MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Solutions #9

1. Define

$$f\colon [0,1]^3 \to \mathbb{R}, \quad (x,y,z) \mapsto \begin{cases} xy, & z \le xy, \\ z, & z \ge xy. \end{cases}$$

Evaluate $\int_{[0,1]^3} f$.

Solution: By Fubini's Theorem, we have

$$\int_{[0,1]^3} f = \int_0^1 \left(\int_0^1 \left(\int_0^1 f(x,y,z)\,dz \right) dy \right) dx.$$

Let $(x,y) \in [0,1]^2$, so that $xy \in [0,1]$. Consequently, we obtain for the innermost integral that

$$\int_0^1 f(x,y,z)\,dz = \int_0^{xy} xy\,dz + \int_{xy}^1 z\,dz = x^2y^2 + \left[\frac{z^2}{2}\right]_{z=xy}^{z=1} = \frac{1}{2}(x^2y^2+1)$$

It follows that

$$\begin{aligned} \int_{[0,1]^3} f &= \int_0^1 \left(\int_0^1 \frac{1}{2}x^2y^2 + 1\,dy \right) dx \\ &= \frac{1}{2}\int_0^1 \left(\int_0^1 x^2y^2\,dy \right) dx + \frac{1}{2} \\ &= \frac{1}{2}\left(\int_0^1 x^2\,dx \right)\left(\int_0^1 y^2\,dy \right) + \frac{1}{2} \\ &= \frac{1}{18} + \frac{1}{2} \\ &= \frac{5}{9}. \end{aligned}$$

2. Let

$$D := \{(x,y) \in \mathbb{R} : x,y \ge 0,\ x^2+y^2 \le 1\},$$

and let

$$f\colon D \to \mathbb{R}, \quad (x,y) \mapsto \frac{4y^3}{(x+1)^2}$$

Evaluate $\int_D f$.

Solution: Define $\phi, \psi\colon [0,1] \to \mathbb{R}$ through

$$\phi(x) = 0 \qquad \text{and} \qquad \psi(x) = \sqrt{1-x^2}$$

for $x \in [0,1]$, so that

$$D = \{(x,y) \in \in \mathbb{R} : x \in [0,1],\ \phi(x) \le y \le \psi(x)\}.$$

It follows that

$$\begin{aligned}
\int_D f &= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \frac{4y^3}{(x+1)^2}\, dy \right) dx \\
&= \int_0^1 \left(\left. \frac{y^4}{(x+1)^2} \right|_{y=0}^{y=\sqrt{1-x^2}} \right) dx \\
&= \int_0^1 \frac{(1-x^2)^2}{(x+1)^2}\, dx \\
&= \int_0^1 (1-x)^2\, dx \\
&= -\left. \frac{(1-x)^3}{3} \right|_{x=0}^{x=1} \\
&= \frac{1}{3}.
\end{aligned}$$

3. Let $I \subset \mathbb{R}^N$ and $J \subset \mathbb{R}^M$ be compact intervals, let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be continuous, and define

$$f \otimes g \colon I \times J \to \mathbb{R}, \quad (x,y) \mapsto f(x)g(y).$$

Then $f \otimes g$ is continuous and thus Riemann integrable. Show that

$$\int_{I \times J} f \otimes g = \left(\int_I f \right) \left(\int_J g \right).$$

Solution: By Fubini's Theorem, we have

$$\begin{aligned}
\int_{I \times J} f \otimes g &= \int_I \left(\int_J f(x)g(y)\, d\mu_M(y) \right) d\mu_N(x) \\
&= \int_I \left(f(x) \int_J g(y)\, d\mu_M(y) \right) d\mu_N(x) \\
&= \left(\int_I f \right) \left(\int_J g \right),
\end{aligned}$$

which proves the claim.

