MATH 336- WINTER 2022

Problem 1. Consider the following equation

$$y' = y(2+y)(1-y)$$

a) Find equilibrium values of the equation and classify them in terms of stability and unstability.

b) Do the concavity analysis and draw a few integral curves of the equation.

c) Use an online software to draw the slope field and a few integral curves.

Problem 2. Solve the following linear first-order equations and determine the domain of the solution

a) $y' + 2y = x$

b) $xy' - y = x^3, y(1) = -1$

Problem 3. Solve the following BERNOULLI's equations

a) $y' + xy = xy^2, y(0) = 1$

b) $y' - y = xy^3$

Problem 4. Integrate the following equations with the given integrating factor

a) $(2xy - 5y^2) + (1 - 5xy)\, y' = 0,\ \mu = \mu(y)$

b) $(x^2y + 4) - x^2(y - x)\, y' = 0,\ \mu = \mu(x)$

c) $(xe^y - 1) + (ye^{-x} + 1)\, y' = 0,\ \mu = \mu(x - y)$

Problem 5. Solve the following initial value problem and draw the solution

$$\begin{cases} y' + y = r(x) \\ y(0) = 0 \end{cases},$$

where $r(x)$ is the following piecewise function.

$$\begin{cases} 1 & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}.$$

Problem 6. Rewrite the following equation as a linear equation with respect to $x = x(y)$

$$(p(y)\, x + q(y))y' = r(y),$$

Use this idea to solve the following equations

$$(x - y^2)y' = y.$$

Problem 7. Consider the following equation

$$y' + a\,y = f(x),$$

where $a > 0$ is a constant, and assume that f is a continuous function and bounded

$$|f(x)| \le M,$$

for some $M > 0$. Prove the following relation

$$\lim_{x\to\infty} |y(x)| \le \frac{M}{a}$$

1 Challenging problems (bonus)

Problem 8. Consider the following equation

$$y' = f(y),$$

where f is a smooth function such that the associated IVP

$$\begin{cases} y' = f(y) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution. Assume that $\bar{y}$ is an asymptotically stable equilibrium of the equation. Prove that if $y_0 \neq \bar{y}$, then there is no finite x such that $y(x) = \bar{y}$.

Problem 9. Suppose that $a > 0$, and f is a bounded function; that is, $\max_x |f(x)| \le M$. Prove: there is a unique initial condition $y(0) = y_0$ such that the solution of the following IVP remain bounded

$$\begin{cases} y' - a\,y = f(t) \\ y(0) = y_0 \end{cases}.$$

Hint: take the initial condition as follows and show the solution of the above IVP is bounded

$$y(0) = -\int_0^\infty e^{-at} f(t)\,\mathrm{d}t,$$

Show also that if f is periodic then the bounded solution is periodic.

Problem 10. Verify that the function

$$\phi(x) = \begin{cases} \frac{1}{4}x^2 & x \ge 0 \\ \frac{-1}{4}x^2 & x < 0 \end{cases},$$

is a solution to the problem

$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}.$$

In particular, you need to show that the given function is continuously differentiable at $x = 0$. The other solution is trivially zero, that is $y(x) = 0$. Find infinitely many other solutions of the problem.