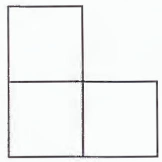
# Lecture 16

## The Trominoes

**Definition:** A (*right*) *tromino* is an object made of three unit squares as shown:



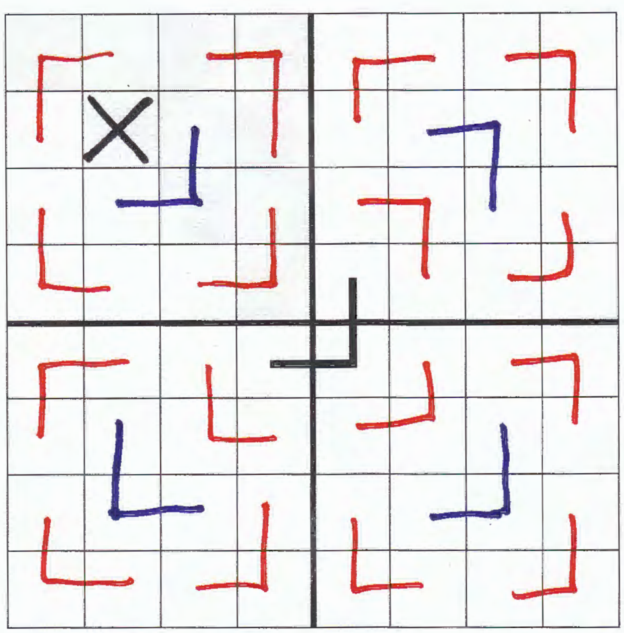
a tromino may appear as above, or it may be rotated through some multiple of 90°.

**Definition:** A board is called *deficient* if one unit squares is missing.

**Definition:** A board is said to be *tiled* if you can fit individual tiles together with no gaps or overlaps to fill the board.

**Note:** Suppose we are given a deficient $n \times n$ board. We can rotate the board so the missing square is in the top left quadrant. Further we could reflect the board along its main diagonal. Using this symmetry would place the missing square above or on the main diagonal (see the example for the $8 \times 8$ board below).

**Note:** Given a deficient $8 \times 8$ board, by symmetry we need only look at cases where the missing square is in the shaded region below. Notice that by dividing the board into four quadrants and placing one tromino in the center as shown we have four deficient $4 \times 4$ boards. Next, divide the board into sixteen $2 \times 2$ boards and place four trominoes to make each one deficient. Finally we can complete the tiling of the $8 \times 8$ board by tiling the sixteen deficient $2 \times 2$ boards.

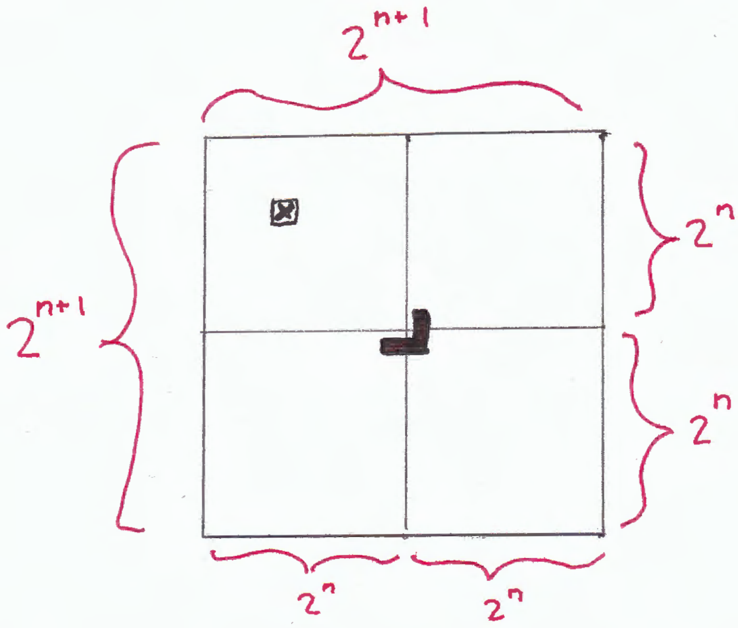


**Example 1:** For $n \geq 0$, show that any deficient $2^n \times 2^n$ board can be tiled by trominoes.

BC  If $n=0 \Rightarrow$ the deficient $1 \times 1$ board is tiled : $\boxed{\times}$

IS  Show : any deficient $2^n \times 2^n$ can be tiled ($=$ tiled by trominoes)

$\Rightarrow$ " " $2^{n+1} \times 2^{n+1}$ " " " for $n \geq 0$.




