**MATH 217** (Fall 2021)

Honors Advanced Calculus, I

***Final Practice Problems***

1. Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be differentiable such that there is $\alpha \in \mathbb{R}$ such that $f(tx) = t^\alpha f(x)$ for all $t > 0$ and all $x \in \mathbb{R}^N$. Show that

$$(\nabla f)(x) \cdot x = \alpha\, f(x)$$

for $x \in \mathbb{R}^N$. (*Hint*: Fix $x \in \mathbb{R}^N$, and compute the first derivative of $(0,\infty) \ni t \mapsto f(tx)$ at $t = 1$ in two different ways.)

*Solution*: For $x \in \mathbb{R}^N$ fixed, let

$$\phi\colon (0,\infty) \to \mathbb{R}, \quad t \mapsto f(tx)$$

be as specified in the hint. As $\phi(t) = t^\alpha f(x)$ for $t > 0$, it is clear that

$$\phi'(t) = \alpha t^{\alpha-1} f(x)$$

for $t > 0$, so that $\phi'(1) = \alpha\, f(x)$. On the other hand, let

$$g\colon (0,\infty) \to \mathbb{R}^N, \quad t \to tx,$$

so that $\phi = f \circ g$ as both both $f$ and $g$ are differentiable, the Chain Rule yields that

$$\phi'(1) = J_{f\circ g}(1) = J_f(g(1))J_g(1) = (\nabla f)(x) \cdot x.$$

This proves the claim.

2. Let $D \subset \mathbb{R}^N$ have content. Show that

$$\mu(D) = \inf \sum_{j=1}^n \mu(I_j) \tag{$*$}$$

holds, where the infimum on the right hand side is taken over all $n \in \mathbb{N}$ and all compact intervals $I_1, \ldots, I_n \subset \mathbb{R}^N$ such that $D \subset I_1 \cup \cdots \cup I_n$.

*Solution*: Let $I_1, \ldots, I_n, I \subset \mathbb{R}^N$ be compact intervals such that

$$D \subset I_1 \cup \cdots \cup I_n \subset I$$

As

$$\mu(D) = \int_I \chi_D \leq \int_I \chi_{\bigcup_{j=1}^n I_j} \leq \int_I \sum_{j=1}^n \chi_{I_j} = \sum_{j=1}^n \int_I \chi_{I_j} = \sum_{j=1}^n \mu(I_j),$$

it clear that $\mu(D)$ is less than or equal to the infimum in $(*)$.

For the reversed inequality, let $\epsilon > 0$, and choose a compact interval $I \subset \mathbb{R}^N$ such that $D \subset I$. As $\mu(D) = \int_I \chi_D$, there is a partition $\mathcal{P}$ of $I$ such that

$$\left| \mu(D) - \sum_\nu \chi_D(x_\nu)\mu(I_\nu) \right| < \epsilon,$$

where $(I_\nu)_\nu$ is the subdivision of $I$ corresponding to $\mathcal{P}$ and $(x_\nu)_\nu$ are any points with $x_\nu \in I_\nu$. Choose $(x_\nu)_\nu$ such that $x_\nu \in D$ whenever $D \cap I_\nu \neq \varnothing$. Let $I_1, \ldots, I_n$ be an enumeration of those $I_\nu$ for which $D \cap I_\nu \neq \varnothing$. It follows that

$$\begin{aligned} \sum_{j=1}^n \mu(I_j) &= \sum_\nu \chi_D(x_\nu)\mu(I_\nu) \\ &\leq \mu(D) + \left| \mu(D) - \sum_\nu \chi_D(x_\nu)\mu(I_\nu) \right| \\ &< \mu(D) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof.

3. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} \frac{\sin(xy)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{otherwise.} \end{cases}$$

Check—and justify— whether or not $f$ is

(a) partially differentiable,

(b) continuous,

(c) totally differentiable,

(d) continuously partially differentiable, and

(e) Riemann integrable on $[-1,1] \times [-1,1]$.

*Solution*:

(a) Clearly, $f$ is partially differentiable at every point of $\mathbb{R}^2 \setminus \{(0,0)\}$. Since

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(h,0) - f(0,0)}{h} = 0,$$

it follows that $\frac{\partial f}{\partial x}(0,0) = 0$; similarly, $\frac{\partial f}{\partial y}(0,0) = 0$ is shown to hold.

(b) Since

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{2}{n^2}} \to \frac{1}{2} \neq 0,$$

$f$ is not continuous at $(0,0)$.

