**MATH 217** (Fall 2021)

Honors Advanced Calculus, I

***Final Model Solutions***

1. Determine and classify all stationary points of

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \quad (x,y,z) \mapsto x^3 - 3x - y^3 + 9y + z^2.$$

If $f$ attains a local minimum or maximum at one of its stationary points, evaluate it there.

*Solution*: The first order partial derivatives of $f$ are computed as

$$\frac{\partial f}{\partial x} = 3x^2 - 3, \qquad \frac{\partial f}{\partial x} = -3y^2 - 9, \qquad \text{and} \qquad \frac{\partial f}{\partial z} = 2z.$$

It is immediate that these derivatives vanish simultaneously at $(x,y,z) \in \mathbb{R}^3$ if and only if $x^2 = 1$, $y^2 = 3$, and $z = 0$. Hence, the critical points of $f$ are $\left(1, \sqrt{3}, 0\right)$, $\left(-1, \sqrt{3}, 0\right)$, $\left(-1, -\sqrt{3}, 0\right)$, and $\left(1, -\sqrt{3}, 0\right)$,

The next step is to compute the second order partial derivatives of $f$. We have

$$\frac{\partial^2 f}{\partial x^2} = -6x, \qquad \frac{\partial^2 f}{\partial y^2} = 6y, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial z^2} = 2,$$

as well as

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial z} = \frac{\partial^2 f}{\partial z\,\partial x} = 0,$$

so that

$$\operatorname{Hess} f = \begin{bmatrix} 6x & 0 & 0 \\ 0 & -6y & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Clearly, $\operatorname{Hess} f$ has the positive eigenvalue 2. Hence, $f$ attains a local minimum at $\left(1, -\sqrt{3}, 0\right)$: the eigenvalues of $\operatorname{Hess} f$ are 6, $6\sqrt{3}$, and 2. At all other critical points, $\operatorname{Hess} f$ has at least one negative eigenvalue, so that $f$ has a saddle at those points. Finally, note that $f\left(1, -\sqrt{3}, 0\right) = -2 - 6\sqrt{3}$.

2. Let $R > 0$, and define, for $0 < \rho < R$,

$$A_{\rho,R} := \{(x,y,z) \in \mathbb{R}^3 : \rho^2 \leq x^2 + y^2 + z^2 \leq R^2\}.$$

Determine

$$\lim_{\rho \to 0} \int_{A_{\rho,R}} \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

*Solution*: Use spherical coordinates. This means that, for $0 < \rho < R$, we have $A_{\rho,R} = \phi(K)$ where

$$K := \left\{ (r, \theta, \sigma) \in \mathbb{R}^3 : r \in [\rho, R],\, \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right],\, \theta \in [0, 2\pi] \right\}.$$

It follows that

$$\begin{aligned}
\int_{A_{\rho,R}} \frac{1}{\sqrt{x^2+y^2+z^2}} &= \int_K \frac{r^2 \cos\theta}{r} \\
&= \int_K r\cos\theta \\
&= \int_\rho^R \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \int_0^{2\pi} r\cos\theta \, d\sigma \right) d\theta \right) dr \\
&= 2\pi \int_\rho^R \left( r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta \right) dr \\
&= 4\pi \int_\rho^R r \, dr \\
&= 2\pi(R^2 - \rho^2) \\
&\overset{\rho \to 0}{\to} 2\pi R^2.
\end{aligned}$$

3. Let $I \subset \mathbb{R}^N$ be a compact interval. Show that

$$\mathcal{A} := \{ A \subset I : A \text{ has content} \}$$

is an *algebra* over $I$, i.e.,

(a) $\varnothing, I \in \mathcal{A}$,

(b) if $A \in \mathcal{A}$, then $I \setminus A \in \mathcal{A}$, and

(c) if $A_1, \ldots, A_n \in \mathcal{A}$, then $A_1 \cup \cdots \cup A_n \in \mathcal{A}$.

*Solution*: As the constant functions $0 = \chi_\varnothing$ and $1 = \chi_I$ are trivially Riemann integrable on $I$, (a) is clear.

Let $A \in \mathcal{A}$, i.e., $\chi_A$ is Riemann integrable on $I$. Consequently, $\chi_{I \setminus A} = \chi_I - \chi_A$ is Riemann integrable, so that $I \setminus A \in \mathcal{A}$.

For (c), we may suppose that $n = 2$. So, let $A, B \in \mathcal{A}$. By Problem 3 on Assignment #8, $\chi_{A \cap B} = \chi_A \chi_B$ is Riemann integrable. As

$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B},$$

it follows that $\chi_{A \cup B}$ is Riemann integrable, i.e., $A \cup B \in \mathcal{A}$.

