Fall 2021, Math 328, Homework 3

Due: End of day on 2021-10-01

1 10 points

Let G be a group.

1. For $g \in G$, prove that the function $\rho(g) : G \to G$ defined by $\rho(g)(h) = g \cdot h \cdot g^{-1}$ is an *automorphism* of G.
2. Prove that the map $\rho : G \to \mathrm{Aut}(G)$ is a homomorphism.
3. Prove that $Z(G)$ is the kernel of ρ.
4. Let $\sigma \in \mathrm{im}(\rho)$ and $\tau \in \mathrm{Aut}(G)$ be given. Prove that $\tau \circ \sigma \circ \tau^{-1} \in \mathrm{im}(\rho)$.

2 10 points

Suppose that G acts on a set X, and let $\rho : G \to \mathrm{Per}(X)$ denote the associated permutation representation. Prove that $\ker(\rho)$ agrees with G_X, the kernel of the action of G on X.

3 10 points

Let G act on a set X. For $x, y \in X$, write $x \sim y$ provided that there exists some $g \in G$ such that $g \cdot x = y$. Prove that $\sim$ is an equivalence relation on X.

Remark: The equivalence class assocaited to $x \in X$ is called the *orbit* of x.

4 10 points

Let H_i, $i \in \mathbb{N}$ be a collection of subgroups of G. Suppose that for all $i \in \mathbb{N}$, one has $H_i \subset H_{i+1}$. Prove that

$$H := \bigcup_{i=0}^{\infty} H_i$$

is a subgroup of G. Give an explicit example showing that this fails without the assumption that $H_i \subset H_{i+1}$ for all i.

5 20 points

For each of the following, a subset of a group is specified. Determine (with proof) whether the given subset is a subgroup.

1. The set of complex numbers of the form $\{a + i \cdot a \mid a \in \mathbb{R}\}$, a subset of $\mathbb{C}$.
2. The set of complex numbers of absolute value 1, a subset of $(\mathbb{C}, +)$.
3. The set of complex numbers of absolute value 1, a subset of $(\mathbb{C}^\times, \times)$.
4. The set of transpositions in S_n, with $n \geq 3$.
5. The set of reflections in D_{2n}, with $n \geq 3$.
6. The set of odd integers in $\mathbb{Z}$.
7. The set of integers which are divisible by n in $\mathbb{Z}$, for $n \in \mathbb{N}$.
8. The set $\{1, r^2, s, sr^2\}$ in D_8.
9. The set $\{1, r^2, sr, sr^3\}$ in D_8.
10. The set $\{1, r^2, s, sr^2\}$ in D_{10}.
11. The set $\{1, r^2, sr, sr^3\}$ in D_{10}.
12. The set $\{1, i, j, k\}$ in Q_8.
13. The set $\{1, i, -1, -i\}$ in Q_8.
14. The subset of S_4 containing the following elements:
    (a) The identity.
    (b) All cycles of length 3.
    (c) The elements $(1,2)(3,4)$, $(1,3)(2,4)$, $(1,4)(2,3)$.