MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Solutions #10

1. Let D in spherical coordinates be given as the solid lying between the spheres given by $r = 2$ and $r = 4$, above the xy-plane and below the cone given by the angle $\theta = \frac{\pi}{3}$. Evaluate the integral $\int_D xyz$.

   Solution: In spherical coordinates, D is described as

$$\left\{(r, \theta, \sigma) \in \mathbb{R}^3 : r \in [2, 4],\, \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right], \sigma \in [0, 2\pi]\right\},$$

   so that

$$\begin{aligned}\int_D xyz &= \int_2^4 \left(\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\int_0^{2\pi} (r\cos\theta\cos\sigma)(r\cos\theta\sin\sigma)(r\sin\theta) r^2 \cos\theta \, d\theta \right) d\sigma \right) dr \\ &= \left(\int_2^4 r^5 \, dr \right) \left(\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^3\theta \sin\theta \, d\theta \right) \left(\int_0^{2\pi} \cos\sigma \sin\sigma \, d\sigma \right).\end{aligned}$$

   Since (substitute $u = \sin\sigma$)

$$\int_0^{2\pi} \sin\sigma\cos\sigma \, d\sigma = \int_0^0 u \, du = 0,$$

   we have $\int_D xyz = 0$.

2. Let K be the triangle with vertices $(1, 8)$, $(2, 7)$, and $(9, 3)$. Evaluate the line integral

$$\int_{\partial K} \sin y \, dx + x \cos y \, dy$$

   where ∂K is positively oriented.

   Solution: By Green's Theorem, we have

$$\begin{aligned}\int_{\partial K} \sin y \, dx + x\cos y \, dy &= \int_K \frac{\partial}{\partial x} x\cos y - \frac{\partial}{\partial y} \sin y \\ &= \int_K \cos y - \cos y \\ &= 0.\end{aligned}$$

3. Let $P, Q \colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$P(x, y) = e^x + y^3 \qquad \text{and} \qquad Q(x, y) = 4xy^2.$$

   Suppose that the force field (P, Q) moves a particle once along the boundary of the ellipse $\left\{(x, y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{4} \leq 1\right\}$ in counterclockwise direction. Compute the work done.

Solution: Let E denote the ellipse in question. The work done is given by the curve integral of (P, Q) along ∂E. Noting that

$$\frac{\partial Q}{\partial x}(x,y) = 4y^2 \qquad \text{and} \qquad \frac{\partial P}{\partial y}(x,y) = 3y^2$$

we have

$$\int_{\partial E} P\,dx + Q\,dy = \int_E \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_E y^2$$

by Green's Theorem. Using the same parameter transformation as in Problem 5 on Assignment #9, we obtain

$$\begin{aligned}
\int_{\partial E} P\,dx + Q\,dy &= \int_E y^2 \\
&= \int_{[0,1]\times[0,2\pi]} (2r\sin\theta)^2 2r \\
&= 8 \int_0^1 \left(\int_0^{2\pi} r^3 (\sin\theta)^2\,d\theta \right) dr \\
&= 8 \left(\int_0^1 r^3\,dr \right) \left(\int_0^{2\pi} (\sin\theta)^2\,d\theta \right) \\
&= 4 \int_0^{2\pi} (\sin\theta)^2\,d\theta \\
&= 4\pi.
\end{aligned}$$

4. Let $a, b > 0$. Use Green's Theorem to compute the area of the ellipse

$$E := \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Solution: By the corollary of Green's Theorem from class, we have

$$\mu(E) = \frac{1}{2} \int_{\partial E} x\,dy - y\,dx.$$

Let

$$\gamma \colon [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto (a\cos t, b\sin t).$$

Then γ is a continuously differentiable curve parametrizing ∂E in counterclockwise orientation. It follows that

$$\begin{aligned}
\mu(E) &= \frac{1}{2} \int_{\partial E} x\,dy - y\,dx \\
&= \frac{1}{2} \int_0^{2\pi} (-b\sin t, a\cos t) \cdot (-a\sin t, b\cos t)\,dt \\
&= \frac{1}{2} \int_0^{2\pi} ab\big(\underbrace{(\sin t)^2 + (\cos t)^2}_{=1} \big)\,dt \\
&= \pi ab.
\end{aligned}$$

5. Let $\varnothing \neq U \subset \mathbb{R}^3$ be open, and let $f, g : U \to \mathbb{R}$ be twice continuously partially differentiable. Show that $\operatorname{div}(\nabla f \times \nabla g) = 0$ on U, where $\times$ denotes the cross product in $\mathbb{R}^3$.

Solution: First, note that

$$\nabla f \times \nabla g = \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}, -\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}, \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \right).$$

It follows that

$$\begin{aligned}
&\operatorname{div}(\nabla f \times \nabla g) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \right) \\
&= \frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial x \partial y} \\
&\qquad - \frac{\partial^2 f}{\partial y \partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y \partial z} + \frac{\partial^2 f}{\partial y \partial z}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial y \partial x} \\
&\qquad\qquad + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial z \partial y} - \frac{\partial^2 f}{\partial z \partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial z \partial x} \\
&= \frac{\partial f}{\partial x}\left(-\frac{\partial^2 g}{\partial y \partial z} + \frac{\partial^2 g}{\partial z \partial y} \right) + \frac{\partial f}{\partial y}\left(\frac{\partial^2 g}{\partial x \partial z} - \frac{\partial^2 g}{\partial z \partial x} \right) + \frac{\partial f}{\partial z}\left(-\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 g}{\partial y \partial x} \right) \\
&\qquad + \frac{\partial g}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) + \frac{\partial g}{\partial y}\left(-\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial z \partial x} \right) + \frac{\partial g}{\partial z}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \\
&= 0
\end{aligned}$$

by Clairaut's Theorem.

6*. Let $D \subset \mathbb{R}^2$ be the trapeze with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$. Evaluate $\int_D \exp\left(\frac{x+y}{x-y}\right)$. (*Hint*: Consider

$$\phi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (u,v) \mapsto \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v) \right)$$

and apply Change of Variables.)

Solution: Let

$$K := \{(u,v) \in \mathbb{R}^2 : 1 \leq v \leq 2, \quad -v \leq u \leq v\}.$$

Then K is compact with content such that $\phi(K) = D$. Obviously, ϕ is injective, and as

$$\det J_\phi(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

the Change of Variables Theorem applies and yields

$$\begin{aligned}
\int_D \exp\left(\frac{x+y}{x-y}\right) &= \frac{1}{2}\int_D \exp\left(\frac{u}{v}\right) \\
&= \frac{1}{2}\int_1^2 \left(\int_{-v}^{v} \exp\left(\frac{u}{v}\right)\, du\right) dv \\
&= \frac{1}{2}\int_1^2 \left(v \exp\left(\frac{u}{v}\right)\Big|_{u=-v}^{u=v}\right) dv \\
&= \frac{1}{2}\int_1^2 \left(e - \frac{1}{e}\right) v\, dv \\
&= \frac{3}{4}\left(e - \frac{1}{e}\right).
\end{aligned}$$