MATH 371
Solutions to Homework Assignment #3

Due date/time: Feb 14, 2022/23:59

a. Write your solutions on paper or pads, and try to keep solutions for different questions on separate pages.

b. Upload scans/photos/pdfs of your solutions to **Assign2** before the due date and time. Make sure to upload solutions to the right slot for each question.

c. Submissions after the deadline will not be graded and will result in a 0 mark.

1. (70 points) Exercises 2.4.13, 2.4.14, 2.4.15, 2.4.16. (Please submit your solutions in separate files for different questions)

Question 2.4.13. Consider the following simple competition model:

$$A_{n+1} = \mu_1 A_n - \mu_3 A_n B_n = f(A_n, B_n),$$
$$B_{n+1} = \mu_2 B_n - \mu_4 A_n B_n = g(A_n, B_n),$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are positive constants.

(a) Find all fixed points.

Solution: Solving systems

$$A_n = \mu_1 A_n - \mu_3 A_n B_n,$$
$$B_n = \mu_2 B_n - \mu_4 A_n B_n,$$

we obtain four fixed points:

$$P_1 = (0, 0), \quad P_2 = (\frac{1}{\mu_1}, 0), \quad , P_3 = (0, \frac{1}{\mu_2}), \quad P_4 = (\frac{\mu_2 - 1}{\mu_4}, \frac{\mu_1 - 1}{\mu_3}).$$

We note that $P_1, P_2, P_3 \in \mathbb{R}^2_+$ for all $\mu_1, \mu_2 > 0$, and $P_4 \in \mathbb{R}^2_+$ only if $\mu_1 > 1$ and $\mu_2 > 1$.

(b). Determine the stability of the fixed points for the specific case $\mu_1 = 1.2, \mu_2 = 1.3, \mu_3 = 0.001$**, and** $\mu_4 = 0.002$.

Solution: The Jacobian matrix of (f, g) with $f(A, B) = \mu_1 A - \mu_3 AB$ and $g(A, B) = \mu_2 B - \mu_4 AB$ is

$$J(A, B) = \begin{bmatrix} \mu_1 - \mu_3 B & -\mu_3 A \\ -\mu_4 B & \mu_2 - \mu_4 A \end{bmatrix}.$$

(1) Stability of $P_1 = (0, 0)$.

$$J(0,0) = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}.$$

The eigenvalues at $\lambda_1 = \mu_1 = 1.2 > 1$, $\lambda_2 = \mu_2 = 1.3 > 1$. Therefore, P_1 is unstable.

(2) Stability of $P_2 = (\frac{1}{\mu_1}, 0)$.

$$J(\frac{1}{\mu_1}, 0) = \begin{bmatrix} \mu_1 & -\frac{\mu_3}{\mu_1} \\ 0 & \mu_2 - \frac{\mu_4}{\mu_1} \end{bmatrix}.$$

The eigenvalues at $\lambda_1 = \mu_1 = 1.2 > 1$, $\lambda_2 = \mu_2 - \frac{\mu_4}{\mu_1} = 1.298 > 1$. Therefore, P_2 is unstable.

(3) Stability of $P_3 = (0, \frac{1}{\mu_2})$.

$$J(0, \frac{1}{\mu_2}) = \begin{bmatrix} \mu_1 - \frac{\mu_3}{\mu_2} & 0 \\ -\frac{\mu_4}{\mu_2} & \mu_2 \end{bmatrix}.$$

The eigenvalues at $\lambda_1 = \mu_2 = 1.3 > 1$, $\lambda_2 = \mu_1 - \frac{\mu_3}{\mu_2} = 1.199 > 1$. Therefore, P_3 is unstable.

(4) Stability of $P_3 = (\frac{\mu_2 - 1}{\mu_4}, \frac{\mu_1 - 1}{\mu_3})$.

$$J(\frac{\mu_2 - 1}{\mu_4}, \frac{\mu_1 - 1}{\mu_3}) = \begin{bmatrix} 1 & -\frac{\mu_3}{\mu_4}(\mu_2 - 1) \\ -\frac{\mu_4}{\mu_3}(\mu_1 - 1) & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1.5 \\ -0.4 & 1 \end{bmatrix}$$

Using Matlab, the eigenvalues at $\lambda_1 = 1.77 > 1$, $\lambda_2 = 0.23 < 1$. Therefore, P_3 is also unstable.

