Advanced Recurrence Relations

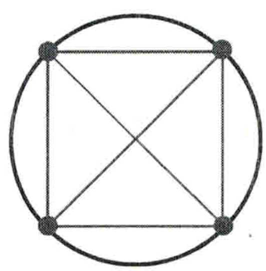
Example 1: There are $n \geq 1$ points on a circle, every two of which are joined by a chord. No three chords pass through a common point which lies inside the circle. Let a_n be the number of regions within the circle. Start by finding $a_1, a_2, a_3, a_4, a_5, a_6$ and then find a formula for a_n.

$a_1 = 1$

$a_1 = 2$

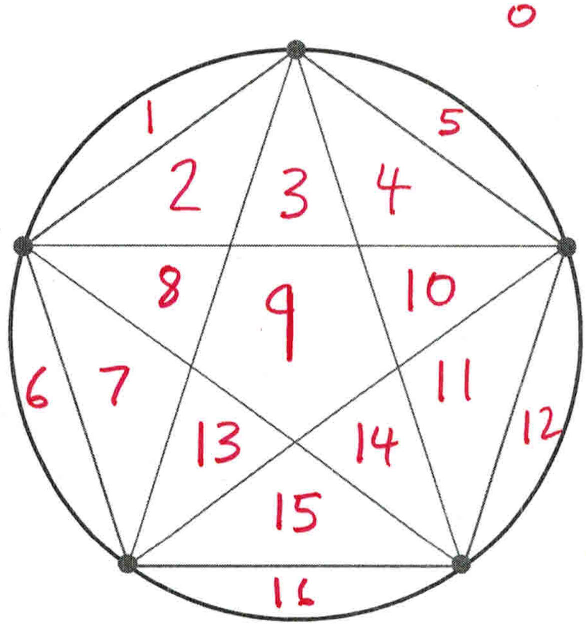
$a_3 = 4$

$a_4 = 8$

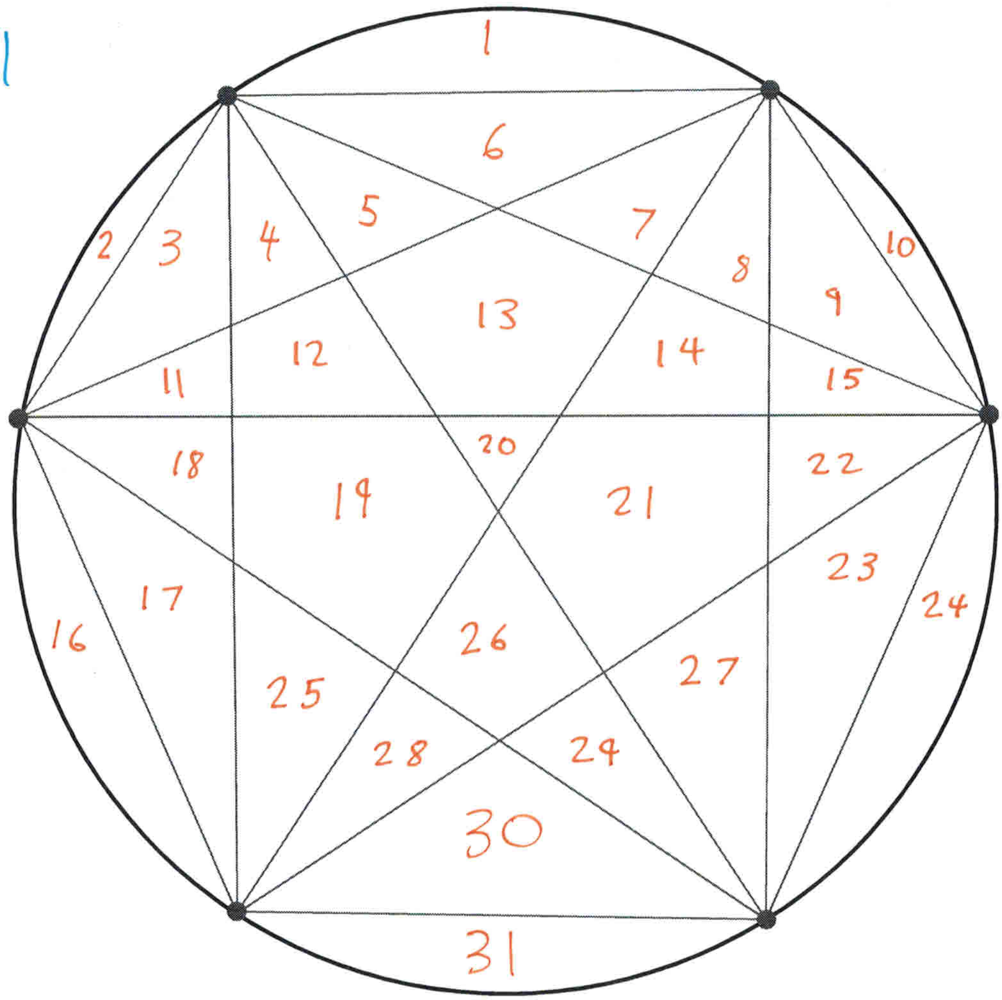


It seems like $a_n = 2^n$?

