Solutions #2

1. For any set S, its *power set* $\mathfrak{P}(S)$ is defined to be the set consisting of all subsets of S. Show that there is *no* surjective map from S to $\mathfrak{P}(S)$. (*Hint*: Assume that there is a surjective map $f: S \to \mathfrak{P}(S)$ and consider the set $\{x \in S : x \notin f(x)\}$.)

2. Which of the following sets are convex:

    (i) $\{(x, y) \in \mathbb{R}^2 : x > y\}$;

    (ii) $\{x \in \mathbb{R}^N : \|x\| > 2\}$;

    (iii) $\mathbb{R} \setminus \mathbb{Q}$;

    (iv) $\{(x, y, z) \in \mathbb{R}^3 : x + y + z \geq 2021\}$?

    Justify your answers.

3. Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^N$. Show that $\bigcap_{C \in \mathcal{C}} C$ is again convex. Is $\bigcup_{C \in \mathcal{C}} C$ necessarily convex?

4. Show that $\mathbb{Z}$ is closed in $\mathbb{R}$, but not open, and that $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

5. Let $\varnothing \neq S \subset \mathbb{R}^N$ be arbitrary, and let $\varnothing \neq U \subset \mathbb{R}^N$ be open. Show that

$$S + U := \{x + y : x \in S, \, y \in U\}$$

    is open.

6* For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, set

$$\|x\|_1 := |x_1| + \cdots + |x_N| \quad \text{and} \quad \|x\|_\infty := \max\{|x_1|, \ldots, |x_N|\}.$$

    (a) Show that the following are true for $j = 1, \infty$, $x, y \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$:

    (i) $\|x\|_j \geq 0$ and $\|x\|_j = 0$ if and only if $x = 0$;

    (ii) $\|\lambda x\|_j = |\lambda| \|x\|_j$;

    (iii) $\|x + y\|_j \leq \|x\|_j + \|y\|_j$.

    (b) For $N = 2$, sketch the sets of those x for which $\|x\|_1 \leq 1$, $\|x\| \leq 1$, and $\|x\|_\infty \leq 1$.

    (c) Show that

$$\|x\|_1 \leq \sqrt{N} \|x\| \leq N \|x\|_\infty$$

    for all $x \in \mathbb{R}^N$.

Due Thursday, September 23, 2020, at 5:00 p.m.; no late assignments.