7 Integration

Integration is a procedure somewhat inverse to differentiation. It is also an analogue for an infinite sum.

Consider a particle moving along a line. Suppose you are sitting "on the particle" and you have a speedometer. Your task is to provide the distance travelled in a certain time interval.

Let $v(t)$ denote the velocity measured at time t. To compute an approximation of the distance travelled, you would sample (ie. measure) the velocity at many time spots $0 = t_0 < t_1 < t_2 < \cdots < t_n < T = t_{n+1}$. An approximation for the distance travelled $D(T)$ would then be

$$D(T) \sim \sum_{i=0}^{n} v(t_{i+1})(t_{i+1} - t_i)$$

where $t_0 = 0$ and $t_{n+1} = T$. The smaller the intervals $[t_i, t_{i+1}]$, the more precise this computation will usually be. Assuming the velocity function is continuous for example, the computation will be exact in the limit as the maximum interval size approaches zero.

Of course, there are problems with that. For example, v may not be constant on each of the intervals $[t_i, t_{i+1}]$ but rather vary considerably. Then our approximation may be really bad. However, if v is close to constant on these intervals, our approximation might be very good. For example, if v is continuous, and we choose our interval sizes small enough, this might be a good assumption (as we shall see).

Now let us turn this around, and suppose the distance travelled at time t, $D(t)$ is a differentiable function. Then $D(0) = 0$ and

$$D(T) = D(T) - D(0) = \sum_{i=0}^{n} D(t_{i+1}) - D(t_i) = \sum_{i=0}^{n} D'(c_i)(t_{i+1} - t_i)$$

where $c_i \in (t_i, t_{i+1})$ is suitably chosen (this uses the MVT). Again, if D' is continuous, say, and the intervals are small, then $D'(c_i) \sim D'(t_{i+1})$, which suggests that $D' = v$ (which we know by definition).

In this chapter we will formalize this kind of thinking. It will turn out, that for "reasonable" functions $v(t)$ (e.g. continuous functions) this procedure does indeed give meaningful results.

7.1 Indefinite integrals

7.1.1 Antiderivatives

We have seen how to compute derivatives of functions. We now turn the question around and ask whether we can go back: given f, is there F such that $F' = f$?

Definition
Let f be a function on an interval I. Then a function $F: I \to \mathbb{R}$ is called an **antiderivative** or an **indefinite integral** of f, if $F' = f$ on I. EOD.

Remark
Convince yourself why it makes no sense to talk about the antidervative at a point $x_0 \in I$. EOR.

Antiderivatives may or may not exist. We typically write $F = \int f$ or $F = \int f dx$ or $F = \int f(x)dx$ (where x is the variable) to indicate that $F' = f$. This notation is somewhat dangerous though, because there are many antiderivatives for a given function f, as long as there is one.

Example

1. $\sin x$ is an antiderivative of $\cos x$ on any interval.
2. $\frac{1}{2}x^2$ is an antiderivative of x on any interval.
3. If $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is a formal power series with radius of convergence $R > 0$, $F(x) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}(x - x_0)^n$ is an antiderviative with the same radius of convergence.

Remark
If F is an antiderivative of f, then so is $F + C$ where $C \in \mathbb{R}$ is any constant. Because of this, many authors insist on writing $\int f(x)dx = F + C$. So, they would write $\int \sin x \, dx = \cos x + C$.
We don't always do that, because the technically as long as C is a given constant, $\cos x + C$ is also just one antiderivative. A technically more precise notation would be $\int \sin x \, dx = \{\cos x + C \mid C \in \mathbb{R}\}$. EOR.

Remark
If the domain of f consists of multiple disconnected intervals, e.g. $(0,1) \cup (2,3)$, then two antiderivatives of f need not differ by a constant (they could differ by different constants on each sub-interval). However, if F, G are two antiderivatives of f on an interval I, then $F - G$ *is* constant because $(F - G)' = 0$ on I (and it is enough that $F' = G'$ on I°). EOR.

We often write $\int f$ or $\int f dx$ also for the **set** of all antiderivatives of f.

7.1.2 Integration rules for indefinite integrals

Let $f, g: I \to \mathbb{R}$ be functions defined on an interval I and suppose they each have antiderivatives, F, G respectively.

Linearity
For any $a, b \in \mathbb{R}$ we have $\int (af + bg) = a\int f + b\int g$.

Note, if the antiderivatives F of f and G of g are given, this does not mean, that a *specific* antiderivative of $af + bg$ is of the form $aF + bG$. It does mean that any antiderivative of $af + bg$ is of the form $aF + bG + C$ for some constant C. Or conversely, it means that any $aF + bG$ is an antiderivative of $af + bg$.

Integration by parts
Let $F' = f$ and $G' = g$. Then $\int Fg dx = FG - \int fG dx$.

One should think of this as an inverse to the product rule. It is easily proved by computing the derivative of FG.

Substitution rule
Suppose h is a function with antiderivative H, defined on an interval J such that $G(I) \subseteq J$. Then $\int (h \circ G) g dx = H \circ G$. Here $g = G'$.

Examples

1. $\int ax + b dx = \frac{a}{2}x^2 + bx + C$

2. $\int (ax+b)db = axb + \frac{1}{2}b^2 + C$
3. $\int a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 x dx = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + \frac{a_1}{2}x^2 + a_0 x$
4. $\int \sin(x)\cos(x) = \sin(x)^2 - \int \cos(\mathrm{x})\sin(x) = \frac{1}{2}\sin^2(x)$. $(F = \sin(x), g = \cos(x))$
5. $\int \sin(x)\cos(x) = \frac{1}{2}\sin^2(x)$ $(h = x, G = \sin(x))$
6. $\int \log x\, dx = \int 1 \cdot \log x\, dx = x\log x - \int x \cdot \frac{1}{x} dx = x\log x - x$
7. $\int xe^x = xe^x - \int e^x = xe^x - e^x$
8. $\int x^2 e^x = x^2 e^x - \int 2xe^x dx = x^2 e^x - 2xe^x + 2e^x$
9. $\int \frac{f'}{f} dx = \int \frac{1}{y} \circ f(x) f'(x) dx = \log|f|$
10. $\int f(ax+b)dx = \frac{1}{a}F(ax+b)$ where $F = \int f(x)dx$.
11. $\int (\arctan x)dx = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2} = x\arctan x - \frac{1}{2}\log(1+x^2)$
12. $\int \cos^2(x)\, dx = \sin(x)\cos(x) + \int \sin^2(x)\, dx$ by integration by parts. On the other hand $\int \cos^2(x)\, dx = \int (1-\sin^2(x))dx = x - \int \sin^2(x)\, dx$.
    Therefore $\sin(x)\cos(x) + \int \sin^2(x)\, dx = x - \int \sin^2(x)\, dx$, or
    $$\int \sin^2(x)\, dx = \frac{1}{2}(x - \sin(x)\cos(x))$$
    Then $\int \cos^2(x)\, dx = x - \int \sin^2(x)\, dx = \frac{1}{2}(\sin(x)\cos(x) + x)$.
    Note from the usual trigonometric identities we have $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, so
    $$\int \cos^2(x)\, dx = \frac{1}{4}(\sin(2x) + 2x)$$

The substitution rule above does not give an antiderivative of f but rather of the expression $(f \circ G)g'$. The antiderivative is then $F \circ G$. To turn it around we must get $F = (F \circ G) \circ G^{-1}$. This is sometimes possible.

Proposition (Substitution Rule)
Suppose the following holds:
1. f is defined on some interval I.
2. g is defined on an interval J and $g(I) = J$.
3. $g'(x) \neq 0$ for all $x \in J$.
4. $(f \circ g)g'$ has an anti-derivative F_0 on J.

Then g^{-1} is defined on I, and $F(x) \coloneqq F_0 \circ g^{-1}(x)$ is an antiderivative for f:
$$\int f(x)dx = \left[\int f(g(y))g'(y)dy\right]_{y=g^{-1}(x)}$$

Proof. If $g'(x) \neq 0$ for all $x \in J^\circ$, the Mean Value Theorem guarantees that g is injective. It is therefore strictly increasing or strictly decreasing. Thus, g is invertible as a function $J \to I$. Then $h = g^{-1}: I \to J$ is also bijective. Moreover, h is differentiable and $h'(x) = \frac{1}{g(h(x))}$.

Now consider $F_0 \circ h$: Then $\frac{d}{dx}(F_0 \circ h) = F_0'(h(x))h'(x) = f \circ g(h(x))g'(h(x))h'(x) = f(x)$. QED.

Example

$\int \frac{1}{\sin x} dx$ on $(-\pi, 0)$ or $(0, \pi)$.
Put $x = 2\arctan t$ on $(-\infty, 0)$ or $(0, \infty)$, respectively. (You should think $x = g(t)$.)
Then $g'(t) = \frac{2}{1+t^2}$.
Then $\arctan t = \frac{x}{2}$, so $t = h(x) = \tan\left(\frac{x}{2}\right)$.
Recall that $\sin(2x) = \sin(x)\cos(x) + \cos(x)\sin(x) = 2\sin(x)\cos(x)$ and hence

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\left(\sin\left(\frac{x}{2}\right)\right)^2 + \left(\cos\left(\frac{x}{2}\right)\right)^2} = \frac{2\left(\tan\frac{x}{2}\right)}{1 + \left(\tan\frac{x}{2}\right)^2} = \frac{2t}{1+t^2}$$

$(\sin(g(t)) = \frac{2t}{1+t^2})$.
Let $f = \frac{1}{\sin x}$, then $f \circ g(t) = \frac{1+t^2}{2t}$. $g'(t) = \frac{2}{1+t^2}$, so $(f \circ g)g' = \frac{2t}{2t}$, and $F_0(t) = \log|t|$.
Then $F_0 \circ h(x) = \log\left|\tan\left(\frac{x}{2}\right)\right|$, and we conclude

$$\int \frac{1}{\sin x} dx = \log\left|\tan\left(\frac{x}{2}\right)\right|$$

EOE.

7.2 Definite Riemann integrals

As mentioned in the introduction to the chapter, it is a natural (and important) question whether given a differentiable function F, say, we can recover the value $F(b)$ at b, provided we know $F(a)$, and the *rate of change* (ie. $F'(x)$) on the interval $[a, b]$. It turns out, the answer is yes in many cases (virtually all cases of interest), and the idea is the idea given in the introduction:

If F changes by $F'(x)\Delta x$ on a small interval $[x, x + \Delta x]$, then we should have

$F(b) = F(a) + \sum_{i=0}^{n} F'(c_i)\Delta x$ where $c_0, c_1, \dots, c_n$ are spaced out in $[a, b]$ in intervals of size Δx. This is absolutely not yet rigorous.

7.2.1 Partitions

In the following we will talk a lot about **partitions** of closed intervals. Let $I = [a, b]$ be a fixed closed interval.

Definition
A **partition** P of $I = [a, b]$ is a finite ordered sequence $a < x_1 < x_2 < \cdots < x_n < b$ of ordered elements of I. So P is just a finite ordered subset of I. P is allowed to be empty. We call n the **size** of P, and denote it by $|P|$ or $\#P$. We always put $x_0 = a$, and $x_{n+1} = b$, and then obtain $n + 1$ associated subintervals $[x_i, x_{i+1}]$ $(i = 0,1, \dots, n)$. The maximum interval size $\max\{\, x_{i+1} - x_i \mid i = 0,1,2, \dots, n\}$ is called the **mesh size** of P, and denoted $m(P)$. We write $\Pi(I)$ or $\Pi(a, b)$ for the set of all partitions of I.

If P is a partition, its elements are denoted $x_1 < x_2$ … unless otherwise noted. Similarly, if P_n is a sequence of partitions, then its elements are denoted $x_{n,1} < x_{n,2} < \cdots$.

Notice that the partitions of I can be *partially ordered* by putting $P \leq Q$ if $P \subseteq Q$. We call such a Q a **refinement** of P.

If P, Q are any two paritions of I, then $P \cup Q$ is a **common refinement**: $P, Q \leq P \cup Q$.

7.2.2 Riemann sums

Definition

Let $I = [a, b]$ and let $P = x_1 < x_2 < \cdots < x_n \in \Pi(I)$ be any partition of size $n \geq 0$. A **tag vector** for P is an element $\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$ such that

$$a = x_0 \leq y_0 \leq x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_n \leq y_n \leq x_{n+1} = b$$

Or, equivalently, $y_i \in [x_i, x_{i+1}]$. We write $T(P)$ for the set of all tag-vectors for P. EOD.

For $f: I \to \mathbb{R}$, $P \in \Pi(I)$, and $\boldsymbol{y} \in T(P)$, we define the corresponding **Riemann sum** as the sum

$$S(P, \boldsymbol{y}, f) \coloneqq \sum_{i=0}^{|P|} f(y_i)(x_{i+1} - x_i)$$

A **Riemann sequence** for f is a sequence of the form $S(P_n, \boldsymbol{y}_n, f)$ where $\lim_{\mathrm{n}\to\infty} m(P_n) = 0$. EOD.

Note that the function f in the definition of a Riemann sequence does not vary with n. Technically, a Riemann sequence is more than its value. The partitions, tag-vectors, and function are all part of the definition.

7.2.3 The Riemann integral

Definition

Let $f: [a, b] \to \mathbb{R}$ be a function (where $a \leq b \in \mathbb{R}$). We say f is **Riemann integrable** or **R-integrable** or **integrable**, if every Riemann sequence for f converges. The set of integrable functions on $[a, b]$ is denoted $\mathcal{R}[a, b]$. EOD.

Lemma

Suppose f is integrable on $[a, b]$. Then all Riemann sequences have the same limit. EOL.

Proof. Let $S(P_n, \boldsymbol{y}_n, f)$ and $S(Q_n, \boldsymbol{z}_n, f)$ be two Riemann sequences. Then

$$S(P_1, \boldsymbol{y}_1, f), S(Q_1, \boldsymbol{z}_1, f), S(P_2, \boldsymbol{y}_2, f), S(Q_2, \boldsymbol{z}_2, f), \dots$$

is again a Riemann sequence.