STEP 1  Divide the board into 4 equal quadrants.

STEP 2  By symmetry of a square we only need to consider cases where the missing square is in the top left quadrant.

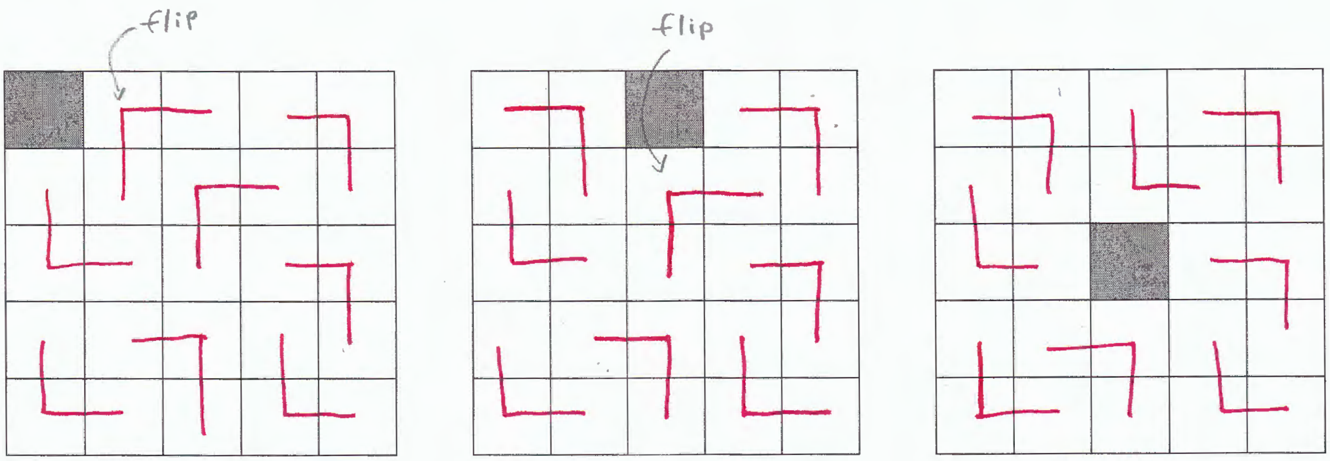
STEP 3  Place one tromino in the center as shown.

STEP 4  There are now four deficient $2^n \times 2^n$ board which can be tiled.

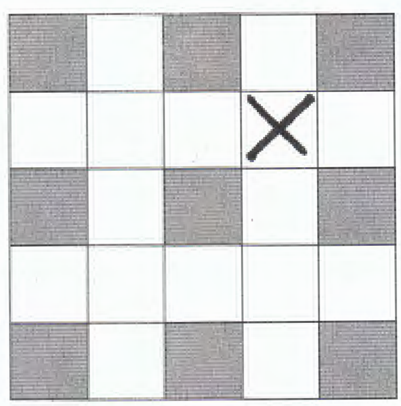
$\therefore$ any deficient $2^{n+1} \times 2^{n+1}$ board can be tiled.

The Deficient 5 × 5 Boards

**Example 2:** a) Tile the following deficient 5 × 5 boards.




b) From the boards above, using symmetry, the black squares represent deficiencies of a 5 × 5 board that can still be tiled with trominoes. Show that a deficient 5 × 5 board made by removing one of the white squares below cannot be tiled with trominoes.



Suppose one white square is missing. The 9 black squares need exactly 9 trominoes to cover them.

