Graph Theory

Warm up problem: **The Odd Doors Problem**. (Section 1.3 of Ecco)

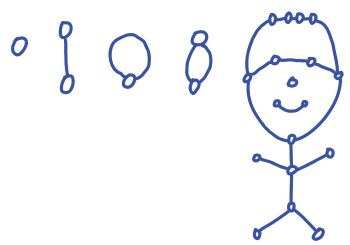
Lawrence Terrence III has a problem. His recently departed father has hidden a cache of jewels in one of two underground labyrinths. Lawrence knows the following facts about the labyrinths:

- The jewels are in a room with an odd number of doors.
- Only one of the labyrinths has a room with an odd number of doors.
- One labyrinth has two doors leading to the outside and the other has three.

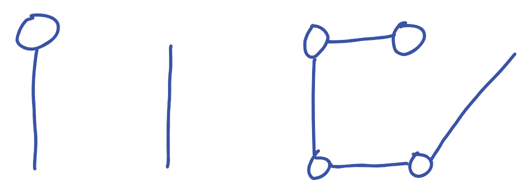
Since it will cost a small fortune to explore each labyrinth, Lawrence Terrence III wants to know which labyrinth he should search.

Definition 1: A *graph* is a collection of dots and lines, with every line terminated by a dot at each end. The dots are called *vertices* and the lines are called *edges*. Note we can have an edge that starts and ends at the same vertex; also, we could have multiple edges joining the same two vertices.

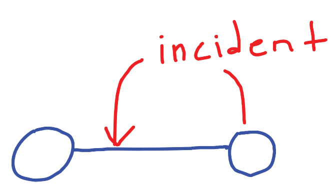
Example(s) of graph(s)



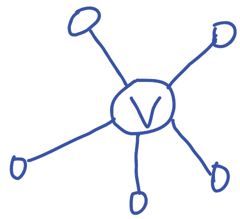
Not Graphs



Definition 2: An edge that is joined to a vertex is said to be *incident* to the vertex, and the vertices that are joined to an edge are *incident* to the edge. More formally, the edge that connects the vertices u and v can be written as: $e = \{u, v\}$. Here we also say that u and v are *adjacent* to each other, and that they are the *endpoints* of the edge $\{u, v\}$. Two edges incident with the same vertex are also said to be *adjacent*.



Definition 3: In a graph, the *degree* of a vertex v is the number of edges incident to v, and is denoted $\deg(v)$. Each edge contributes 1 to the degree of each of the two vertices incident with it.



$\deg(v) = 5$

Theorem 1 (The Parity Theorem): The sum of the degrees of all vertices of a graph is equal to twice its number of edges. That is:

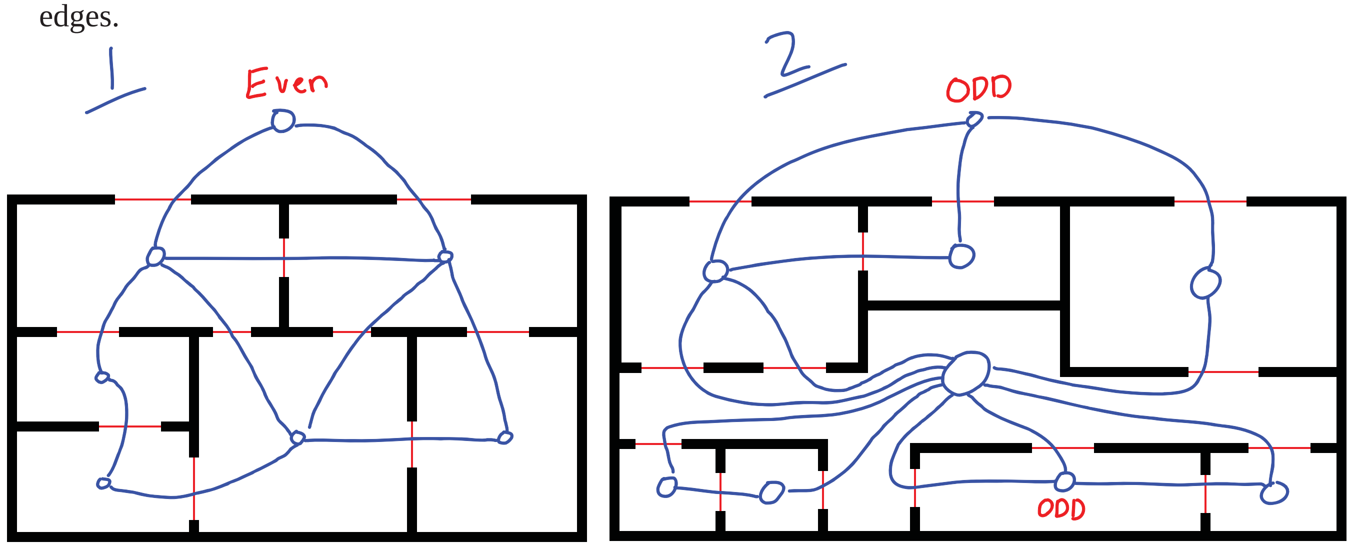
$$2E = \sum_{v \in G} \deg(v) \equiv 0 \pmod{2}$$