$a_5 = 16$



$a_6 = 31$



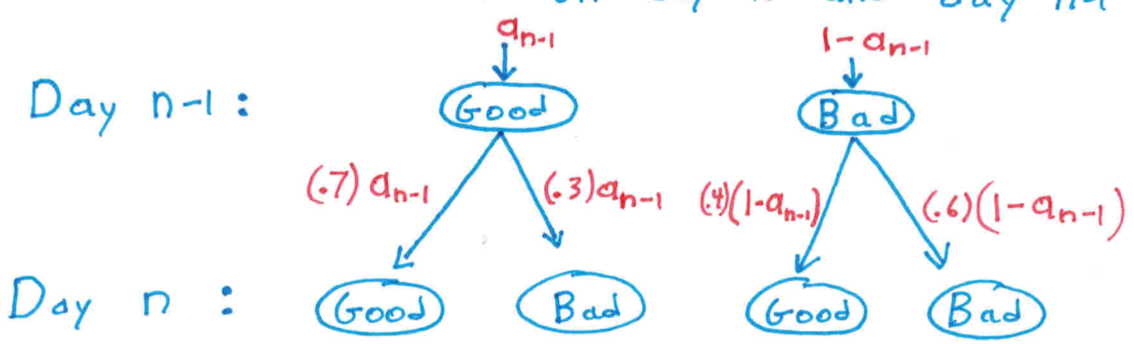
$a_n = 1 +$ "the number of lines" $+$ "the number of $\cap$ pts"

- There is one line for every two points: 
  $\therefore$ there are $\binom{n}{2}$ lines.
- There is one $\cap$ pt for every 4 points: 
  $\therefore$ there are $\binom{n}{4}$ $\cap$ pts.

$$\therefore \boxed{a_n = 1 + \binom{n}{2} + \binom{n}{4} \quad , n \geq 1}$$

Example 2: On the planet Jubilation, the weather each day is either good or bad. If it is good, then it remains good the next day 70% of the time. If it is bad, then it remains bad the next day 60% of the time. Let a_n be the probability that weather is good on day n. Find a recurrence relation and solve the recurrence. If you landed on Jubilation on a random day what are the chances of having good weather (in other words, what happens as n goes to ∞)?

Consider the weather on day n and day n-1 :



There are 2 ways the weather will end up being good on day n :

- " % the weather on day n-1 is good and ______ n ______ " $= (.7)\,a_{n-1}$

OR

- " % the weather on day n-1 is bad and ______ n – good " $= (.4)(1-a_{n-1})$

$$\therefore\ a_n = (.7)\,a_{n-1} + (.4)(1-a_{n-1})$$

$$= (.3)\,a_{n-1} + (.4)$$

Letting $a_0 = p$ the recurrence relation is :

$$\boxed{\begin{aligned} a_0 &= p \\ a_n &= (.3)\, a_{n-1} + (.4) \end{aligned}}$$

solve

$$a_0 = p$$

$$a_1 = (.3)p + (.4)$$

$$a_2 = (.3)^2 p + (.3)(.4)$$

$$a_3 = (.3)^3 p + (.3)^2 (.4) + (.4)$$

$$\vdots$$

$$a_n = (.3)^n p + (.3)^{n-1} (.4) + (.3)^{n-2} (.4) + \cdots + (.3)(.4) + (.4)$$

$$= (.3)^n p + (.4)\left((.3)^{n-1} + (.3)^{n-2} + \cdots + (.3) + 1\right)$$

$$= (.3)^n p + (.4) \frac{1-(.3)^n}{1-(.3)} \quad \text{(by Lecture 12)}$$

$$= \boxed{(.3)^n p + \frac{(.4)(1-(.3)^n)}{(.7)}}$$

as $n \to \infty$

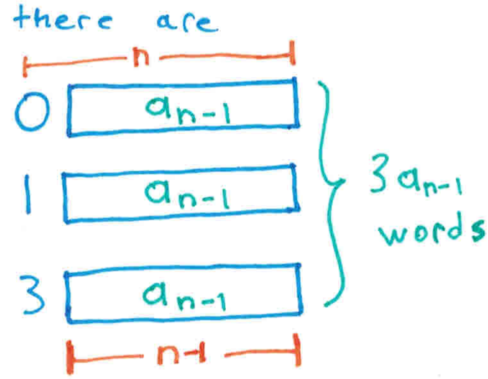
$$a_n = \frac{.4}{.7} \cong \boxed{57\%}$$

Example 3: Let a_n be the number of words of length $n \geq 1$ containing the digits $\{0, 1, 2, 3\}$ with an even number of 2's. Start by calculating $a_1, a_2,$ and a_3 then find recurrence relation and solve it.