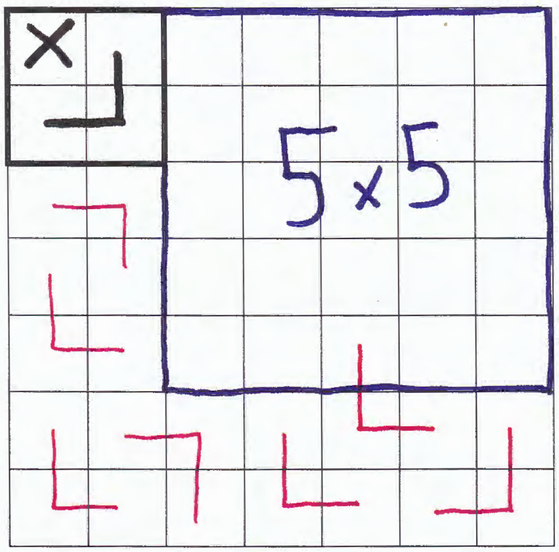
But, on a deficient 5×5 board in terms of area there is room for only

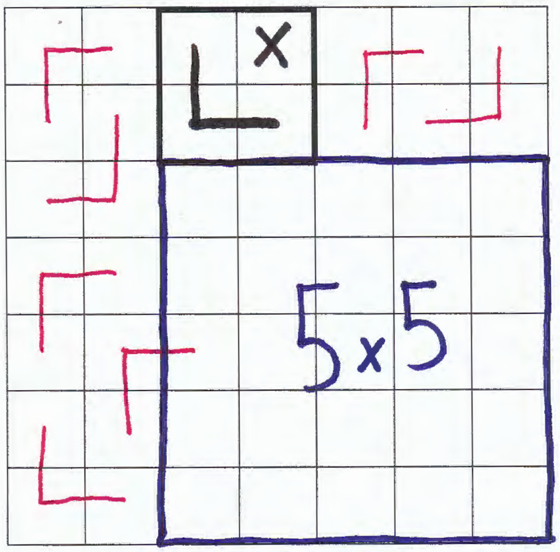
$$\frac{5^2-1}{3} = \frac{24}{3} = 8 \text{ trominoes}.$$

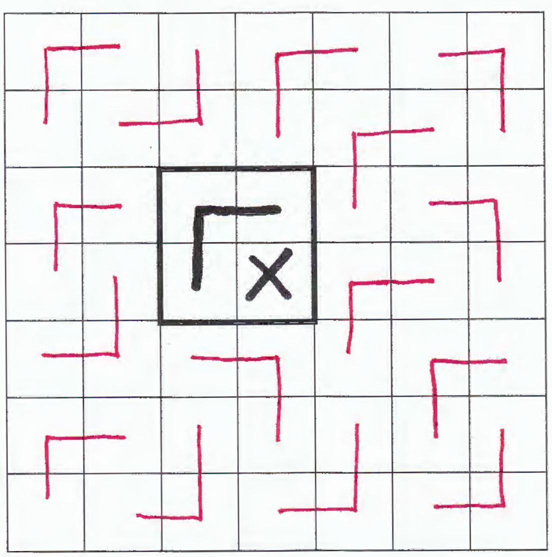
$\therefore$ the above board with one white square missing can not be tiled.

The Deficient 7 × 7 Boards

**Example 3:** Show that any deficient $7 \times 7$ board can be tiled by trominoes.



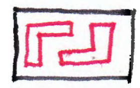



- The first two tilings can be completed using Ex. 2.
- By symmetry, we only need to consider the cases where the missing square is in one of the bold 2x2 squares.
- By rotating each two by two bold square we get all the cases.

**Proposition 1:** Given that $n \equiv 0 \pmod 3$ and $m \equiv 0 \pmod 2$ an $n \times m$ board can be tiled with trominoes.