Proof. Each edge contributes 2 to the total degree.

The Parity Theorem is also called the Handshaking Lemma and is stated as follows:

Theorem 2 (The Handshaking Lemma): Every graph has an even number of vertices of odd degree.

Example 1: Solve the Odd Doors Problem. Two example labyrinths are suggested below. You can make a graph out of these two examples by letting the rooms be vertices and the doors be edges.



1 could have all even vertices (as the example above shows)

2 must have two (or more) odd vertices (by Theorem 1,2)
$\therefore$ 2 must have a room with an odd # of doors

Example 2: The University of Two Hills has 25 professors each with a telephone. If any professor collaborates over the phone with more than 5 others confusion spreads through the university and nothing gets done. To maximize collaboration without risking confusion they have decided to connect each phone to exactly 5 other phones. Can this be done?

Vertices = phones

edges = connected phones

$\boxed{\text{No}}$ since:

degree of each vertex

# of vertices

$$0 \equiv 2E \neq \sum_{v \in G} \deg(v) = 5V = 5 \cdot 25 \equiv 1 \cdot 1 \equiv 1 \pmod 2$$

Example 3: In the Kingdom of Glee roads (edges) do not intersect nor do they lead to dead ends.

a) If there are 100 cities (vertices), and four roads lead out of each city, how many roads are there altogether in the kingdom?

b) If 3 roads lead out of each city can the kingdom have exactly 100 roads?

a) $V = 100$, degree of each vertex is 4.

