Chapter 7

Matched expansions, Boundary Layer Theory, WKB method.

7.1 Boundary layer theory - regular and singular perturbation problems.

In this section we will consider boundary layer and WKB theory for obtaining asymptotic solutions to differential equations whose highest derivatives are multiplied by a small parameter ϵ. We will find that the solutions change rapidly in thin regions as $\epsilon \to 0$. A **singular perturbation** problem is characterised by the fact that the $\epsilon = 0$ problem has quite different solution properties as compared to the $0 < \epsilon << 1$ problem. In a **regular perturbation** problem as $\epsilon \to 0$ the solution tends to the solution for $\epsilon = 0$. This is best illustrated by looking at some simple examples.

Example

Consider

$$y'' + 2\epsilon y' - y = 0, \quad y(0) = 0, \quad y(1) = 1 \tag{7.1.1}$$

and $0 < \epsilon << 1$. The general solution is

$$y(x, \epsilon) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where

$$m_1 = -\epsilon + \sqrt{1 + \epsilon^2}, \quad m_2 = -\epsilon - \sqrt{1 + \epsilon^2}.$$

As $\epsilon \to 0$ we have

$$y(x) \to \frac{\sinh(x)}{\sinh(1)},$$

and everything seems ok.

We can also obtain a solution as follows: Write

$$y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \dots.$$

Substitution into the equation (7.1.1) and equating coefficents of like powers of ϵ to zero gives

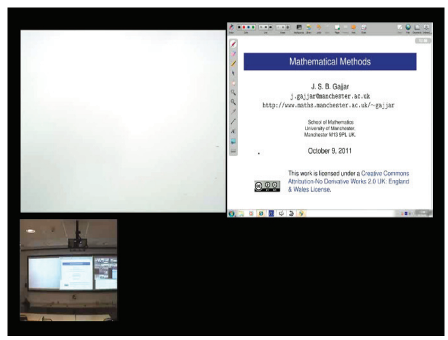
$$\begin{aligned} Y_0'' - Y_0 &= 0, \quad Y_0(0) = 0, \quad Y_0(1) = 1 \\ Y_1'' - Y_1 &= -2Y_0', \quad Y_1(0) = 0, Y_1(1) = 0. \end{aligned} \tag{7.1.2}$$

Solving (7.1.2) gives

$$Y_0 = \frac{\sinh(x)}{\sinh(1)}, \quad Y_1 = (1-x)\frac{\sinh(x)}{\sinh(1)}.$$

Again there are no problems - we have a regular perturbation problem.

Video clip for regular perturbation problem example. Click here to open video clip in external player.



Example

Consider

$$\epsilon y'' + 2y' - y = 0, \quad y(0) = 0, \quad y(1) = 1, \tag{7.1.3}$$

for $0 < \epsilon << 1$. The solution is as before

$$y(x, \epsilon) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where now

$$m_1 = \frac{1}{\epsilon}(-1 + \sqrt{1+\epsilon}), \quad m_2 = \frac{1}{\epsilon}(-1 - \sqrt{1+\epsilon}).$$

As $\epsilon \to 0$ we have

$$m_1 \to \frac{1}{2}, \quad m_2 \sim -\frac{2}{\epsilon}.$$

Note that as $\epsilon \to 0$

$$y \sim \frac{1}{(e^{\frac{1}{2}} - e^{-\frac{2}{\epsilon}})}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}) \sim e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}).$$

Clearly there are two distinct regions:

- $\frac{x}{\epsilon} = O(1)$, and then
$$y \sim e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}).$$
- $x >> \epsilon$ and then
$$y \sim e^{-\frac{1}{2}}e^{\frac{x}{2}}.$$

The analytic solution for different values of ϵ is shown in Fig. 7.1. Note that

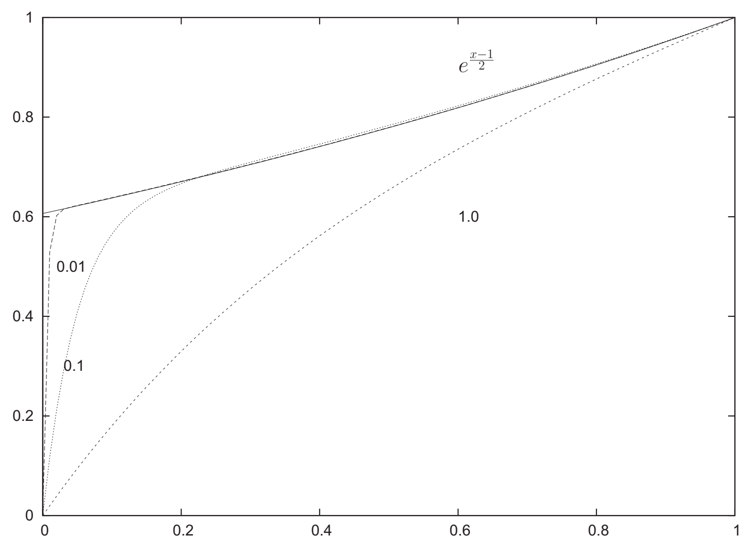


Figure 7.1: Solution $y(x, \epsilon)$ for different values of ϵ.

the solution changes rapidly in the region $x = O(\epsilon)$. We have an example of a singular limit as $\epsilon \to 0$. The region $x = O(\epsilon)$ is called a *boundary layer.*

Suppose we try solving the equation as before. Put

$$y = Y_0 + \epsilon Y_1 + \ldots.$$

This gives after substitution into (7.1.3)

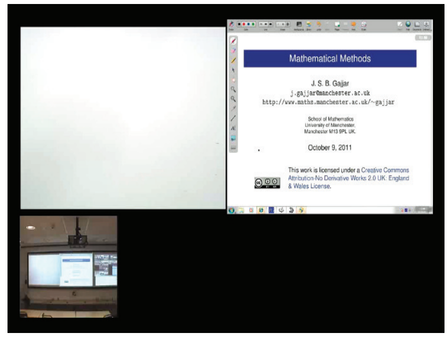
$$2Y_0' - Y_0 = 0, \quad 2Y_1' - Y_1 = Y_0'', \tag{7.1.4}$$

and boundary conditions

$$Y_0(0) = 0, \quad Y_0(1) = 1,$$

$$Y_1(0) = 0, \quad Y_1(1) = 0,$$

etc. Now there is a problem! The order of the equations (7.1.4) is reduced, ie we now have first order equations for the Y_i. Consequently which boundary conditions do we choose? The exact solution suggests we can satisfy the condition at $x = 1$. Let us continue with the boundary condition at $x = 1$.



Video clip showing working for singular perturbation example.Click here to open video clip in external player.

Solution of first order problem

$$2Y_0' - Y_0 = 0, \quad Y_0(1) = 1,$$

gives

$$Y_0 = e^{\frac{x-1}{2}}.$$

Clearly this solution is not valid for all x since the condition at $x = 0$ is not satisfied. When x is small the solution fails and we need to examine this region in more detail. The Y_0 solution is the leading order *outer solution*. Now when x is small we have

$$Y_0 \sim e^{-\frac{1}{2}}(1 + \frac{x}{2}) = O(1).$$

Put $x = \epsilon^n X$ say where $n > 0$ is to be found. The variable X is called the inner variable and is $O(1)$ in the *inner region* of thickness $O(\epsilon^n)$. The differential equation (7.1.3) in terms of X is

$$\epsilon^{1-2n}\frac{d^2y}{dX^2} + 2\epsilon^{-n}\frac{dy}{dX} - y = 0. \tag{7.1.5}$$

For $n > 0$ the dominant terms are the first two terms and these balance if

$$1 - 2n = -n \quad \Longrightarrow n = 1.$$

A quick consistency check shows that this is ok, (other choices for n eg $n = 1/2$ are not). In the inner region if we put

$$y = y_0(X) + \epsilon^\alpha y_1(X) + \ldots$$

with $\alpha > 0$ and substitute into (7.1.5) (with $n = 1$) we find that the leading order problem is

$$\frac{d^2y_0}{dX^2} + 2\frac{dy_0}{dX} = 0,$$

and one boundary condition is $y_0(X = 0) = 0$.