(c) Since total differentiability entails continuity, $f$ is not totally differentiable.

(d) Since continuously partially differentiable functions are totally differentiable, $f$ is not continuously partially differentiable.

(e) Clearly, $f$ is discontinuous only at $(0,0)$. It is therefore sufficient to show that $f$ is bounded on $[-1,1] \times [-1,1]$. Let $(x,y) \in ([-1,1] \times [-1,1]) \setminus \{(0,0)\}$, and note that

$$\begin{aligned}
|f(x,y)| &= \frac{|\sin(xy)|}{x^2+y^2} \\
&\leq \frac{|xy|}{x^2+y^2} \\
&= \frac{\sqrt{x^2y^2}}{x^2+y^2} \\
&\leq \frac{1}{2}\frac{x^2+y^2}{x^2+y^2}, \\
&\quad \text{by the inequality between geometric and arithmetic mean,} \\
&= \frac{1}{2}.
\end{aligned}$$

Consequently, $f$ is Riemann integrable on $[-1,1] \times [-1,1]$.

4. Let $D \subset \mathbb{R}^3$ be the region in the first octant, i.e., with $x, y, z \geq 0$, which is bounded by the cylinder given by $x^2 + y^2 = 16$ and the plane given by $z = 3$. Evaluate

$$\int_D xyz.$$

*Solution*: We have

$$D = \{(x,y,z) \in \mathbb{R}^2 : x, y, z \geq 0,\ x^2 + y^2 \leq 16,\ z \leq 3\}.$$

Use cylindrical coordinates, so that so that $D = \phi(K)$ with

$$K = \left\{(r,\theta,z) : r \in [0,4],\ \theta \in \left[0, \frac{\pi}{2}\right], z \in [0,3]\right\}.$$

The change of variables formula yields

$$\begin{aligned}
\int_D xyz &= \int_K r^3(\cos\theta)(\sin\theta)z \\
&= \int_0^4 \left( \int_0^{\frac{\pi}{2}} \left( \int_0^3 r^3(\cos\theta)(\sin\theta)z \, dz \right) d\theta \right) dr \\
&= \frac{9}{2} \int_0^4 \left( \int_0^{\frac{\pi}{2}} r^3(\cos\theta)(\sin\theta) \, d\theta \right) dr \\
&= \frac{9}{2} \int_0^4 \left( \int_0^1 r^3 u \, du \right) dr \\
&= \frac{9}{4} \int_0^4 r^3 \, dr \\
&= \frac{9}{4} \frac{4^4}{4} \\
&= 144.
\end{aligned}$$

5. Let $K$ be the triangle with vertices $(0,0)$, $(4,2)$, and $(4,-8)$. Evaluate the curve integral

$$\int_{\partial K} x^2y^2 \, dx + (yx^3 + y^2) \, dy,$$

where $\partial K$ is the positively oriented boundary of $K$.

*Solution*: Set

$$P(x,y) := x^2y^2 \qquad \text{and} \qquad Q(x,y) := yx^3 + y^2$$

for $(x,y) \in \mathbb{R}^2$, and apply Green's Theorem:

$$\begin{aligned}
\int_{\partial K} x^2y^2 \, dx + (yx^3 + y^2) \, dy &= \int_{\partial K} P \, dx + Q \, dy \\
&= \int_K \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_K 3yx^2 - 2yx^2 = \int_K yx^2.
\end{aligned}$$

Noting that

$$K = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le x \le 4, \; -2x \le y \le \frac{x}{2} \right\},$$

we obtain

$$
\begin{aligned}
\int_{\partial K} x^2y^2\,dx + (yx^3+y^2)\,dy &= \int_K yx^2 \\
&= \int_0^4 \int_{-2x}^{\frac{x}{2}} yx^2\,dy\,dx, \qquad \text{by Fubini's Theorem,} \\
&= \int_0^4 \frac{1}{2}y^2x^2\Big|_{-2x}^{\frac{x}{2}}\,dx \\
&= -\frac{15}{8}\int_0^4 x^4\,dx \\
&= -\frac{3x^5}{8}\Big|_0^4 \\
&= -384.
\end{aligned}
$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable, let $c > 0$ and $v \in \mathbb{R}^N$ be arbitrary, and let $\omega := c\|v\|$. Show that

$$F\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, \quad (x,t) \mapsto f(x \cdot v - \omega t)$$