Question 2.4.14. Consider the following model for the spread of an infectious disease (such as the flu or the common cold) through a population of size N:

$$I_{n+1} = I_n + kI_n(N - I_n), \quad I_0 = 1,$$

where I_n is the number of infected (and infectious) individuals on day n, and k is a measure of the infectivity and how well the population mixes.

(a) What does the model predict? You may assume that $kN < 2$.

Solution: We can rewrite the model as

$$I_{n+1} = I_n(1 + kN - kI_n) = (1 + kN)I_n(1 - \frac{1}{1 + kN}I_n)$$
$$= rI_n(1 - \frac{I_n}{K}),$$

where $r = 1 + kN < 3$ and $K = (1 + kN)/k$. We observe that this is similar (identical) to the logistic model in Section 2.2. If we assume that $K = 1$, then we expect the same dynamics and bifurcations as shown in Figure 2.12 in the textbook (shown below) when $r < 3$.

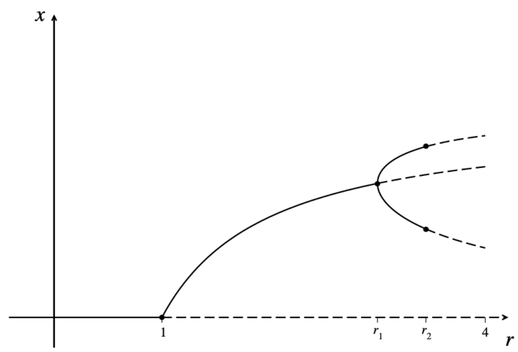


Figure 2.12. *Updated bifurcation diagram for the discrete logistic equation shown earlier in Figure 2.7. Shown are the fixed points, as well as the 2-cycle for values of $r > r_1 = 3$. The 2-cycle is stable up to $r_2 = 1 + \sqrt{6}$, and unstable thereafter.*

(b) Modify the model to incorporate immunity. Explain (justify) your model. What additional assumptions have you made?

Solution: To incorporate immunity into the the model, we assume that an individual recovers from infection in exactly d days and remains immune to re-infection for life. Then there is fraction $\frac{1}{d}$ of I_n recovers per day. This is an additional outflow from the I compartment. The revised model is given by

$$I_{n+1} = I_n + kI_n(N - I_n) - \frac{I_n}{d}, \quad I_0 = 1,$$

where $k, d, N > 0$.

Question 2.4.15. Jury conditions. Let J be the Jacobian matrix, (2.54), corresponding to the general two-dimensional discrete-time system, (2.42)-(2.43).

(a) Show that the characteristic polynomial for J can be written as

$$p(\lambda) = \lambda^2 - \text{tr J}\,\lambda + \det J = 0.$$

Solution: Let λ_1, λ_2 be the eigenvalues of J. Then

$$\operatorname{tr} J = \lambda_1 + \lambda_2, \quad \det J = \lambda_1 \lambda_2.$$

The characteristic polynomial of J can be written as

$$|\lambda I - A| = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = \lambda^2 - \text{tr J}\,\lambda + \det J.$$

(b) Show that necessary and sufficient conditions for both eigenvalues of J to have magnitude less than 1 are the following Jury conditions:

$$|\operatorname{tr} J| < 1 + \det J < 2.$$

Solution: The Jury conditions can be rewritten equivalently as the following three conditions

(1) $\det J < 1$.

(2) $\operatorname{tr} J < 1 + \det J$.