*Proof.* Such a board can be tiled by $2 \times 3$ blocks made by two trominoes:



**Proposition 2:** If $n \equiv 1$ (*mod* 3) a deficient $n \times n$ board can be tiled with trominoes.

*Proof.* Let $P_n$ be the statement for $n \geq 1$.

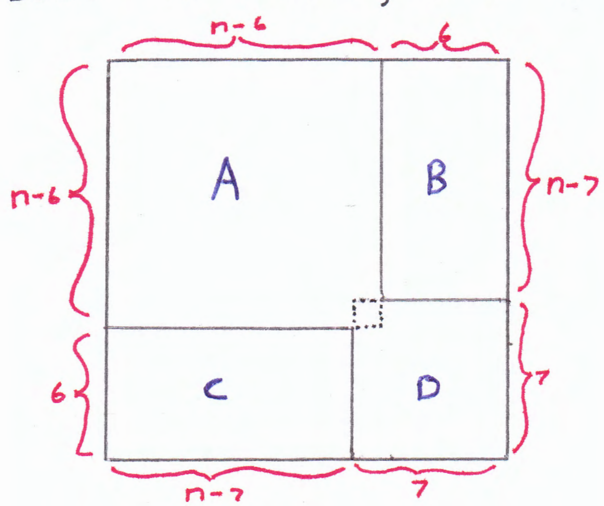
<u>BC</u> • $P_1$ is true by Ex 1

• $P_4$ " " " "

• $P_7$ " " " Ex 3

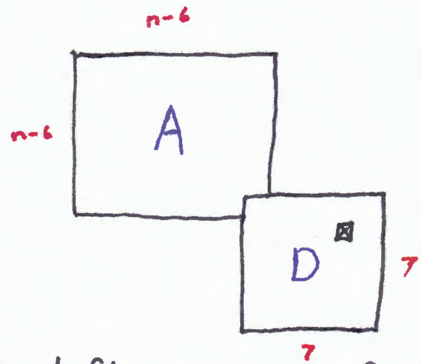
<u>IS</u> Show: $P_k$ is true for all $k < n \Rightarrow P_n$ is true, for $n > 7$.

Divide an $n \times n$ board as follows, where $n \equiv 1 \bmod 3$ and $n > 7$:



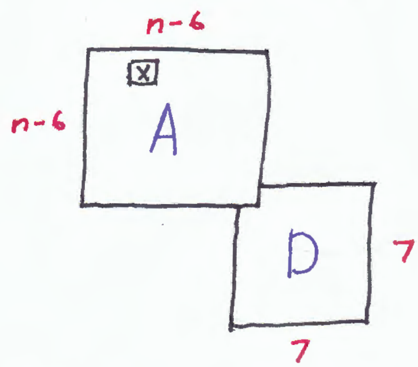

By symmetry we only need to consider the cases where the missing square is in section A or D. If $n = 10, 13$ the missing square is in section D otherwise it is in section A.

If $n = 10, 13$ the missing square is in section D:




use the top left corner of section D to make section A into a $(n-6)\times(n-6)$ deficient board.

If $n \geq 16$ the missing square is in section A:




Use the bottom right corner of section A to make section D into a $7\times 7$ deficient board.

Now, each section can be tiled (in either of the above cases):

- Section A is an $(n-6)\times(n-6)$ deficient board:

  since $n-6 \equiv n-0 \equiv n \equiv 1 \mod 3$

  $\Rightarrow P_{n-6}$ is true

  $\Rightarrow$ Section A can be tiled

- Section D is a $7\times 7$ deficient board which can be tiled by Ex. 3.