solves the *wave equation*

$$\Delta F - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0,$$

where $\Delta$ denotes the *spatial* Laplace operator, i.e.,

$$\Delta F = \sum_{j=1}^N \frac{\partial^2 F}{\partial x_j^2}.$$

*Solution*: Since

$$F(x_1,\ldots,x_N,t) = f(x_1v_1 + \cdots + x_Nv_N - \omega t)$$

for $x_1,\ldots,x_N,t \in \mathbb{R}$, it follows that

$$\frac{\partial F}{\partial x_j}(x_1,\ldots,x_N,t) = v_j f'(x_1v_1 + \cdots + x_Nv_N - \omega t)$$

for $j = 1,\ldots,N$ and therefore

$$\frac{\partial^2 F}{\partial x_j^2}(x_1,\ldots,x_N,t) = v_j^2 f''(x_1v_1 + \cdots + x_Nv_N - \omega t)$$

for $j = 1,\ldots,N$. It follows that

$$(\Delta F)(x_1,\ldots,x_N,t) = \|v\|^2 f''(x_1v_1 + \cdots + x_Nv_N - \omega t).$$

On the other hand, we have

$$\frac{\partial F}{\partial t}(x_1, \ldots, x_N, t) = -\omega f'(x_1 v_1 + \cdots + x_N v_N - \omega t)$$

and

$$\begin{aligned}\frac{\partial^2 F}{\partial t^2}(x_1, \ldots, x_N, t) &= \omega^2 f''(x_1 v_1 + \cdots + x_N v_N - \omega t) \\ &= c^2 \|v\|^2 f''(x_1 v_1 + \cdots + x_N v_N - \omega t),\end{aligned}$$

so that

$$\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}(x_1, \ldots, x_N, t) = \|v\|^2 f''(x_1 v_1 + \cdots + x_N v_N - \omega t) = (\Delta F)(x_1, \ldots, x_N, t).$$

This yields the claim.

7. For $N \geq 2$, set

$$\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : \|x\| = 1\}.$$

Show that $\mathbb{S}^{N-1}$ is path connected. (*Hint*: Use Midterm Problem 3(b) and induction on $N$.)

*Solution*: Suppose that $N = 2$. As $[0, 2\pi] \subset \mathbb{R}$ is convex and therefore path connected, and since

$$f : [0, 2\pi] \to \mathbb{R}^2, \quad \theta \mapsto (\cos\theta, \sin\theta)$$

is continuous, it follows from Midterm Problem 3(b), that $f([0, 2\pi]) = \mathbb{S}^1$ is path connected.

Suppose that $N \geq 2$ is such that $S^{N-1}$ is path connected. Define

$$g : [0, 2\pi] \times \mathbb{S}^{N-1} \to \mathbb{R}^{N+1}, \quad (\theta, x) \mapsto ((\cos\theta)x, \sin\theta).$$

By the induction hypothesis, $\mathbb{S}^{N-1}$ is path connected, as is $[0, 2\pi] \times \mathbb{S}^{N-1}$ by Midterm Problem 3(a). Since $g$ is continuous, this means that $g([0, 2\pi] \times \mathbb{S}^{N-1})$ is also path connected by Midterm Problem 3(b). We claim that $g([0, 2\pi] \times \mathbb{S}^{N-1}) = \mathbb{S}^N$.

Let $(\theta, x) \in [0, 2\pi] \times \mathbb{S}^{N-1}$, and note that

$$\|g(\theta, x)\|^2 = (\cos\theta)^2 \|x\|^2 + (\sin\theta)^2 = (\cos\theta)^2 + (\sin\theta)^2 = 1.$$

It follows that $g([0, 2\pi] \times \mathbb{S}^{N-1}) \subset \mathbb{S}^N$.

For the converse inclusion, let $y = (y_1, \ldots, y_N, y_{N+1}) \in \mathbb{S}^N$; set $y' := (y_1, \ldots, y_N)$.

*Case* 1: $y' = 0$. As $\|y\| = 1$, this means that $y_{N+1}^2 = 1$, i.e., $y_{N+1} = \pm 1$. Choose $\theta \in [0, 2\pi]$ such that $\sin\theta = y_{N+1}$; it follows that $\cos\theta = 0$. Hence, $y = g(\theta, x)$ holds for any choice of $x \in \mathbb{S}^{N-1}$.