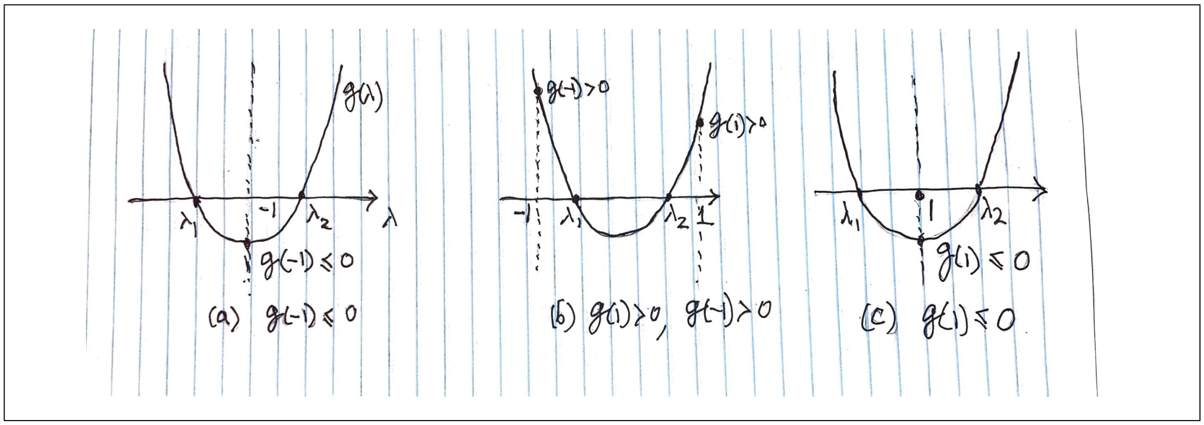
(3) $-\operatorname{tr} J < 1 + \det J$.

We first show that, if λ_1 and λ_1 are complex, then $|\lambda_1| < 1$ and $|\lambda_2| < 1$ is equivalent to condition (1). This is clear from the relation $\det J = \lambda_1\lambda_2 = |\lambda_1|^2 = \lambda_2|^2$.

Next, we assume that λ_1 and λ_2 are real numbers. Then it is clear that Jury condition (1) is mutually exclusive to $|\lambda_1| > 1$ and $\lambda_2| > 1$.

For the remaining cases of λ_1 and λ_2, we note that Jury conditions (2) and (3) are equivalents to $g(1) > 0$ and $g(-1) > 0$, respectively. Also note that the graph of the characteristic polynomial is a parabola, opening up, with zeroes at λ_1 and λ_2. Jury conditions (2) and (3) can be shown based on the relative positions of λ_1, λ_2 and $\lambda = \pm 1$. The following Figure shows the remaining possible relative positions among λ_1, λ_2 and $\lambda = \pm 1$. We can see that in cases (a) and (c), we will have $g(1) \leq 0$ and $g(-1) \leq 0$ (the equal sign holds when $\lambda_1 = \lambda_2$), contradicting Jury conditions (2) and (3), respectively. In case (b), we have $g(-1) > 0$ and $g(1) > 0$ and both roots satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

This establish the proof.



Exercise 2.4.16: Romeo and Juliet in love/hate-preserving mode. Consider the discrete- time model developed for the relationship between Romeo and Juliet, (2.40)-(2.41), and assume that the amount of love/hate that Romeo and Juliet feel for each other initially is preserved on all subsequent days, that is, $a_R + p_J = 1$ and $a_J + p_R = 1$.

(a) Show that det$(A) = 0$, where the matrix A is defined in (2.70).

Solution: We recall that

$$A = \begin{bmatrix} a_R - 1 & p_R \\ p_J & a_J - 1 \end{bmatrix}$$

Under the assumptions $a_R + p_J = 1$ and $a_J + p_R = 1$, we know that the sum of columns are zero, and thus the two columns are proportional, and thus $\det A = 0$.

(b) Show that the two eigenvalues of the Jacobian matrix are $\lambda_1 = 1$, $\lambda_2 = a_R + a_J - 1$.

Solution: The Jacobian matrix is, as given in (2.71),

$$J = \begin{bmatrix} a_R & p_R \\ p_J & a_J \end{bmatrix}$$

and we see that $A = J - I_{2\times 2}$, where $I_{2\times 2}$ is the 2×2 identity matrix. From $|A| = 0$ we know $|J - I_{2\times 2}| = 0$, and thus $\lambda_1 = 1$ is an eigenvalue of J. Also, $\mathrm{tr} J = a_R + a_J = \lambda_1 + \lambda_2 = 1 + \lambda_2$. This gives $\lambda_2 = a_R + a_J - 1$.

2. (30 points) Computation exercises 8.2.4 - 8.2.7. Please hand in your code with an output (e.g. generated by Word or LaTex) for each question (You may use any software package to do the exercises).