$$2E = \sum_{v \in G} \deg(v) = 4V = 400 \Rightarrow \boxed{E = 200}$$

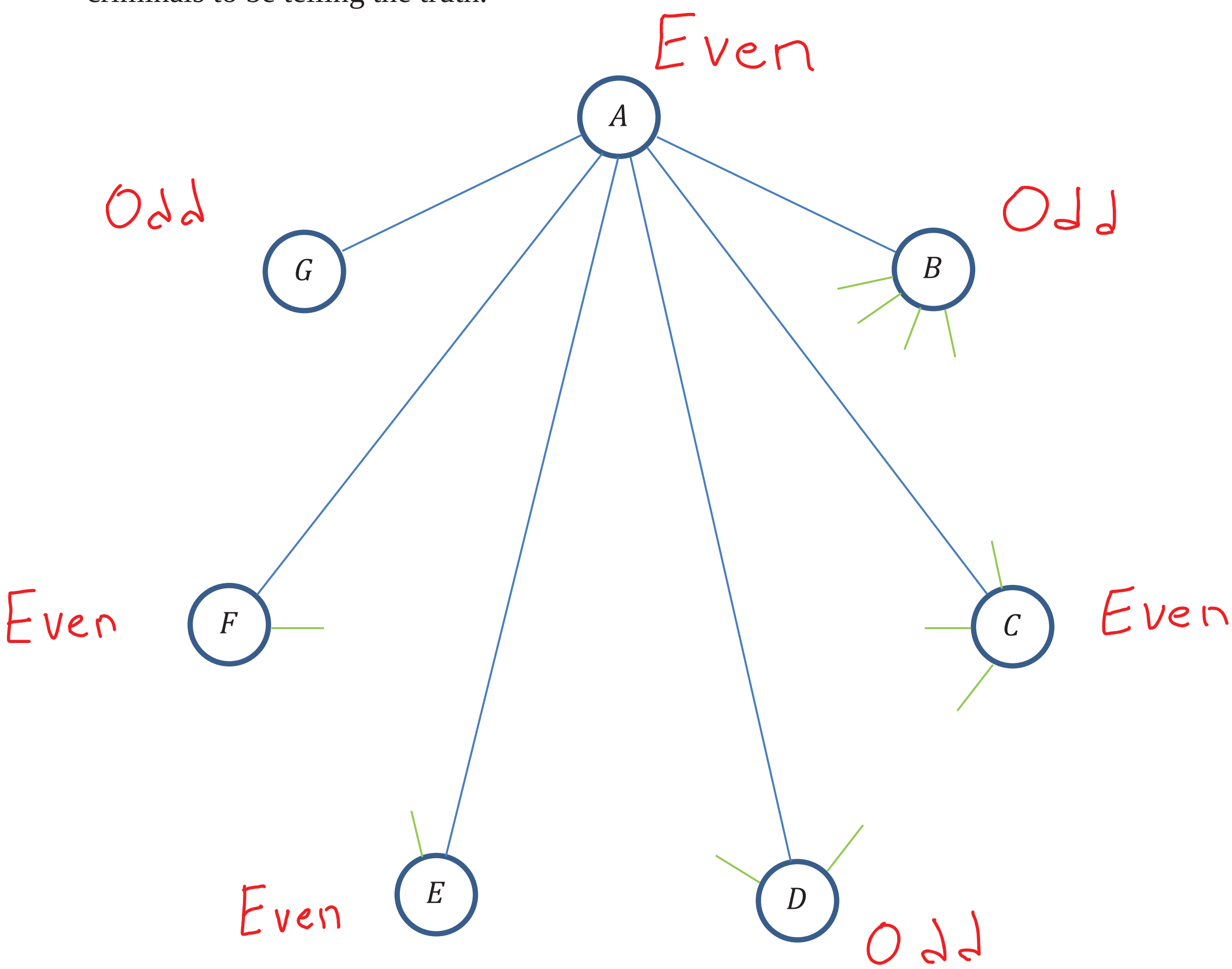
b) $E = 100$, degree of each vertex is 3.

No since:

$$2 \equiv 200 = 2E \neq \sum_{v \in G} \deg(v) = 3V \equiv 0 \pmod 3$$

Example 4: (Ecco 2.1) The police have captured seven criminals **A**lice, **B**ob, **C**ourtney, **D**an, **E**lla, **F**rank, and **G**ary. When questioned by the police, Alice admitted to having known all of the other six criminals. Bob admitted to having known five, Courtney admitted to having known four, Dan to having known three, Ella to having known two, Frank to having known two, and Gary to having known one.

a) Part of a graph is given below where criminals are vertices and edges connect criminals that know each other. Use this partial graph to explain if it is possible for all of the seven criminals to be telling the truth.



There are 3 odd vertices

an odd #

$\therefore$ no graph represents this situation (by Theorem 1)

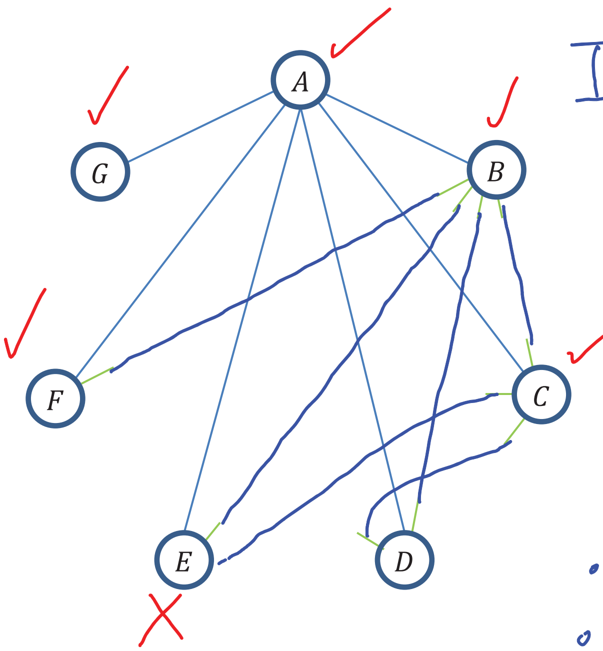
$\therefore$ Someone is lying!

b) Though further investigations it has been determined that at exactly one of the six criminal is lying. The criminal that is lying will claim to know fewer criminals than they actually do. In addition it is known that Frank is telling the truth. Which of criminals could possibly be lying?