The other condition must come from *matching* with the outer solution taking X large. Solving yields

$$y_0(X) = A + Be^{-2X}$$

and $y_0(0) = 0$ implies that $A = -B$. Thus

$$y_0(X) = A(1 - e^{-X}).$$

To obtain the constant A we match the inner solution just derived with the outer solution.

$$y_0(X) \sim A \quad \text{for} \quad X >> 1,$$

and

$$Y_0(x) \sim e^{-\frac{1}{2}} \quad \text{for} \quad x \to 0.$$

This gives $A = e^{-\frac{1}{2}}$ and

$$y_0(X) = e^{-\frac{1}{2}}(1 - e^{-2X}).$$

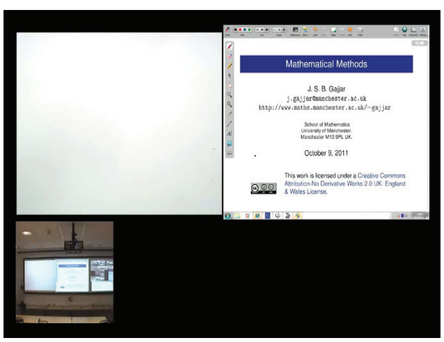
Summary so far: 1 term inner and 1 term outer expansions.

outer:

$$x = O(1), \quad y = Y_0(x) + \epsilon Y_1(x) + \dots,$$
$$Y_0(x) = e^{-\frac{1}{2}} e^{\frac{x}{2}}.$$

inner:

$$x = \epsilon X, \quad y = y_0(X) + \epsilon^{\alpha} y_1(X) + \dots,$$
$$y_0(X) = e^{-\frac{1}{2}}(1 - e^{-2X}).$$



Video clip for discussion boundary layer solution. Click here to open video clip in external player.

These are the basics of boundary layer theory and matched asymptotic expansions. The solution can be continued to higher order. Notice that the outer solution expanded for small x gives

$$y \sim e^{-\frac{1}{2}}(1 + \frac{x}{2} + \dots) + \epsilon Y_1(x) + \dots.$$

When written in terms of $x = \epsilon X$ this suggests that the inner solution should proceed as

$$y = y_0 + \epsilon y_1 + \dots.$$

We had assumed that the outer expansion proceeded in powers of ϵ but this does not have to be the case. One needs to proceed on a term by term basis matching the inner and outer solutions systematically and this will inform how the addtional terms behave. We will continue to the next order for both the inner and outer solutions. Now for the outer solution

$$y = Y_0 + \epsilon Y_1 + \ldots,$$

and the problem for Y_1 is

$$2Y_1' - Y_1 = Y_0'' = \frac{1}{4}e^{\frac{x-1}{2}}, \quad Y_1(1) = 0.$$

Solving gives

$$Y_1 = \frac{(x-1)}{8}e^{\frac{x-1}{2}}.$$

For the inner problem, we have $x = \epsilon X$ and

$$y = y_0(X) + \epsilon y_1(X) + \ldots.$$

The problem for y_1 is

$$\frac{d^2 y_1}{dX^2} + 2\frac{dy_1}{dX} = y_0 = e^{-\frac{1}{2}}(1 - e^{-2X}), \quad y_1(X=0) = 0. \tag{7.1.6}$$

The solution of (7.1.6) gives

$$y_1 = A(1 - e^{-2X}) + \frac{1}{2}X(1 + e^{-2X})e^{-\frac{1}{2}},$$

where we have incorporated the boundary condition and A is an arbitrary constant to be determined from matching with the outer solution. The outer solution expanded for small x gives

$$y_{outer} = e^{\frac{x-1}{2}} + \epsilon\frac{1}{8}(x-1)e^{\frac{x-1}{2}} + \ldots,$$

$$\sim e^{-\frac{1}{2}}(1 + \frac{x}{2} + \ldots) + \epsilon e^{-\frac{1}{2}}(\frac{(x-1)}{8}(1 + \frac{x}{2} + \ldots)).$$

Written in terms of inner variables this is

$$y_{outer} \sim e^{-\frac{1}{2}} + \epsilon e^{-\frac{1}{2}}(\frac{X}{2} - \frac{1}{8}) + \ldots.$$

The two term inner solution is

$$y_{inn} = e^{-\frac{1}{2}}(1 - e^{-2X}) + \epsilon[A(1 - e^{-2X}) + \frac{1}{2}X(e^{-2X} + 1)e^{-\frac{1}{2}}] + \ldots,$$

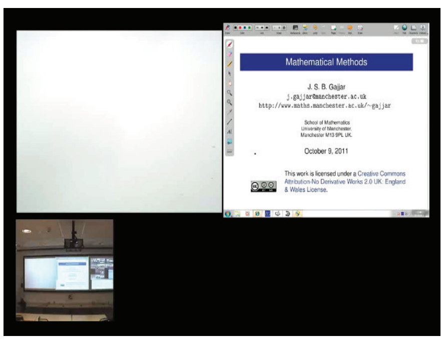
$$\sim e^{-\frac{1}{2}} + \epsilon(A + \frac{1}{2}Xe^{-\frac{1}{2}}) + \dots, \tag{7.1.7}$$

as $X \to \infty$. This has to match with the two term outer solution written in terms of inner variables, i.e.,

$$y_{outer} \sim e^{-\frac{1}{2}} + \epsilon e^{-\frac{1}{2}}(\frac{X}{2} - \frac{1}{8}) + \dots. \tag{7.1.8}$$

A match is only possible if $A = -\frac{1}{8}e^{-\frac{1}{2}}$. Thus

$$y_1 = -\frac{1}{8}e^{-\frac{1}{2}}(1 - e^{-2X}) + \frac{X}{2}e^{-\frac{1}{2}}(1 + e^{-2X}).$$



Video clip showing example of higher-order matching.Click here to open video clip in external player.

7.2 Uniform approximations

A uniform approximation to the solution valid in the whole region is defined by

$$y_{unif} = Y_{outer} + y_{inn} - y_{match}$$

where y_{match} is the approximation to $y(x)$ in the matching region.

For the above problem we had

$$Y_{outer} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + \epsilon\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}} + O(\epsilon^2).$$

$$y_{inn} = e^{-\frac{1}{2}}(1 - e^{-\frac{2x}{\epsilon}}) + \epsilon[-\frac{1}{8}e^{-\frac{1}{2}}(1 - e^{-\frac{2x}{\epsilon}}) + \frac{e^{-\frac{1}{2}}}{2}\frac{x}{\epsilon}(1 + e^{-\frac{2x}{\epsilon}})] + \dots.$$

The matching region is $X(= x/\epsilon) >> 1$ and $x << 1$, ie,

$$\epsilon << x << 1.$$

Thus a one-term uniform approximation is

$$y_{unif} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + e^{-\frac{1}{2}}(1 - e^{-\frac{2x}{\epsilon}}) - e^{-\frac{1}{2}}$$

ie

$$y_{unif} = e^{-\frac{1}{2}}[e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}].$$

A two term uniform approximation is

$$y_{unif} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + \epsilon\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}}$$

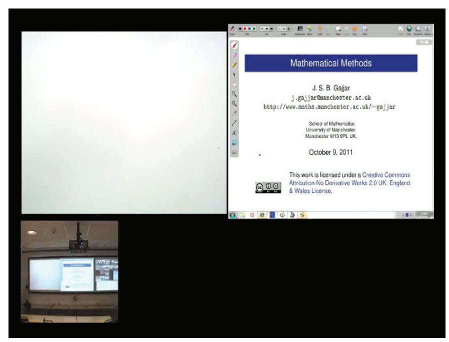
$$+e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) + \epsilon[-\frac{1}{8}e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) + \frac{e^{-\frac{1}{2}}}{2}\frac{x}{\epsilon}(1+e^{-\frac{2x}{\epsilon}})]$$

$$-[e^{-\frac{1}{2}} + \epsilon(-\frac{1}{8} + \frac{x}{2\epsilon})e^{-\frac{1}{2}}].$$

ie

$$y_{unif} = e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}} + \frac{x}{2}e^{-\frac{2x}{\epsilon}}) + \epsilon(\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}} + \frac{e^{-\frac{1}{2}}}{8}e^{-\frac{2x}{\epsilon}}).$$

Video clip for uniform approximations. Click here to open video clip in external player.