*Case* 2: $y' \neq 0$. Set $x := \frac{y'}{\|y'\|}$, so that $x \in \mathbb{S}^{N-1}$. As $\|y'\|^2 + y_{N+1}^2 = \|y\|^2 = 1$, i.e., $(\|y'\|, y_{N+1}) \in \mathbb{S}^1$, there is $\theta \in [0, 2\pi]$ such that $\cos\theta = \|y'\|$ and $\sin\theta = y_{N+1}$. With these choices of $x$ and $\theta$, it follows that

$$g(\theta, x) = ((\cos\theta)x, \sin\theta) = \left(\|y'\|\frac{y'}{\|y'\|}, y_{N+1}\right) = (y', y_{N+1}) = y.$$

All in all $\mathbb{S}^N$ is path connected.

8. Let $r > 0$, and let $P, Q, R \colon \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$P(x,y,z) := x(\cos y)^2 + \arctan(yz),$$
$$Q(x,y,z) := y + e^z, \qquad \text{and} \qquad R(x,y,z) := z\sin^2 y$$

for $(x,y,z) \in \mathbb{R}$. Evaluate

$$\int_S P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$$

where $S$ is the sphere with radius $r$ centered at $(0,0,0)$, with the normal vector pointing outward.

*Solution*: Let $V = B_r[(0,0,0)]$, so that $S = \partial V$. With $f := (P, Q, R)$, Gauß' Theorem asserts that

$$\int_S P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy = \int_V \operatorname{div} f.$$

As

$$\operatorname{div} f = \frac{\partial}{\partial x}(x\cos^2 y + \arctan(yz)) + \frac{\partial}{\partial y}(y + e^z) + \frac{\partial}{\partial z} z\sin^2 y = \cos^2 y + 1 + \sin^2 y = 2,$$

this means that

$$\int_S f \cdot n\,d\sigma = 2\,\mu(V) = \frac{8}{3}r^3\pi.$$

9. Let

$$\mathbb{F} := \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

be equipped with addition and multiplication of matrices. Show that $\mathbb{F}$ is a field. (*Hint*: Many properties of a field follow immediately from corresponding properties of addition and multiplication of matrices.)

*Solution*: It is clear that $\mathbb{F}$ is closed under $+$. To see closedness under $\cdot$, observe that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -ad - bc \\ bc + ad & -bd + ac \end{bmatrix} \in \mathbb{F}.$$

Commutativity for $+$ is clear, and since

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \cdot \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} ca - db & -cb - da \\ da + cb & -db + ca \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

commutativity holds for $\cdot$ as well.

Associativity, distributivity, the existence of neutral elements, as well as the existence of an inverse for $+$ are clear from the corresponding properties of matrix addition and multiplication.

Let

$$A := \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \neq 0,$$

so that $a^2 + b^2 \neq 0$. Let

$$B := \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}.$$

It follows that

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

All in all, $\mathbb{F}$ is a field.

10. Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, and let $f : U \to \mathbb{R}^M$ be continuously partially differentiable.

   (a) Let $x \in U$, and let $\xi \in \mathbb{R}^N$ be such that $\{x + t\xi : t \in [0,1]\} \subset U$. Show that

   $$f(x+\xi) - f(x) = \int_0^1 J_f(x + t\xi)\xi \, dt.$$

   (b) Suppose that $U$ is convex, and that $\{|||J_f(x)||| : x \in U\}$ is bounded. Show that $f$ is Lipschitz continuous.

*Solution*:

   (a) We may suppose without loss of generality that $M = 1$. Define

   $$g\colon [0,1] \to \mathbb{R}, \quad t \mapsto f(x + t\xi).$$

   Then $g$ is continuously differentiable, and the chain rule yields

   $$g'(t) = J_f(x + t\xi)\xi$$

   for $t \in [0,1]$. From the Fundamental Theorem of Calculus, we obtain

   $$f(x+\xi) - f(x) = g(1) - g(0) = \int_0^1 g'(t)\, dt = \int_0^1 J_f(x+t\xi)\xi \, dt.$$

(b) Recall that, for any $x \in U$, the inequality

$$\|J_f(x)\xi\| \leq |||J_f(x)|||\|\xi\|$$

holds for all $\xi \in \mathbb{R}^N$.