Length	Words	a_n
1	0 1 3	$a_1 = 3$
2	00 10 30 01 11 31 03 13 33 22	$a_2 = 10$
3	000 100 300 001 101 301 003 103 303 010 110 310 011 111 311 013 113 313 030 130 330 031 131 331 033 133 333 022 122 322 220 221 223 202 212 232	$a_3 = 36$

$a_n = ?$

CASE 1 The first digit is not a 2. Then there are $n-1$ remaining digits that have an even # of 2's. Since there are 3 choices for the first digit there are $3a_{n-1}$ words in CASE 1.

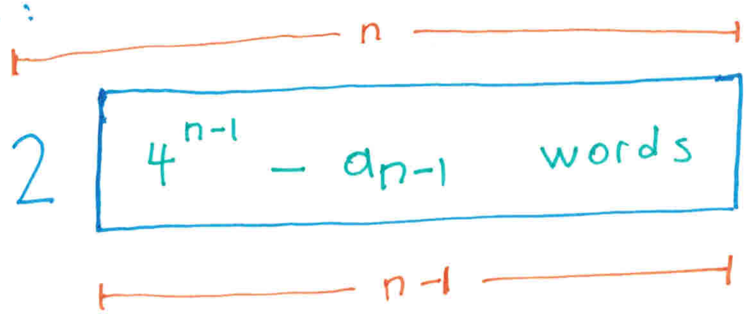


CASE 2 The first digit is a 2. There is an odd # of 2's among the remaining $n-1$ digits. The number of words of length $n-1$ with an odd # of 2's is:

"Total # of words" − "number of words with an even amount of 2's"

$$= 4^{n-1} - a_{n-1}$$

picture:



$$\therefore \boxed{\begin{array}{l} a_1 = 3 \\ a_n = 2a_{n-1} + 4^{n-1} \end{array}}$$ is the r.r.

Solve: $a_1 = 3$
$a_n = 2a_{n-1} + 2^{2(n-1)}$

$a_1 = 3$

$a_2 = 3 \cdot 2 + 2^{2(1)}$

$a_3 = 3 \cdot 2^2 + 2^3 + 2^{2(3-1)}$

$a_4 = 3 \cdot 2^3 + 2^4 + 2^5 + 2^6$

$a_5 = 3 \cdot 2^4 + 2^5 + 2^6 + 2^7 + 2^8$

$\vdots$

$$a_n = 3 \cdot 2^{n-1} + 2^n + 2^{n+1} + \cdots + 2^{2(n-1)}$$

$$= 3 \cdot 2^{n-1} + (1 + 2^1 + 2^2 + \cdots + 2^{n-1}) + 2^n + \cdots + 2^{2(n-1)} - (1 + 2^1 + 2^2 + \cdots + 2^{n-1})$$

$$= 3 \cdot 2^{n-1} + \left(\frac{2^{2n-1} - 1}{2-1}\right) - \left(\frac{2^n - 1}{2-1}\right)$$ (by lecture 12)

$$= 2^{n-1}(3 + 2^n - 2)$$

$$= \boxed{2^{n-1}(2^n + 1)}$$

Note:

$$\underbrace{\begin{pmatrix}\text{\# of words} \\ \text{with an even} \\ \text{\# of 2's}\end{pmatrix}}_{2^{n-1}(2^n+1)} + \underbrace{\begin{pmatrix}\text{\# of words} \\ \text{with an odd} \\ \text{\# of 2's}\end{pmatrix}}_{2^{n-1}(2^n-1)} = \underbrace{\begin{pmatrix}\text{\#} \\ \text{of} \\ \text{words}\end{pmatrix}}_{4^n}$$

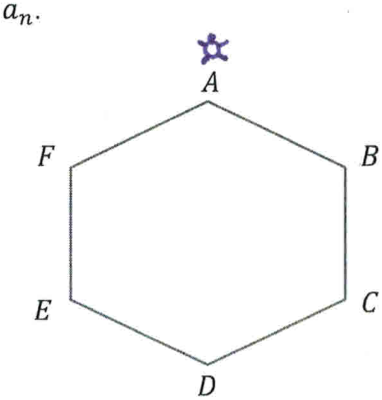
This works since:

$$\begin{aligned}
&2^{n-1}(2^n+1) + 2^{n-1}(2^n-1) \\
&= 2^{n-1}\left(2^n+1 + 2^n - 1\right) \\
&= 2^{n-1}\left(2 \cdot 2^n\right) \\
&= 2^{n-1} \cdot 2^{n+1} \\
&= 2^{2n} \\
&= 4^n
\end{aligned}$$