7.3 More on matching and intermediate variables

In the previous example we constructed an outer solution with x fixed and ϵ tending to zero, and an inner expansion with $X = x/\epsilon$ fixed and ϵ going to zero. Grapically the process may be represented as in fig. 7.2 with the region A representing the outer solution and region B the inner solution. The figure also shows an overlap region where the two solutions agree. However closer examination of the figure might suggest that there is a possibility of a region C not accessible by the inner or outer solutions. In reality the actual domains of validity of the two solutions may be larger than the above limiting process allows. The difficulty here is arises from the way the matching is done.

A different way to match the two solutions is to introduce an intermediate variable, say $x = \epsilon^\alpha \xi$ with (in the above example) $0 < \alpha < 1$. We have $X = x/\epsilon = \epsilon^{-1+\alpha}\xi$ and so as $\epsilon \to 0$ with ξ fixed gives $X \to \infty$ and $\epsilon \to 0$ with ξ fixed also gives $x \to 0$. Thus ξ is an *intermediate* variable and it is in this variable that we attempt to match the inner and outer solutions. The region defined by

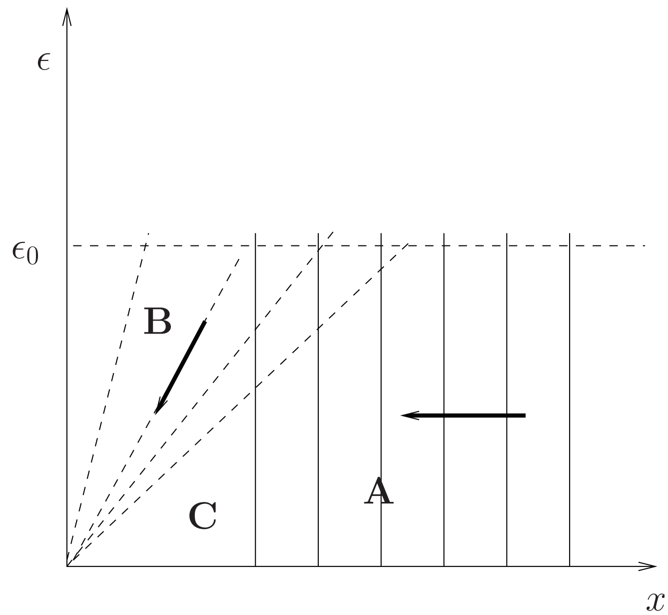


Figure 7.2: Outer solution represented by region **A** with $\epsilon \to 0$ x fixed, and inner solution by **B** with $\epsilon \to 0$ with $X = x/\epsilon$ fixed.

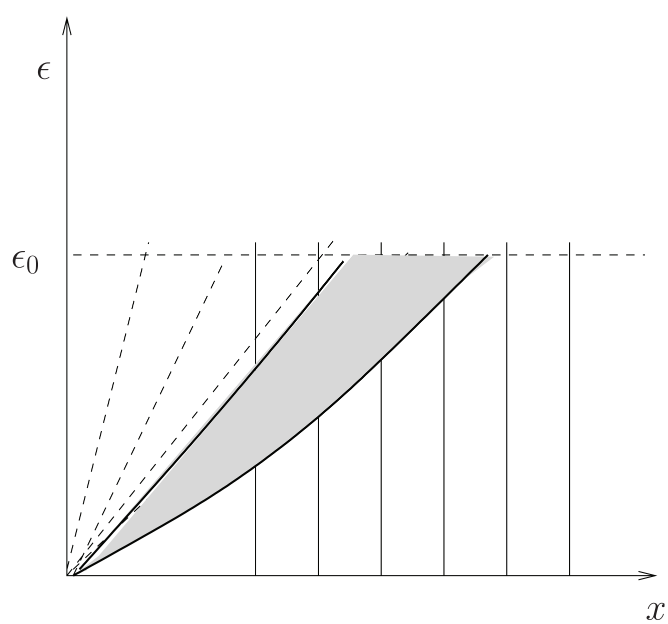
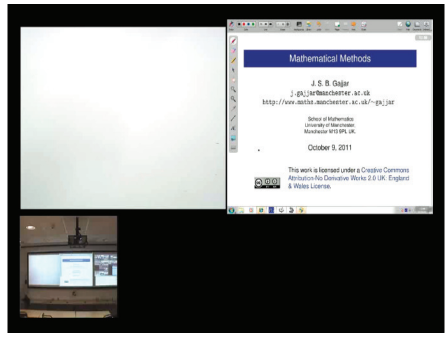


Figure 7.3: Overlap region (shaded) working in terms of intermediate variables $x = \epsilon^{\alpha}\xi$ and $X = \epsilon^{-1+\alpha}\xi$ with $0 < \alpha < 1$, and $\epsilon \to 0+$.



Video clip showing use of intermediate variables for previous example.Click here to open video clip in external player.

$\xi = O(1)$ is *an overlap region* for the two solutions, as shown schematically in fig. 7.3.

We will show how this works with another example in which the differential equation is nonlinear.

Example Consider

$$\epsilon y'' + y' + y^2 = 0, \quad y(0) = 0, y(1) = 1/2. \tag{7.3.1}$$

Suppose we look for an outer solution of the form

$$y = y_0 + \epsilon y_1 + \dots.$$

Then from (7.3.1) we obtain

$$y_0' + y_0^2 = 0, \quad y_0'' + y_1' + 2y_0 y_1 = 0. \tag{7.3.2}$$

The solution of the outer problem shows that

$$-\frac{y_0'}{y_0^2} = 1, \quad \frac{1}{y_0} = x + k,$$

and so

$$y_0 = \frac{1}{x+k}.$$

The boundary layer occurs at $x = 0$ (why?) and so we need to use the condition $y_0(1) = 1/2$ giving $k = 1$, and so

$$y_0 = \frac{1}{x+1}.$$

At next order

$$y_1' + 2y_0 y_1 + y_0'' = 0, \quad y_1(1) = 0.$$

Substituting for $y_0 = 1/(x+1)$ gives

$$y_1' + \frac{2}{x+1} y_1 = \frac{-2}{(1+x)^3}.$$

Hence

$$((1+x)^2 y_1)' = -\frac{2}{1+x},$$

$$(1+x)^2 y_1 + k_1 = -2\log(x+1).$$

Applying the condition $y_1(1) = 0$ gives $k_1 = -2\log 2$ and thus

$$y_1 = \frac{2\log(\frac{2}{1+x})}{(1+x)^2}.$$

For the inner solution we need to seek a solution in terms of an inner variable say $x = \epsilon^n X$ and substitution in (7.3.1) shows that $n = 1$ for a distinguished limit. The inner solution may be expanded as

$$y = Y_0(X) + \epsilon Y_1(X) + \ldots.$$

After substitution into (7.3.1) and using $x = \epsilon X$ we obtain

$$Y_0'' + Y_0' = 0, \quad Y_1'' + Y_1' + Y_0^2 = 0.$$

The boundary conditions are

$$Y_0(0) = 0, \quad Y_1(0) = 0.$$

Solving for Y_0 yields

$$Y_0 = A_0 + B_0 e^{-X}, \quad \text{and} \quad A_0 + B_0 = 0.$$

Thus

$$Y_0 = A_0(1 - e^{-X}).$$

To find A_0 we match with intermediate variables and put $x = \epsilon^\alpha \xi, X = \epsilon^{-1+\alpha}\xi$, and $0 < \alpha < 1$ with $\xi = O(1)$. The one term outer solution written in terms of ξ is

$$y = y_0(x) + \cdots \sim \frac{1}{1+\epsilon^\alpha \xi} \sim 1 - \epsilon^\alpha \xi + \ldots. \tag{7.3.3}$$

Similarly the outer solution in terms of ξ is

$$y = Y_0(X) + \cdots \sim A_0(1 - e^{-\epsilon^{-1+\alpha}\xi}) \sim A_0.$$

Thus matching with (7.3.3) shows that $A_0 = 1$ with error $O(\epsilon^\alpha)$.

