MATH 217 (Fall 2021)

Honors Advanced Calculus, I

Solutions #6

1. Determine the Jacobians of

$$\mathbb{R}^3 \to \mathbb{R}^3, \quad (r, \theta, \phi) \mapsto (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$$

and

$$\mathbb{R}^3 \to \mathbb{R}^3, \quad (r, \theta, z) \mapsto (r \cos\theta, r \sin\theta, z).$$

Solution: The first Jacobian is

$$\begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

and the second one

$$\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. An $N \times N$ matrix X is *invertible* if there is $X^{-1} \in M_N(\mathbb{R})$ such that $XX^{-1} = X^{-1}X = I_N$ where I_N denotes the unit matrix.

   (a) Show that $U := \{X \in M_N(\mathbb{R}) : X \text{ is invertible}\}$ is open. (*Hint*: $X \in M_N(\mathbb{R})$ is invertible if and only if $\det X \neq 0$.)

   (b) Show that the map

   $$f \colon U \to M_N(\mathbb{R}), \quad X \mapsto X^{-1}$$

   is totally differentiable on U, and calculate $Df(X_0)$ for each $X_0 \in U$. (*Hint*: You may use that, by Cramer's Rule, f is continuous.)

Solution:

(a) Since $\det \colon M_N(\mathbb{R}) \to \mathbb{R}$ is continuous and $\mathbb{R} \setminus \{0\}$ is open, $U = \det^{-1}(\mathbb{R} \setminus \{0\})$ is open.

(b) Let $X_0 \in U$. Since U is open by (i), $X_0 + H \in U$ for $\|H\|$ sufficiently small. Note that

$$\begin{aligned}(X_0 + H)^{-1} - X_0^{-1} &= -X_0^{-1}((X_0 + H) - X_0)(X_0 + H)^{-1} \\ &= -X_0^{-1}H(X_0 + H)^{-1}.\end{aligned}$$

Define

$$T\colon M_N(\mathbb{R}) \to M_N(\mathbb{R}), \quad X \mapsto -X_0^{-1} X X_0^{-1}.$$

For $\|H\|$ sufficiently small, we have

$$\begin{aligned}\frac{\|f(X_0+H) - f(X_0) - TH\|}{\|H\|} &= \frac{1}{\|H\|}\left\|X_0^{-1}H(X_0+H)^{-1} - X_0^{-1}HX_0^{-1}\right\| \\ &= \left\|X_0^{-1}\frac{H}{\|H\|}\left((X_0+H)^{-1} - X_0^{-1}\right)\right\|.\end{aligned}$$

As $\|H\| \to 0$, the term $X_0^{-1}\frac{H}{\|H\|}$ stays bounded whereas $(X_0+H)^{-1} - X_0^{-1} \to 0$ by the continuity of f. It follows that

$$\lim_{\|H\|\to 0} \frac{\|f(X_0+H) - f(X_0) - TH\|}{\|H\|} = 0.$$

Hence, f is differentiable at X_0 and $Df(X_0) = T$.

3. Let

$$p\colon (\mathbb{R}\setminus\{0\}) \times \mathbb{R} \to \mathbb{R}^2, \quad (r,\theta) \mapsto (r\cos\theta, r\sin\theta),$$

let $\varnothing \neq U \subset \mathbb{R}^2$ be open, and let $f : U \to \mathbb{R}$ be twice continuously partially differentiable. Show that

$$(\Delta f)\circ p = \frac{\partial^2(f\circ p)}{\partial r^2} + \frac{1}{r}\frac{\partial(f\circ p)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(f\circ p)}{\partial\theta^2}$$

on $p^{-1}(U)$. (*Hint*: Apply the chain rule twice.)

Solution: First, note tht

$$J_p(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

The chain rule implies that

$$\begin{aligned}&\left(\frac{\partial(f\circ p)}{\partial r}(r,\theta), \frac{\partial(f\circ p)}{\partial\theta}(r,\theta)\right) \\ &= J_{f\circ p}(r,\theta) \\ &= J_f(p(r,\theta))J_p(r,\theta) \\ &= \left(\cos\theta\frac{\partial f}{\partial x}(p(r,\theta)) + \sin\theta\frac{\partial f}{\partial y}(p(r,\theta)), -r\sin\theta\frac{\partial f}{\partial x}(p(r,\theta)) + r\cos\theta\frac{\partial f}{\partial y}(p(r,\theta))\right),\end{aligned}$$

so that

$$\frac{\partial(f\circ p)}{\partial r}(r,\theta) = \cos\theta\frac{\partial f}{\partial x}(p(r,\theta)) + \sin\theta\frac{\partial f}{\partial y}(p(r,\theta))$$

and

$$\frac{\partial(f\circ p)}{\partial\theta}(r,\theta) = -r\sin\theta\frac{\partial f}{\partial x}(p(r,\theta)) + r\cos\theta\frac{\partial f}{\partial y}(p(r,\theta)).$$