Formally, we form a new Riemann sequence $S(R_n, \boldsymbol{x}_n, f)$ where

$$(R_n, \boldsymbol{x}_n) = \left\{\begin{matrix} (P_k, \boldsymbol{y}_k) & n = 2k - 1 \\ (Q_k, \boldsymbol{z}_k) & n = 2k \end{matrix}\right\}$$

This is again a Riemman sequence and hence convergent. Since the original ones are subsequences, they have the same limit. QED.

Definition

Let f be integrable on $[a, b]$. The **Riemann integral** or **R-integral** or **integral** of f on $[a, b]$ is defined as the common limit of its Riemann sequences. It is usually denoted as

$\int_a^b f$ or $\int_a^b f(x)dx$ or $\int_a^b f dx$. EOD.

Example

Consider $f = \chi_{\mathbb{Q}}$,the characteristic function of $\mathbb{Q}$. Then f is not integrable on any bounded interval $[a, b]$ with $a < b$. Recall that $f(x) = 0$ if $x \notin \mathbb{Q}$ and $f(x) = 1$ if $x \in \mathbb{Q}$.

Consider any Riemann sequence $S(P_n, \boldsymbol{y}_n, f)$ where all the tag points in $\boldsymbol{y}_n$ are rational. Then $S(P_n, \boldsymbol{y}_n, f) = b - a$, and the sequence converges.
If on the other hand all tag points are irrational, then $S(P_n, \boldsymbol{y}_n, f) = 0$ and again the sequence converges. As the limits are not equal, it follows that f cannot be integrable. EOE.

7.2.4 The Fundamental Theorem of Calculus (Part I)

As mentioned in the introduction, if F is a differentiable function (e.g. D above), then it is natural to ask whether we can recover F from $F' = f$. A partial answer to this question is given by the following theorem:

Theorem (First Fundamental Theorem of Calculus, or FTC 1)
Let $F: [a, b] \to \mathbb{R}$ be a continuous function, differentiable on at least (a, b) and suppose $f: [a, b] \to \mathbb{R}$ is a function such that:

1. f is integrable or continuous.
2. f agrees with F' on at least (a, b).

Then $F(b) - F(a) = \int_a^b f(x)dx$, or equivalently

$$F(b) = F(a) + \int_a^b f(x)dx$$

EOT.

Note that the theorem asserts that if F' is continuous then it is integrable. It does not say a priori that every continuous function is integrable: it only asserts that for those that have an indefinite integral.

Proof. Let $P_n = x_{n0} = a < x_{n1} < x_{n2} < \cdots < x_{n|P_n|} < x_{n|P_n|+1} = b$ be a sequence of partitions such that $m(P_n) \to 0$.

Then

$$F(b) - F(a) = \sum_{i=0}^{|P_n|} \left(F(x_{n,i+1}) - F(x_{ni})\right) = \sum_{i=0}^{|P_n|} F'(c_{ni})(x_{n,i+1} - x_{ni})$$

where $c_{ni} \in (x_{ni}, x_{n,i+1})$ is suitably chosen (this uses the MVT).

Put $\boldsymbol{y}_n = (c_{n0}, c_{n1}, \dots, c_{n,|P_n|}) \in \mathbb{R}^{|P_n|+1}$. Then $\boldsymbol{y}_n$ is a tag vector for P, and the right hand side is equal to $S(P_n, \boldsymbol{y}_n, f)$ (since $f = F'$). This is a Riemann sequence, and since $F(b) - F(a) = S(P_n, \boldsymbol{y}_n, f)$ is a constant sequence we get

$$F(b) - F(a) = \lim_{n\to\infty} S(P_n, \boldsymbol{y}_n, f)$$

If f is integrable, the right hand side is equal to $\int_a^b f(x)dx$, which finishes the proof in this case.

If f is continuous we must show it is integrable. Let $\boldsymbol{z}_n = (z_{n0}, z_{n1}, \dots, z_{n,|P_n|})$ be any tag vector for P_n. Note that as $[a, b]$ is closed and bounded, f is *uniformly continuous* (see below). That is, for any $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in [a, b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

Now $|S(P_n, \boldsymbol{y}_n, f) - S(P_n, \boldsymbol{z}_n, f)| \leq \sum_{i=0}^{|P_n|} |f(c_{ni}) - f(z_{ni})|(x_{n,i+1} - x_{ni})$

$$\leq \max_{0\leq i\leq |P_n|} |f(c_{ni}) - f(z_{ni})|(b-a)$$

Let $\varepsilon > 0$ and let $\delta > 0$ be such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x-y| < \delta$. Then there is n_0 such that for all $n > n_0$, $m(P_n) < \delta$. But then

$$|S(P_n, \boldsymbol{y}_n, f) - S(P_n, \boldsymbol{z}_n, f)| \leq \max_{0\leq i\leq |P_n|} |f(c_{ni}) - f(z_{ni})|(b-a) \leq \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$$

It follows that $S(P_n, \boldsymbol{y}_n, f) - S(P_n, \boldsymbol{z}_n, f)$ is a zero sequence, and therefore

$$\lim_{n\to\infty} S(P_n, \boldsymbol{z}_n, f) = \lim_{n\to\infty} S(P_n, \boldsymbol{y}_n, f) = F(b) - F(a)$$

Since we started with an arbitrary sequence of partitions P_n (for which $m(P_n) \to 0$), this shows that for any Riemman sequence for f, we get one (and in fact the same) limit, namely $F(b) - F(a)$. This shows that f is integrable. QED.

Note again, that in this proof, in the case of continuous f, we needed to know hat *a priori* for every sequence P_n with $m(P_n) \to 0$ there is at least one associated sequence of tag-vectors such that the associated Riemann sequence converges. This worked because we knew that f had an anti-derivative.

To complete the proof of the FTC 1, we recall the definition of uniform continuity.

Definition
Let I be an interval. A function f on I is called **uniformly continuous** if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in I$ with $|x-y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. EOD.

Uniformly continuous functions are obviously continuous. The converse is not always true. But it is true if the interval is closed and bounded, ie. of the form $I = [a, b]$ where $a \leq b \in \mathbb{R}$.

Example/Exercise
Recall that a function f defined on an interval I is called *Lipschitz continuous*, if there is a constant L such that for all $x, y \in I$, $|f(x) - f(y)| \leq L|x-y|$.
Show that a Lipschitz continuous function is uniformly continuous. EOE.

Example
Consider $I = \mathbb{R}$ and $f(x) = e^x$. Then f is not uniformly continuous.
Let $\varepsilon > 0$ and suppose there is $\delta > 0$ such that for all x, y with $|x-y| < \delta$, $|e^x - e^y| < \varepsilon$.
Suppose $x < y$, and $y = x + h$ with $h < \delta$. Then $e^y = e^h e^x$, and $e^y - e^x = (e^h - 1)e^x$. Keeping $h > 0$ constant, but increasing x (and $y = x + h$), we can achieve $e^y - e^x > \varepsilon$ (since e^x is unbounded and $e^h - 1 > 0$). EOE.

Exercise
Let I be a bounded interval and suppose f is uniformly continuous on I. Show that f is bounded on I. It follows that e.g. $\frac{1}{x}$ on $(0,1)$ is not uniformly continuous.
(*Hint*: Let $\varepsilon = 1$ and consider the corresponding $\delta > 0$. Let $a \leq b$ be the boundary points of I. Then finitely many intervals of the form $(x_i - \delta, x_i + \delta)$ cover I.) EOE.

Exercise
Let $a > 0$. Show that $\ln x$ is uniformly continuous on $[a, \infty)$. EOE.

Proposition
Let $I = [a, b]$ be a *closed* bounded interval. If f is a continuous function on I, then it is uniform continuous. EOP.

Proof. Let $\varepsilon > 0$. As f is continuous for every $x_0 \in I$, there exists $\delta(x_0)$ such that for all $x \in I$ with $|x - x_0| < \delta(x_0)$ we have $|f(x) - f(x_0)| < \varepsilon$. We must show that there exists a δ that works for all x_0.

Suppose not. Then for $\delta_n = \frac{1}{n}$ there is a pair x_n, y_n such that $|x_n - y_n| < \delta_n$ but $|f(x_n) - f(y_n)| > \varepsilon$. Note x_n, y_n are each a bounded sequence so contain convergent subsequences. We may therefore replace x_n with a convergent subsequence. Then y_n is automatically also convergent, because we still have $|x_n - y_n| < \frac{1}{n}$. As $[a, b]$ is closed, it must contain the common limit x_0 of x_n, y_n.

Then $0 = |f(x_0) - f(x_0)| = \lim_{n\to\infty} |f(x_n) - f(y_n)| \geq \varepsilon > 0$, which is obviously a contradiction. QED.

Returning to the FTC1:

Warning
The theorem does not make any statement about cases where F' is *not* integrable. There are functions with non-integrable derivative.
Conversely, even if a function is integrable it need not have an antidervative.
This is an unlimited source of mistakes. EOW.

Example

1. Whenever F has a continuous derivative, the FTC1 applies:
2. $\log x = \log 1 + \int_1^x \frac{1}{t} dt = \int_1^x \frac{1}{t} dt$ for all $x > 1$. (This also holds for $x \leq 1$, but we haven't discussed this yet.)
3. $e^x = 1 + \int_0^x e^t dt$ for $x > 0$.
4. $\sin x = \int_0^x \cos t \, dt$ for $x > 0$.
5. Take any *continuous* function f on $[a, b]$ which has an antiderivative F there. Then $F(b) = F(a) + \int_a^b f(x) \, dx$

Notation
If a function f is integrable on $[a, b]$ and has an antiderivative F there, it is common to write $[F]_a^b$ for $F(b) - F(a)$.
Thus, for example

$$\int_a^b \cos x \, dx = [\sin x]_a^b$$

EON.

7.2.5 Linearity of integration

For now, let us record that if f, g are integrable, then so are all linear combinations:

Lemma (Linearity of integration)
Suppose $f, g: [a, b] \to \mathbb{R}$ are integrable. Then $\alpha f + \beta g$ is integrable for all $\alpha, \beta \in \mathbb{R}$ and

$$\int_a^b \alpha f(x) + \beta g(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

EOL.

Proof. This immediately follows from the fact that for all Riemann sequences here we have $S(P_n, \boldsymbol{y}_n, cf + dg) = cS(P_n, \boldsymbol{y}_n, f) + dS(P_n, \boldsymbol{y}_n, g)$, together with the linearity of limits of sequences. QED.

If you know linear algebra, the lemma shows that $\mathcal{R}[a, b]$ is a linear subspace of $\mathcal{F}[a, b]$.

7.2.6 Integrable functions are bounded

For an interval I we denote the set of bounded functions on I by $\mathcal{B}I$ or $\mathcal{B}(I)$. So $\mathcal{B}[a, b]$ is the set of bounded functions on $[a, b]$.

Theorem

$$\mathcal{R}[a, b] \subseteq \mathcal{B}[a, b]$$

In other words, every integrable function on $[a, b]$ is bounded. EOT.

Proof. Suppose f is not bounded. Then $\sup f = \infty$ or $\inf f = -\infty$. We will assume the first and show that f is not integrable. The second case then follows by applying the first to $-f$ (which is also not integrable by the linearity lemma for integration 7.2.3).

We may assume there is a sequence $x_n \in [a, b]$ such that $\lim_{n\to\infty} f(x_n) = \infty$. By Bolzano-Weierstrass such a bounded sequence has a convergent subsequence x_{n_k} with limit $x_0 \in [a, b]$. Then still $\lim_{k\to\infty} f(x_{n_k}) = \infty$, so we may replace the original sequence with this subsequence and assume $x_n \to x_0 \in [a, b]$.

After again restricting to a subsequence, we may assume that $f(x_n) > n^2$. (Indeed, to construct the subsequence x_{n_k} simply recursively put $n_k = \min\{m \mid m > n_{k-1}, f(x_m) > k^2\}$).

Similarly, again choosing a subsequence we may also assume that $|x_n - x_0| < \frac{b-a}{n^3}$. (Note that for any subsequence x_{n_k} of x_n we have $f(x_{n_k}) > n_k^2 \geq k^2$.)

Let P_n be a partition such that $m(P_n) \to 0$ and such that for each n (large enough), P_n contains an interval of size $\frac{b-a}{n}$, say $[x_{n,i(n)}, x_{n,i(n)+1}]$, which contains x_0. We may also assume that if $x_0 \neq a, b$, then x_0 is the midpoint of the interval.

Let $S(P_n, \boldsymbol{y}_n, f)$ be a Riemann sequence. We may assume that $y_{n,i(n)} = x_0$.

Let $\boldsymbol{y}'_n \in \mathbb{R}^{|P_n|+1}$ be obtained from $\boldsymbol{y}_n$ by replacing $y_{n,i(n)} = x_0$ with x_n which is in the same interval (if $n \geq 2$). This uses that $|x_n - x_0| < \frac{b-a}{n^3}$ and that the interval length is $\frac{b-a}{n}$.

Then $S(P_n, \boldsymbol{y}'_n, f) - S(P_n, \boldsymbol{y}_n, f) = \big(f(x_n) - f(x_0)\big)\big(x_{n,i(n)+1} - x_{n,i(n)}\big) = \frac{f(x_n)-f(x_0)}{n}(b-a) > n - \frac{f(x_0)}{n} \to \infty$, contradicting the fact that f is integrable. QED.

Example/Exercise

Let $f(x) = \vdash \begin{cases} x\sqrt{x}\sin\frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$

defined on $[0, \infty)$. Then f is differentiable everywhere:

$f'(x) = \frac{3}{2}\sqrt{x}\sin\left(\frac{1}{x}\right) - \frac{x\sqrt{x}}{x^2}\cos\frac{1}{x} = \frac{3}{2}\sqrt{x}\sin\left(\frac{1}{x}\right) - \frac{1}{\sqrt{x}}\cos\frac{1}{x}$ on $(0, \infty)$
and $f'(0) = 0$.
Verify these assertions. Then show that f' is not bounded on any interval $[0, b]$ and hence not integrable on any such interval. EOE.