Before we match to second order we need to find Y_1 which satisfies

$$Y_1'' + Y_1' + Y_0^2 = 0, \quad Y_1(0) = 0.$$

Thus

$$Y_1'' + Y_1' = -(1 - e^{-X})^2.$$

Solving and applying the condition on $X = 0$ gives (check)

$$Y_1(X) = A_1(1 - e^{-X}) + \frac{1}{2}(1 - e^{-2X}) - X(1 + 2e^{-X}).$$

Next we write the outer and inner expansions in terms of the intermediate variables and do the matching. The outer expansion written in terms of ξ is

$$\begin{aligned} y_{out} &= \frac{1}{1+x} + \epsilon\frac{1}{(1+x)^2}2\log(\frac{2}{1+x}) + \ldots, \\ &= \frac{1}{1+\epsilon^\alpha\xi} + \epsilon\frac{1}{(1+\epsilon^\alpha\xi)^2}2\log(\frac{2}{1+\epsilon^\alpha\xi}) + \ldots, \\ &\sim 1 - \epsilon^\alpha\xi + \epsilon^{2\alpha}\xi^2 + \cdots + 2\log 2(\epsilon - 2\epsilon^{\alpha+1}\xi + O(\epsilon^{2\alpha})) - 2\epsilon(1 - 2\epsilon^\alpha\xi)(\epsilon^\alpha\xi - O(\epsilon^{2\alpha})). \end{aligned} \tag{7.3.4}$$

Next the inner solution written in terms of ξ is

$$\begin{aligned} y_{inn} &= (1 - e^{-X}) + \epsilon(A_1(1 - e^{-X}) + \frac{1}{2}(1 - e^{-2X}) - X(1 + 2e^{-X})) + \ldots, \\ &= (1 - e^{-\epsilon^{\alpha-1}\xi}) + \epsilon\left[A_1(1 - e^{-\epsilon^{\alpha-1}\xi}) + \frac{1}{2}(1 - e^{-2\epsilon^{\alpha-1}\xi}) - \epsilon^{\alpha-1}\xi(1 + 2e^{-\epsilon^{\alpha-1}\xi})\right] + \ldots, \\ &\sim 1 + \epsilon A_1 + \frac{\epsilon}{2} - \epsilon^\alpha\xi + \ldots \end{aligned} \tag{7.3.5}$$

In (7.3.4) if we keep terms to order ϵ and assuming that the terms $O(\epsilon^{2\alpha})$ are smaller than terms of $O(\epsilon)$ we require $0 < \alpha < 1/2$. This gives

$$y_{out} \sim 1 - \epsilon^\alpha\xi + \epsilon 2\log 2 + O(\epsilon^2, \epsilon^{1+\alpha}, \epsilon^{2\alpha}). \tag{7.3.6}$$

Comparing (7.3.6) and (7.3.5) we see that the terms of $O(\epsilon^\alpha)$ match automatically and to match the $O(\epsilon)$ terms we require

$$\epsilon A_1 + \frac{\epsilon}{2} = 2\epsilon\log 2,$$

giving

$$A_1 = -\frac{1}{2} + 2\log 2.$$

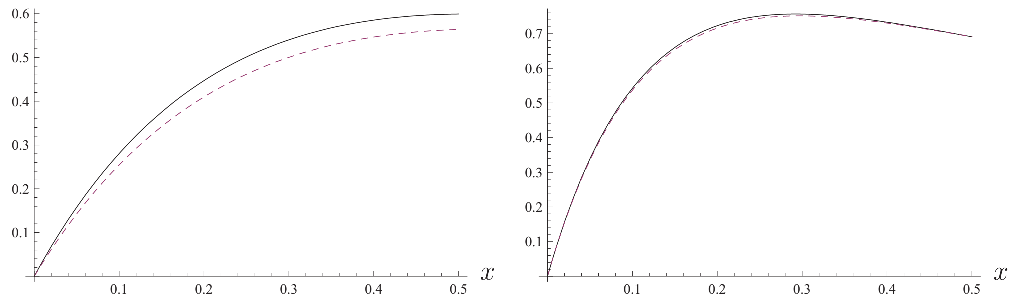
At the next order of matching the terms of $O(\epsilon^{2\alpha})$ match automatically.

The composite solution to $O(\epsilon^2)$ is

$$y_{comp} = y_{out} + y_{inn} - y_{match}.$$

In the above example we find

$$y_{comp} = \frac{1}{x+1} + \frac{2\epsilon}{(x+1)^2}\log\frac{2}{x+1} + (1 - e^{-\frac{x}{\epsilon}}) +$$



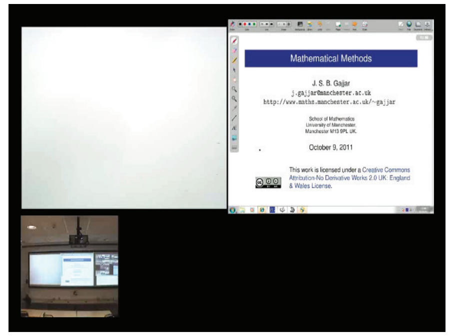
(a) (b)

Figure 7.4: A comparison of the numerical solution of (7.3.1) (solid lines) with the composite solution given by (7.3.7) (dashed line) taking (a) $\epsilon = 0.2$, and (b) $\epsilon = 0.1$

$$\epsilon\left[(-\frac{1}{2}+2\log 2)(1-e^{-\frac{x}{\epsilon}})+\frac{1}{2}(1-e^{-\frac{2x}{\epsilon}})-\frac{x}{\epsilon}(1+2e^{-\frac{x}{\epsilon}})\right] - (1+\epsilon(-\frac{1}{2}+2\log 2+\frac{1}{2}-\frac{x}{\epsilon})). \tag{7.3.7}$$

A comparison of the numerical solution of (7.3.1) with the composite solution is shown in Fig. (7.4) and shows excellent agreement for ϵ small.

7.4 Interior boundary layers



Video clip covering an example of hidden boundary layer. Click here to open video clip in external player.

Consider

$$\epsilon y'' + a(x)y' + b(x)y = 0, \quad y(0) = A, y(1) = B. \tag{7.4.1}$$

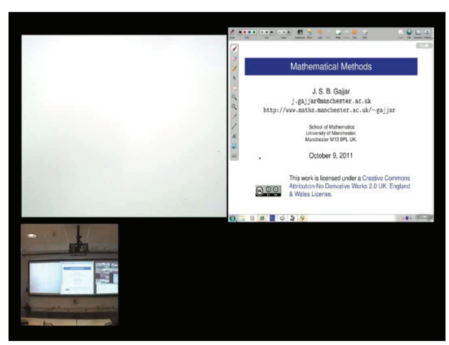
The outer problem (set $\epsilon = 0$) is just

$$a(x)y' + b(x)y = 0.$$

Take $a(x) > 0$ and then

$$y' = -\frac{b(x)}{a(x)}y, \quad y = Ce^{-\int_{x_0}^{x} \frac{b(s)}{a(s)}\, ds}.$$

Again there are two boundary conditions to satisfy and so there must be a boundary layer, but where is the boundary located?



Video clip for example of boundary layer not at $x = 0$. Click here to open video clip in external player.

