.7 of Liao's book: Beyond Perturbation,
CRC Press.
*****************************************************************************)
hp[F_, m_, n_ ] := Block[{i, k, dF, temp, q },
dF[0] = F[0];
For[k = 1, k <= m + n, k = k + 1, dF[k] = Expand[F[k] - F[k - 1]]];
temp = dF[0] + Sum[dF[i] * q ^ i, {i, 1, m+n }];
Pade[temp, {q, 0, m, n }] /. q -> 1
];

(*****************************************************************************
Boyd's one-point analytic approximation
*****************************************************************************)
Boyd1[A_ ] := 3.2 * A * Exp[-0.64 * A];

(*****************************************************************************
Boyd's three-point analytic approximation
*****************************************************************************)
Boyd3[A_ ] := Module[{ temp, B, C, G },
G = 0.2763 + Exp[0.463 * A] + 0.0483 * Exp[-0.209 * A];
B = A * (0.829 - 0.566 * Exp[0.463 * A] - 0.0787 * Exp[-0.209 * A])/G;
C = A * (-1.934 + 0.514 * Exp[0.463 * A] + 1.9750 * Exp[-0.209 * A])/G;
temp[1] = 2.667 * A + 4.830 * B + 0.127 * C;
temp[2] = 0.381 * A + 0.254 * B + 0.018 * C;
temp[1] * Exp[-temp[2]]
];

(*****************************************************************************
Main Code
*****************************************************************************)
```

```
ham[m0_, m1_ ] := Module[{ temp, k, j },
For[k = Max[1, m0], k <= m1, k = k + 1,
Print[" k = ", k];
GetGAMMA[k - 1];
GetRHS[k];
GetwSpecial[k];
Getw[k];
U[k] = U[k - 1] + w[k]//Simplify;
Getdelta[k];
DELTA[k - 1] = Simplify[Sum[delta[j], { j, 0, k - 1 }]]//Expand;
lambda[k - 1] = DELTA[k - 1] * Exp[-A];
Print[k - 1, " th approximation of lambda = ", N[lambda[k - 1],10]];
];
Print["Successful !"];
];

(*******************************************************************************
Define the parameters
*******************************************************************************)
hbar = -1;
beta = 0;
A = . ;
alpha[2] = 0;
alpha[4] = -1;

(*Get 10th-order HAM approximation of the 2D Gelfand equation*)
ham[1,11];
```

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