7.2.7 Darboux integration

The definition of the Riemann integral is an awful way to show that a given function is integrable. The main issue is that we don't have to show convergence of a single Riemann sequence, which is not too bad for reasonable functions, but for *all* of them.

This is important because the convergence of a single, or even infinitely many Riemann sequences is not enough in general.

The notion of Darboux integration is a bit better behaved for functions which have easily determined suprema and infima. But in the end, it will turn out that this is not a new notion.

Let $f \in \mathcal{B}[a, b]$ and let $P \in \Pi(a, b)$. As f is bounded, on each subinterval $[x_i, x_{i+1}]$ of $[a, b]$, f as a supremum M_i and an infimum m_i.

We put $\mathcal{U}(P, f) \coloneqq \sum_{i=0}^{|P|} M_i(x_{i+1} - x_i)$, and $\mathcal{L}(P, f) \coloneqq \sum_{i=0}^{|P|} m_i(x_{i+1} - x_i)$.

Clearly $\mathcal{U}(P, f) \geq \mathcal{L}(P, f)$. And for every tag vector $\boldsymbol{y} \in T(P)$ we have

$$\mathcal{L}(P, f) \leq S(P, \boldsymbol{y}, f) \leq \mathcal{U}(P, f)$$

Note that $\mathcal{U}(P, f) \geq (\inf f)(b - a)$ and $\mathcal{L}(P, f) \leq (\sup f)(b - a)$. We can therefore define:

$$\mathcal{U}(f) \coloneqq \inf_{P \in \Pi(a,b)} \mathcal{U}(P, f)$$

and

$$\mathcal{L}(f) \coloneqq \sup_{P \in \Pi(a,b)} \mathcal{L}(P, f)$$

In the literature you also find the notation

$$\int_{\underline{a}}^{b} f(x)dx = \mathcal{L}(f)$$

And

$$\int_{a}^{\bar{b}} f(x)dx = \mathcal{U}(f)$$

Note we will show below that always $\mathcal{L}(f) \leq \mathcal{U}(f)$.

Definition
A function $f \in \mathcal{B}[a, b]$ is called **Darboux-integrable** if $\mathcal{L}(f) = \mathcal{U}(f)$. For such a function this common value is called its **Darboux integral**. We denote it by $\mathcal{D} - \int_a^b f$. EOD.

One of the advantages of this notion is that if $P \leq Q$ is a refinement, then the lower and upper are well behaved:

Fact
If $P \leq Q$, then

$$\mathcal{U}(P,f) \geq \mathcal{U}(Q,f)$$

and

$$\mathcal{L}(P,f) \leq \mathcal{L}(Q,f)$$

EOF.

We show the first inequality, the second follows by applying the first to $-f$, or by similar reasoning.

If $P \leq Q$, then every interval of Q is a subinterval of some interval of P. Therefore, the supremum of f on this subinterval is *smaller* (or equal) than the supremum of f on the interval of P containing it.

Proof of the Fact. We first assume Q is obtained from P by adding a single element p at position $i+1$, say:

$$P = x_1 < \cdots < x_i < x_{i+1} < \cdots < x_n$$

and

$$Q = x_1 < \cdots < x_i < p < x_{i+1} < \cdots < x_n$$

As before let $M_j = \sup\limits_{[x_j, x_{j+1}]} f(x)$ $(j = 0,1,\ldots,n)$.

Let $A = \sum_{j=0}^{i-1} M_j\big(x_{j+1} - x_i\big) + \sum_{j=i+1}^{n} M_j\big(x_{j+1} - x_j\big)$.

Then $\mathcal{U}(P,f) = A + M_i(x_{i+1} - x_i)$. Note that the intervals of Q are those of P except the ith and $(i+1)$st one: those are $[x_i, p]$ and $[p, x_{i+1}]$. Let $N_1 = \sup\limits_{[x_i,p]} f(x)$ and $N_2 = \sup\limits_{[p,x_{i+1}]} f(x)$. Then $N_1, N_2 \leq M_i$ and

$$\mathcal{U}(Q,f) = A + N_1(p - x_i) + N_2(x_{i+1} - p) \leq A + M_i\big((p - x_i) + (x_{i+1} - p)\big) = \mathcal{U}(P,f)$$

as claimed. The general case then follows easily by induction on $k \coloneqq |Q| - |P|$. We just dealt with the case $k = 1$. Now suppose we have shown that for a given k, whenever $P \subseteq Q$ and $|Q| = |P| + k$, the claim holds. If now Q contains P and $k+1$ other points, then $P \subseteq Q' \subseteq Q$ where Q' contains k points not in P. By the induction assumption $\mathcal{U}(P,f) \leq \mathcal{U}(Q',f)$ and by the case $k = 1$, $\mathcal{U}(Q',f) \leq \mathcal{U}(Q,f)$. QED.

Corollary
If P, Q are two partitions of $[a,b]$, then always

$$\mathcal{L}(P,f) \leq \mathcal{U}(Q,f)$$

EOC.

Proof. Note that P, Q have a common refinement, namely e.g. $R \coloneqq P \cup Q$. Then

$$\mathcal{L}(P,f) \leq \mathcal{L}(R,f) \leq \mathcal{U}(R,f) \leq \mathcal{U}(Q,f)$$

QED.

It follows that as promised that $\mathcal{L}(f) \leq \mathcal{U}(f)$ because for any P, $\mathcal{U}(P, f)$ is an upper bound for $\{\mathcal{L}(Q, f) \mid Q \in \Pi[a, b]\}$. Therefore $\mathcal{L}(f) \leq \mathcal{U}(P, f)$ for all P, which in turn means $\mathcal{L}(f) \leq \mathcal{U}(f)$.

Lemma (Cauchy Criteron for Darboux Integration)
$f \in \mathcal{B}[a, b]$ is Darboux integrable iff for every $\varepsilon > 0$, there is a partition P such that $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \varepsilon$. EOL.

Proof. "If": suppose such a P exists for each $\varepsilon > 0$. Note that $0 \leq \mathcal{U}(f) - \mathcal{L}(f) \leq \mathcal{U}(P, f) - \mathcal{L}(P, f)$ for every P. Therefore, $\mathcal{U}(f) - \mathcal{L}(f) < \varepsilon$ for every $\varepsilon > 0$, and hence $\mathcal{U}(f) = \mathcal{L}(f)$.

"only if": Suppose $\mathcal{U}(f) = \mathcal{L}(f)$ and let $\varepsilon > 0$. Then there is a partition Q such that $\mathcal{U}(Q, f) < \mathcal{U}(f) + \frac{\varepsilon}{2}$ and a partition R such that $\mathcal{L}(R, f) > \mathcal{L}(f) - \frac{\varepsilon}{2}$. By the fact above this still holds if we replace Q, R by a common refinement P. But then $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \mathcal{U}(f) + \frac{\varepsilon}{2} - \mathcal{L}(f) + \frac{\varepsilon}{2} = \varepsilon$. QED.

This criterion is sometimes also referred to as the Riemann Criterion for integration (once one knows that Darboux integrable functions are Riemann integrable).

Corollary
$f \in \mathcal{B}[a, b]$ is Darboux integrable if and only if there is a sequence of partitions P_n with $\lim_{n\to\infty} \big(\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f)\big) = 0$, and if that is the case then

$$\mathcal{D} - \int_a^b f(x)dx = \lim_{n\to\infty} \mathcal{U}(P_n, f) = \lim_{n\to\infty} \mathcal{L}(P_n, f)$$

EOC.

Proof. Suppose P_n exists. Then f is Darboux-integrable by the lemma (for any $\varepsilon > 0$ pick n large enough such that $\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) < \varepsilon$). Conversely, if f is Darboux-integrable, then for each n there is P_n such that $\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) < \frac{1}{n}$. This shows the first part.

Now suppose $\lim_{n\to\infty} \big(\mathcal{U}(P_n. f) - \mathcal{L}(P_n, f)\big) = 0$. Then the result follows from the following lemma (put $c = \mathcal{D} - \int_a^b f(x)dx$, which exists by the first part). QED.

Lemma
Let a_n, b_n be sequences such that $\lim_{n\to\infty} (a_n - b_n) = 0$ and there exists c such that $a_n \geq c \geq b_n$ for all n. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$. EOL.

Proof. If one of the sequences converges, so does the other, and the limits coincide. If $a_n - b_n \to 0$, then $0 \leq a_n - c \leq a_n - b_n \to 0$. QED.

Example
Let $f(x) = x$ defined on $[a, b]$.
Let P_n be the partition $a + \frac{L}{n} < a + \frac{2L}{n} < \cdots < a + \frac{(n-1)L}{n}$ where $L = b - a$.
Then

$$\mathcal{U}(P_n, f) = \sum_{i=0}^{n-1}\left(a + \frac{(i+1)L}{n}\right)\frac{L}{n} = \frac{n}{n}aL + \frac{L^2}{n^2}\sum_{i=0}^{n-1} i + 1 = \frac{n}{n}aL + \frac{\frac{L^2}{n^2}(n+1)n}{2}$$

This converges to $aL + \frac{L^2}{2} = a(b-a) + \frac{(b-a)^2}{2} = ab - a^2 + \frac{1}{2}b^2 - ab + \frac{1}{2}a^2 = \frac{1}{2}(b^2 - a^2)$.
Note replacing $i+1$ in the summation above by i gives $\mathcal{L}(P_n, f)$ but does not change the limit. Hence f is Darboux integrable and

$$\mathcal{D} - \int_a^b x dx = \frac{1}{2}(b^2 - a^2) = \int_a^b x dx$$

by the FTC 1. EOE.

Theorem
Let $f \in \mathcal{B}[a,b]$ and suppose P_n is a sequence of partitions with $m(P_n) \to 0$. Then f is Darboux integrable iff $\lim_{n\to\infty} \mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = 0$.
In particular, if f is Darboux integrable, then for every sequence of partitions P_n with $m(P_n) \to 0$, we have $\lim_{n\to\infty} \mathcal{U}(P_n, f) = \lim_{n\to\infty} \mathcal{L}(P_n, f) = \mathcal{D} - \int_a^b f(x)dx$. EOT.

Proof. Suppose $\lim_{n\to\infty} \mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = 0$. Then f is Darboux integrable by the Cauchy criterion above. For the converse let f be Darboux integrable, $L = \mathcal{D} - \int_a^b f\, dx$, and let $\varepsilon > 0$. Then there is a partition P such that $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \frac{\varepsilon}{2}$.

We now compare $\mathcal{U}(P_n, f)$ and $\mathcal{U}(P, f)$. For large enough n, so that the mesh size $m(P_n)$ is small compared to the smallest interval length of P, most intervals of P_n must be properly contained in an interval of P. Suppose $m = |P|$, so P has $m+1$ intervals. Then there are at most $2m$ intervals of P_n that are not properly contained in any interval of P (namely those that contain one of the points in P). Let $P = y_1 < y_2 < \cdots < y_m$.

We get

$$\mathcal{U}(P_n, f) = \sum_{i=0}^{|P_n|} M_{ni}(x_{i+1} - x_i) = \sum_{\substack{i=0 \\ P\cap[x_i, x_{i+1}]=\emptyset}}^{|P_n|} M_{ni}(x_{i+1} - x_i) + \sum_{\substack{i=0 \\ P\cap[x_i, x_{i+1}]\neq\emptyset}}^{|P_n|} M_{ni}(x_{i+1} - x_i)$$

where $M_{ni} = \sup_{[x_{ni}, x_{n,i+1}]} f(x)$.

Similarly, we have

$$\mathcal{L}(P_n, f) = \sum_{\substack{i=0 \\ P\cap[x_{ni}, x_{n,i+1}]\neq\emptyset}}^{|P_n|} m_{ni}(x_{i+1} - x_i) + \sum_{\substack{i=0 \\ P\cap[x_i, x_{i+1}]=\emptyset}}^{|P_n|} m_{ni}(x_{i+1} - x_i)$$

Then

$$\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)\le \sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]=\emptyset}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)+\sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]\neq\emptyset}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)$$

$$\le \sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]=\emptyset}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)+\sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]\neq\emptyset}}^{|P_n|}(M-I)(x_{i+1}-x_i)$$

$$\sum_{\substack{i=0\\ P\cap[x_i,x_{i+1}]=\emptyset}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)+(M-I)2mm(P_n)$$

where $M=\sup f$ and $I=\inf f$. (Note all summands are nonnegative, and there at most $2m$ intervals with $P\cap[x_i,x_{i+1}]\neq\emptyset$.)

In the sum on the left, each occurring interval $[x_{ni},x_{n,i+1}]$ is properly contained in an interval of P. If $A_j=\sup_{[y_j,y_{j+1}]}f$ and $a_j=\inf_{[x_j,x_{j+1}]}f$ then

$$\mathcal{U}(P,f)-\mathcal{L}(P,f)=\sum_{j=0}^{m}(A_j-a_j)(y_{j+1}-y_j)$$

It follows that

$$\sum_{\substack{i=0\\ Q\cap[x_i,x_{i+1}]=\emptyset}}^{n}(M_{ni}-m_{ni})(x_{i+1}-x_i)=\sum_{j=0}^{m}\sum_{\substack{i=0\\ [x_{ni},x_{n,i+1}]\subseteq[y_j,y_{j+1}]}}^{|P_n|}(M_{ni}-m_{ni})(x_{i+1}-x_i)\le$$

$$\le\sum_{j=0}^{m}\sum_{\substack{i=0\\ [x_{ni},x_{n,i+1}]\subseteq[y_j,y_{j+1}]}}^{|P_n|}\left(A_j-a_j\right)(x_{i+1}-x_i)\le\sum_{j=0}^{m}(A_j-a_j)(y_{j+1}-y_j)$$

Thus $\mathcal{U}(P_n,f)-\ \mathcal{L}(P_n,f)\le\mathcal{U}(P,f)-\mathcal{L}(P,f)+2m(M-I)m_n(P)<\frac{\varepsilon}{2}+2m(M-I)m_n(P)\to\frac{\varepsilon}{2}$ (for $n\to\infty$).

Thus, for n large, $\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)<\varepsilon$, and so $\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)\to 0$ as needed. QED.