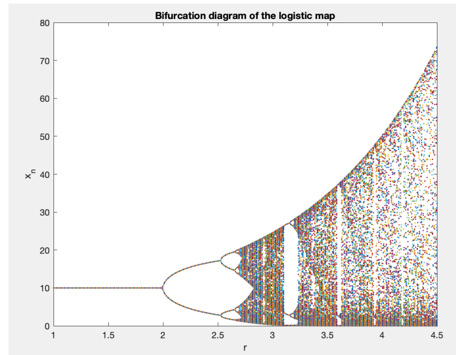
Questions 8.2.4, 8.2.5

Solution: The Matlab codes shown below are better than the ones suggested in the questions. The bifurcation diagram for the Ricker model in (2.24) is shown below.

```
%Plot the bifurcation digram for the Ricker model (2.24) in Chapter 2.

close all;
clear all;

%K=10
Npre = 200; Nplot = 100;
x = zeros(Nplot,1);
for r = 1:0.005:4.5
  x(1) = 0.5;
  for n = 1:Npre
    x(1) = x(1)*exp(r*(1-0.1*x(1)));  %Change the equation to yours
  end
  for n = 1:Nplot-1
    x(n+1) = x(n)*exp(r*(1-0.1*x(n)));  %Change the equation to yours
  end
  plot(r*ones(Nplot,1), x, '.', 'markersize', 2);
  hold on;
end
title('Bifurcation diagram of the logistic map');
xlabel('r');  ylabel('x_n');
set(gca, 'xlim', [1 4.5]);
hold off;
```



Questions 8.2.6, 8.2.7

Solution: We want to show that, for the simplified Ricker model

$$x_{n+1} = ax_n e^{-x_n} = f(x_n), \quad a > 0,$$

where $f(x) = axe^{-x}$., when $a = 8$,

(1) Trajectories converge to a 2-cycle $\{u, v\}$;

(2) Estimate the values of u, v.

(3) Identify u, v as fixed points of the second iterate f^2.

(4) Verify that $\{u, v\}$ is a two cycle by showing that $f(u) = v, f(v) = u$.

Matlab codes used to carry out these operations are included in a single m file shown below.

We solve the system of equations

$$f(u) = v, \quad f(v) = u$$

in Matlab using "vpasolve" and found

$$u = 1.3862943611198906188344642429164, \quad v = 2.7725887222397812376689284858327.$$

We can also solve the equation

$$f^2(x) = x$$

for fixed points of the second iterate f^2, and we obtained the same solutions as the preceding step:

$$v = 1.3862943611198906188344642429164, \quad u = 2.7725887222397812376689284858327.$$

Note that there are two other fixed points of f^2 that are also fixed points of f.

These two points are plotted as the intersections (fixed points) of $y = f^2(x)$ and $y = x$. This finishes steps (2) - (4).

```
%Plotting trajectories of simplified Ricker Model
% x(n+1) = a*x(n)*exp(-x(n)),
% for different values of a and 0<x(0)<1.

close all; %Close all previous plots

x(1) = 0.9;    % initial value

for n=1:50 % n is the iteration variable
x(n+1) = a*x(n)*exp(-x(n));
end

%Plot trajectories
plot(x, 'ob')
title('A trajectory converges to a 2-cycle');
xlabel('n');  ylabel('x_n');

%solving the system
%u = f(v), v = f(u), (u, v) are the two points in the 2-cycle

syms u v;

a = 8;
eqn1 = a*u*exp(-u) == v;
eqn2 = a*v*exp(-v) == u;

[solu, solv] = vpasolve([eqn1, eqn2], [u, v], [1,3]);

solu
solv

```

```
syms w;

eqn = rickerm(rickerm(w)) == w;

solw1 = vpasolve(eqn, w, [2,3]);
solw2 = vpasolve(eqn, w, [1,1.5]);

solw1
solw2

%Plot the graphs of y=f^2(x) and y=x to visualize the intersections

t = 0 : 0.01 : 3.5;
f = @(t) rickerm(t);
y = f(f(t));
plot(t, y);
hold on;
y1 = t;
plot(t, y1);
hold off
```