Q1 $X \to$ random variable

$\Omega = \{a, b, c\}$ $p(a) = 0.1$, $p(b) = 0.2$ & $p(c) = 0.7$

$$f(x) \begin{cases} 10 & x = a \\ 5 & x = b \\ 10/7 & x = c \end{cases}$$

(a) $E[f(x)]$

By definition of $E[f(x)]$

$$E[f(x)] \begin{cases} \sum_{x \in X} f(x)\, p(x) & X \text{ discrete} \\ \int_{x \in X} f(x)\, p(x)\, dx & X \text{ continuous} \end{cases}$$

Here X is a discrete space

$$E[f(x)] = \sum_{x \in X} f(x)\, p(x) = 10 \cdot p(a) + 5 \cdot p(b) + \frac{10}{7} \cdot p(c)$$

$$= 10 \cdot 0.1 + 5 \cdot 0.2 + \frac{10}{7} \cdot 0.7$$

$$E[f(x)] = 1 + 1 + 1 = 3$$

(b) $E[1/p(x)]$ sps we assume $1/p(x) = g(x)$ ← a function

$$\therefore E[g(x)] = \sum_{x \in \Omega} g(x)\, p(x)$$

$$= \sum_{x \in \Omega} \frac{1}{p(x)}\, p(x) = \sum_{x \in \Omega} 1 = 3 \cdot 1 = 3$$

(as $\Omega = \{a, b, c\}$)

(c) For any arbitrary pmf p, $E[1/p(x)]$?

$$E[1/p(x)] = \sum_{x \in \mathcal{X}} 1/p(x) \cdot p(x) = \sum_{x \in \mathcal{X}} 1 = n \cdot 1 = n$$

where $n = |\mathcal{X}|$

size of space (discrete)

(d). $E[f(x)^2]$ & $E[f(x)]^2$.

$$E[f(x)^2] = \sum_{x \in \Omega} f(x)^2 \cdot p(x)$$

$$= 10^2 \cdot 0.1 + 5^2 \cdot 0.2 + \left(\frac{10}{7}\right)^2 \cdot 0.7$$

$$= 10 + 5 + 10/7 = \frac{115}{7}$$

$$E[f(x)]^2 = (E[f(x)])^2 = 3^2 = 9$$

Q2 (a) 3 coins A $P(H) = 0.75$ $P(T) = 0.25$

B $P(H) = 0.5$ $P(T) = 0.5$

C $P(H) = 0.25$ $P(T) = 0.75$

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} ← outcome space for 3 coins

$P(HHH) = \cancel{0.75 \times 0.25 \times 0.2} \; 0.75 \times 0.5 \times 0.25$ — 1 head

$P(HHT) = 0.75 \times 0.5 \times 0.75$

$P(HTH) = 0.75 \times 0.5 \times 0.25$ } 2 heads $\quad \sum P(\text{2 heads}) = \frac{26}{64} = \frac{13}{32}$

$P(THH) = 0.25 \times 0.5 \times 0.25$

$P(HTT) = 0.75 \times 0.5 \times 0.75$

$P(THT) = 0.25 \times 0.5 \times 0.75$ } 1 head

$P(TTH) = 0.25 \times 0.5 \times 0.25$

$P(TTT) = 0.25 \times 0.5 \times 0.75$ — 0 head.

$x \to$ no of heads	0	1	2	3
$P(X)$	$\frac{6}{64}$	$\frac{26}{64}$	$\frac{26}{64}$	$\frac{6}{64}$

$$E[x] = \sum_{x \in X} x\,p(x) = 0 \cdot \frac{6}{64} + 1 \cdot \frac{26}{64} + 2 \cdot \frac{26}{64} + 3 \cdot \frac{6}{64}$$

$$= \frac{96}{64} = \cancel{\frac{48}{32}}\;\cancel{\frac{24}{16}}\;\frac{63}{42} = 1.5$$

$\cancel{E[E[x]] =}$

(b) Let $D =$ 3 heads & 2 tails

$\therefore P(D \mid C)$

coin

Need to find $P(C \mid D) = ?$

By Bayes $P(C|D) = \dfrac{P(D|C)\,P(C)}{P(D)}$

$P(D|C) = \dfrac{P(D \cap C)}{P(C)}$

P(getting with C) = 1/3

$P(D \cap C) = ?$

~~W α P getting heads.~~

3 heads & 2 tails = {HHHTT, HHTTH, HTTHH, TTHHH, HHTHT, HTHTH, THTHH, HTHHT, THHTH, THHHT} → 10. possible outcomes.