It follows that

$$
\begin{aligned}
&\frac{\partial^2 (f\circ p)}{\partial r^2}(r,\theta)\\
&= \cos\theta \frac{\partial}{\partial r}\frac{\partial f}{\partial x}(p(r,\theta)) + \sin\theta \frac{\partial}{\partial r}\frac{\partial f}{\partial y}(p(r,\theta))\\
&= (\cos\theta)^2 \frac{\partial^2 f}{\partial x^2}(p(r,\theta)) + \cos\theta\sin\theta \frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))\\
&\qquad + \sin\theta\cos\theta \frac{\partial^2 f}{\partial x\partial y}(p(r,\theta)) + (\sin\theta)^2 \frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&= (\cos\theta)^2 \frac{\partial^2 f}{\partial x^2}(p(r,\theta)) + 2\cos\theta\sin\theta \frac{\partial^2 f}{\partial x\partial y}(p(r,\theta)) + (\sin\theta)^2 \frac{\partial^2 f}{\partial y^2}(p(r,\theta))
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial^2 (f\circ p)}{\partial \theta^2}(r,\theta)\\
&= \frac{\partial}{\partial\theta}\left(-r\sin\theta\frac{\partial f}{\partial x}(p(r,\theta)) + r\cos\theta\frac{\partial f}{\partial y}(p(r,\theta))\right)\\
&= -r\cos\theta\frac{\partial f}{\partial x}(p(r,\theta)) - r\sin\theta\frac{\partial}{\partial\theta}\frac{\partial f}{\partial x}(p(r,\theta))\\
&\qquad - r\sin\theta\frac{\partial f}{\partial y}(p(r,\theta)) + r\cos\theta\frac{\partial}{\partial\theta}\frac{\partial f}{\partial y}(p(r,\theta))\\
&= \; - r\cos\theta\frac{\partial f}{\partial x}(p(r,\theta)) + r^2(\sin\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta)) - r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))\\
&\qquad - r\sin\theta\frac{\partial f}{\partial y}(p(r,\theta)) - r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta)) + r^2(\cos\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&= \; - r\frac{\partial (f\circ p)}{\partial r}(r,\theta) + r^2(\sin\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta)) - 2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))\\
&\qquad + r^2(\cos\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta)).
\end{aligned}
$$

This means that

$$
\begin{aligned}
&r^2\frac{\partial^2 (f\circ p)}{\partial r^2}(r,\theta) + r\frac{\partial (f\circ p)}{\partial r}(r,\theta) + \frac{\partial^2 (f\circ p)}{\partial \theta^2}(r,\theta)\\
&= r^2(\cos\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta)) + 2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta)) + r^2(\sin\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&\qquad + r^2(\sin\theta)^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta)) - 2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(p(r,\theta))\\
&\qquad + r^2(\cos\theta)^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&= r^2((\cos\theta)^2 + (\sin\theta)^2)\frac{\partial^2 f}{\partial x^2}(p(r,\theta)) + r^2((\cos\theta)^2 + (\sin\theta)^2)\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&= r^2\frac{\partial^2 f}{\partial x^2}(p(r,\theta)) + r^2\frac{\partial^2 f}{\partial y^2}(p(r,\theta))\\
&= r^2(\Delta f)(p(r,\theta)).
\end{aligned}
$$

Division by r^2 then yields the claim.

4. Let

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \begin{cases} \frac{xy^3}{x^2+y^4}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Show that:

(a) f is continuous at $(0,0)$;

(b) for each $v \in \mathbb{R}^2$ with $\|v\| = 1$, the directional derivative $D_v f(0,0)$ exists and equals 0;

(c) f is not totally differentiable at $(0,0)$.

(*Hint for* (c): Assume towards a contradiction that f is totally differentiable at $(0,0)$, and compute the first derivative of $\mathbb{R} \ni t \mapsto f(t^2,t)$ at 0 first directly and then using the chain rule. What do you observe?)

Solution:

(a) Note that, for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have

$$|f(x,y)| = |y| \frac{\sqrt{x^2y^4}}{x^2+y^4} \leq |y| \frac{1}{2} \frac{x^2+y^4}{x^2+y^4} = \frac{|y|}{2}.$$

Hence, if $(x_n,y_n) \to 0$, it follows that $|f(x_n,y_n)| \leq \frac{|y_n|}{2} \to 0 = f(0,0)$.

