MATH 217 (Fall 2020)

Honors Advanced Calculus, I

Solutions #8

1. Let I be a compact interval, and let $f = (f_1, \ldots, f_M) : I \to \mathbb{R}^M$. Show that f is Riemann integrable if and only if $f_j : I \to \mathbb{R}$ is Riemann integrable for each $j = 1, \ldots, M$ and that, in this case,

$$\int_I f = \left(\int_I f_1, \ldots, \int_I f_M \right)$$

holds.

Solution: Suppose that f is Riemann integrable. Fix $k \in \{1, \ldots, M\}$, and let $y = (y_1, \ldots, y_M)$ be the Riemann integral of f over I. Let $\epsilon > 0$. Then there is a partition $\mathcal{P}_\epsilon$ of I such that, for each refinement $\mathcal{P}$ of $\mathcal{P}_\epsilon$ and each associated Riemann sum $S(f, \mathcal{P})$, we have

$$|S(f_k, \mathcal{P}) - y_k| \leq \|S(f, \mathcal{P}) - y\| < \epsilon.$$

This means that f_k is Riemann integrable with $\int_I f_k = y_k$.

Conversely, suppose that f_j is Riemann integrable with integral y_j for $j = 1, \ldots, M$. Set $y := (y_1, \ldots, y_M)$. Let $\epsilon > 0$. For each $j = 1, \ldots, M$, there is a partition $\mathcal{P}_j$ of I such that, for each refinement $\mathcal{P}$ of $\mathcal{P}_j$, we have

$$|S(f_j, \mathcal{P}) - y_j| < \frac{\epsilon}{\sqrt{M}}$$

for each Riemann sum $S(f_j, \mathcal{P})$. Let $\mathcal{P}_\epsilon$ be a common refinement of $\mathcal{P}_1, \ldots, \mathcal{P}_M$. Then for every refinement $\mathcal{P}$ of $\mathcal{P}_\epsilon$ and each Riemann sum $S(f, \mathcal{P})$, we obtain

$$\|S(f, \mathcal{P}) - y\| \leq \sqrt{M} \max_{j=1,\ldots,M} |S(f_j, \mathcal{P}) - y_j| < \sqrt{M} \frac{\epsilon}{\sqrt{M}} = \epsilon.$$

Consequently, f is Riemann integrable with $\int_I f = y$.

2. Let $I \subset \mathbb{R}^N$ be a compact interval, and let $f : I \to \mathbb{R}^M$ be Riemann integrable. Show that f is bounded.

Solution: Assume towards a contradiction that f is not bounded.

Let $\mathcal{P}$ be a partition of I—with corresponding subdivision $(I_\nu)_\nu$ of I—such that

$$\left\| S(f, \mathcal{P}) - \int_I f \right\| < 1$$

for each Riemann sum $S(f, \mathcal{P})$ of f corresponding to $\mathcal{P}$. In particular, this means that

$$\|S(f, \mathcal{P})\| \leq 1 + \left\| \int_I f \right\| =: C$$

for each such Riemann sum $S(f,\mathcal{P})$. Since f is assumed to be unbounded and since $I=\bigcup_\nu I_\nu$, there is at least one ν_0 such that f is unbounded on I_{ν_0}. Choose $x_{\nu_0}\in I_{\nu_0}$ such that

$$\|f(x_{\nu_0})\| > \frac{1}{\mu(I_{\nu_0})}\left(C+\left\|\sum_{\nu\neq\nu_0} f(x_\nu)\mu(I_\nu)\right\|\right).$$

For the Riemann sum

$$S_0(f,\mathcal{P}) := \sum_\nu f(x_\nu)\mu(I_\nu),$$

we thus obtain

$$\begin{aligned}
\|S_0(f,\mathcal{P})\| &= \left\|\sum_\nu f(x_\nu)\mu(I_\nu)\right\| \\
&\geq \left|\|f(x_{\nu_0})\|\mu(I_{\nu_0}) - \left\|\sum_{\nu\neq\nu_0} f(x_\nu)\mu(I_\nu)\right\|\right| \\
&= \|f(x_{\nu_0})\|\mu(I_{\nu_0}) - \left\|\sum_{\nu\neq\nu_0} f(x_\nu)\mu(I_\nu)\right\| \\
&> C.
\end{aligned}$$

which is impossible.

3. Let $\varnothing\neq D\subset\mathbb{R}^N$ be bounded, and let $f,g\colon D\to\mathbb{R}$ be Riemann-integrable. Show that $fg\colon D\to\mathbb{R}$ is Riemann-integrable.

   Do we necessarily have

$$\int_D fg = \left(\int_D f\right)\left(\int_D g\right)?$$

   (*Hint*: First, treat the case where $f=g$ and then the general case by observing that $fg=\frac{1}{2}((f+g)^2-f^2-g^2.)$

   Solution: Without loss of generality suppose that D is a compact interval I.