Suppose that we have a boundary layer at $x = \bar{x}$ of thickness $\gamma(\epsilon)$. We write

$$x = \bar{x} + \gamma(\epsilon)X, \quad \text{where} \quad X = O(1).$$

Then substituting into (7.4.1) with $y = Y$ gives

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2} + \frac{a(\bar{x} + \gamma X)}{\gamma}\frac{dY}{dX} + b(\bar{x} + \gamma X)Y = 0.$$

Now expand a, b as

$$a(\bar{x}+\gamma X) = a(\bar{x}) + \gamma X a'(\bar{x}) + \dots, \quad b(\bar{x}+\gamma X) = b(\bar{x}) + \gamma X b'(\bar{x}) + \dots,$$

to get

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2} + \frac{a(\bar{x})}{\gamma}\frac{dY}{dX} + b(\bar{x})Y + \cdots = 0. \tag{7.4.2}$$

For $|\gamma| << 1$ the dominant terms are

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2}, \quad \frac{a}{\gamma}\frac{dY}{dX}.$$

For a balance we require

$$\frac{\epsilon}{\gamma^2} \sim \frac{1}{\gamma} \quad \implies \gamma = O(\epsilon).$$

Hence set $\gamma = \epsilon$ ie $x = \bar{x} + \epsilon X$. From (7.4.2) the reduced inner equation is

$$\epsilon^{-1}\left[\frac{d^2Y}{dX^2} + a(\bar{x})\frac{dY}{dX}\right] + b(\bar{x})Y + \cdots = 0. \tag{7.4.3}$$

The leading order inner problem is

$$\frac{d^2Y}{dX^2} + a(\bar{x})\frac{dY}{dX} = 0,$$

giving

$$Y = C_0 + C_1 e^{-a(\bar{x})X}.$$

Now we have assumed that $a(\bar{x}) > 0$. If $\bar{x} > 0$ we need to match as we go out of the boundary layer, ie we need limits $X \to \pm\infty$.

As $X \to \infty$ everything is ok, but as $X \to -\infty$ it suggests that C_1 must be zero to avoid exponential growth.

But $C_1 = 0$ implies no boundary layer. Hence $\bar{x} = 0$ and the boundary layer is at $x = 0$ if $a(x) > 0$.

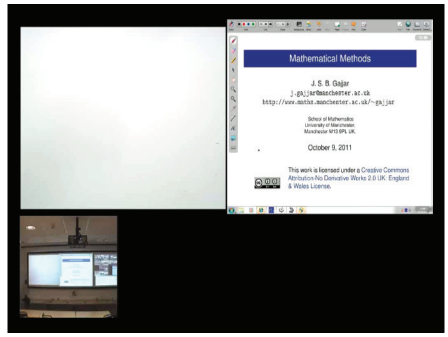
Similarly if $a(x) < 0$ then we have a bounday layer at $x = 1$. If $a(x) = 0$ inside the region we have an internal boundary layer. The above analysis also breaks down.

7.4.1 Further Examples including interior layers

Consider

$$\epsilon y'' + xy' - (\epsilon^2 x^3 + 1)y = 0, \quad y(-1) = 1, y(1) = 2$$

and $-1 \le x \le 1, \quad 0 < \epsilon << 1$. The above discussion suggests an interior layer at $x = 0$.



Video clip for above section. Click here to open video clip in external player.

For the outer solution put

$$y = y_0 + \epsilon y_1 + \dots,$$

to get

$$x y_0' - y_0 = 0.$$

Thus

$$y_0 = Ax.$$

Here we have a new difficulty. Which boundary condition do we choose? We can show that there are no boundary layers near $x = \pm 1$. We write

$$y = A_{\pm} x$$

where the $+$ stands for $x > 0$ and $-$ for $x < 0$.

From the boundary conditions it suggests that

$$A_+ = 2, \quad A_- = -1.$$

When x is small the $\epsilon y''$ term is not negligble, and hence we look for an interior layer at $x = 0$ and write

$$x = \gamma(\epsilon) X, \quad \gamma(\epsilon) << 1.$$

This gives with $y = Y$

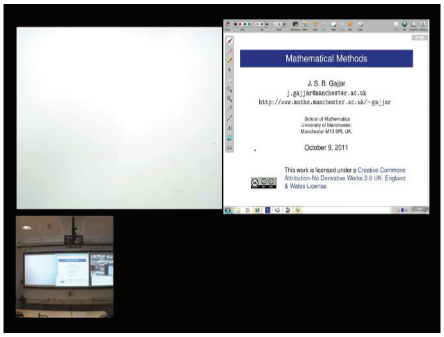
$$\frac{\epsilon}{\gamma^2}\frac{d^2 Y}{dX^2} + \frac{\gamma X}{\gamma}\frac{dY}{dX} - Y + \dots = 0.$$

For a dominant balance this suggests that

$$\frac{\epsilon}{\gamma^2} \sim O(1) \qquad \Longrightarrow \gamma = O(\epsilon^{\frac{1}{2}}).$$

Hence set $x = \epsilon^{\frac{1}{2}} X$ and from the outer solution it suggests that we expand the inner solution as

$$y = \epsilon^{\frac{1}{2}} Y_0 + \epsilon Y_1 + \dots.$$



Video clip for interior layer problem, outer solution for above example. Click here to open video clip in external player.

Substituting into the equation gives

$$\epsilon\epsilon^{-1}\left(\epsilon^{\frac{1}{2}}\frac{d^2Y_0}{dX^2}+\dots\right)+\epsilon^{\frac{1}{2}}X\epsilon^{-\frac{1}{2}}\left(\epsilon^{\frac{1}{2}}\frac{dY_0}{dX}+\dots\right)-\epsilon^{\frac{1}{2}}Y_0+\dots=0.$$

Hence the leading order problem is

$$\frac{d^2Y_0}{dX^2}+X\frac{dY_0}{dX}-Y_0=0. \tag{7.4.4}$$

The boundary conditions suggest that we must match with the outer solution as $X\to\pm\infty$. This suggests that

$$Y_0\sim A_{\pm}X \quad \text{as} \quad X\to\pm\infty. \tag{7.4.5}$$

The equation (7.4.4) can be solved in terms of parabolic cylinder functions. If we put

$$Y_0=e^{-\frac{X^2}{4}}W_0$$

then W_0 satisfies

$$W_0''+(\frac{1}{2}-2-\frac{X^2}{4})W_0=0.$$

Note that two linearly independent solutions of the equation

$$W''+(\frac{1}{2}+\nu-\frac{X^2}{4})W=0,$$

are the parabolic cylinder functions $W=D_\nu(X)$ and $D_{-\nu-1}(iX)$.

In order to do the matching we require the behaviours of $D_\nu(x)$ for $|x|$ large. The properties of $D_\nu(z)$ are summarized below (see for example Abramovitz & Stegun[1]:

$$D_\nu(z)\sim z^\nu e^{-\frac{z^2}{4}}\sum_{n=0}^{\infty}(-1)^n a_n z^{-2n} \quad \text{as} \quad z\to\infty, \quad |\arg(z)|<\frac{3\pi}{4}. \tag{7.4.6}$$

[1]M. Abramovitz and I. A. Stegun *Handbook of Mathematical Function*, Dover. [web version also available]

$$D_\nu(z) \sim z^\nu e^{-\frac{z^2}{4}} \sum_{n=0}^{\infty} (-1)^n a_n z^{-2n} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{\frac{z^2}{4}} \sum_{n=0}^{\infty} b_n z^{-2n}$$

$$\text{as} \quad z \to \infty, \quad \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4}. \tag{7.4.7}$$

Here $a_0 = b_0 = 1$, and

$$a_n = \frac{\nu(\nu-1)\dots(\nu-2n+1)}{2^n n!} \quad b_n = \frac{(\nu+1)(\nu+2)\dots(\nu+n)}{2^n n!}.$$

Hence we can write the solution of (7.4.4) as

$$Y_0 = e^{-\frac{X^2}{4}}(CD_{-2}(X) + ED_1(iX)).$$

Now using

$$D_{-2}(X) \sim X^{-2} e^{-\frac{X^2}{4}}, \quad D_1(iX) \sim (iX) e^{\frac{X^2}{4}} \quad \text{as} \quad X \to \infty$$

and

$$D_{-2}(X) \sim X^{-2} e^{-\frac{X^2}{4}} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(2)} e^{2\pi i} X e^{\frac{X^2}{4}}, \quad \text{as} \quad X \to -\infty$$

$$D_1(iX) \sim (iX) e^{\frac{X^2}{4}} \quad \text{as} \quad X \to -\infty,$$

we find that

$$Y_0 \sim e^{-\frac{X^2}{4}} \left[\frac{C}{X^2} e^{-\frac{X^2}{4}} + E(iX) e^{\frac{X^2}{4}} \right] \quad X \to \infty,$$

ie

$$Y_0 \sim EiX \quad \text{as} \quad X \to \infty.$$

Hence

$$Ei = A_+.$$

Similarly

$$Y_0 \sim e^{-\frac{X^2}{4}} \left[-C\sqrt{(2\pi)} X e^{\frac{X^2}{4}} + E(iX) e^{\frac{X^2}{4}} \right] \quad X \to -\infty.$$