- Sections B and C are $(n-7)\times 6$ boards:

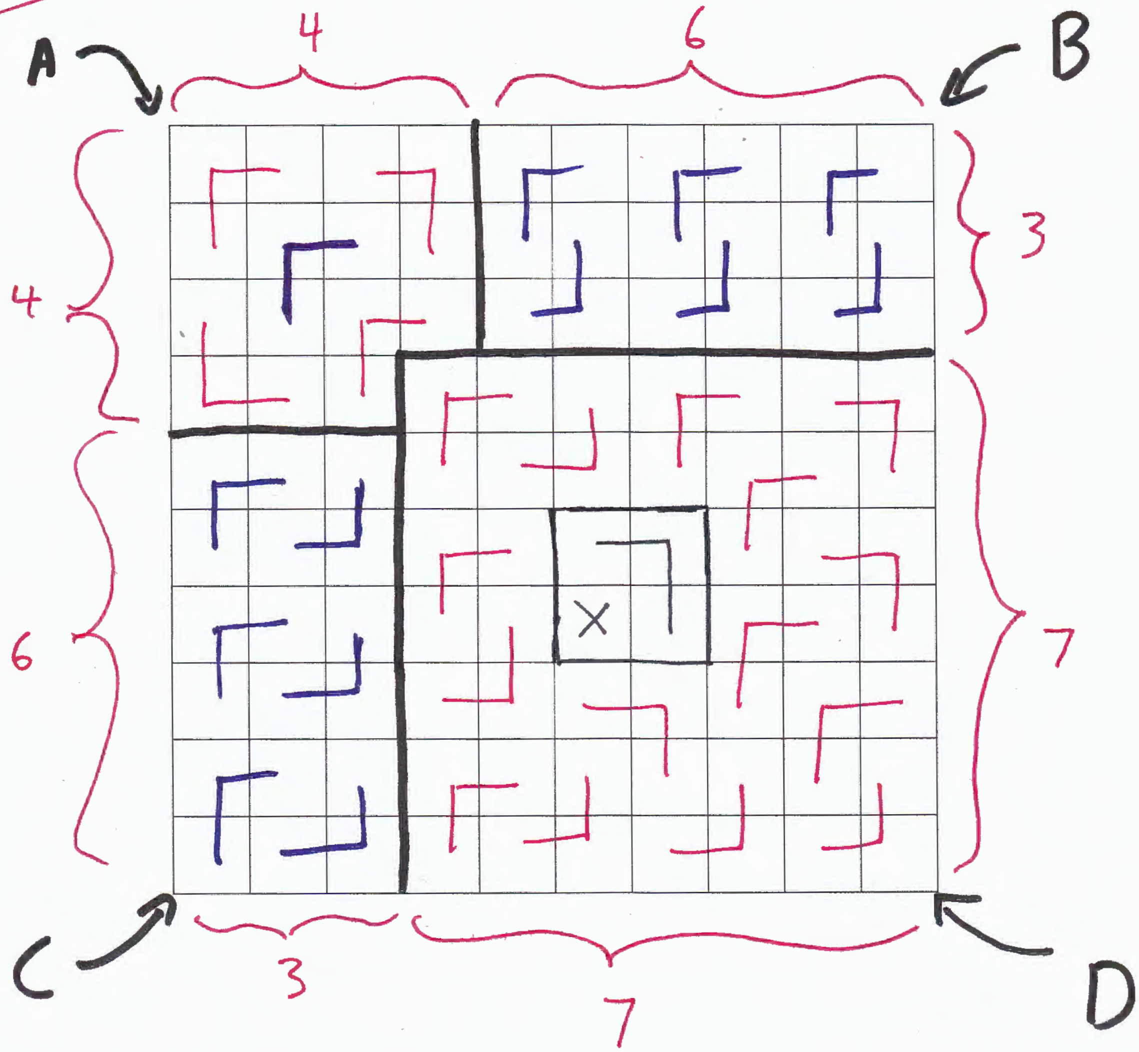
  now $n-7 \equiv 1-7 \equiv -6 \equiv 0 \mod 3$
  $6 \equiv 0 \mod 2$ $\Big| \Rightarrow$ Sections B and C can be tiled.

$\therefore P_n$ is true.

$\blacksquare$

**Example 4:** Pick any deficient $10 \times 10$ board and use the proof proposition 2 to tile the board with trominoes.

$n=10$



Another Example

**Example 4:** Pick any deficient $10 \times 10$ board and use the proof proposition 2 to tile the board with trominoes.

$n=10$