$\therefore P(D \cap C) = \alpha^3 (1-\alpha)^2 \times 10$

$\alpha \to$ P(getting H) for C

$= \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 \times 10$ ← possible outcomes

$P(D|C) = \dfrac{(1/4)^3 (3/4)^2 \times 10}{1/3} = \dfrac{3}{\cancel{64}\,32} \times \dfrac{9}{16} \times \cancel{10}\,5$

$= \dfrac{135}{512}$

$P(C|D) = \dfrac{P(D|C) \cdot P(C)}{P(D)} = \dfrac{\frac{135}{512} \times \frac{1}{3}}{P(D)}$

P(3 heads & 2 tails) = 10. P(3 heads & 2 tails from A) + 10 (3 heads & tails from B) + 10 (" " from C)

$= 10 \left\{ \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^5 + \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 \right\}$

$$= \frac{10}{4^5}\left\{27+32+9\right\} = \frac{680}{4^5}$$

$$\therefore P(D|C) = \frac{\frac{135}{512}\times\frac{1}{3}}{\frac{680}{4^5}} = \frac{135\times 1024}{512\times 3\times 680}$$

$$= \frac{138240}{1044480} = \underline{\underline{0.1323}}$$

Q3 <10 hours

$X \rightarrow$ time (hours)

density $p(x) \begin{cases} 1/10 & \text{if } 0<x<10 \\ 0 & \text{otherwise} \end{cases}$

cost of electrical breakdown is x^3

~~pb E~~ let $f(x) = x^3$

$$E[f(x)] = \int_X f(x)\,p(x)\,dx$$

$$= \int_0^{10} x^3/10\,dx + \int 0 \cdot x^3\,dx \quad x \notin [0,10)$$

$$= \left.\frac{x^4}{40}\right|_0^{10} + 0 = \frac{10^4}{40} = \frac{1000}{4} = \underline{250}$$

Q4 (b)

$\cancel{M_1} = 0.092 = V_1$

$\cancel{M_2} = 0.095 = V_2$

$\cancel{M_3} = 0.084 = V_3$

$\cancel{M_4} = 0.0739 = V_4$

$\cancel{M_5} = 0.061 = V_5$

$M_1 = 0.52$

$M_2 = 0.31$

$M_3 = 0.44$

$M_4 = 0.39$

$M_5 = 0.48$

(c)

$V_1 = 0.085$

$V_2 = 0.090$

$V_3 = 0.077$

$V_4 = 0.070$

$V_5 = 0.084$

They are lower than the variance from 10 samples

(d) $\sigma^2 = 10.0$

One sample Avg $-$ ~~$M_1 = 0.50$~~

$M_1 = 0.50 = \bar{x}$

$n = 30$

We known As Gaussian

$$\varepsilon = 1.96\sigma/\sqrt{n}$$

$$= \frac{1.96\sqrt{10}}{\sqrt{30}} = 1.96\sqrt{1/3}$$

$$= \frac{1.96}{\sqrt{3}} = \frac{\sqrt{3}\cdot(1.96)}{(3)}$$

$$= \frac{1.96\sqrt{3}}{3}$$

$$Pr\left(|\bar{x}-\mu| \geq \frac{1.96\sqrt{3}}{3}\right) = 0.95$$

$$|\bar{x}-\mu| \geq \frac{1.96\sqrt{3}}{3} \Rightarrow$$

~~$\bar{x} - \frac{1.96\sqrt{3}}{3} \leq 0.50 \leq \bar{x} + \frac{1.96\sqrt{3}}{3}$~~

$$\bar{x} - \frac{1.96\sqrt{3}}{3} \leq \mu \leq \bar{x} + \frac{1.96\sqrt{3}}{3}$$

$$= 0.50 - \frac{1.96\sqrt{3}}{3} \leq \mu \leq 0.5 + \frac{1.96\sqrt{3}}{3}$$

$$= -0.61 \leq \mu \leq 1.61$$

$$\mu \in [-0.61, 1.61]$$

(e) sps not Gaussian

$\sigma^2 = 10.0$

$\mu = 0.5 = \bar{X}$

$n = 30$

since we want a 95% $\therefore \delta = 0.05$

Using Chebyshev's Inequality

$$\delta = \frac{\sigma^2}{n\varepsilon^2} \Rightarrow \varepsilon = \sqrt{\frac{\sigma^2}{\delta n}}$$

