MATH 336- Winter 2022

Assignment 3

Problem 1.

a) Assume that p, q are continuous functions. Show that any nontrivial solution of the equation

$$y'' + p(x)\, y' + q(x) y = 0,$$

can not be tangent to the x-axis.

b) Assume that p, q are continuous functions and $p(x) > 0$. Suppose that y_1 and y_2 are two linearly independent solutions to the following equation

$$y'' + p(x)\, y' + q(x) y = 0.$$

Show that y_1 , y_2 can not have same inflection point.

Problem 2. Find the recursive formula for the power series solution of the following equation and write down a series containing at least 5 nonzero terms

$$\begin{cases} y'' + (1+x) y = 0 \\ y(0) = 1,\, y'(0) = 0 \end{cases}.$$

Problem 3. Consider the following equation

$$x y'' + 2(1-x) y' + (x-2) y = 0$$

a) Find a series solution for the equation around $x_0 = 0$. Verify that the obtained series is the expansion of $\phi_1 = e^x$.

b) Use the reduction of order method to find $\phi_2(x)$, the second solution of the equation.

c) Use variation of parameters method to find the general solution to the following equation:

$$x y'' + 2(1-x) y' + (x-2) y = x e^x$$

Problem 4. Use the change of variable $x = e^t$ for the following equi-dimensional equation

$$x^3 y''' + a_0\, x^2 y'' + b_0\, x y' + c_0\, y = 0,$$

and transform it to an equation with constant coefficients.

Problem 5. The following equation is called Hermite equation

$$y'' - 2x y' + \lambda y = 0,$$

where λ is a constant.

a) Show that if $\lambda = 2n$, where n is an integer, then one solution of the equation is a polynomial of order n. This polynomial is called the HERMITE polynomial and is denoted by $H_n(x)$.

b) Show that the HERMITE equation can be written as

$$\frac{d}{dx}(e^{-x^2} y') = -\lambda e^{-x^2} y.$$

Use the above relation and prove the following relation

$$\int_0^\infty e^{-x^2} H_n(x)\, H_m(x) = \frac{W(H_m, H_n)(0)}{2(m-n)}, n \neq m.$$

Conclude that if n, m are both even or both odd, then

$$\int_0^\infty e^{-x^2} H_n(x)\, H_m(x) = 0, n \neq m.$$

Problem 6. The following equation is called CHEBYSHEV equation

$$(1-x^2)y'' - xy' + \lambda y = 0.$$

a) Show that if $\lambda = n^2$ an integer, then one solution of the equation is a polynomial of order n. This polynomials is denoted by $T_n(x)$.

b) Use the substitution $x = \cos(\theta)$ in the equation and derive the following one

$$\frac{d^2 y}{d\theta^2} + \lambda y = 0,$$

and thus for $\lambda = n^2$, show that one solution is $T_n(\theta) = \cos(n\theta)$.

c) Use the following identity

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(\theta)\cos(n\theta),$$

and derive the relation

$$T_{n+1}(x) = 2x T_n(x) + T_{n-1}(x).$$

d) Show the following relation

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} T_n(x)\, T_m(x) = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \end{cases}.$$

Problem 7. The following equation is called LEGENDRE equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0.$$

a) Show that if $\lambda = n(n+1)$, for integer n, then one solution is a polynomial of order n. This polynomial is called LEGNDRE polynomial and is denoted by $P_n(x)$.

b) Show that the equation can be written as

$$\frac{d}{dx}[(1-x^2)\,y'] = -\lambda y.$$

Use the above form and prove the following relation

$$\int_{-1}^{1} P_n(x)\,P_m(x) = 0, n \neq m.$$

Problem 8. The following equation is called BESSEL equation

$$x^2 y'' + x\,y' + (x^2 - \lambda^2) y = 0.$$

Note that the equation is singular at $x_0 = 0$. Actually, $x_0 = 0$ is a regular-singular point for the equation

a) Let $\lambda = 4$. For $c_0 = \frac{1}{8}$, use the method we described in lectures to find one series solution of the equation. Use MatLab and draw the series solution for a few terms and compare it with the graph of the BESSEL function $J_2(x)$.

b) Use the method we used in lecture and find a few terms of the second solution. Compare the graph of the solution with the BESSEL function $Y_2(x)$.