Let $x, y \in U$. Since $U$ is convex, we have $\{x + t(y-x) : t \in [0,1]\} \subset U$. By the previous problem, we have

$$\begin{aligned} \|f(x) - f(y)\| &= \|f(y) - f(x)\| \\ &= \left\| \int_0^1 J_f(x + t(y-x))(y-x)\, dt \right\| \\ &\leq \int_0^1 \|J_f(x + t(y-x))(y-x)\|\, dt \\ &\leq \int_0^1 |||J_f(x + t(y-x))|||\|y - x\|\, dt. \end{aligned}$$

Let $C := \sup\{|||J_f(x) : ||: x \in U\}$. Then we obtain

$$\|f(x) - f(y)\| \leq \int_0^1 C\|x - y\|\, dt = C\|x - y\|.$$

This proves that $f$ is Lipschitz continuous.

11. Let $S \subset \mathbb{R}^N$. Show that $\partial S$ is closed, conclude that $\partial(\partial S) \subset \partial S$, and give an example of a set $S$ such that $\partial S \not\subset \partial(\partial S)$.

*Solution*: We will show that $\mathbb{R}^N \setminus \partial S$ is open. Let $x \in \mathbb{R}^N \setminus \partial S$. Then there is $\epsilon > 0$ such that $B_\epsilon(x) \cap S = \varnothing$ or $B_\epsilon(x) \cap S^c \neq \varnothing$.

*Case* 1: $B_\epsilon(x) \cap S = \varnothing$.

Let $y \in B_\epsilon(x)$. As $B_\epsilon(x)$ is open, there is $\delta > 0$ such that $B_\delta(y) \subset B_\epsilon(x) \subset S^c$. It follows that $y \notin \partial S$. As $y \in B_\epsilon(x)$ was arbitrary, this means that $B_\epsilon(x) \subset \mathbb{R}^N \setminus \partial S$.

*Case* 2: $B_\epsilon(x) \cap S^c = \varnothing$.

This is dealt with like Case 1, with $S$ replaced by $S^c$.

All in all, $\partial S$ is closed, so that $\partial(\partial S) \subset \partial S$.

On the other hand, $\partial\mathbb{Q} = \mathbb{R}$ whereas $\partial(\partial\mathbb{Q}) = \partial\mathbb{R} = \varnothing$.

12. Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, and let $f : U \to \mathbb{R}$ be partially differentiable such that $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}$ are bounded. Show that $f$ is continuous.

*Solution*: Let $x \in U$, and choose $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Let $\xi \in \mathbb{R}^N$ with $\|\xi\| < \epsilon$, so that $x + \xi \in B_\epsilon(x)$. For $j = 1, \ldots, N$, let

$$x^{(j)} := x + \sum_{\nu=1}^{j-1} \xi_j e_j,$$

so that $x^{(0)} = x$ and $x^{(N)} = x + \xi$. For $j = 1, \ldots, N$, define

$$g^{(j)} \colon [0,1] \to \mathbb{R}, \quad t \mapsto f\left(x^{(j-1)} + t\xi_j e_j\right),$$

so that $g^{(j)}(0) = f\left(x^{(j-1)}\right)$ and $g^{(j)}(1) = f\left(x^{(j)}\right)$. By the Mean Value Theorem, there is $\theta_j \in (0,1)$ such that

$$f\left(x^{(j)}n\right) - f\left(x^{(j-1)}\right) = g^{(j)}(1) - g^{(j)}(0) = \frac{dg^{(j)}}{dt}(\theta_j) = \frac{\partial f}{\partial x_j}\left(x^{(j-1)} + \theta_j \xi_j e_j\right)\xi_j.$$

Let $C \geq 0$ be such that

$$\left|\frac{\partial f}{\partial x_j}(x)\right| \leq C$$

for $j = 1, \ldots, N$ and $x \in U$. We then obtain

$$\begin{aligned}
|f(x+\xi) - f(x)| &= \left|\sum_{j=1}^{N} f\left(x^{(j)}\right) - f\left(x^{(j-1)}\right)\right| \\
&\leq \sum_{j=1}^{N} \left|f\left(x^{(j)}\right) - f\left(x^{(j-1)}\right)\right| \\
&= \sum_{j=1}^{N} \left|\frac{\partial f}{\partial x_j}\left(x^{(j-1)} + \theta_j \xi_j e_j\right)\xi_j\right| \\
&\leq C \sum_{j=1}^{N} |\xi_j|.
\end{aligned}$$

It is clear that the right hand side of this inequality tends to zero as $\xi \to 0$. Hence, the left hand side does the same, i.e. $f$ is continuous at $x$.