   Let $C\geq 0$ such that $|f(x)|\leq C$ for $x\in I$. Let $\epsilon>0$ and let $\mathcal{P}_\epsilon$ be a partition of I such that

$$|S_1(f,\mathcal{P}_\epsilon)-S_2(f,\mathcal{P}_\epsilon)| < \frac{\epsilon}{2(C+1)}$$

   for all Riemann sums $S_1(f,\mathcal{P}_\epsilon)$ and $S_2(f,\mathcal{P}_\epsilon)$ corresponding to $\mathcal{P}_\epsilon$. Let $(I_\nu)_\nu$ the subdivision of I induced by $\mathcal{P}_\epsilon$, and let $x_\nu,y_\nu\in I_\nu$ be support points. As in the proof of Proposition 4.2.12(iii), one sees that

$$\sum_\nu |f(x_\nu)-f(y_\nu)|\mu(I_\nu) < \frac{\epsilon}{2(C+1)}.$$

It follows that

$$\begin{aligned}\sum_\nu |f(x_\nu)^2 - f(y_\nu)^2|\mu(I_\nu) &= \sum_\nu |f(x_\nu)+f(y_\nu)||f(x_\nu)-f(y_\nu)|\mu(I_\nu)\\ &\leq \sum_\nu 2C|f(x_\nu)-f(y_\nu)|\mu(I_\nu)\\ &< 2C\frac{\epsilon}{2(C+1)}\\ &< \epsilon.\end{aligned}$$

Hence, f^2 is Riemann-integrable by Corollary 4.2.6.

For Riemann-integrable $f, g \colon I \to \mathbb{R}$, we have

$$fg = \frac{1}{2}((f+g)^2 - f^2 - g^2),$$

so that fg is also Riemann-integrable.

However, we have, for instance,

$$\int_0^1 x^2\,dx = \frac{1}{3} \neq \frac{1}{4} = \left(\int_0^1 x\,dx\right)^2.$$

4. Let $\varnothing \neq D \subset \mathbb{R}^N$ have content zero, and let $f \colon D \to \mathbb{R}^M$ be bounded. Show that f is Riemann-integrable on D such that

$$\int_D f = 0.$$

Solution: Let $C \geq 0$ be such that $\|f(x)\| \leq C$ for $x \in D$.

Let $I \subset \mathbb{R}^N$ be a compact interval such that $D \subset I$, and extend f to $\tilde{f} \colon I \to \mathbb{R}^M$ as pointed out in class. Let $\epsilon > 0$, and choose a partition $\mathcal{P}$ of I with corresponding subdivision $(I_\nu)_\nu$ of I such that

$$\sum_{I_\nu \cap D \neq \varnothing} \mu(I_\nu) < \frac{\epsilon}{C+1}.$$

Let $\mathcal{Q}$ be a refinement of $\mathcal{P}$ with corresponding subdivision $(J_\lambda)_\lambda$. It follows that

$$\sum_{J_\lambda \cap D \neq \varnothing} \mu(J_\lambda) < \frac{\epsilon}{C+1}.$$

For each λ, pick a support point $y_\lambda \in J_\lambda$. Then we have

$$\left\|\sum_\lambda \tilde{f}(y_\lambda)\mu(J_\lambda)\right\| = \left\|\sum_{J_\lambda \cap D \neq \varnothing} f(y_\lambda)\mu(J_\lambda)\right\| \leq C \sum_{J_\lambda \cap D \neq \varnothing} \mu(J_\lambda) < \epsilon.$$

It follows that $\int_D f = 0$.

5. Let $\varnothing \neq U \subset \mathbb{R}^N$ be open with content, and let $f : U \to [0, \infty)$ be bounded and continuous such that $\int_U f = 0$. Show that $f \equiv 0$ on U.

Solution: Assume that there is $x_0 \in U$ such that $f(x_0) \neq 0$, i.e., $f(x_0) > 0$. By the continuity of f, there is $\delta > 0$, such that $B_\delta(x_0) \subset U$ and $f(x) > \frac{f(x_0)}{2}$ for all $x \in B_\delta(x_0)$. Let

$$J := \left[x_{0,1} - \frac{\delta}{3\sqrt{N}}, x_{0,1} + \frac{\delta}{3\sqrt{N}}\right] \times \cdots \times \left[x_{0,N} - \frac{\delta}{3\sqrt{N}}, x_{0,N} + \frac{\delta}{3\sqrt{N}}\right],$$

so that $J \subset B_\delta(x_0)$. We thus obtain

$$\int_I f \geq \int_I f\chi_J = \int_J f \geq \int_J \frac{f(x_0)}{2} = \frac{f(x_0)}{2}\mu(J) > 0,$$

which is a contradiction.