Hence

$$Y_0 \sim (-\sqrt{(2\pi)}C + iE)X + O(1) \quad X \to -\infty,$$

giving

$$-\sqrt{(2\pi)}C + iE = A_-.$$

Using the given values for $A_\pm$ leads to

$$C = \frac{3}{\sqrt{2\pi}}, \quad E = -2i,$$

and the inner solution as

$$Y_0 = \left(\frac{3}{\sqrt{2\pi}} D_{-2}(X) - 2iD_1(iX) \right) e^{-\frac{X^2}{4}}.$$

A uniform approximation can be calculated to give

$$y_{unif} = \epsilon^{\frac{1}{2}} \left(\frac{3}{\sqrt{2\pi}} D_{-2}(\frac{x}{\sqrt{\epsilon}}) - 2iD_1(\frac{ix}{\sqrt{\epsilon}}) \right) e^{-\frac{x^2}{4\epsilon}}.$$

A comparison of the uniform approxmation

$$y_{unif} = \epsilon^{\frac{1}{2}} \left(\frac{3}{\sqrt{2\pi}} D_{-2}(\frac{x}{\sqrt{\epsilon}}) - 2iD_1(\frac{ix}{\sqrt{\epsilon}}) \right) e^{-\frac{x^2}{4\epsilon}}.$$

with a numerical solution of the differential equation

$$\epsilon y'' + xy' - (\epsilon^2 x^3 + 1)y = 0, \quad y(-1) = 1, y(1) = 2$$

and $-1 \leq x \leq 1, \quad 0 < \epsilon << 1$, for $\epsilon = 0.05$ is shown in Fig. 7.5 below.

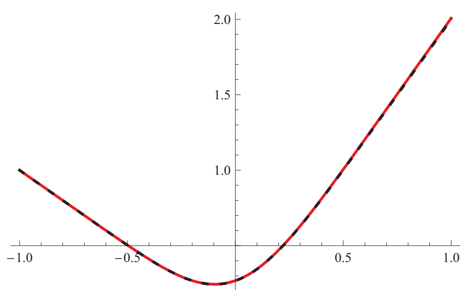
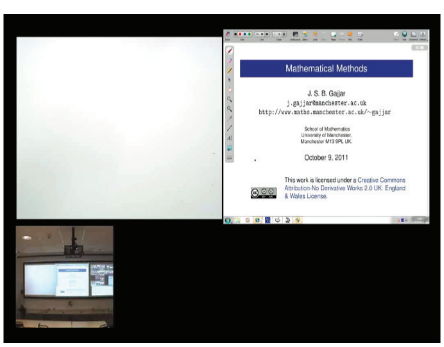


Figure 7.5: A comparison of the (exact) numerical solution to the full equation as compared with the uniform approximation (dashed line) for $\epsilon = 0.05$.



Video clip for interior layer problem, inner solution for above example. Click here to open video clip in external player.

7.5 The LG approximation, WKBJ Method

Boundary layer theory fails when we have a rapid variation in the solution throughout the region rather than locally at some location.

Example Consider

$$\epsilon y'' + by = 0, \quad y(0) = 0, \quad y(1) = 1,$$

where $b > 0$ and $0 < \epsilon << 1$. Note that the general solution is

$$y = \frac{\sin(x\sqrt{\frac{b}{\epsilon}})}{\sin(\sqrt{\frac{b}{\epsilon}})}.$$

The outer solution is just $y = 0$. For the inner solution, suppose we set

$$x = \bar{x} + \gamma(\epsilon)X, \quad \gamma << 1.$$

Then the equations gives

$$\frac{\epsilon}{\gamma^2}\frac{d^2y}{dX^2} + by = 0.$$

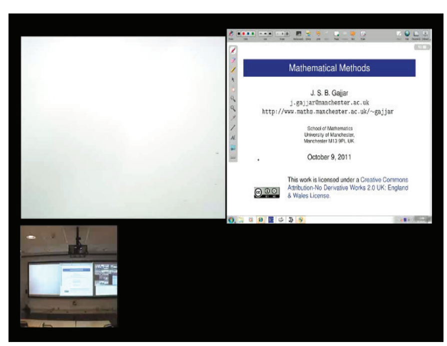
A dominant balance gives $\gamma = \epsilon^{\frac{1}{2}}$ and the resulting inner problem is

$$\frac{d^2y}{dX^2} + by = 0.$$

The solution gives

$$y = A\sin(\sqrt{b}X) + B\cos(\sqrt{b}X).$$

We can choose any $\bar{x}$ but note that for any choice of $\bar{x}$ the solution is not of boundary layer form and cannot be matched to the outer solution as $X \to \pm\infty$ because the inner solution oscillates.



Video clip for above example. Click here to open video clip in external player.

Boundary layer theory fails for these types of singular perturbation problems in which we have wavelike behaviour (as opposed to dissipative or dispersive behaviour). The LG approximation or WKBJ theory is ideal for these classes of problems. The technique we describe below leads to an approximation which was obtained by Liouville (1837) and Green (1837). In fact as noted earlier, Carlini (1817) had also used the same ideas.

The method is more commonly known as the WKBJ method after Wentzel (1926), Kramers (1926), Brillouin (1926), and Jeffreys (1924). (Theoretical physicists call it the WKB method). However it is more correct it to call it the LG approximation which was used by Jeffreys, Wentzel, Kramers and Brillouin, to derive the connection formula in the presence of turning points (see later).

Consider

$$\epsilon y'' = Q(x)y, \quad Q(x) \neq 0. \tag{7.5.1}$$

The basic idea of the theory is that for $\epsilon \to 0$ we look for a solution to (7.5.1) of the form

$$y \sim A(x,\delta)e^{\frac{s(x,\delta)}{\delta}}, \quad \delta(\epsilon) \to 0$$

where $A(x,\delta), s(x,\delta)$ are slowly varying functions of x, but note the rapid variation of the solution because of the exponential factor. We can absorb the A into the exponential by writing

$$y = e^{\frac{S(x,\delta)}{\delta}}. \tag{7.5.2}$$

Substitution (7.5.2) into the equation (7.5.1) gives

$$\epsilon\left[\frac{S'^2}{\delta^2} + \frac{S''}{\delta}\right] - Q(x) = 0,$$

where primes denote differentiation with respect to x.

For a dominant balance we have $\delta = \epsilon^{\frac{1}{2}}$, and the equation for S reduces to

$$S'^2 - Q(x) = -\epsilon^{\frac{1}{2}} S''. \tag{7.5.3}$$

This suggests that we write

$$S = \sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}} S_n, \quad \epsilon \to 0.$$

Substitution into (7.5.3) gives

$$(S_0' + \epsilon^{\frac{1}{2}} S_1' + \dots)^2 - Q(x) = -\epsilon^{\frac{1}{2}}(S_0'' + \epsilon^{\frac{1}{2}} S_1'' + \dots). \tag{7.5.4a}$$

Equating like powers of ϵ in (7.5.4a) to zero gives

$$(S_0')^2 = Q(x), \tag{7.5.4b}$$

$$2S_0'S_1' = -S_0'', \tag{7.5.4c}$$

$$2S_0'S_n' + \sum_{j=1}^{n-1} S_j'S_{n-j}' = -S_{n-1}'', \quad n \geq 2. \tag{7.5.4d}$$

We can solve (7.5.4b) to obtain

$$S_0 = \pm \int^x Q^{\frac{1}{2}}\, dx,$$

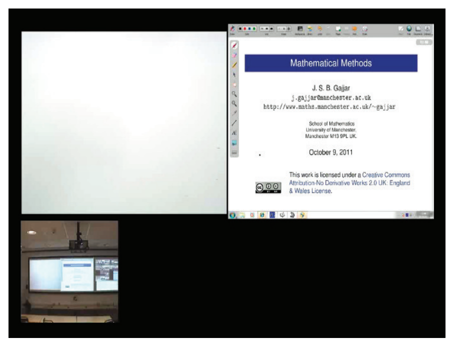
$$S_1' = -\frac{S_0''}{2S_0'} \quad \Longrightarrow S_1 = -\frac{1}{4}\log|Q|.$$

Hence the leading order behaviour of the solution can be written down as

$$y \sim |Q|^{-\frac{1}{4}} \left[C_1 \exp\left(\frac{1}{\epsilon^{\frac{1}{2}}} \int_a^x Q^{\frac{1}{2}}\right) + C_2 \exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}} \int_a^x Q^{\frac{1}{2}}\right)\right], \tag{7.5.5}$$

where C_1, C_2, a are determined from the boundary conditions. This is the LG approximation to the solution. The approximation with just the leading order term S_0 gives what the physicists like to call the *geometrical optics* approximation. The approximation (7.5.5) is also referred to as the *physical optics* approximation.