This theorem is important: to test whether a function is Darboux integrable and compute the integral, all we need to do is to check whether the two limits $\lim_{n\to\infty}\mathcal{U}(P_n,f)=\lim_{n\to\infty}\mathcal{L}(P_n,f)$ exist, are finite, and coincide for *one* choice of sequence P_n with $m(P_n)\to 0$. Contrast that with a Riemann sequence $S(P_n,\boldsymbol{y}_n,f)$, where convergence of a single one does not mean that f is integrable.

7.2.8 Equivalence of Darboux and Riemann integration

Theorem

Let $f\in\mathcal{B}[a,b]$. Then f is Darboux integrable if and only if it is integrable. If f is integrable then

$$\mathcal{D}-\int_a^b f=\int_a^b f$$

Proof. Suppose f is integrable, and let $S(P_n,\boldsymbol{y}_n,f)$ a Riemann sequence as follows: P_n is a sequence of partitions such that $m(P_n)\to 0$. For each i let $y_{n,i}\in[x_{n,i},x_{n,i+1}]$ be chosen such that $f(y_{n,i})>\sup_{[x_{ni},x_{n,i+1}]} f(x)-\frac{1}{n(|P_n|+1)(x_{n,i+1}-x_{n,i})}$.

Let $\boldsymbol{z}_n$ be defined such that for $z_{n,i}\in[x_{ni},x_{n,i+1}]$, $f(z_i)<\inf_{[x_{n,i},x_{n,i+1}]} f(x)+\frac{1}{n|P_n|(x_{n,i+1}-x_{n,i})}$.

Then $\mathcal{L}(P_n,f)\le S(P_n,\boldsymbol{y}_n,f), S(P_n,\boldsymbol{z}_n,f)\le \mathcal{U}(P_n,f)$. Note that

$$\mathcal{U}(P_n,f)-S(P_n,\boldsymbol{y}_n,f)<\sum_{i=0}^{|P_n|}\frac{1}{n(|P_n|+1)(x_{n,i+1}-x_{n,i})}(x_{n,i+1}-x_{ni})=\frac{1}{n}$$

We similarly conclude that $S(P_n,\boldsymbol{z}_n,f)-\mathcal{L}(P_n,f)<\frac{1}{n}$.

For n large enough, both Riemann sums are “close” to $I=\int_a^b f$, and then necessarily $\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)$ must be “small”. To be precise, let $\varepsilon>0$, and let n be large enough such that $|S(P_n,\boldsymbol{y}_n,f)-I|, |S(P_n,\boldsymbol{z}_n,f)-I|<\frac{\varepsilon}{8}$ and also assume $\frac{1}{n}<\frac{\varepsilon}{4}$. Then

$$\mathcal{U}(P_n,f)-\mathcal{L}(P_n,f)\le \mathcal{U}(P_n,f)-S(P_n,\boldsymbol{y}_n,f)+$$
$$|S(P_n,\boldsymbol{y}_n,f)-S(P_n,\boldsymbol{z}_n,f)|+S(P_n,\boldsymbol{z}_n,f)-\mathcal{L}(P_n,f)<\varepsilon$$

This shows f is Darboux integrable, and moreover, that $\mathcal{D}-\int_a^b f(x)dx=\int_a^b f(x)dx$.

To show the converse, we know that if P_n is any sequence of partitions such that $m(P_n)\to 0$, then

$$\mathcal{U}(P_n,f),\mathcal{L}(P_n,f)\to \mathcal{D}-\int_a^b f(x)\,dx$$

On the other hand, for every Riemann sequence $S(P_n,\boldsymbol{t}_n,f)$ we have

$$\mathcal{L}(P_n,f)\le S(P_n,\boldsymbol{t}_n,f)\le \mathcal{U}(P_n,f)$$

The squeeze principle shows that every such Riemann sequence converges, and that the common limit is equal to $\mathcal{D}-\int_a^b f$. QED.

We therefore drop the $\mathcal{D}-$ from the notation and treat the two notions of integral as equivalent.

7.2.9 Continuous functions are integrable

Theorem

Let f be continuous on $[a,b]$. Then f is integrable. EOT.

Proof. Let P_n be a sequence of partitions of $[a,b]$ for which $m(P_n)\to 0$.

Note that for each interval $[x_{ni},x_{n,i+1}]$ of P_n, there are $y_{ni},z_{ni}\in[x_{ni},x_{n,i+1}]$ such that $f(y_{ni})=\sup_{[x_{ni},x_{n,i+1}]} f(x)=:M_{ni}$ and $f(z_{ni})=\inf_{[x_{ni},x_{n,i+1}]} f(x)=:m_{ni}$. This uses that f is continuous on $[a,b]$.

Then $\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = \sum_{i=0}^{|P_n|}(M_{ni} - m_{ni})\left(x_{n,i+1} - x_{ni}\right)$.

But f is uniformly continuous on $[a, b]$. So, for a given $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ for $|x - y| < \delta$. So, as soon as n is large enough such that $m(P_n) < \delta$, we have

$$\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) < \sum_{i=0}^{|P_n|} \frac{\varepsilon}{b-a}\left(x_{n,i+1} - x_{ni}\right) = \frac{\varepsilon}{b-a}\sum\left(x_{n,i+1} - x_i\right) = \frac{\varepsilon}{b-a}\,(b-a) = \varepsilon$$

The Cauchy Criterion for Darboux integrals does the rest. QED.

This closes a gap in our discussion of the Fundamental Theorem of Calculus Part 1. There, we only proved that continuous functions with an indefinite integral are integrable. Now we know that this was no restriction: all continuous functions are integrable on closed and bounded intervals.

7.2.10 Monotone functions are integrable

As an example of our discussion so far, we show that monotone functions are integrable.

Theorem
Let f be monotone on $[a, b]$. Then f is integrable. EOT.

Proof. We will prove the theorem for the case f is increasing. If f is monotone decreasing, the result follows by applying the monotone increasing result to $-f$ (and observing that f is integrable iff $-f$ is integrable (see Linearity of integration 7.2.5).

We may assume that f is not constant (as we know that constant functions are integrable). Then $f(b) > f(a)$.

For each partition P we conclude that

$$\mathcal{U}(P, f) = \sum_{i=0}^{|P|} f(x_{i+1})(x_{i+1} - x_i)$$

and

$$\mathcal{L}(P, f) = \sum_{i=0}^{|P|} f(x_i)(x_{i+1} - x_i)$$

Now suppose that P is a partition with $m(P) < \delta$. Then

$$\begin{aligned}\mathcal{U}(P, f) - \mathcal{L}(P, f) < \sum_{i=0}^{|P|}\big(f(x_{i+1}) - f(x_i)\big)(x_{i+1} - x_i) &< \delta\sum_{i=0}^{|P|} f(x_{i+1}) - f(x_i) \\ &= \delta\big(f(b) - f(a)\big)\end{aligned}$$

For $\varepsilon > 0$, if we choose $\delta < \frac{\varepsilon}{f(b)-f(a)}$, then for every partition P with $m(P) < \delta$, we get $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \varepsilon$. QED.

7.2.11 Properties of the definite integral

For a closed interval $[a, b]$, we denote the set of integrable functions on $[a, b]$ by $\mathcal{R}[a, b]$. We have seen that this is a vector space.

Moreover, the map $\mathcal{R}[a, b] \to \mathbb{R}$ defined by $f \mapsto \int_a^b f$ is a linear transformation.

Lemma (First inequality for integrals)
$f, g \in \mathcal{R}[a, b]$ and $f \leq g$. Then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. EOL.

Proof. If $f \leq g$, then for any partition P of $[a, b]$ and any tag-vector $\boldsymbol{y}$ we have

$$S(P, \boldsymbol{y}, f) \leq S(P, \boldsymbol{y}, g)$$

from which the result follows immediately after considering Riemann sequences. QED.

Corollary
For any $f \in \mathcal{R}[a, b]$, we have

$$\inf f \cdot (b - a) \leq \int_a^b f(x)dx \leq \sup f \cdot (b - a)$$

EOC.

Proof. Observe that constant functions are integrable so put $I(x) = \inf f$ and $S(x) = \sup f$ on $[a, b]$, then $I \leq f \leq S$ and $\int_a^b I(x)dx = \inf\ f \cdot (b - a) \leq \int_a^b f(x)dx \leq \sup f \cdot (b - a)$. QED.

Lemma (Fundamental inequality for integrals)
Let $f \in \mathcal{R}[a, b]$, then $\left|\int_a^b f(x)dx\right| \leq \|f\|_\infty (b - a)$. EOC.

Here, $\|f\|_\infty := \sup|f|$.

Proof. This is immediate from the observation that for every Riemann sum $S(P, \boldsymbol{y}, f)$ we have

$$|S(P, \boldsymbol{y}, f)| \leq \sup|f| \cdot (b - a)$$

(triangle inequality). As the integral is the limit of a sequence of Riemann sums, the result follows. QED.

Remark
Note we would also like to have a result of the form

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx \leq \sup|f| \cdot (b - a)$$

The problem is, we haven't shown that for an integrable function f, the absolute value function $|f|$ is also integrable. While this is true, this is not immediate. For continuous f, however, $|f|$ is also continuous, and therefore integrable. Then the above inequality is true. EOR.

Lemma
Let $f \in \mathcal{B}[a, b]$ and $c \in (a, b)$. Then $f \in \mathcal{R}[a, b]$ if and only if $f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$, and if either is the case then

$$\int_a^b f = \int_a^c f + \int_c^b f$$

EOL.

Note that technically we should use different labels for the restrictions of f to $[a, c]$ and $[c, b]$. However, this would not add anything useful to the discussion.

Proof. We use Darboux integration: Suppose f is integrable on $[a, b]$. We must find sequences of partitions of $[a, c]$ and $[c, b]$ such that the corresponding Darboux integrals converge. To be precise, let P_n be a partition such that $\lim\limits_{n\to\infty}\big(\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f)\big) = 0$. We may assume that $c \in P_n$, otherwise we just add it without changing the limit (refinements make the difference smaller). Then $Q_n \coloneqq P_n \cap (a, c)$, viewed as a partition of $[a, c]$ and $R_n \coloneqq P_n \cap (c, b)$ viewed as a partition of $[c, b]$, are partitions of $[a, c]$ and $[c, b]$ respectively.

But then

$$\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = \big(\mathcal{U}(Q_n, f) - \mathcal{L}(Q_n, f)\big) + \big(\mathcal{U}(R_n, f) - \mathcal{L}(R_n, f)\big)$$

The left hand side, and the two bracketed terms on the right are all nonnegative. Thus the squeeze principle dictates that each bracketed term on the right is a zero sequence.

Conversely, if f is integrable on $[a, c]$ and $[c, b]$, then choosing again sequences of partitions Q_n of $[a, c]$ and R_n of $[c, b]$, we get a sequence of partitions $P_n \coloneqq Q_n \cup \{c\} \cup R_n$ of $[a, b]$, and the above equation still holds. Now the right hand side converges to zero, so the left hand side does.

Regarding the value of the integral: note that both sides of the above equation converge to 0, then we know that

a. $\mathcal{U}(P_n, f) = \mathcal{U}(Q_n, f) + \mathcal{U}(R_n, f)$ for all n
b. $\lim\limits_{n\to\infty} \mathcal{U}(P_n, f) = \int_a^b f(x)\,dx$, $\lim\limits_{n\to\infty} \mathcal{U}(Q_n, f) = \int_a^c f(x)dx$, and $\lim\limits_{n\to\infty} \mathcal{U}(R_n, f) = \int_c^b f(x)dx$.

Our standard results on limits of sequences then tell us that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

A technicality are some degenerate cases (where $c = a$, or $c = d$, or $d = b$). The result still holds if we define for any function f defined in some $\alpha \in \mathbb{R}$ that $\int_\alpha^\alpha f(x)dx = 0$. QED.

Corollary
Let $c < d \in [a, b]$. If $f \in \mathcal{R}[a, b]$, then f is also integrable on $[c, d]$.

Proof. We first apply the lemma to conclude that f is integrable on $[a, c]$ and $[c, b]$. Applying the lemma again to the restriction of f to $[c, b]$, we find that f is integrable on $[c, d]$. QED.

Definition
- Let $[a, b]$ be an interval. If $b = a$, we define $\int_a^b f(x)dx = 0$ for all functions f defined in $a = b$.
- Let $a > b$, we define $\mathcal{R}[a, b] = \mathcal{R}[b, a]$ and $\int_a^b f(x)dx = -\int_b^a f(x)dx$ for all functions integrable on $[b, a]$. EOD.

With these definitions we have for all $c, d, e \in [a, b]$ that

$$\int_c^d f(x)dx + \int_d^e f(x)dx = \int_c^e f(x)dx$$

and we do not need to have that $c \leq d \leq e$. (Convince yourself of that.)

For example if $d > c > e$, then $\int_e^c f(x)dx + \int_c^d f(x)dx = \int_e^d f(x)dx$. But then $\int_c^e f = -\int_e^c f(x)dx = \int_c^d f - \int_e^d f = \int_c^d f + \int_d^e f$.

Definition
A function f defined on an interval of the form $[-a, a]$ is called **odd** if for all $x \in [-a, a]$ $f(-x) = -f(x)$. It is called **even** if $f(-x) = f(x)$.

Corollary
Let f be an integrable function on $[-a, a]$.

1. If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.
2. If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

Proof. f is integrable on $[-a, 0]$ and $[0, a]$ and it suffices to show that $\int_{-a}^{0} f = -\int_0^a f$ if f is odd, and $\int_{-a}^{0} f = \int_0^a f$ if f is even.

Let $P_n = \frac{a}{n} < \frac{2a}{n} < \cdots < \frac{(n-1)a}{n}$, which is a partition of $[0, a]$. We write $-P_n$ for the partition $-\frac{(n-1)a}{n} < -\frac{n-2}{a} < \cdots < -\frac{a}{n}$ of $[-a, 0]$. Any tag-vector $\boldsymbol{y}_n$ for P_n gives rise to the tag vector $-\boldsymbol{y}'_n$ of $-P_n$, where we write $\boldsymbol{y}'$ for the vector $\boldsymbol{y}$ *ordered in reverse*. For example, $(1,2,3)' = (3,2,1)$.