4. Let $\varnothing \neq D \subset \mathbb{R}^N$. A point $x_0 \in D$ is called an *isolated point* of $D$ if there is $\epsilon > 0$ such that $B_\epsilon(x_0) \cap D = \{x_0\}$. Show that the following are equivalent for $x_0 \in D$:

   (i) $x_0$ is an isolated point of $D$;

   (ii) $x_0$ is not a cluster point of $D$;

   (iii) every sequence $(x_n)_{n=1}^\infty$ in $D$ such that $\lim_{n\to\infty} x_n = x_0$ is eventually constant, i.e., there is $n_0 \in \mathbb{N}$ such that $x_n = x_0$ for all $n \geq n_0$;

   (iv) every function $f\colon D \to \mathbb{R}$ is continuous at $x_0$.

   *Solution*: (i) $\iff$ (ii) is clear by the very definitions of a cluster point and of an isolated point, respectively.

   Let $x_0 \in D$ be an isolated point of $D$, and let $f\colon D \to \mathbb{R}$ be a function. Let $(x_n)$

   (i) $\Longrightarrow$ (iii): Let $(x_n)_{n=1}^\infty$ be a sequence in $D$ such that $\lim_{n\to\infty} x_n = x_0$. Choose $\epsilon > 0$ such that $B_\epsilon(x_0) \cap D = \{x_0\}$. As $\lim_{n\to\infty} x_n = x_0$, there is $n_\epsilon \in \mathbb{N}$ such that $\|x_n - x_0\| < \epsilon$ for all $n \geq n_\epsilon$ and therefore $x_n = x_0$ for all $n \geq n_\epsilon$.

   (iii) $\Longrightarrow$ (iv): Let $f : D \to \mathbb{R}$ be a function, and let $(x_n)_{n=1}^\infty$ be a sequence in $D$ such that $\lim_{n\to\infty} x_n = x_0$. Then $(x_n)_{n=1}^\infty$ is eventually constant as is, consequently, $(f(x_n))_{n=1}^\infty$, so that $\lim_{n\to\infty} f(x_n) = f(x_0)$. Therefore, $f$ is continuous at $x_0$.

   (iv) $\Longrightarrow$ (i): Define

$$f\colon D \to \mathbb{R}, \quad x \mapsto \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

   As $f$ is continuous at $x_0$, there is $\delta > 0$ such that $|f(x) - f(x_0)| < 1$ for all $x \in D$ with $|x - x_0| < \delta$, which is possible only if $x = x_0$ for all $x \in D$ with $|x - x_0| < \delta$, i.e., if $B_\delta(x_0) \cap D = \varnothing$.

5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} \frac{e^{xy}-1}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{otherwise.} \end{cases}$$

   Check—and justify— whether or not $f$ is

   (a) partially differentiable,

   (b) continuous,

   (c) totally differentiable,

   (d) continuously partially differentiable, or

   (e) Riemann integrable on $[-1,1] \times [-1,1]$.

   *Solution*:

(a) Clearly, $f$ is partially differentiable at every point of $\mathbb{R}^2 \setminus \{(0,0)\}$. Since

$$\frac{f(h,0)-f(0,0)}{h} = 0 = \frac{f(0,h)-f(0,0)}{h}$$

for $h \neq 0$, it is clear that $f$ is partially differentiable at $(0,0)$ as well.

(b) Since

$$f\left(\frac{1}{n},\frac{1}{n}\right) = \frac{e^{\frac{1}{n^2}}-1}{\frac{2}{n^2}} \to \frac{1}{2} \neq 0,$$

$f$ is not continuous at $(0,0)$.

(c) Since total differentiability implies continuity, $f$ is not totally differentiable.

(d) Since continuously partially differentiable functions are totally differentiable, $f$ is not continuously partially differentiable.

(e) Clearly, $f$ is discontinuous only at $(0,0)$. It is therefore sufficient to show that $f$ is bounded on $[-1,1]\times[-1,1]$. First note that, since $\lim_{\substack{h\to 0\\ h\neq 0}} \frac{e^h-1}{h} = 1$, there is $C \geq 0$ such that $|e^h - 1| \leq C|h|$ for all $h \in [-1,1]$. For $(x,y) \in ([-1,1]\times[-1,1]) \setminus \{(0,0)\}$, we obtain

$$\begin{aligned}
|f(x,y)| &= \frac{|e^{xy}-1|}{x^2+y^2} \\
&\leq C\frac{|xy|}{x^2+y^2} \\
&= C\frac{\sqrt{x^2y^2}}{x^2+y^2} \\
&\leq C\frac{1}{2}\frac{x^2+y^2}{x^2+y^2}, \\
&\qquad \text{by the inequality between geometric and arithmetic mean,} \\
&= \frac{C}{2}.
\end{aligned}$$

Consequently, $f$ is Riemann integrable on $[-1,1]\times[-1,1]$.