Hint: the problem gives us very little information about Gary so making an assumption about Gary will be very helpful.

A knows everyone ∴ A is not lying

Case 1: Gary is telling the truth.



If B lies then G is lying

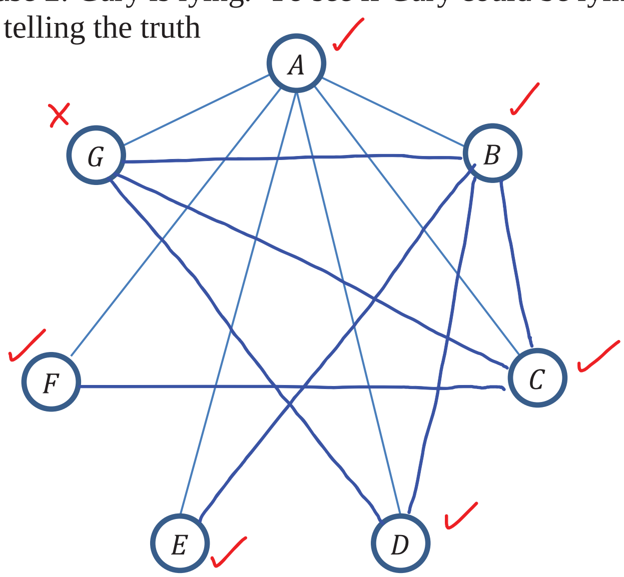
∴ B knows C, D, E, F

If C lies then G or F lies

∴ C knows D, E

∴ $\boxed{\text{E is lying}}$

Case 2: Gary is lying. To see if Gary could be lying try to find an example where everyone else is telling the truth



Therefore the criminals that could be lying are: $\boxed{\text{E or G}}$

Definition 4: A *subgraph* of a graph is a subset of its vertices and edges, provided that all vertices incident with edges in the subgraph are included. In other words, a subgraph is a subset of the vertices and edges that itself forms a graph.

Certain types of subgraphs have specific names:

Definition 5: A *walk* is a subgraph that consists of a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ such that for $1 \leq i \leq n$ the edge e_i joins vertices v_{i-1} and v_i .

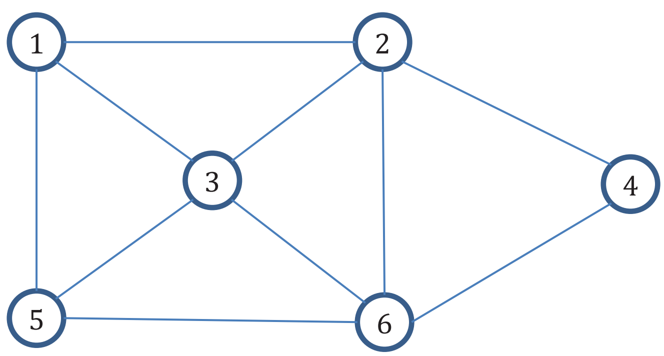
Definition 6: A *trail* is a walk in which no edges are repeated.

Definition 7: A *path* is a trail in which no vertices are repeated except perhaps for the first and last vertex.

Definition 8: A *circuit* is a trail that's first and last vertices are the same.

Definition 9: A *cycle* is a path that's first and last vertices are the same.

Example 5: Find an example of a walk, trail, path, circuit, and cycle in the graph below:



Walk: 1 2 4 6 3 2 4

trail: 1 2 6 3 2 4

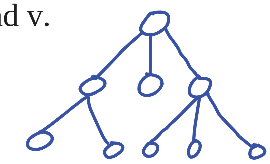
path: 1 2 3 6 4

Circuit: 1 3 2 4 6 3 5 1

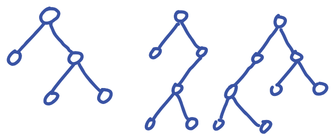
cycle: 1 2 4 6 3 5 1

Definition 10: Two vertices of a graph that are joined by a path are said to belong to the same *component* of the graph. If the whole graph is one component, then it is said to be *connected*.