It is then clear that $S(-P_n, -\boldsymbol{y}_n, f) = S(P_n, \boldsymbol{y}_n, f)$ if f is even, and $S(-P_n, -\boldsymbol{y}_n, f) = -S(P_n, \boldsymbol{y}_n, f)$ if f is odd. Since $m(P_n) \to 0$, the claim follows. QED.

7.2.12 Variable integration boundaries

By the above if f is integrable on $[a, b]$, for a fixed $c \in [a, b]$ and any $x \in [a, b]$, we may define

$$F(x) = \int_c^x f(t)dt$$

In other words, we treat the upper boundary as a variable. This defines a function on $[a, b]$.

Note that if f happens to have an anti-derivative on $[a, b]$, F_0, say, then F_0 is also an antiderivative on $[c, x]$ if $x > c$, and on $[x, c]$ if $x < c$.

In the first case $F(x) = F_0(x) - F_0(c)$ by the FTC1. In the second case $F(x) = -\int_x^c f(t)dt = -\big(F_0(c) - F_0(x)\big) = F_0(x) - F_0(c)$.

Finally if $x = c$, then $F(x) = 0$ by definition and also $F_0(x) - F_0(c) = 0$, so in all cases we have

$$F(x) = F_0(x) - F_0(c)$$

This means F and F_0 differ by a constant (namely $\pm F_0(c)$) and so F is differentiable and $F'(x) = f(x)$ on $[a, b]$.

The upshot is that if f has an antiderivative, then F as defined above is one.

We haven't yet precisely discussed which functions do have antiderivatives, and how to find one.

But it seems natural to look among functions of the form $F(x) = \int_c^x f(t)dt$ at least for integrable functions. This is what we will do next.

7.2.13 Second Fundamental Theorem of Calculus

The first observation may be a little bit surprising:

Lemma
Let f be integrable on $[a, b]$ and $F(x) = \int_c^x f(t)dt$ on $[a, b]$. Then F is continuous. EOL.

Proof. First, note that by the discussion in 7.2.11 we have for any fixed $x_0 \in [a, b]$

$$F(x) - F(x_0) = \int_c^x f - \int_c^{x_0} f = \int_c^x f + \int_{x_0}^c f = \int_{x_0}^x f(t)dt$$

To show that F is continuous at x_0 it therefore is enough to show that

$$\lim_{x \to x_0} \int_{x_0}^x f(t)dt = 0$$

Now recall the Fundamental Inequality for Integrals:

$$\left| \int_{x_0}^x f(t)dt \right| \leq S(x - x_0)$$

where $S = \sup\limits_{[x_0, x]} f$. But $0 \leq S \leq \sup\limits_{[a,b]} f$. Therefore $\lim\limits_{x \to x_0} S(x - x_0) = 0$. QED.

Note that this holds even if f itself is not continuous. Integration seems to "smooth" out jumps.

Theorem (Second Fundamental Theorem of Calculus)
Let f be continuous on $[a, b]$. For any $c \in [a, b]$, the function $F(x) = \int_c^x f(t)dt$ is an anti-derivative of f:

$$\frac{d}{dx} \int_c^x f(t)dt = f(x)$$

EOT.

The theorem is an immediate consequence of the following stronger result:

Lemma
Let f be integrable on $[a, b]$, $c \in [a, b]$, and f continuous at $x_0 \in [a, b]$. Let $F(x) = \int_c^x f(t)dt$. Then

$$F'(x_0) = f(x_0)$$

EOL.

Proof. We must show that $\lim\limits_{h \to 0} \frac{1}{h} \left(F(x_0 + h) - F(x_0)\right) = f(x_0)$.

$F(x_0 + h) - F(x_0) = \int_{x_0}^{x_0+h} f(t)dt$. Let $\varepsilon > 0$. As f is continuous at x_0, there is $\delta > 0$ such that $|f(t) - f(x_0)| < \frac{\varepsilon}{2}$ for all $t \in [a, b]$ with $|t - x_0| < \delta$.

Observe that for all h such that $x_0 + h \in [a, b]$ we have $\int_{x_0}^{x_0+h} f(x_0)dt = f(x_0)h$. And thus for such h with $h \neq 0$

$$\frac{1}{h}\big(F(x_0 + h) - F(x_0)\big) - f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} f(t) - f(x_0)dt$$

Then $\left|\frac{1}{h}\int_{x_0}^{x_0+h}\big(f(t) - f(x_0)\big)dt\right| \leq \frac{1}{h} \cdot \frac{\varepsilon}{2} \cdot h < \varepsilon$ as long as $|h| < \delta$ by the Fundamental Inequality. QED.

7.2.14 Rules of definite integration

The rules for indefinite integrals have mirror images for definite integrals.

We already have seen that if f, g are integrable in $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable on $[a, b]$ and

$$\int_a^b (\alpha f + \beta g)dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$$

Product Rule
Let f be continuous on $[a, b]$ and let G be continuously differentiable on $[a, b]$. Let F be any antiderivative of f
Then fG is integrable and

$$\int_a^b fg dx = [FG]_a^b - \int_a^b FG' dx$$

EOL.

Proof. By the FTC2, we know that F always exists. We also know that FG' is continuous and therefore integrable.

Now let $U(x) = F(x)G(x) - \int_a^x F(t)G'(t)dt$. Then U is an antiderivative of fg:

$$U'(x) = F'(x)G(x) + F(x)G'(x) - F(x)G'(x) = f(x)G(x)$$

by the FTC2. Thus, by the FTC1, $\int_a^b fg dx = U(b) - U(a)$ wich is what is claimed. QED.

Substitution Rule
Let f be continuous on $[a, b]$. Let g be continuously differentiable on $[\alpha, \beta]$ and suppose $g([\alpha, \beta]) \subseteq [a, b]$ and $a = g(\alpha)$ and $b = g(\beta)$. Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f\big(g(t)\big)g'(t)dt$$

EOL.

Proof. By FTC2 f has an antiderivative F on $[a, b]$. Then $F(g(t))$ is an antiderivative of $f\big(g(t)\big)g'(t)$ and hence by the FTC1, we have

$$\int_\alpha^\beta f\big(g(t)\big)g'(t)dt = F\big(g(\beta)\big) - F\big(g(\alpha)\big) = F(b) - F(a) = \int_a^b f(x)dx$$

QED.

The condition that $g(\alpha) = a$ and $g(\beta) = b$ forces $g([\alpha, \beta]) = [a, b]$ by the IVT.

Note that the substitution rule also works if $a > b$ or $\alpha > \beta$:

For example

$$\int_1^0 \sin(t)\cos(t)\, dt = \int_{\sin 1}^0 x dx = \left[\frac{1}{2}x^2\right]_{\sin 1}^0 = -\frac{1}{2}\sin^2(1)$$

7.2.15 Some examples

We have seen that all continuous functions are integrable and that continuous functions have antiderivatives.

However, the converse of both statements has counterexamples:

Let $f(x) = \vdash\begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

Then f is integrable on every closed and bounded subinterval. But f has no antiderivative on any interval that contains 0 and also negative points: indeed, if $0 \in [a, b]$ and $a < 0 \leq b$, then f takes both values 0 and 1 on $[a, b]$. By the Intermediate Value Theorem for derivatives (this was a homework assignment) if $F' = f$, then f must take all values between 0,1.

But f is integrable since it is monotone increasing. One could also show this directly by dealing with the one jump using a clever partition.

f is also not continuous at 0. But it may not be surprising that f is integrable (because we know that we can built up an integral by integration over a partition into subintervals). So as long as f has only finitely many jumps we should be fine (even though one has to deal with the jumps).

Here is a more surprising function, defined on $[0, \infty)$

$$g(x) = \vdash\begin{cases} \frac{1}{n} & x = \frac{m}{n} \in \mathbb{Q}, m, n \in \mathbb{N}_0, n \neq 0, \gcd(m, n) = 1 \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(It is possible to extend g to all of $\mathbb{R}$ by allowing negative numerators.) Also note that 1 is the only positive integer with a gcd of 1 with 0, so $g(0) = 1$.

Fact
g is continuous at exactly the irrational positive numbers. EOF.

Proof. If $x_0 \in \mathbb{Q}$ and $x_0 \geq 0$, there is a sequence $x_n \geq 0$ all of its elements are irrational, and such that $x_n \to x_0$ (the irrational numbers are dense). But to be concrete you could use $x_n = x_0 + \frac{\sqrt{2}}{n}$.

Then $g(x_n) = 0$ for all n, so $g(x_n) \to 0 \neq g(x_0)$. So g is not continuous at x_0.

If on the other hand $x_0 \notin \mathbb{Q}$, we must show that for every sequence $x_n \to x_0$ we have $g(x_n) \to 0$.

We discussed this in our online lecture, but admittedly not in a very clear manner. Let me try to be precise here: First off, if $x_n \notin \mathbb{Q}$, then $f(x_n) = 0$ we can restrict our attention to the subsequence of x_n of elements in $\mathbb{Q}$, and hence may assume that $x_n \in \mathbb{Q}$ for all n.

But then, if we denote the subsequence by $x_n = \frac{p_n}{q_n}$, we must have that $\lim_{n\to\infty} q_n = \infty$. (This would show hat $f(x_n) = \frac{1}{q_n} \to 0 = f(x_0)$.)

Indeed, if $\lim x_n = x_0$, then for all but finitely many n, $x_n \in (x_0 - \varepsilon, x_0 + \varepsilon)$ for any given $\varepsilon > 0$.

(I'll phrase the proof below so that it would also work if we extend the definition of f to all of $\mathbb{R}$. For $[0, \infty)$, the absolute value can be omitted.)

Let $y = |x_0| + \varepsilon$. Then there are at most finitely many natural numbers in the interval $\left(0, \frac{1}{\varepsilon}\right)$. So $q < \frac{1}{\varepsilon}$ for only finitely many $q \in \mathbb{N}$. But then there are only finitely many rational numbers $\frac{p}{q}$ in the interval $[0, y)$ with $q < \frac{1}{\varepsilon}$ (as for each q there are only finitely many $p > 0$ such that $\frac{p}{q} < y$). Now all but finitely many $|x_n|$ are contained in $[0, y)$, and as $|x_n| \to |x_0|$, we must have that for all but finitely many n, $|x_n|$ is not equal to one of the finitely many rational numbers with denominator $q < \frac{1}{\varepsilon}$ (each of those rational numbers has a positive distance from $|x_0|$), and therefore, for only finitely many n we can have $q_n < \frac{1}{\varepsilon}$. But that means $q_n \to \infty$. QED.

One can show (and we will see), that this means that g is integrable on any interval $[a, b]$ with $0 < a < b$.

7.2.16 Oscillation

Definition
Let f be a bounded function on an interval I. We define the **oscillation** of f on I, denoted $\Omega_f(I)$ as

$$\Omega_f(I) \coloneqq \sup_I f - \inf_I f$$

EOD.

Note that if $I \subset J$ and f is defined on J, then $\Omega_f(I) \leq \Omega_f(J)$.

Let now f be defined and bounded on $[a, b]$ and let $x \in [a, b]$. For $\delta > 0$ we define

$$\Omega_{f,x}(\delta) \coloneqq \Omega_f\big([a, b] \cap (x - \delta, x + \delta)\big)$$

(that is, the oscillation of f on the "small" δ interval around x).

$\Omega_{f,x}(\delta)$ is a monotone increasing function on $(0, \infty)$ and bounded below. It follows that

$$\omega_f(x) \coloneqq \lim_{\delta\to 0^+} \Omega_{f,x}(\delta)$$

exists and is ≥ 0. It is called the **oscillation** of f **at** x.

Proposition
$f \in \mathcal{B}[a, b]$ is continuous at $x \in [a, b]$ iff $\omega_f(x) = 0$. EOP.

Proof. Homework exercise.

7.2.17 Sets of measure zero

Definition

A subset $N \subseteq \mathbb{R}$ is called a **set of measure zero**, if for very $\varepsilon > 0$ there are (at most) countably infinitely many open intervals $I_1, I_2, \ldots$ such that $N \subseteq I_1 \cup I_2 \cup \ldots$, and such that $|I_1| + |I_2| + \cdots = \sum_{k=1}^{\infty} |I_k| < \varepsilon$ EOD.

Here for any interval I of the form $(a,b), [a,b], (a,b], [a,b)$ we put $|I| = b - a$.

Terminology
We say a property of points $x \in I$ holds **almost everywhere** in a set I, if the set where the property fails is a set of measure zero. EOT.

For example, f defined on I is *almost everywhere continuous* on I if the set $\{ x \in I \mid f \text{ is not continuous at } x\}$ is a zero set.

Exercise
Show that $\mathbb{Q} \subseteq \mathbb{R}$ or more generally any countably infinite set in $\mathbb{R}$ is a set of measure zero. EOE.

Lemma

1. Subsets of a zero set are zero sets.
2. Any finite or countable union of zero sets is again a zero set.

EOL.

Proof. 1. is straight forward: any open cover of a set of of measure zero is also an open cover of any subset.

As for 2. the case of a finite union is covered by the case of a countably infinite union (why?).

Let Z_k ($k \in \mathbb{N}$) be a countably infinite collection of sets of measure zero, and let $\varepsilon > 0$.

Then for each k, Z_k may be covered by a countably infinite union of open intervals $I_{k\ell}$ such that

$$\sum_{\ell=1}^{\infty} |I_{k\ell}| < \frac{\varepsilon}{2^k}$$

Let $Z = \bigcup_{k=1}^{\infty} Z_k$. Then $Z \subseteq \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} I_{k\ell} = \{x \in \mathbb{R} \mid \exists k, \ell : x \in I_{k,\ell}\}$.

Note that by the Cauchy Double Series Theorem $\sum_{k,\ell=1}^{\infty} |I_{k\ell}| = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} |I_{k\ell}|$ if the right hand side converges (and this then shows that summing up the $|I_{k\ell}|$ in any order is bounded by either of that sum).