$$= \sqrt{\frac{10}{0.05 \times 30}} = \sqrt{6.67} = 2.58$$

$$\cancel{= \frac{10}{0.05 \times 30} = \frac{100}{15}\ \frac{20}{3} = 6.67}$$

$$\Pr(|\bar{X} - E[\bar{X}]| \geq \cancel{6.67}\ 2.58)$$

$$|\bar{X} - E[\bar{X}]| \geq \cancel{6.67}\ 2.58$$

$$\Rightarrow \bar{X} - \varepsilon \leq E[\bar{X}] \leq \bar{X} + \varepsilon$$

$$\Rightarrow 0.5 - \cancel{6.67}\ 2.58 \leq E[\bar{X}] \leq 0.5 + \cancel{6.67}\ 2.58$$

$$\cancel{\Rightarrow -6.17 \leq E[\bar{X}] \leq 7.17}$$

$$\Rightarrow -2.08 \leq E[\bar{X}] \leq 3.08$$

Q5

(a) $E[\bar{v}] = \sigma^2$

$E[\bar{v}_b] = \left(1 - \frac{1}{n}\right)\sigma^2 \leftarrow$ To prove

Well $\bar{v}_b = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$$\therefore E[\bar{v}_b] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\underbrace{E[(x_i - \bar{x})^2]}_{A}$$

$$= \left(\frac{n-1}{n-1}\right)\frac{1}{n}\cdot A$$

$$= \left(\frac{n-1}{n}\right)\cdot\frac{1}{n-1}\cdot A$$

But $\frac{A}{n-1} = E[\bar{v}] = \sigma^2$

$$\Rightarrow$$

$$\therefore E[\bar{v}_b] = \left(\frac{n-1}{n}\right)\sigma^2 = \left(1 - \frac{1}{n}\right)\sigma^2$$

Q5(b)

Well Given

$(\Pr[Y - E[\bar{Y}] < \varepsilon) > 1 - V/\varepsilon^2$

$\iff ((\Pr[Y - E[Y]]) \geq \varepsilon) \leq (V/\varepsilon^2) = \delta$

~~fix δ~~ $\therefore \delta = V/\varepsilon^2$

Here says $Var[\bar{Y}] = \left(\frac{2(n-1)}{n^2}\right)\sigma^4 = V$

$\therefore \delta = \frac{V}{\varepsilon^2} \Rightarrow \varepsilon = \frac{\sqrt{V}}{\sqrt{\delta}}$

$= \varepsilon = \sqrt{\frac{(2(10)-1)\sigma^4}{10^2 \, \delta}}$

$\varepsilon = \sqrt{\frac{18}{10^2}\frac{\sigma^4}{\delta}} = \frac{3\sqrt{2}}{10}\frac{\sigma^2}{\sqrt{\delta}}$

$E[\bar{Y}] = \sigma^2 \Rightarrow \delta = \frac{V}{\varepsilon^2} = \varepsilon = \sqrt{\frac{\sigma^2}{\delta}} = \frac{\sigma}{\sqrt{\delta}}$

$\therefore$ one estimate is

$(\Pr[Y - E(Y) \geq \varepsilon) \leq \frac{V}{\varepsilon^2}$ E_1

Let $\varepsilon = \frac{3\sqrt{2}}{10}\frac{\sigma^2}{\sqrt{\delta}}$ (Considering $Var[Y] = \frac{2(n-1)\sigma^4}{n^2}$)

$\therefore \Pr[Y - E[Y] \geq \frac{3\sqrt{2}}{10}\frac{\sigma^2}{\sqrt{\delta}}$

$\Rightarrow \frac{3\sqrt{2}\sigma^2}{10\sqrt{\delta}} \leq Y - E[\bar{Y}] < \frac{3\sqrt{2}}{10}\frac{\sigma^2}{\sqrt{\delta}}$

$\therefore \Rightarrow E[\bar{Y}] \in \left(Y - \frac{3\sqrt{2}}{10}\frac{\sigma^2}{\sqrt{\delta}}, Y + \frac{3\sqrt{2}\sigma^2}{10\sqrt{\delta}}\right)$

another estimate is

$(\Pr[Y - E[Y]] > \varepsilon) < V/\varepsilon^2$ (Considering $E[Y] = \sigma^2$)

$\Rightarrow Y - E[Y] < \frac{\sigma}{\sqrt{\delta}}$

$\Rightarrow E[\bar{Y}] \in \left(Y - \frac{\sigma}{\sqrt{\delta}}, Y + \frac{\sigma}{\sqrt{\delta}}\right)$

$\therefore$ We can't really conclude which has a tighter interval as it is dependent of σ sometimes it's $\bar{x}$ & sometimes $\bar{y}$