Video clip for WKB method- general theory.
Click here to open video clip in external player.



Example Consider again

$$\epsilon y'' + by = 0, \quad y(0) = 0, y(1) = 1,$$

and $b > 0$. Here $Q(x) = -b$, and so

$$S_0 = \pm i\sqrt{b}x.$$

Hence using (7.5.5)

$$y \sim b^{-\frac{1}{4}}(C_1 e^{ix\sqrt{\frac{b}{\epsilon}}} + C_2 e^{-ix\sqrt{\frac{b}{\epsilon}}}),$$

or

$$y \sim A_1 \sin(\sqrt{\frac{b}{\epsilon}}x) + A_2 \cos(\sqrt{\frac{b}{\epsilon}}x).$$

Applying the boundary conditions gives the exact solution

$$y = \frac{\sin(\sqrt{\frac{b}{\epsilon}}x)}{\sin(\sqrt{\frac{b}{\epsilon}})}.$$

Example Consider

$$\epsilon y'' - (1+x^2)^2 y = 0, \quad y(0) = 0, y'(0) = 1.$$

If we look for a solution

$$y \sim \exp\left(\frac{1}{\delta}\sum_0^\infty \delta^n S_n\right)$$

then again with $\delta = \epsilon^{\frac{1}{2}}$ we obtain

$$S_0'^2 = (1+x^2)^2, \quad S_0 = \pm\left(\frac{x^3}{3} + x\right).$$

Next

$$S_1 = -\frac{1}{4}\log(1+x^2)^2 = -\frac{1}{2}\log(1+x^2).$$

Thus

$$y \sim (1+x^2)^{-\frac{1}{2}}\left(C_1 \exp\left(\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}}\right) + C_2 \exp\left(-\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}}\right)\right). \tag{7.5.6}$$

To find the constants we need to apply the boundary conditions. The condition $y(0) = 0$ leads to (after substituting $x = 0$ in (7.5.6)

$$0 = C_1 + C_2.$$

Now assuming that differentition of (7.5.6) is valid, we find that

$$y'(x) \sim \frac{(1+x^2)^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}}\left(C_1 \exp\left(\frac{(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}\right) - C_2 \exp\left(\frac{-(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}\right)\right)$$

$$-x(1+x^2)^{-\frac{3}{2}}\left(C_1 \exp\left(\frac{(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}\right) + C_2 \exp\left(\frac{-(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}\right)\right).$$

Hence applying $y'(0) = 1$ gives

$$1 = \epsilon^{-\frac{1}{2}}(C_1 - C_2).$$

Solving for C_1, C_2 gives

$$C_1 = \frac{1}{2}\epsilon^{\frac{1}{2}}, \quad C_2 = -\frac{1}{2}\epsilon^{\frac{1}{2}}.$$

Hence a WKB approximation to the solution is

$$y \sim \epsilon^{\frac{1}{2}}(1+x^2)^{-\frac{1}{2}}\sinh\left(\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}}\right), \quad \epsilon \to 0.$$

The WKB method can also be used for certain eigenvalue problems.

Example Consider

$$y'' + \lambda p(x) y = 0, \quad y(0) = 0, \quad y(\pi) = 0. \tag{7.5.7}$$

This equation has nontrivial solutions only for certain discrete values of λ say $(\lambda_1, \lambda_2, \dots)$. We can obtain an approximation to the eigenvalues and eigenfunction for large λ.

Look for an asymptotic solution in WKB form as

$$y \sim \exp(\lambda^{\frac{1}{2}} \sum_{n=0}^{\infty} \lambda^{-n/2} S_n(x)).$$

Substitution into the equation (7.5.7) gives

$$\lambda(S_0' + \lambda^{-\frac{1}{2}} S_1' + \dots)^2 + \lambda^{\frac{1}{2}} (S_0'' + \lambda^{-\frac{1}{2}} S_1'' + \dots) + \lambda p(x) = 0.$$

Solving for S_0, S_1 gives

$$S_0 = \pm i \int^x (p(x))^{\frac{1}{2}}\, dx, \quad S_1 = -\frac{1}{4} \log |p(x)|.$$

Hence

$$y \sim |p|^{-\frac{1}{4}} \left[C_1 \sin(\lambda^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx) + C_2 \cos(\lambda^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx) \right].$$

The boundary conditions in (7.5.7)imply

$$C_2 = 0,$$

and

$$\sin(\lambda^{\frac{1}{2}} \int_0^\pi (p(x))^{\frac{1}{2}}\, dx) = 0.$$

Hence

$$\lambda^{\frac{1}{2}} = \frac{\pm n\pi}{\int_0^\pi (p(x))^{\frac{1}{4}}\, dx} \quad .$$

Thus

$$\lambda \sim \lambda_n = \frac{n^2\pi^2}{[\int_0^\pi (p(x))^{\frac{1}{4}}\, dx]^2}, \quad n >> 1,$$

and approximate solution to (7.5.7) is

$$y \sim |p|^{-\frac{1}{4}} C_n \sin(\lambda_n^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx).$$

7.5.1 Additional notes

Implicit in the use of the WKB (LG) method ie

$$y \sim \exp(\sum_{n=0}^{\infty} \delta^{n-1} S_n(x))$$

is that the series

$$\sum_{n=0}^{\infty} \delta^{n-1} S_n(x), \quad \text{as} \quad \delta \to 0$$

is an asymptotic series, uniformly valid for all x throughout the interval. This requires that

$$\delta^n S_{n+1}(x) = o(\delta^{n-1} S_n(x)), \quad n = 1, 2, \ldots,$$

holds uniformly in x.

Since we take the exponential of the above series, for the WKB (LG) approximation to be a good approximation, if we truncate the series at $n = M - 1$ say, then we should have

$$\delta^M S_{M+1}(x) = o(1) \quad \delta \to 0$$

since

$$\exp(\delta^M S_{M+1}(x)) = 1 + O(\delta^M S_{M+1}(x)), \quad \text{as} \quad \delta \to 0.$$

7.5.2 Turning points and connection formulae

So far in

$$\epsilon y'' - Q(x, \epsilon) y = 0$$

we have taken $Q(x, \epsilon) > 0$ in the interval.