Now $\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} |I_{k\ell}| \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \left(\frac{1}{1-\frac{1}{2}} - 1 \right) = \varepsilon$. But note as e.g. $\sum_{\ell=1}^{\infty} |I_{1\ell}| < \frac{\varepsilon}{2}$, we actually have strict inequality. Thus, Z is a set of measure zero. QED.

7.2.18 Compact sets and the Heine Borel Theorem

If $S \subseteq \mathbb{R}$ is any subset, an **open cover** or **open covering** of S is a family $\{U_i\}_{i \in I}$ of open sets $U_i \subseteq \mathbb{R}$ such that $S \subseteq \bigcup_{i \in I} U_i = \{x \in \mathbb{R} \mid \exists i : x \in U_i\}$.

For example, $U_n \coloneqq \left(\frac{1}{n}, 1\right)$ defines an open covering of $(0,1)$:

$$(0,1) \subseteq \bigcup_{n=1}^{\infty} U_n$$

In fact, we have equality in this case.

A **subcover** or **subcovering** is then determined by a subset $J \subseteq I$ such that still $S \subseteq \bigcup_{i \in J} U_i$. So $\{U_i\}_{i \in J}$ defines an open cover where each open set also appears in the original cover (with the same index).

A very important case are sets S where we can *always* find a subcover consisting of *finitely many* open sets.

Definition
A subset $K \subseteq \mathbb{R}$ is called **compact**, if every open covering has a finite subcover. In other words, whenever

$$K \subseteq \bigcup_{i \in I} U_i$$

then there are $i_1, i_2, \dots, i_n \in I$ such that

$$K \subseteq U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n}$$

EOD.

Note that $(0,1)$ above is *not* compact: the given covering $\{U_n\}_{n \in \mathbb{N}}$ has no finite subcover: indeed, we have $U_1 \subseteq U_2 \subseteq \cdots$. Therefore, for any finite list of indices $n_1 < n_2 < \cdots < n_k$, we have

$$U_{n_1} \cup U_{n_2} \cup \dots \cup U_{n_k} = U_{n_k} = \left(\frac{1}{n_k}, 1\right)$$

which does not contain $(0,1)$ as a subset.

This leaves the question how we can characterize compact subsets of $\mathbb{R}$. The definition is certainly unwieldy, and it is not at all clear how we could ever determine how a subset might be compact.

Here is a first hint at what the definition means for subsets of $\mathbb{R}$:

Lemma
Let K be a compact subset of $\mathbb{R}$. Then every sequence $x_n \in K$ has a convergent subsequence with limit in K. EOL.

Proof. Let $x_n \in K$ be a sequence. Suppose no $x \in K$ is the limit of a subsequence. Then for each x, there is $\varepsilon_x > 0$ such that there are only finitely many indices n such that $x_n \in (x - \varepsilon_x, x + \varepsilon_x)$. Otherwise, if for every ε, $x_n \in (x - \varepsilon, x + \varepsilon)$ for infinitely many n, then there is a subsequence converging to x.

Let $U_x = (x - \varepsilon_x, x - \varepsilon_x)$. Then $K \subseteq \bigcup_{x \in K} U_x$. This is an open covering. As K is compact, there are finitely many $y_1, y_2, \dots, y_k \in K$ such that $K \subseteq U_{y_1} \cup U_{y_2} \cup \dots \cup U_{y_k}$.

But then for at least one i we must have infinitely many n such that $x_n \in U_{y_i}$ (as there are infinitely many $n \in \mathbb{N}$; pigeonhole principle). This is a contradiction to the definition of U_{y_i}. QED.

Corollary
Let K be a compact subset of $\mathbb{R}$. Then K is closed and bounded. EOC.

Proof. First, K must be bounded, otherwise we can easily construct a strictly monotone increasing or decreasing sequence with limit ∞, respectively $-\infty$. Such a sequence has no convergent subsequence, a contradiction to the lemma.

Let $x_n \in K$ be any sequence that has a limit $x_0 \in \mathbb{R}$. Then any subsequence also converges to x_0. The lemma says there is a subsequence converging to an element of K, and hence $x_0 \in K$. This shows that K is closed. QED.

We have already gone one third of the way of the Heine-Borel Theorem. It turns out that "closed and bounded" is the characterization of compact sets we were looking for:

Theorem (Heine-Borel)
A subset K of $\mathbb{R}$ is compact if and only if K is bounded and closed. EOT.

This theorem holds also for subsets of $\mathbb{R}^n$ and is fundamentally important in Analysis. However, it does not hold in arbitrary "topological spaces" where this statement might make sense.

Also, I said "one third" above, because the "if"-direction is significantly more technical to prove than the "only if"-direction.

Proof. We have just seen that a compact set is closed and bounded. So we must show the converse.

Let $K \subseteq \bigcup_{i\in I} U_i$ be an open covering. The U_i are open but need not be open intervals a priori. To get better control over the type of open sets let for any $x \in K$, $i(x)$ be an index in I such that $x \in U_{i(x)}$. Note hat $i(x)$ need not be unique[1]. Then there is[2] $\delta_x > 0$ such that $V_x := (x - \delta_x, x + \delta_x) \subseteq U_{i(x)}$. We define $W_x := (x - \varepsilon_x, x + \varepsilon_x)$ with $\varepsilon_x := \frac{\delta_x}{2}$. It will turn out important that we have this additional buffer. Note that $W_x \subseteq V_x \subseteq U_{i(x)}$.

Then we have $K \subseteq \bigcup_{x\in K} W_x$.

First, we want to get some control on the "size" of the ε_x needed. Ideally, they would be bounded below by a positive ε.

To this end, we put

$$W_\varepsilon = \bigcup_{\substack{x\in K \\ \varepsilon_x \geq \varepsilon}} W_x$$

First, W_ε is open, and second, $W_\varepsilon \subseteq W_\mu$ if $\mu < \varepsilon$.

Claim. There is $\varepsilon > 0$ such that $K \subseteq W_\varepsilon$.

To prove the claim, suppose it is false. Then for every $n \in \mathbb{N}$ and $\varepsilon = \frac{1}{n}$ there is $x_n \in K$ such that $x_n \notin W_{\frac{1}{n}}$.

As K is bounded and closed, by BW, there is a subsequence x_{n_k} converging to some $x_0 \in K$.

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[1] In fact, the existence of a well-defined $i(x)$ needs the Axiom of Choice.
[2] Again, Axiom of Choice.

Let $n_0 > \frac{1}{\varepsilon_{x_0}}$. Then if $n_k > n_0$, we have $\frac{1}{n_k} < \varepsilon_{x_0}$, and therefore $x_{n_k} \notin W_{x_0} \subseteq W_{\varepsilon_{x_0}} \subseteq W_{\frac{1}{n_k}}$.

But for $k \to \infty$ we must have $n_k \to \infty$, so for almost all[3] k we have $n_k > n_0$ and therefore $x_{n_k} \notin W_{x_0}$. This contradicts the fact that $x_{n_k} \to x_0$ and proves the claim.

For a nonempty bounded and closed subset K we define its *diameter* $d(K)$ as $d(K) = b - a$, where $a = \min K$ and $b = \max K$ (note that as K is bounded, it has a sup and an inf. Since it is closed, both are elements of K, so $\sup K = \max K$ and $\inf K = \min K$).

For any $x \in \mathbb{R}$ and $\delta > 0$ let $V_x(\delta) = (x - \delta, x + \delta)$.

For fixed $\varepsilon > 0$, we will now show by induction on $n \in \mathbb{N}$ that if a nonempty closed and bounded subset A has $d(A) \leq n\varepsilon$, and $A \subseteq \bigcup_{j \in J} V_{y_j}(\varepsilon_j)$ is an open cover where for each y_j, $\varepsilon_j \geq \varepsilon$ then there are finitely many $j_1, j_2, \ldots, j_k \in J$ such that

$$A \subseteq V_{y_{j_1}}(2\varepsilon_{j_1}) \cup V_{y_{j_2}}(2\varepsilon_{j_2}) \cup \ldots \cup V_{y_{j_k}}(2\varepsilon_{j_k})$$

Let first $n = 1$. Let $a = \min A$ and $b = \max A$. Then $b - a \leq \varepsilon$. Let $x \in A$ be arbitrary. There is $j \in J$ such that $x \in V_{y_j}(\varepsilon_j)$. Then $|x - y_j| < \varepsilon_j$. And for any $x' \in A$, we have

$$|x' - y_j| \leq |x' - x| + |x - y_j| < \varepsilon + \varepsilon_j \leq 2\varepsilon_j$$

Therefore $x' \in V_{y_j}(2\varepsilon_j)$ and hence $A \subseteq V_{y_j}(2\varepsilon_j)$. This shows the base case.

Now suppose $d(A) \leq (n+1)\varepsilon$, and suppose the assertion is true for any closed and bounded subset of diameter at most $n\varepsilon$.

Let $A \subseteq \bigcup_{j \in J} V_{y_j}(\varepsilon_j)$. Let $a = \min A$, and let $A_1 = A \cap [a, a + n\varepsilon]$. Let $A_2 = A \cap [a + n\varepsilon, a + (n+1)\varepsilon]$. We may assume that both A_1, A_2, are nonempty (why?). Then $d(A_1) \leq n\varepsilon$ and $d(A_2) \leq \varepsilon$.

Both A_1, A_2 are closed and bounded. By the induction hypothesis for A_1 and the case $n = 1$ for A_2, there are finitely many $j_1, j_2, \ldots, j_k, j_0 \in J$, such that

$A_1 \subseteq V_{y_{j_1}}(2\varepsilon_{j_1}) \cup V_{y_{j_2}}(2\varepsilon_{j_2}) \cup \ldots \cup V_{y_{j_k}}(2\varepsilon_{j_k})$ and $A_2 \subseteq V_{y_{j_0}}(2\varepsilon_{j_0})$.

But then $A = A_1 \cup A_2 \subseteq V_{y_{j_1}}(2\varepsilon_{j_1}) \cup V_{y_{j_2}}(2\varepsilon_{j_2}) \cup \ldots \cup V_{y_{j_k}}(2\varepsilon_{j_k}) \cup V_{y_{j_0}}(2\varepsilon_{j_0})$. This proves the claim.

Now returning to our original closed and bounded subset K. Then with $J = K$ we have

$$K \subseteq \bigcup_{x \in J} W_x$$

and therefore there are finitely many $x_1, x_2, \ldots, x_k \in K$ such that

$$K \subseteq V_{x_1} \cup V_{x_2} \cup \ldots V_{x_k}$$

(recall $W_x = V_x(\varepsilon_x)$ and $V_x = V_x(2\varepsilon_x)$). But then

[3] All but finitely many.

$$K \subseteq U_{i(x_1)} \cup U_{i(x_2)} \cup \dots \cup U_{i(x_k)}$$

QED.

As an application we show:

Lemma
Suppose K is a compact set. Then K is a set of measure zero, if and only if for every $\varepsilon > 0$ there exist finitely many open intervals $I_1, I_2, \dots, I_n$ with $K \subseteq I_1 \cup I_2 \cup \dots \cup I_n$ and $\sum |I_k| < \varepsilon$. EOL.

Proof. If K has measure zero, then for each $\varepsilon > 0$ there exists a countable open cover $K \subseteq \bigcup_k I_k$ with $\sum |I_k| < \varepsilon$. As K is compact there is a finite subcover $K \subseteq I_{n_1} \cup I_{n_2} \cup \dots \cup I_{n_\ell}$, and then $\sum I_{n_i} < \varepsilon$. The if-direction is immediate. QED.

Corollary
If N is a set of measure zero, then N does not contain any interval (a, b) where $a < b$. EOC.

This should be intuitively clear.

Proof. If $(a, b) \subseteq N$, then (a, b) is also a set of measure zero. Let $a' < b' \in (a, b)$. Then $[a', b'] \subseteq (a, b) \subseteq N$. So $K = [a', b']$ is a set of measure zero. It is also compact.

Thus, if $\varepsilon > 0$, there exist n open intervals $I_1, I_2, \dots, I_n$ such that $K \subseteq I_1 \cup I_2 \cup \dots \cup I_n$ and such that $\sum_{k=1}^{n} |I_k| < \varepsilon$. Then the same remains true if we replace each open interval $I_k = (a_k, b_k)$ with its closure $\overline{I_k} = [a_k, b_k]$.

If $\varepsilon < b' - a'$, this is impossible: an easy induction on n shows that if $K \subseteq I_1 \cup I_2 \cup \dots \cup I_n$ is a compact interval for closed intervals I_k with $\sum |I_k| < \varepsilon$ then $d(K) < \varepsilon$. Indeed, this is clear if $n = 1$. Suppose for any union of n closed intervals I_k with $\sum |I_k| < \varepsilon$, any compact subinterval has diameter $< \varepsilon$.

Let $K \subseteq I_1 \cup I_2 \cup \dots \cup I_{n+1}$ with $\sum |I_k| < \varepsilon$. We may assume that $\max K \in I_{n+1}$.

Then $K = K_1 \cup K_2$, where $K_1 = K \cap (I_1 \cup I_2 \cup \dots \cup I_n)$ and $K_2 = K \cap I_{n+1}$. Note that K_2 is a compact interval of length at most $|I_{n+1}|$ (because $\max K \in I_{n+1}$). Let $K' = \{x \in K \mid x < \min K_2\}$. Then $K' \subseteq K_1$, and therefore also its closure $K_0 = \overline{K'} \subseteq K_1$. K_0 is a compact interval, and by induction $d(K_0) < \varepsilon - |I_{n+1}|$. Since $d(K) \leq d(K_0) + d(K_2)$ the claim follows. QED.

Corollary
If I is an interval and $N \subseteq I$ is a set of measure zero. Then $I \setminus N$ is dense in I. EOC.

Proof. N does not contain any interval by the previous corollary. Let $x \in I$. Then for each $n \in \mathbb{N}$, there exists $x_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap I$ such that $x_n \notin N$. Then $x_n \to x$. QED.