Definition 11: A *tree* is defined as a graph T such that for any two vertices u and v in T, there is exactly one path which joins u and v.



Definition 12: A collection of disjoint trees is called a *forest*.



Theorem 3: (Trees). A tree has the following three properties.
1. It is connected.
2. It has no cycles.
3. It satisfies the Tree Formula $V = E + 1$, where V and E are the numbers of vertices and edges respectively.

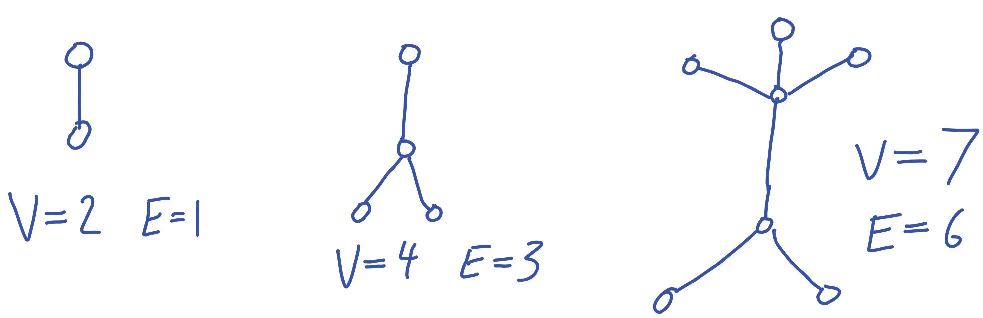
Proof.

1. Since we can go from any vertex to any other vertex in a tree, it must be connected.

2. If the graph has a cycle, then for any two vertices on the cycle we can go from one to the other in at least two ways, which is a contradiction. Therefore there are no cycles.

3. For every edge added to a tree we must add a vertex or else there will be a cycle.

Ex

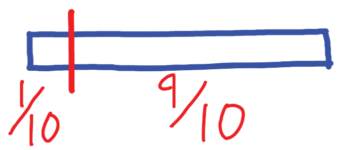


Theorem 4: (Two out of three is a tree). A graph is a tree if it has any two of the following properties:
1. It is connected.
2. It has no cycles.
3. It satisfies the Tree Formula $V = E + 1$, where V and E are the numbers of vertices and edges respectively.

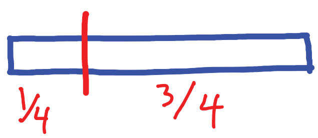
Example 6: Kids and Chocolate bars. What are the possible values of $n > 9$ such that n children can equally share 9 identical chocolate bars, with the restriction that no bar be cut into more than two pieces.

Step 1. Guess as many different possibilities for n as you can. Let C be the (equal) amount of chocolate that each child gets; note $C = 9/n$. In each case what does C equal?

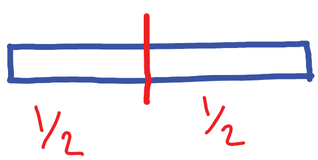
Cut each bar:



$C = 9/10$ $\boxed{n=10}$



$C = 3/4$ $\boxed{n=12}$

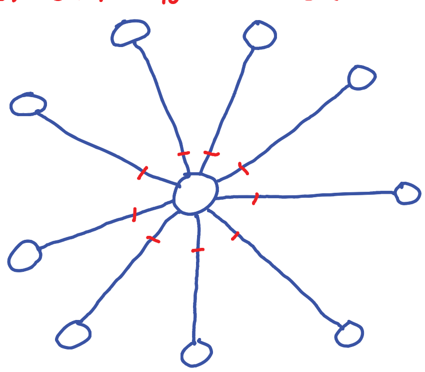


$C = 1/2$ $\boxed{n=18}$

Step 2. Represent the examples from step 1 as a graph, where kids are vertices and kids that share a chocolate bar have edges between them.

