Fall 2021, Math 328, Homework 6

Due: End of day on 2021-11-01

1 10 points

Let H be a subgroup of a group G and assume that $[G : H] = 2$. Prove that H is normal in G. Give an explicit example showing that this fails if 2 is replaced by an integer which is larger than 2.

2 10 points

Let G be a group.

1. Show that $Z(G)$ is a normal subgroup of G.
2. Show that the following are equivalent:
   (a) G is abelian.
   (b) $G = Z(G)$.
   (c) $G/Z(G)$ is trivial.
   (d) $G/Z(G)$ is cyclic.
3. Assume that G is a finite group of order $p \cdot q$ where p, q are (not necessarily distinct) primes. Prove that either G is abelian or $Z(G)$ is trivial.

3 10 points

Let G be a group. Assume that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

Hint: Use problem 2.

4 10 points

Let G be a group and let H and K be two subgroups of G.

1. Assume that H and K are finite, and that their orders are coprime. Prove that $H \cap K$ is trivial.
2. Assume that H and K have finite index in G. Prove that
$$\operatorname{lcm}([G : H], [G : K]) \leq [G : H \cap K] \leq [G : H] \cdot [G : K].$$
Deduce that if $[G : H]$ and $[G : K]$ are coprime, then $[G : H \cap K] = [G : H] \cdot [G : K]$.
3. Assume that $H \leq K \leq G$. Prove that $[G : H] = [G : K] \cdot [K : H]$.

5 10 points

Let G be a group which acts on a set X, and let $x \in X$ be a fixed element. Recall that the stabilizer of x in G is the subgroup of G defined as

$$\operatorname{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}.$$

Define the *orbit* of x as the set

$$\operatorname{Orb}_G(x) = \{g \cdot x \mid g \in G\}.$$

1. Consider the map
$$\delta_x : G \to \operatorname{Orb}_G(x)$$
defined by $g \mapsto g \cdot x$. Prove that this map is surjective.
2. For $g, h \in G$, prove that $\delta_x(g) = \delta_x(h)$ if and only if $g \cdot \operatorname{Stab}_G(x) = h \cdot \operatorname{Stab}_G(x)$.
3. Prove that the map
$$g \cdot \operatorname{Stab}_G(x) \mapsto g \cdot x$$
induces a bijection between $G/\operatorname{Stab}_G(x)$ and $\operatorname{Orb}_G(x)$.
4. Assume that G is finite. Prove that
$$\#G = \#\operatorname{Stab}_G(x) \cdot \#\operatorname{Orb}_G(x).$$
You have proved the the so-called *Orbit-Stabilizer Theorem*.