7.2.19 The Lebesgue Criterion for Integrability

Theorem (Lebesgue Criterion)
A function $f \in \mathcal{B}[a, b]$ is integrable iff f is continuous almost everywhere. EOT.

Proof (following *Heuser*, *Lehrbuch der Analysis I, 1992, p471f)*. Suppose f is continuous almost everywhere. Let N be the set of points in I where f is not continuous. Let $\varepsilon > 0$. As N is a set of measure zero, there exist open intervals $J_1, J_2, \ldots$ such hat $N \subseteq J_1 \cup J_2 \cup \ldots$, and $\sum_{i=1}^{\infty} |J_i| < \varepsilon$. We write $I = [a, b]$.

For any $x \in I \setminus N$, f is continuous at x and therefore $\omega_f(x) = 0$. We may therefore find an open interval U_x containing x such that

$$\Omega_f\left(\overline{U}_x \cap I\right) < \varepsilon$$

Then $I \subseteq \bigcup_{\{k=1\}}^{n} J_k \cup \bigcup_{x \notin N} U_x$ is a covering of I by open intervals. Heine Borel tells us that there must be finitely many of those intervals doing the trick. After relabeling, we may therefore assume that

$$I \subseteq \bigcup_{k=1}^{m} J_k \cup \bigcup_{k=1}^{n} U_k$$

This is still the case if we replace all J_k and all U_k by their closures $\overline{J}_k, \overline{U}_k$.

Let P be a partition of $[a, b]$ with small enough mesh size such that all intervals of P are contained in one of the $\overline{J}_k$ or one of the $\overline{U}_k$ (why is that possible?). We may then write

$$\mathcal{U}(P, f) - \mathcal{L}(P, f) = S_1 + S_2$$

where S_1 is the sum of all $(M_i - m_i)(x_{i+1} - x_i)$ for which $[x_i, x_{i+1}] \subseteq \overline{J}_k$ for some k, and S_2 is the sum of all others. Thus, if $[x_i, x_{i+1}]$ is not contained in any $\overline{J_k}$, $(M_i - m_i)(x_{i+1} - x_i)$ appears in S_2, and then $[x_i, x_{i+1}] \subseteq \overline{U}_k$ for some k. Now f is bounded, so $|S_1| \leq (S - I) \sum_{k=1}^{m} \left|\overline{J}_k\right| < (S - I)\varepsilon$ where $S = \sup f$ and $I = \inf f$.

On the other hand if $[x_i, x_{i+1}] \subseteq \overline{U}_k$, then $M_i - m_i \leq \Omega_f\left(\overline{U}_k \cap I\right) < \varepsilon$. Therefore $|S_2| < \varepsilon(b - a)$. As we can do that for *every* $\varepsilon > 0$, the Cauchy criterion for Darboux integrals shows that f is integrable.

Now suppose f is integrable, and let N be the set where f is not continuous. Then for all $x \in N$, $\omega_f(x) > 0$.

We may write $N = \bigcup_n \Omega_n$ with

$$\Omega_n = \left\{ x \in N \;\middle|\; \omega_f(x) > \frac{1}{n} \right\}$$

It is then enough that Ω_n is a set of measure zero.

Let $\varepsilon > 0$ and let P be a partition such that $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \frac{\varepsilon}{2n}$. Then

Let $M = \left\{ i \mid N \cap \left(x_i, x_{(i+1)}\right) \neq \emptyset \right\}$. Let $i \in M$ and let $x \in \Omega_n \cap (x_i, x_{i+1})$.

Then there is $\delta > 0$ such that $\Omega_{f,x}(\delta) > \frac{1}{n}$ and $(x - \delta, x + \delta) \subseteq [x_i, x_{i+1}]$. In that case we have for sure that $M_i - m_i > \frac{1}{n}$.

But then $\frac{1}{n} \sum_{i \in M} |(x_i, x_{i+1})| = \frac{1}{n} \sum_{i \in M} (x_{i+1} - x_i) \leq \sum_{i \in M} (M_i - m_i)(x_{i+1} - x_i) < \frac{\varepsilon}{2n}$

Note that $N \subseteq \bigcup_{i\in M}(x_i, x_{i+1}) \cup \{a, x_1, x_2, \dots, x_{|P|}, b\}$. For $x \in P \cup \{a, b\}$ choose an open interval J_x containing x such that $\sum_{x\in P\cup\{a,b\}} |J_x| < \frac{\varepsilon}{2}$.

Then $N \subseteq \bigcup_{i\in M}(x_i, x_{i+1}) \cup J_a \cup J_{x_1} \cup \dots \cup J_{x_{|P|}} \cup J_b$. The length of all these intervals sums up to something $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore, N is a set of measure zero. QED.

7.2.20 Consequences of the Lebesgue Criterion

If f, g are integrable on $[a, b]$, then so are

1. $|f|$
2. fg
3. $\min\{f, g\}$ and $\max\{f, g\}$
4. f^+ and f^-

Here $\min\{f, g\}(x) = \min\{f(x), g(x)\}$ and $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.

$$f^+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) < 0 \end{cases}$$

Similarly,

$$f^-(x) = \begin{cases} -f(x) & f(x) < 0 \\ 0 & f(x) \geq 0 \end{cases}$$

Note that $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

If f, g are almost everywhere continuous, then so are all of the functions listed here, which means they are then integrable.

Note that we need both direction of the Lebesgue Criterion here: if f, g are integrable, then they are almost everywhere continuous, and therefore so are the functions listed. And because of that, these functions are then integrable.

It is a good exercise to convince yourself that if $NC(h)$ denotes the set of points where a function is not continuous, then $NC(fg) \subseteq NC(f) \cup NC(g)$, $NC(f^+) \subseteq NC(f)$ and so on.

To prove directly (using the definition) that any of the functions in 1.,2.,3.,4. is integrable, is rather cumbersome and requires ad-hoc arguments. The admittedly technical Lebesgue Criterion provides a uniform and immediate argument.

It is a good exercise (just to appreciate the point) to try to prove that fg is integrable without using the Lebesgue Criterion.

Triangle inequality for integrals

Let f be integrable on $[a, b]$. Then we have $\left|\int_a^b f dx\right| \leq \int_a^b |f| dx \leq \|f\|_\infty (b - a)$. EOL.

Proof. The second inequality is simply the Fundamental Inequality applied to $|f|$, which is integrable by the Lebesgue Criterion.

The first inequality is immediate from the fact that for any partition P and tag vector $\boldsymbol{y}$, we have

$$|S(P, \boldsymbol{y}, f)| \leq S(P, \boldsymbol{y}, |f|)$$

by the usual triangle inequality for finite sums. Now take limits over a Riemann sequence. QED.

Lemma

Let f be integrable on $[a, b]$ and $F(t) = \int_c^t f(x)dx$ for some $c \in [a, b]$. Then F is almost everywhere differentiable. EOL.

Proof. F is differentiable certainly where f is continuous. As f is almost everywhere continuous, the lemma follows. QED.

Corollary

Let $f \geq 0$ be integrable on $[a, b]$. If there is $x_0 \in [a, b]$ where f is continuous and $f(x_0) > 0$, then $\int_a^b f dx > 0$. EOC.

Proof. Let $F(t) = \int_a^t f(x)dx$. Then F is monotone increasing. It is also differentiable at x_0 and $F'(x_0) = f(x_0) > 0$. But then there is $x > x_0$ such that $F(x) > F(x_0) \geq 0$, and $F(b) \geq F(x) > 0$. QED.

Lemma

Let $f \geq 0$ be integrable on $[a, b]$. Then $\int_a^b f(x)dx = 0$ if and only if $f(x) = 0$ almost everywhere on $[a, b]$. EOL.

Proof. The only if part is clear: let $N = \{x \in [a, b] \mid f(x) > 0\}$. If N is not a set of measure zero, then N is not a subset of $NC(f)$, and there must be $x \in N$ where f is also continuous. By the previous corollary, $\int_a^b f(x)dx > 0$.

Now suppose N is a set of measure zero. And let $U = \{x \in [a, b] \mid x \notin N\}$. Then $f(x) = 0$ for all $x \in U$. Also U is dense in $[a, b]$. In other words, every sub-interval of $[a, b]$ contains elements of U (as N does not contain any interval). Let P_n a sequence of partitions with $m(P_n) \to 0$. We may then choose tagvectors as follows: for each n, $y_{ni} \in [x_{ni}, x_{n,i+1}] \cap U \neq \emptyset$. Then $S(P_n, \boldsymbol{y}_n, f) = 0$ by construction. It also converges to $\int_a^b f(x)dx$. QED.

Note the assumption that f *is* integrable is crucial: the Dirichlet function $\chi_{\mathbb{Q}}$ is not integrable but equal to zero almost everywhere.

7.3 Improper integrals

So far, we discussed integration on closed and bounded intervals. This was possible for *bounded* functions. What about unbounded functions? And what about unbounded intervals?

Consider the following example from classical mechanics:

Two masses m_1 and m_2 enact a gravitational force of $F(r) = \frac{Gm_1m_2}{r^2}$ on each other, where G is some constant, and r is the distance between the centre of masses.

For example, if the first body is Earth, and the second body is Enterprise, then to take Enterprise out of Earth's gravity field you must increase r to infinity (or reasonably close to that).

To do so, you must spend work (energy) (force times distance). However, the force is not constant.

So going form R_0 to R, you could compute $S(P, y, F) = \sum F(y_i)(r_{i+1} - r_i)$ as an approximation of the energy necessary to increase the distance from r to R where P is a partition of $[R_0, R]$ and y is a tag vector, so that $F(y)$ is an approximation of the average force on the interval $[r_i, r_{i+1}]$. It is reasonable to assume that the finer the mesh size of P the better the approximation of the actual energy needed.

Then $E(R) = \int_{R_0}^{R} \frac{Gm_1m_2}{r^2} dr = \left[-\frac{Gm_1m_2 1}{r}\right]_{R_0}^{R} = Gm_1m_2\left(\frac{1}{R_0} - \frac{1}{R}\right)$.

To completely escape Earth's gravity (which is of course impossible in this model), you therefore need an amount of energy equal to

$$E = \lim_{R\to\infty}\left(Gm_1m_2\left(\frac{1}{R_0} - \frac{1}{R}\right)\right) = \frac{Gm_1m_2}{R_0}$$

It is clearly natural to define

$$E = \int_{R_0}^{\infty} F(x)dx$$

7.3.1 Definition of improper integrals

Suppose f is integrable in an interval $[a, b]$. Then we have seen that

$$G(t) = \int_a^t f dx$$

is a continuous function on $[a, b]$, and in particular,

$$\lim_{t\to b^-} G(t) = G(b) = \int_a^b f dx$$

We may use the left hand side to define the right hand side, if f is not integrable on $[a, b]$. More precisely consider the following definition.

Definition

Let f be a function defined on an interval $[a, b)$ (and $b = \infty$ is allowed) and suppose f is integrable on all intervals $[a, t]$ where $a < t < b$. Then the **improper integral** $\int_a^b f dx$ is defined as

$$\int_a^b f dx \coloneqq \lim_{t\to b^-} \int_a^t f dx$$

if that limit exists (finite or infinite). Likewise, if f is defined on $(a, b]$ (and $a = -\infty$ is allowed) and integrable on all $[t, b]$ for $a < t < b$, then

$$\int_a^b f dx \coloneqq \lim_{t\to a+} \int_t^b f dx$$

if that limit exists.

If the limt is finite, we say the integral **converges**, and **diverges** otherwise. EOD.

Example

$$\int_0^1 \frac{1}{t} dt = \infty$$

Indeed, for any $t \in (0,1]$, we have $\int_t^1 \frac{1}{x} dx = \log(1) - \log(t) = -\log t \to \infty$ for $t \to 0$. EOE.

Remark

If f is defined on all of $[a, b]$, and is integrable, then this is not a new definition and the improper integral $\int_a^b f dx$ is the Riemann integral $\int_a^b f dx$. In fact, if f is bounded on $(a, b]$ say, and the improper integral $\int_a^b f\, dx$ exists, then, if we define $f(a)$ to be any number, f is bounded on $[a, b]$, and integrable, and its integral agrees with the improper integral.

This is a consequence of the Lebesgue criterion: Let f_n be the restriction of f to $[a + \frac{1}{n}, b]$. Then $NC(f) \subseteq \{a\} \cup \bigcup_{n \in \mathbb{N}} NC(f_n)$ is a set of measure zero.

The argument is similar if f is defined and bounded on $[a, b)$ and we choose any value for $f(b)$.

Therefore, improper integrals over bounded intervals are only interesting for unbounded functions.

EOR.

We can combine the two types of improper integrals above:

Definition

Let f be defined on (a, b) (and$a = -\infty$, or $b = \infty$ is allowed), and suppose for every $a < s \leq t < b$ f is integrable on $[s, t]$.

If for some $c \in (a, b)$ both improper integrals $\int_a^c f dx$ and $\int_c^b f dx$ converge, we put

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

EOD.

Note the existence is independent of the particular choice of c: for example, if $c_1 < c_2 \in (a, b)$ then

$$\int_a^{c_1} f dx + \int_{c_1}^{c_2} f dx = \int_a^{c_2} f dx$$

if one of the two sides exists. This is an elementary computation involving sums of limits. The same holds for the improper integrals $\int_{c_1}^b f dx$ and $\int_{c_2}^b f dx$ (if one exists so does the other, and $\int_{c_1}^b f dx = \int_{c_1}^{c_2} f dx + \int_{c_2}^b f dx$.

In particular, in the situation of the definition we have

$$\lim_{s \to a^+} \lim_{t \to b^-} \int_s^t f dx = \lim_{t \to b^-} \lim_{s \to a^+} \int_s^t f dx = \int_a^b f dx$$

Why?

7.3.2 Integration over unbounded intervals

Example

$$\int_0^\infty e^{-x} dx = 1$$

Indeed, $\int_0^t e^{-x} dx = [-e^{-x}]_0^t = e^0 - e^{-t} \to 1$ for $t \to \infty$. EOE.

Such integrals behave in a similar way as infinite series as far as convergence is concerned.