We will now consider

$$\epsilon y'' - Q(x) y = 0, \quad a < x < b, \quad Q(x_0) = 0, \quad Q'(x_0) > 0, \quad a < x_0 < b. \tag{7.5.8}$$

We will assume that there is only one zero in the $a < x < b$. A WKB approximation to the equation (7.5.8) is

$$y \sim C|Q(x)|^{-\frac{1}{4}} \exp\left(\pm \frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds.\right).$$

Thus for $x > x_0$ we write

$$y \sim |Q(x)|^{-\frac{1}{4}} \left[A_1 \exp\left(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds\right) + A_2 \exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds.\right)\right], \tag{7.5.9}$$

and for $x < x_0$ we have

$$y \sim |Q|^{-\frac{1}{4}} \left[B_1 \cos(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x |Q(s)|^{\frac{1}{2}}\, ds) + B_2 \sin(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x |Q(s)|^{\frac{1}{2}}\, ds) \right]. \tag{7.5.10}$$

The above approximation fails near $x = x_0$, where we have

$$Q(x) \sim (x - x_0)Q'(x_0) + \dots. \tag{7.5.11}$$

If we put $x = x_0 + \epsilon^\gamma X$ and substitute into the differential equation (7.5.8) and use (7.5.11) we obtain

$$\epsilon\epsilon^{-2\gamma}\frac{d^2y}{dX^2} - (\epsilon^\gamma XQ'(x_0)y + \dots) = 0.$$

For a dominant balance we require

$$\epsilon^{1-2\gamma} \sim \epsilon^\gamma, \qquad \Longrightarrow \gamma = \frac{1}{3}.$$

The dominant equation in this region reduces to Airy's equation

$$\frac{d^2y}{dX^2} - Xcy = 0, \quad c = Q'(x_0) > 0.$$

This has the solution

$$y_{inn} = D_1\mathrm{Ai}(c^{\frac{1}{3}}X) + D_2\mathrm{Bi}(c^{\frac{1}{3}}X), \tag{7.5.12}$$

which is the inner solution. We need to match this with the outer solution (7.5.9.7.5.10) as $X \to \pm\infty$ or $x \to x_0 \pm$. Now

$$\mathrm{Ai}(X) \sim \frac{1}{2\sqrt{\pi}}X^{-\frac{1}{4}}e^{-\frac{2}{3}X^{\frac{3}{2}}}, \quad X \to \infty,$$

$$\mathrm{Bi}(X) \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}e^{\frac{2}{3}X^{\frac{3}{2}}}, \quad X \to \infty.$$

Thus for $X \to +\infty$ from (7.5.12)

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}c^{-\frac{1}{12}}\left(\frac{D_1}{2}e^{-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}} + D_2e^{\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}}\right).$$

Also

$$\mathrm{Ai}(X) \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}\sin(\frac{2}{3}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) \quad X \to -\infty,$$

$$\mathrm{Bi}(X) \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}\cos(\frac{2}{3}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) \quad X \to -\infty.$$

Hence from (7.5.12) for $X \to -\infty$

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}c^{-\frac{1}{12}}\left[D_1\sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) + D_2\cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4})\right]$$

Now if we take the lower limit in (7.5.9) to be equal to x_0 (this is not necessary but it simplifies the expressions) then

$$\int_{x_0}^{x} (Q(s))^{\frac{1}{2}}\, ds = \int_0^{x-x_0} [Q(x_0+T)]^{\frac{1}{2}}\, dT,$$

$$\sim \int_0^{x-x_0} \left[cT + \frac{Q''(x_0)}{2}T^2 + \dots\right]^{\frac{1}{2}} dT,$$

$$\sim \int_0^{x-x_0} c^{\frac{1}{2}}T^{\frac{1}{2}} \left[1 + \frac{Q''(x_0)}{4c}T + \dots\right] dT.$$

Hence

$$\int_{x_0}^{x} (Q(s))^{\frac{1}{2}}\, ds \sim \frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}}$$

Hence as $x \to x_0+$ the outer solution behaves as

$$y_{out}^{+} \sim [c(x-x_0)]^{-\frac{1}{4}} \left[A_1 \exp(\frac{1}{\epsilon^{\frac{1}{2}}}\frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}}) + A_2 \exp(-\frac{1}{\epsilon^{\frac{1}{2}}}\frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}})\right],$$

ie

$$y_{out}^{+} \sim c^{-\frac{1}{4}}\epsilon^{\frac{-1}{12}}X^{-\frac{1}{4}} \left[A_1 \exp(\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}) + A_2 \exp(-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}})\right].$$

Also

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}c^{-\frac{1}{12}} \left(\frac{D_1}{2}e^{-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}} + D_2 e^{\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}}\right). \tag{7.5.13}$$

To match with the inner solution (7.5.13) as $X >> 1$ we must have

$$\frac{D_1}{2\sqrt{\pi}}c^{-\frac{1}{12}} = A_2 c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}, \quad \frac{D_2}{\sqrt{\pi}}c^{-\frac{1}{12}} = A_1 c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}.$$

Similarly as $x \to 0-$ we have

$$\int_{x_0}^{x} (-Q(s))^{\frac{1}{2}}\, ds \sim -\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}}.$$

Thus

$$y_{out}^{-} \sim c^{-\frac{1}{4}}(x_0-x)^{-\frac{1}{4}} \left[B_1 \cos(\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}}) - B_2 \sin(\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}})\right],$$

$$y_{out}^{-} \sim c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}|X|^{-\frac{1}{4}} \left[B_1 \cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}}) - B_2 \sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}})\right]. \tag{7.5.14}$$

And

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}c^{-\frac{1}{12}} \left[D_1 \sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) + D_2 \cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4})\right] \tag{7.5.15}$$

To match (7.5.14, 7.5.15) we must have

$$c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}B_1 = \frac{c^{-\frac{1}{12}}}{\sqrt{2\pi}}(D_1 + D_2) = (\frac{A_1}{\sqrt{2}} + A_2\sqrt{2})c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}},$$

$$-c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}B_2 = \frac{c^{-\frac{1}{12}}}{\sqrt{2\pi}}(D_1 - D_2) = -(\frac{A_1}{\sqrt{2}} - A_2\sqrt{2})c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}.$$

Hence solving for B_1, B_2 gives

$$B_1 = \frac{A_1}{\sqrt{2}} + A_2\sqrt{2},$$

$$B_2 = \frac{A_1}{\sqrt{2}} - A_2\sqrt{2}.$$

Summary: For $x > x_0$

$$y \sim |Q(x)|^{-\frac{1}{4}}\left[A_1 \exp\left(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}(Q(s))^{\frac{1}{2}}\,ds\right)\right.$$
$$\left.+A_2 \exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}(Q(s))^{\frac{1}{2}}\,ds.\right)\right], \tag{7.5.16a}$$

and for $x < x_0$ we have

$$y \sim |Q|^{-\frac{1}{4}}\left[A_1 \sin(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}|Q(s)|^{\frac{1}{2}}\,ds + \frac{\pi}{4})\right.$$
$$\left.+2A_2 \cos(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}|Q(s)|^{\frac{1}{2}}\,ds + \frac{\pi}{4})\right]. \tag{7.5.16b}$$

For $(x - x_0) << 1$

$$y \sim \sqrt{\pi}c^{-\frac{1}{6}}\epsilon^{-\frac{1}{12}}\left[2A_2\mathrm{Ai}(c^{\frac{1}{3}}\epsilon^{-\frac{1}{3}}(x-x_0)) + A_1\mathrm{Bi}(c^{\frac{1}{3}}\epsilon^{-\frac{1}{3}}(x-x_0))\right].$$

The formulae (7.5.16) are known as the connection formulae. The constants A_1, A_2 are determined by the boundary conditions. ** CHECK video**

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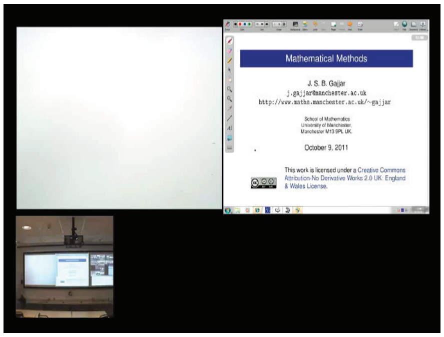
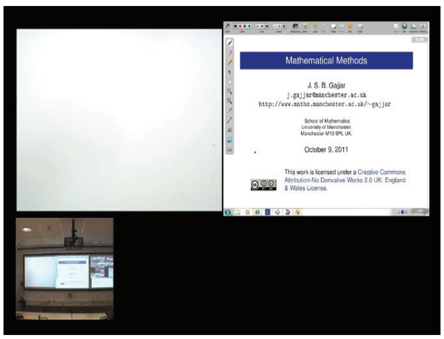


Video clip discussing the theory for a single turning point. Click here to open video clip in external player.



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