6*. The function

$$f : [0,1] \times [0,1] \to \mathbb{R}, \quad (x,y) \mapsto xy$$

is continuous and thus Riemann integrable. Evaluate $\int_{[0,1]\times[0,1]} f$ using only the definition of the Riemann integral, i.e., in particular, without using Fubini's Theorem.

Solution: For $n \in \mathbb{N}$, let

$$\mathcal{P}_n := \left\{\frac{j}{n} : j = 0, \ldots, n\right\} \times \left\{\frac{k}{n} : k = 0, \ldots, n\right\}.$$

For $(j,k) \in \{0, \ldots, n\}$, let $x_{j,k} := \left(\frac{j}{n}, \frac{k}{n}\right)$. The corresponding Riemann sum is then

$$\begin{aligned}
S_n(f, \mathcal{P}_n) &= \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{jk}{n^2}\frac{1}{n^2} \\
&= \frac{1}{n^4}\left(\sum_{j=1}^{n} j\right)\left(\sum_{k=1}^{n} k\right) \\
&= \frac{1}{n^4}\frac{n^2(n+1)^2}{4} \\
&\to \frac{1}{4}.
\end{aligned}$$

We claim that $\int_{[0,1]^2} f = \frac{1}{4}$.

Let $\epsilon > 0$, and choose $\delta > 0$ such that $|(f(x,y) - f(x',y')| < \frac{\epsilon}{3}$ for all $(x,y), (x',y') \in [0,1]^2$ such that $\|(x,y) - (x',y')\| < \delta$. Choose a partition $\mathcal{P}_0$ of I such that the following are true for the corresponding subdivision $(I_\nu)_\nu$ of $[0,1]^2$:

- if $(x,y), (x',y') \in I_\nu$ for some ν, then $\|(x,y) - (x',y')\| < \delta$;

- if $\mathcal{P}$ is any refinement of $\mathcal{P}_0$, then $\left|S(f,\mathcal{P}) - \int_I f\right| < \frac{\epsilon}{3}$ for any Riemann sum $S(f,\mathcal{P})$ corresponding to $\mathcal{P}$).

Choose $n_0 \in \mathbb{N}$ be such that the following are true for the corresponding subdivision $(J_\mu)_\mu$ of $[0,1]^2$:

- if $(x,y),(x',y') \in J_\mu$ for some μ, then $\|(x,y)-(x',y')\| < \delta$;
- for any $n \geq n_0$, we have $\left|\frac{1}{4} - S_n(f,\mathcal{P}_n)\right| < \frac{\epsilon}{3}$.

Let $\mathcal{Q}$ be any common refinement of $\mathcal{P}_0$ and $\mathcal{P}_{n_0}$, and let $(K_\lambda)_\lambda$ be the corresponding partition of $[0,1]^2$, and let $S(f,\mathcal{Q})$ be a corresponding Riemann sum. Then we have

$$\left|\frac{1}{4} - \int_{[0,1]^2} f\right| \leq \underbrace{\left|\frac{1}{4} - S_{n_0}(f,\mathcal{P}_{n_0})\right|}_{<\frac{\epsilon}{3}} - |S_{n_0}(f,\mathcal{P}_{n_0}) - S(f,\mathcal{Q})| + \underbrace{\left|S(f,\mathcal{Q}) - \int_{[0,1]^2 f}\right|}_{<\frac{\epsilon}{3}}$$

$$< \frac{2}{3}\epsilon + |S_{n_0}(f,\mathcal{P}_{n_0}) - S(f\mathcal{Q})|$$

Let $S(f,\mathcal{Q}) = \sum_\lambda f(x_\lambda)\mu(K_\lambda)$ with $x_\lambda \in K_\lambda$), and $S_{n_0}(f,\mathcal{P}_{n_0}) = \sum_\nu f(y_\nu)\mu(I_\nu)$. It follows that

$$\begin{aligned} |S_{n_0}(f,\mathcal{P}_{n_0}) - S(f\mathcal{Q})| &= \left|\sum_\nu f(y_\nu)\mu(I_\nu) - \sum_\lambda f(x_\lambda)\mu(K_\lambda)\right| \\ &\leq \sum_\nu \sum_{K_\lambda \subset I_\nu} \underbrace{|f(y_\nu) - f(x_\lambda)|}_{<\frac{\epsilon}{3}} \mu(K_\lambda) \\ &< \frac{\epsilon}{3}, \end{aligned}$$

so that, all in all, $\left|\frac{1}{4} - \int_{[0,1]^2} f\right| < \epsilon$. As $\epsilon > 0$ was arbitrary, this means that $\int_{[0,1]^2} f = \frac{1}{4}$ as claimed.