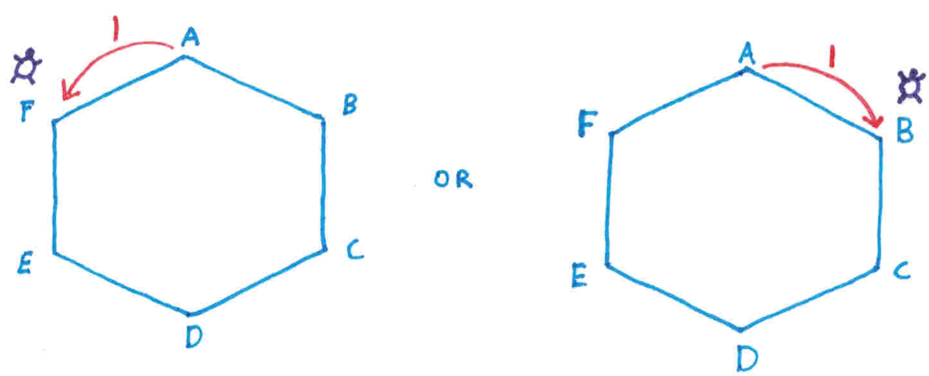
Example 4: A bug starts at vertex A of the regular hexagon below and each minute travels to an adjacent vertex. There is a spider web on vertex D if the bug moves there it is stuck. Let

a_n be the number of different ways the bug can travel from vertex A to vertex D after n minutes.
b_n be the number of different ways the bug can travel from vertex B to vertex D after n minutes.
f_n be the number of different ways the bug can travel from vertex F to vertex D after n minutes.

Set up a recurrence relation for a_n.



The bugs first move is to F or B:



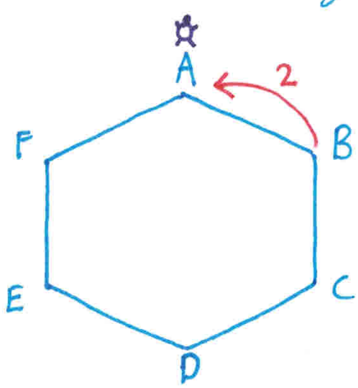
$$\therefore \; a_n = f_{n-1} + b_{n-1}$$

By the vertical symmetry of the hexagon we have:

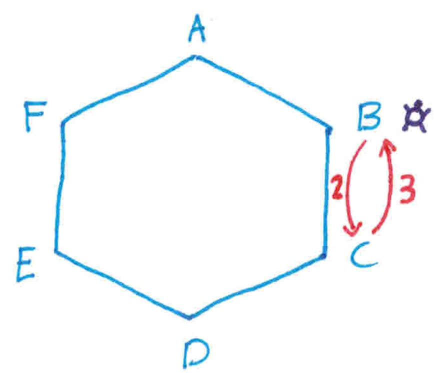
$$f_n = b_n \quad \text{for all } n > 0.$$

$$\therefore \; a_n = 2\, b_{n-1} \qquad n>0 \qquad (1)$$

If the bugs first move is to B then the second move is to A or C. If the bugs second move is to C and $n > 3$ then the bug must return to B or else it gets stuck at D too soon.



OR



$$b_{n-1} = a_{n-2} + b_{n-3} \qquad n>3 \qquad (2)$$

Now, sub equation (2) into equation (1)

$$a_n = 2(a_{n-2} + b_{n-3}) \qquad \text{for } n>3$$

$$= 2a_{n-2} + 2b_{n-3}$$

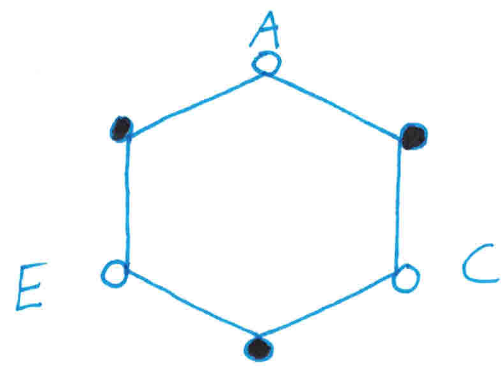
$$= 2a_{n-2} + a_{n-2} \qquad \text{by equation (1)}$$

$$= 3a_{n-2}$$

$$\therefore \boxed{\begin{array}{l} a_2 = 0 \\ a_3 = 2 \\ a_n = 3a_{n-2} \quad \text{for } n>3 \end{array}}$$

is the r.r.

Note: if the bug makes an even # of moves it will never be on vertex D. To see why we can colour the vertices:



$\therefore$ after an even # of moves the bug is on vertex A, C, or E.