4. Let $a < b$, let $f \colon [a,b] \to [0,\infty)$ be continuous, and let

$$D := \{(x,y) : x \in [a,b],\ y \in [0, f(x)]\}.$$

Show that D has content and that

$$\mu(D) = \int_a^b f(x)\,dx.$$

Solution: Note that

$$\partial D = \{(a,y) : y \in [0, f(a)]\} \\ \cup \{(x, f(x)) : x \in [a,b]\} \cup \{(b,y) : y \in [0, f(b)]\} \cup \{(x,0) : x \in [a,b]\}.$$

Each of the sets on the right hand side of this equality has content zero, so that ∂D has content zero, and D has content.

From Fubini's Theorem, we obtain that

$$\begin{aligned}\mu(D) &= \int_D 1 \\ &= \int_a^b \left(\int_0^{f(x)} dy \right) dx \\ &= \int_a^b f(x)\,dx.\end{aligned}$$

5. Let $a, b > 0$. Determine the area of the ellipse

$$E := \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Solution: Use the following coordinate transformation:

$$\phi \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (r, \theta) \mapsto (ra\cos\theta, rb\sin\theta),$$

so that $E = \phi([0,1] \times [0, 2\pi])$. Since

$$J_\phi(r,\theta) = \begin{bmatrix} a\cos\theta & -ra\sin\theta \\ b\sin\theta & rb\cos\theta \end{bmatrix}$$

and thus

$$\det J_\phi(r,\theta) = abr,$$

change of variables yields

$$\begin{aligned}\mu(E) &= \int_E 1 \\ &= \int_{[0,1]\times[0,2\pi]} abr \\ &= ab \int_0^1 \left(\int_0^{2\pi} r\,d\theta \right) dr \\ &= 2\pi ab \int_0^1 r\,dr \\ &= \pi ab.\end{aligned}$$

6*. Define $f\colon [0,1]\times[0,1]\to\mathbb{R}$ by letting

$$f(x,y)=\begin{cases} 2^{2n}, & \text{if } (x,y)\in[2^{-n},2^{-n+1})\times[2^{-n},2^{-n+1}) \text{ for some } n\in\mathbb{N},\\ -2^{2n+1}, & \text{if } (x,y)\in[2^{-n-1},2^{-n})\times[2^{-n},2^{-n+1}) \text{ for some } n\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases}$$

Show that the iterated integrals

$$\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx \qquad \text{and} \qquad \int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy$$

both exist, but that

$$\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx \neq \int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy.$$

Why doesn't this contradict Fubini's Theorem?

Solution: Fix $y_0\in[0,1)$; let $n\in\mathbb{N}$ be such that $y_0\in[2^{-n},2^{-n+1})$. We then have that

$$f(x,y_0)=\begin{cases} 2^{2n}, & \text{if } x\in[2^{-n},2^{-n+1}),\\ -2^{2n+1}, & \text{if } x\in[2^{-n-1},2^{-n}),\\ 0, & \text{otherwise}\end{cases}$$

and therefore

$$\int_0^1 f(x,y_0)\,dx=\int_{2^{-n}}^{2^{-n+1}}2^{2n}\,dx-\int_{2^{-n-1}}^{2^{-n}}2^{2n+1}\,dx=2^n-2^n=0.$$

All in all,

$$\int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy=0$$

holds. Similarly, if $x_0\in\left[0,\frac{1}{2}\right)$, we obtain

$$\int_0^1 f(x_0,y)\,dy=0.$$

If, however, $x_0\in\left[\frac{1}{2},1\right)$, we get

$$f(x_0,y)=\begin{cases} 4, & \text{if } y\in\left[\frac{1}{2},1\right)\\ 0, & \text{otherwise}\end{cases}$$

Therefore, we have

$$\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx=\int_{\frac{1}{2}}^1\left(\int_{\frac{1}{2}}^1 4\,dy\right)dx=1.$$

As f is unbounded, it cannot be Riemann integral. Hence, Fubini's Theorem does not apply.