Note
In the following we will assume throughout that f is defined on $[a, \infty)$ and integrable on all closed subintervals $[a, t]$. EON.

Example
$\int_1^\infty \frac{1}{x^a} dx$ converges if and only if $a > 1$.
Indeed, if $a \neq 1$, then $\int_1^t \frac{1}{x^a} dx = \left[\frac{x^{1-a}}{1-a}\right]_1^t = \frac{t^{1-a}-1}{1-a}$ which has a finite limit only if $a > 1$, and this limit is then $\frac{1}{a-1}$.
If $a = 1$, then $\int_1^t \frac{1}{x} dx = \log t$, which diverges. EOE.

Cauchy Criterion for Integrals
$\int_a^\infty f dx$ converges if and only if for every ε, there is $t_0 > a$ such that for all $s > r > t_0$ we have

$$\left|\int_r^s f dx\right| < \varepsilon$$

EOL.

Compare this with the Cauchy criterion for infinite series: $\sum a_n$ converges if for every $\varepsilon > 0$ there is n_0 such hat for all $\ell > k > m_0$, $\left|\sum_{n=k}^{\ell} a_k\right| < \varepsilon$.

Proof. To simplify notation let $F(t) = \int_a^t f dx$. Then $\int_r^s f dx = F(s) - F(r)$. Suppose $L = \lim_{t\to\infty} F(t)$ exists and is finite. Let $\varepsilon > 0$. Then there is t_0 such that for all $t > t_0$, we have $|F(t) - L| < \frac{\varepsilon}{2}$. For $s > r > t_0$, we then have

$$|F(s) - F(r)| \leq |F(s) - L| + |L - F(r)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For the converse suppose that t_0 exists for each $\varepsilon > 0$.

We must show that $\lim_{t\to\infty} F(t)$ exists and is finite. Equivalently, it suffices to show that for every sequence $x_n \in [a, \infty)$ with $\lim_{n\to\infty} x_n = \infty$, we have $F(x_n)$ is a Cauchy sequence. For then each $F(x_n)$ converges to a finite limit. If x_n, y_n are two such sequences then z_n forms as $x_1, y_1, x_2, y_2, \ldots$ is a third sequence and $F(z_n)$ has a finite limit. But $F(x_n)$ and $F(y_n)$ are subsequences and hence must have the same limit.

So let $\varepsilon > 0$ and let t_0 as in the statement. There is n_0 such that $x_n > t_0$ for all $n > t_0$, and then $|F(x_m) - F(x_n)| < \varepsilon$ for all $m, n > n_0$. (If $x_m > x_n$ put $s = x_m, r = x_n$, if $x_m < x_n$, put $s = x_n, r = x_m$, and if $x_m = x_n$ this is clear.) QED.

Example
Consider $\int_0^\infty \frac{\sin x}{x} dx$. Technically this is doubly improper integral as $\frac{\sin x}{x}$ is not defined at 0. But $\lim_{x\to 0} \frac{\sin x}{x} = 1$, so what we mean here is the integral of f defined as $f(0) = 1$ and $f(x) = \frac{\sin x}{x}$ for $x > 0$. (As f is bounded around 0, there the value of f at 0 is irrelevant.)

Then for $s > r > 0$, $\int_r^s \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x}\right]_r^s - \int_r^s \frac{\cos x}{x^2} dx$ by the product rule.
Taking absolute values we find

$$\left|\int_r^s \frac{\sin x}{x} dx\right| \leq \frac{1}{s} + \frac{1}{r} + \int_r^s \frac{1}{x^2} dx = \frac{1}{s} + \frac{1}{r} + \left[-\frac{1}{x}\right]_r^s = \frac{2}{r}$$

Where we used the First Inequality and the Triangle Inequality to conclude

$$\left|\int_r^s \frac{\cos x}{x^2} dx\right| \leq \int_r^s \frac{|\cos x|}{x^2} dx \leq \int_r^s \frac{1}{x^2} dx$$

EOE.

Monotone criterion

Let $f \geq 0$. Then $\int_a^\infty f dx$ converges if and only if there is B such that for all $t > a$, $\int_a^t f dx < B$. EOL.

Proof. The function $F(t) = \int_a^t f dx$ is monotone increasing as $f \geq 0$, and therefore $F(t) - F(s) = \int_s^t f dx \geq 0$ if $t > s$ (this uses the First Inequality). As F is monotone increasing, it has a finite limit for $t \to \infty$ iff it is bounded above, and then $\lim_{t\to\infty} F(t) = \sup F$. QED.

Example

$\int_0^\infty x^x e^{-x^2} dx$ converges.
Note that here we put $0^0 = 1$ as we know that $\lim_{x\to 0^+} x^x = 1$. Also, $x^x e^{-x^2} = e^{x\log x - x^2} = e^{x(\log x - x)}$.
Now $\lim_{x\to\infty} (\log x - x) = -\infty$, from which we conclude that there is $t_0 > 0$ such that for $x > t_0$, $\log x - x < -1$.

$$\int_{t_0}^t x^x e^{-x^2} dx \leq \int_{t_0}^t e^{-x} dx \leq \int_0^\infty e^{-x} dx$$

The monotone criterion asserts that $\int_{t_0}^\infty x^x e^{-x^2} dx$ converges and therefore also

$$\int_0^\infty x^x e^{-x^2} dx = \int_0^{t_0} x^x e^{-x^2} dx + \int_{t_0}^\infty x^x e^{-x^2} dx$$

converges. EOE.

Definition

We say $\int_a^\infty f dx$ is absolutely convergent, if $\int_a^\infty |f| dx$ converges. EOD.

Lemma (Triangle Inequality)

If $\int_a^\infty f dx$ is absolutely convergent, then it is convergent and

$$\left|\int_a^\infty f dx\right| \leq \int_a^\infty |f|\, dx$$

EOL.

Proof. If the integral is absolutely convergent, the Cauchy Criterion applies to $\int_a^\infty |f| dx$. Given $\varepsilon > 0$, there is t_0 such that for all $s > r > t_0$ $\int_r^s |f| dx < \varepsilon$ (we do not need to take the absolute value of the integral as $|f| \geq 0$ and hence the integral is always nonnegative).

But then also

$$\left|\int_r^s f dx\right| \le \int_r^s |f| dx < \varepsilon$$

Thus, $\int_a^\infty f dx$ converges. The triangle inequality then follows from taking limits on both sides of the inequality

$$\left|\int_a^t f dx\right| \le \int_a^t |f| dx$$

QED.

Example

$\int_0^\infty \frac{\sin x}{x} dx$ does not converge absolutely.

For $n \in \mathbb{N}$, consider $\int_0^{n\pi} \frac{\sin x}{x} dx$. Then

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \ge \sum_{k=0}^{n-1} \frac{1}{(k+1)\pi} \cdot \int_0^\pi \sin x \, dx = \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \cdot [-\cos x]_0^\pi$$

Limit Criterion

Let $f, g > 0$ be defined on $[a, \infty)$ and suppose $L = \lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is finite.

1. If $L > 0$, then $\int_a^\infty f dx$ converges iff $\int_a^\infty g dx$ does.
2. If $L = 0$, then $\int_a^\infty f dx$ converges if $\int_a^\infty g dx$ does.

EOL.

Proof. Suppose $\int_a^\infty g dx$ converges. There is x_0 such that for all $x \ge x_0 \ge a$, we have $\frac{f(x)}{g(x)} < L + 1$. Then $f(x) < (L+1)g(x)$ and $\int_{x_0}^t f(x)dx \le (L+1)\int_{x_0}^t g(x)dx \le (L+1)\int_{x_0}^\infty g(x)dx$. By the Monotone Criterion we then have $\int_a^\infty f(x)dx$ converges.

If $L > 0$, then $\frac{g(x)}{f(x)} \to L^{-1}$ for $x \to \infty$ and we can repeat the argument if with the roles of f and g interchanged if $\int_a^\infty f dx$ converges to conclude that $\int_a^\infty g dx$ converges. QED.

Remark

The similarity between infinite series and such integrals is not absolute: if $\int_a^\infty f(x)dx$ converges it does not necessarily mean that f is bounded or that $\lim\limits_{x\to\infty} f(x) = 0$.

As an example define f on $[2, \infty)$ as follows:

For a natural number $n \ge 2$, on $[n, n+1)$ f is defined as $f(x) = n$ if $n \le x \le n + \frac{1}{n^3}$, and $f(x) = 0$ otherwise.

Then f is integrable on any bounded interval of the form $[2, t]$. By the monotone criterion we must show $\int_2^t f(x)dx$ is bounded.

Let $N > t$ be a natural number, then f is a step function on $[n, n+1]$ and

$$\int_2^t f(x)dx \le \int_2^N f(x)dx = \sum_{n=2}^{N-1} \int_n^{n+1} f(x)dx = \sum_{n=2}^{N-1} \frac{n}{n^3} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

EOR.

7.3.3 The integral criterion

Theorem (Integral criterion)
Let $f \geq 0$ be defined and monotone descending on $[m, \infty)$ where $m \in \mathbb{N}$.
Then $\int_m^\infty f(x)dx$ converges if and only if the infinite series $\sum_{n=m}^\infty f(n)$ converges. EOT.

Proof. Since $f \geq 0$ and f is monotone descending we have for all $n \geq m$ in $\mathbb{N}$,

$$f(n) \geq \int_n^{n+1} f(x)dx \geq f(n+1)$$

Since $\int_m^{N+1} f(x)dx = \sum_{n=m}^N \int_n^{n+1} f(x)dx$ we conclude

$$\sum_{n=m}^N f(n) \geq \int_m^{N+1} f(x)dx \geq \sum_{n=m}^N f(n+1)$$

The monotone criterion then shows that the integral converges if the series converges. And if the integral converges the series is bounded and hence convergent. QED.

Note we do not need to assume that f is integrable as this is automatic as it is monotone.

Example
$\sum_{n=1}^\infty \frac{1}{n^a}$ converges iff $a > 1$.
Indeed, $\sum_{n=1}^\infty \frac{1}{n^a}$ converges iff $\int_1^\infty \frac{1}{x^a} dx$ converges. EOE.

7.3.4 Improper integrals over bounded intervals

Finally, all criteria for convergence of integrals over unbounded intervals apply essentially unchanged to improper integrals of unbounded functions on bounded intervals.

Rather than reproving everything, we illustrate this by means of a few examples.

Suppose $a \neq b$.

$\int_a^b \frac{1}{(x-a)^c} dx$ converges iff $c < 1$.

If $c \neq 1$, then $\int \frac{1}{(x-a)^c} = \frac{(x-a)^{1-c}}{1-c}$. Hence the integral becomes $\lim_{t \to a} \frac{(b-a)^{1-c} - (t-a)^{1-c}}{1-c}$ which is finite only if $1 - c > 0$. And in this case, it is equal to $\frac{(b-a)^{1-c}}{1-c}$.

If $c = 1$, the integral is $\log(b-a) - \lim_{t \to a} \log(t-a)$ which is not finite.

$\int_0^1 \frac{\log x}{\sqrt{x}} dx$ converges (absolutely).

We use the limit criterion: let $f(x) = \frac{|\log x|}{\sqrt{x}}$ and $g(x) = \frac{1}{x^{\frac{3}{4}}}$ defined and positive on $(0,1]$.

Then $\frac{f(x)}{g(x)} = x^{\frac{1}{4}}|\log x|$. By LH we know that $\lim_{x\to 0} x^a \log x = 0$ whenever $a > 0$. (Indeed, $\frac{\log x}{\frac{1}{x^a}} \to 0$ because $\frac{\frac{1}{x}}{(-a)x^{-a-1}} = \frac{x^a}{-a} \to 0$).

Therefore $\int_0^1 f(x)dx$ converges if $\int_0^1 g(x)\,dx$ converges. The latter converges by the previous example.

7.3.5 Step functions*

It is possible to phrase Darboux-integration entirely in terms of step functions. This is relevant as the more general notion of Lebesgue integration can also be based on step functions.

The idea is to take a function f and "approximate" it by functions that are constant on some subintervals.

Definition
Let $[a,b]$ be a closed interval, and let I be one of $[a,b], (a,b), [a,b), (a,b]$. A function $\varphi: I \to \mathbb{R}$ is called a **step function** if there is a partition $P = x_1 < x_2 < \cdots < x_n \in \Pi(a,b)$, such that φ Is constant on each of the intervals (x_i, x_{i+1}) $(i = 0,1,\ldots,n)$. If φ is a step function, and P is a partition with this property then φ, P are said to be **compatible**. For any P, we denote by $\Phi(P)$ the set of all compatible step functions on $[a,b]$. For any step function φ we denote by $\Pi(\varphi)$ the set of all compatible partitions in $\Pi(a,b)$. EOD.

We do not care what the values of φ are at the boundaries of the intervals. Any piecewise constant function is a step function.

Lemma
Step functions on $[a,b]$ are integrable. In fact, if φ is a step function and P is a compatible partition, then

$$\int_a^b \varphi dx = \sum_{i=0}^{|P|} \alpha_i (x_{i+1} - x_i)$$

where $\varphi(x) = \alpha_i$ on the open interval (x_i, x_{i+1}). EOL.

Proof. One can prove this directly using Darboux integration (see the homework problem solution). However, now that we have more theory at our hands, we give a quick and dirty proof:

Note φ is integrable because it is bounded and almost everywhere continuous. Therefore,

$$\int_a^b \varphi dx = \sum_{i=0}^{|P|} \int_{x_i}^{x_{i+1}} \varphi dx$$

It therefore suffices to show that $\int_{x_i}^{x_{i+1}} \varphi dx = \alpha_i(x_{i+1} - x_i)$. But this is clear: as φ is integrable, its integral coincides with the *improper* integral on (x_i, x_{i+1}). Therefore

$$\int_{x_i}^{x_{i+1}} \varphi dx = \lim_{r\to x_i^+} \lim_{s\to x_{i+1}^-} \int_r^s \varphi dx = \lim_{r\to x_i^+} \lim_{s\to x_{i+1}^-} \alpha_i(s-r) = \alpha_i(x_{i+1} - x_i)$$

QED.