6. Let $f\colon [a,b] \to (0,\infty)$ be continuous. Show that

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{1}{f(x)}\,dx\right) \geq (b-a)^2.$$

(*Hint*: Apply Fubini's Theorem to $[a,b]^2 \ni (x,y) \mapsto \frac{f(x)}{f(y)}$.)

*Solution*: Set $I := [a,b]^2$, and define

$$F\colon I \to \mathbb{R}, \quad (x,y) \mapsto \frac{f(x)}{f(y)}$$

Then $F$ is continuous and thus Riemann integrable. By Fubini's Theorem, we have

$$\begin{aligned}\int_I F &= \int_a^b \left( \int_a^b \frac{f(x)}{f(y)}\, dy \right) dx \\ &= \int_a^b \left( f(x) \int_a^b \frac{1}{f(y)}\, dy \right) dx \\ &= \left( \int_a^b f(x)\, dx \right) \left( \int_a^b \frac{1}{f(x)}\, dx \right)\end{aligned}$$

and similarly

$$\int_I \frac{1}{F} = \left( \int_a^b f(x)\, dx \right) \left( \int_a^b \frac{1}{f(x)}\, dx \right).$$

Consequently, we obtain

$$\begin{aligned}&\left( \int_a^b f(x)\, dx \right) \left( \int_a^b \frac{1}{f(x)}\, dx \right) \\ &= \frac{1}{2} \int_I \left( F + \frac{1}{F} \right) \\ &= \int_I \frac{1}{2} \left( F + \frac{1}{F} \right) \\ &\geq \int_I \sqrt{F \frac{1}{F}}, \qquad \text{by the inequality between arithmetic and geometric mean,} \\ &= \int_I 1 \\ &= (b-a)^2.\end{aligned}$$

7. Let $A \in M_N(\mathbb{R})$ be symmetric. Show that

$$f \colon \mathbb{R}^N \to \mathbb{R}, \quad x \mapsto Ax \cdot x$$

is totally differentiable, and that

$$(Df)(x)\xi = 2Ax \cdot \xi$$

for $x, \xi \in \mathbb{R}^N$.

*Solution*: Let $x, \xi \in \mathbb{R}^N$, and note that

$$\begin{aligned}f(x+\xi) &= A(x+\xi) \cdot (x+\xi) \\ &= Ax \cdot x + Ax \cdot \xi + A\xi \cdot x + A\xi \cdot \xi \\ &= f(x) + 2Ax \cdot \xi + A\xi \cdot \xi, \qquad \text{as } A \text{ is symmetric.}\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\|f(x+\xi)-f(x)-2Ax\cdot\xi\|}{\|\xi\|} &= \frac{\|A\xi\cdot\xi\|}{\|\xi\|} \\
&\leq \frac{\|A\xi\|\|\xi\|}{\|\xi\|}, \qquad \text{by Cauchy–Schwarz,} \\
&= \|A\xi\| \\
&\overset{\|\xi\|\to 0}{\to} 0, \qquad \text{because linear maps are continuous.}
\end{aligned}$$

Hence, $f$ is totally differentiable at $x$, and

$$(Df)(x)\xi = 2Ax\cdot\xi$$

for $\xi \in \mathbb{R}^N$.

8. A force field $f = (P,Q)$ with $P,Q\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$P(x,y) = ye^x - y^3 + \arctan x \qquad \text{and} \qquad Q(x,y) = e^x + x^3 - e^{y^{2021}}$$

for $x,y \in \mathbb{R}$ moves a particle along the curve—in counterclockwise orientation—consisting of the line segment $\{(x,0) : x \in [-1,1]\}$ followed by the arc $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1,\ y \geq 0\}$. Determine the work done.

*Solution*: Let

$$D := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1,\ y \geq 0\},$$

i.e., the part of the closed unit disc lying above the $x$-axis. It is clear that the curve described is $\partial D$ in counterclockwise orientation. So, by Green's Theorem, the work done is

$$\int_{\partial D} P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

As

$$\frac{\partial Q}{\partial x} = e^x + 3x^2 \qquad \text{and} \qquad \frac{\partial P}{\partial y}(x,y) = e^x - 3y^2$$

for $x, y \in \mathbb{R}$, this means that

$$\begin{aligned}
&\int_{\partial D} P\,dx + Q\,dy \\
&= 3\int_D x^2 + y^2 \\
&= 3\int_{[0,1]\times[0,\pi]} ((r\cos\theta)^2 + (r\sin\theta)^2)r, \qquad \text{passing to polar coordinates,} \\
&= 3\int_{[0,1]\times[0,\pi]} r^3 \\
&= 3\int_0^1 \left(\int_0^\pi r^3\,d\theta\right) dr, \qquad \text{by Fubini's Theorem,} \\
&= 3\pi\int_0^\pi r^3\,dr \\
&= \frac{3\pi}{4}.
\end{aligned}$$