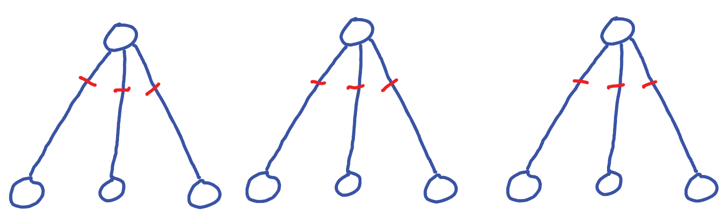
$n=10$

cut off 1/10 of each bar:



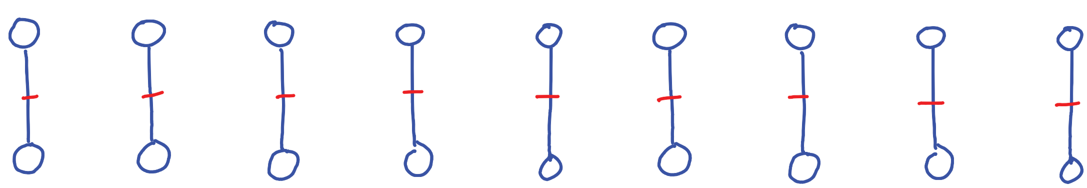
$n=12$

cut off 1/4 of each bar:



$n=18$

cut each bar in half:



Step 3. Explain why this graph has no cycles. $n =$ "number of kids" > 9

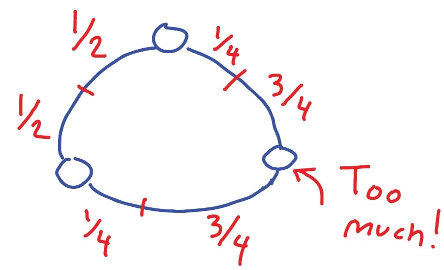
Say there is a cycle of size 3. (we could pick any size cycle and show it can not exist in this graph)

Each kid gets $9/n$ of a bar. So the 3 kids on our cycle should share:

$$3\left(\frac{9}{n}\right) \text{ of chocolate.}$$

Note: $3\left(\frac{9}{n}\right) < 3\left(\frac{9}{9}\right) = 3$. So the 3 bars on the cycle is too much.

$\underbrace{\text{Ex}}$



Step 4. Explain why this $\underbrace{\text{graph}}_{G}$ is a tree or a forest.

There are 2 cases:

CASE 1 G is connected. Now we can use theorem 4:

two out of three

$\left.\begin{array}{l} G \text{ is connected} \\ G \text{ has no cycles} \end{array}\right\} \overset{T4}{\Longrightarrow} G \text{ is a tree}$

CASE 2 G is disconnected

Each connected component has no cycles and therefore (again by Theorem 4) each connected component is a tree.

$\therefore$ G is a forest

# of Kids or vertices

Step 5. Find all the possible choices for n.

Case 1: The graph is a tree. In this case we can use part 3 of theorem 3.

$$n = V = E + 1 = 9 + 1 = \boxed{10}$$

Case 2: The graph is a forest. In this case start by considering the following variables:

$E_i =$ "the number of edges on the i^{th} tree"
$V_i =$ "the number of vertices on the i^{th} tree"
$T =$ "the number of trees that make the forest.

Since the chocolate is equally shared among everyone the chocolate is equally shared on each tree. So that:

$$C = \frac{E_i}{V_i} \quad (E_i \text{ bars of chocolate are divided among } V_i \text{ kids})$$

for each choice of i.

$$\Rightarrow \frac{E_i}{V_i} = C = \frac{E_j}{V_j}$$

$$\Rightarrow E_i V_j = E_j V_i$$

$$\Rightarrow E_i(E_j+1) = E_j(E_i+1) \quad (\text{by the tree formula})$$

$$\Rightarrow \cancel{E_i E_j} + E_i = \cancel{E_j E_i} + E_j$$

$$\Rightarrow E_i = E_j \quad \therefore V_i = V_j \text{ for all trees } i, j$$

$$E = T E_i$$

$$\Rightarrow 9 = T E_i$$

$$\Rightarrow T E_i = 3 \cdot 3 \text{ OR } 9 \cdot 1 \quad \left(\begin{array}{c}\text{since} \\ T > 1\end{array}\right)$$

$$\therefore n = V = T V_i = T(E_i + 1)$$

$$= 3(3+1) \text{ OR } 9(1+1)$$

$$= \boxed{12 \text{ or } 18}$$