MATH 336- Winter 2022

Assignment 5

Problem 1. Consider the following linear system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

and assume that $a_{ij}(t)$ are continuous functions for all $i = 1, 2$, $j = 1, 2$. Prove that there are *exactly* two linearly independent solutions to the above system.

Generalize the claim for n-dimensional systems, that is, if $a_{ij}(t)$ are continuous functions for $i = 1, .., n$, $j = 1, .., n$, the following system has *exactly* n linearly independent solutions.

$$\frac{d}{dt}\boldsymbol{y} = [a_{ij}(t)]_{n\times n}\,\boldsymbol{y},\ \boldsymbol{y} \in \mathbb{R}^n$$

Problem 2. The equation of a damped pendulum is as follows

$$\frac{d^2\theta}{dt^2} + \varepsilon\theta' + \frac{g}{l}\sin\theta = 0,$$

where $\varepsilon > 0$.

a) Rewrite the above equation as a first-order system.

b) Find the equilibrium of the system for $-\pi < \theta < \pi$.

c) Linearize the system at the obtained equilibrium.

d) Show that there are three cases for the type of the equilibrium point: 1) $\varepsilon^2 > \frac{4g}{l}$, 2) $\varepsilon^2 = \frac{4g}{l}$, and 3) $\varepsilon^2 < \frac{4g}{l}$. Determine the type of the equilibrium in each case.

   The first case is called over-damped, the second one is called critical-damped, and the third case is called under-damped.

e) Show that in all above cases, $\theta(t)$ approaches the equilibrium in long terms, as $t \to \infty$.

Problem 3. Solve the following systems, classify the origin, and draw the trajectories in the phase plane.

a)

$$\begin{cases} y_1' = y_1 + 3y_2 \\ y_2' = y_1 - y_2 \end{cases}$$

b)

$$\begin{cases} y_1' = -y_1 - 2y_2 + e^t \\ y_2' = 2y_1 - y_2 + 1 \end{cases}$$

Problem 4. Assume that $A_{n\times n}$ is a symmetric matrix, that is,

$$\langle Au, v\rangle = \langle u, Av\rangle,$$

for any vectors $u, v \in \mathbb{R}^n$.

a) Show that all eigenvalues of A are real and there are n mutually orthogonal eigenvectors for A.

b) If Q is the matrix of eigenvectors of A, that is, $Q = [v_1|v_2|\cdots|v_n]$, show that

$$Q^{-1}AQ = \mathrm{diag}(\lambda_1, .., \lambda_n).$$

c) Solve the following second-order system

$$\frac{d^2}{dt^2}\boldsymbol{y} = A\boldsymbol{y}, \boldsymbol{y}(0) = \mathrm{Id}.$$

Problem 5. Consider the following system for $x(t) \geq 0, y(t) \geq 0$

$$\begin{cases} \frac{dx}{dt} = 2x - kx^2 - 0.5xy \\ \frac{dy}{dt} = -0.5y + 0.5xy \end{cases}.$$

a) Find all equilibrium point of the system. Notice that some of equilibriua depends on the parameter k.

b) Let k increases from $k = 0$. At what value(s) a new equilibrium is generated? These point are called bifurcation points.

c) Use Matlab and draw phase portrait of the system for $k = 1, k = 3$.

Problem 6. Consider the following matrix

$$S = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}.$$

a) Describe the transformation of matrix S to an arbitrary vector $u \in \mathbb{R}^2$ in terms of rotation and scaling.

b) If $A_{2\times 2}$ is a matrix with eigenvalues $\lambda = \sigma \pm i\omega$, show that A and S are similar.

c) Show the following relation

$$e^{St} = e^{\sigma t}\begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

and conclude that the fundamental matrix of A is

$$e^{At} = e^{\sigma t}Q\begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}Q^{-1},$$

for some matrix Q.

Problem 7. Assume that $A_{2\times 2}$ is a matrix with repeated eigenvalue λ and only one eigenvector v.

a) If w is a generalized eigenvector for A, and $Q = [v|w]$, show the relation

$$Q^{-1}AQ = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

b) Show the relation

$$e^{\left[\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right]t} = e^{\lambda t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix},$$

and conclude that the solution of the system

$$\frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y},$$

is $e^{At} = e^{\lambda t}(w + tv)$.

Problem 8. Consider the following system

$$x' = Ax, x \in \mathbb{R}^n$$

a) Show that

$$\operatorname{div}(Ax) = \operatorname{tr}(A).$$

b) The solution of the system can be considered as $\Phi(t; x_0)$ where x_0 is an initial condition for the system. The mapping $\Phi_t: \mathbb{R}^n \to \mathbb{R}^n$ for $t \geq 0$ is called the flow of the system. For $n = 2$, show that if $\operatorname{tr}(A) > 0$ and if S_0 is a unit square centered at the origin, then $\text{Area}\{\Phi_t(S_0)\} > 1$ for $t > 0$.

c) Assume A is an $n \times n$ matrix with n real distinct eigenvalues. Show that if $\operatorname{tr}(A) > 0$ and if S_0 is a unit cube centered at the origin, then $\operatorname{vol}(\Phi_t(S_0)) > 0$ for $t > 0$.

Problem 9. (Bonus)

a) Show that all norms in $\mathbb{R}^n$ are equivalent, that is, if $\|.\|_1, \|.\|_2$ are any two norms in $\mathbb{R}^n$, then there are positive constants m, M such that

$$m\,\|x\|_2 \leq \|x\|_1 \leq M\,\|x\|_2,$$

for all $x \in \mathbb{R}^n$.

b) Show that the following are norms in the space of $n \times n$ matrices

$$\|A\|_\infty = \max_i \left\{\sum_{j=1}^n |a_{ij}|\right\}, \|A\|_1 = \max_j \left\{\sum_{i=1}^n |a_{ij}|\right\}.$$