(b) Let $v = (v_1,v_2)$ have norm one. For $t \neq 0$, we have

$$f(tv_1,tv_2) = \frac{t^4 v_1 v_2^3}{t^2(v_1^2 + t^2 v_2^4)} = t^2 \frac{v_1 v_2^3}{v_1^2 + t^2 v_2^4},$$

so that

$$\frac{f((0,0)+tv) - f(0,0)}{t} = t \frac{v_1 v_2^3}{v_1^2 + t^2 v_2^4}.$$

It follows that

$$D_v f(0,0) = \lim_{\substack{t \to 0 \\ t \neq 0}} \frac{f((0,0)+tv) - f(0,0)}{t} = 0.$$

(c) Let

$$g\colon \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (t^2,t),$$

so that

$$(f \circ g)(t) = \frac{t^2 t^3}{t^4+t^4} = \frac{t}{2}$$

for $t \in \mathbb{R}$ and thus $\frac{d(f\circ g)}{dt}(t)\Big|_{t=0} = \frac{1}{2}$.

Assume that f is totally differentiable at $(0,0)$. From (b), it is clear that $Df(0,0) = (0,0)$. The chain rule then yields that

$$\frac{d(f \circ g)}{dt}(t)\bigg|_{t=0} = Df(g(0))Dg(0) = (0,0)Dg(0) = 0,$$

which is a contradiction.

5. Let $x, y \in \mathbb{R}$. Show that there is $\theta \in [0,1]$ such that

$$\sin(x+y) = x + y - \frac{1}{2}(x^2 + 2xy + y^2)\sin(\theta(x+y)).$$

Solution: Let

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \sin(x+y).$$

By Taylor's Theorem, there is $\theta \in [0,1]$, such that

$$f(x,y) = f(0,0) + (\text{grad } f)(0,0) \cdot (x,y) + \frac{1}{2}(\text{Hess } f)(\theta x, \theta y)(x,y) \cdot (x,y).$$

Clearly, $f(0,0) = 0$ holds. Since

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = \cos(x+y),$$

we have

$$(\text{grad } f)(0,0) \cdot (x,y) = (1,1) \cdot (x,y) = x + y.$$

Moreover, since

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = -\sin(x+y)$$

we also have

$$\begin{aligned}
&(\text{Hess } f)(\theta x, \theta y)(x,y) \cdot (x,y) \\
&= \left(\begin{bmatrix} -\sin(\theta(x+y)) & -\sin(\theta(x+y)) \\ -\sin(\theta(x+y)) & -\sin(\theta(x+y)) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \cdot \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -\sin(\theta(x+y))(x+y) \\ -\sin(\theta(x+y))(x+y) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \\
&= -(x^2 + 2xy + y^2)\sin(\theta(x+y)).
\end{aligned}$$

Hence,

$$\sin(x+y) = x + y - \frac{1}{2}(x^2 + 2xy + y^2)\sin(\theta(x+y))$$

holds.

6*. Let $\varnothing \neq C \subset \mathbb{R}^N$ be open and connected, and let $f : C \to \mathbb{R}$ be differentiable such that $\nabla f \equiv 0$. Show that f is constant. (*Hint*: First, treat the case where C is convex using the chain rule; then, for general C, assume that f is not constant, let $x, y \in C$ such that $f(x) \neq f(y)$, and show that $\{U, V\}$ with $U := \{z \in C : f(z) = f(x)\}$ and $V := \{z \in C : f(z) \neq f(x)\}$ is a disconnection for C.)

Solution: First, suppose that C is convex, and assume that f is not constant, i.e., there are $x, y \in C$ with $f(x) \neq f(x)$. Since C is convex, $\{x + t(y - x) : t \in [0, 1]\}$ is contained in C. Define

$$g : [0,1] \to \mathbb{R}, \quad t \mapsto f(x + t(y - x)).$$

Then g is continuous and differentiable on $(0, 1)$. The chain rule yields

$$g'(t) = (\nabla f(x + t(y - x))) \cdot (y - x) = 0$$

for $t \in (0, 1)$. From one variable calculus, we know that this means that g is constant. However, we have $g(0) = f(x) \neq f(y) = g(1)$, which is a contradiction.

For the general case, assume that f is not constant, and let $x, y \in C$ such that $f(x) \neq f(y)$. Define

$$U := \{z \in C : f(z) = f(x)\} \qquad \text{and} \qquad V := \{z \in C : f(z) \neq f(x)\}.$$

As f is continuous, there is an open set $\tilde{V} \subset \mathbb{R}^N$ such that $V = C \cap \tilde{V}$. Since C is also open, this means that V is open.

We claim that U is open as well. Let $z \in U$, and choose $\epsilon > 0$ such that $B_\epsilon(z) \subset C$. As $B_\epsilon(z)$ is convex, it follows from the convex case that f is constant on $B_\epsilon(z)$, i.e., $f(z') = f(x)$ for all $z' \in B_\epsilon(z)$, so that $B_\epsilon(x) \subset U$. As $z \in U$ is arbitrary, this proves the claim.

By definition, $U \neq \varnothing \neq V$, $U \cap V = \varnothing$, and $U \cup V = C$. Hence, $\{U, V\}$ is a disconnection for C, which is a contradiction.