MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Solutions #5

1. Let $D := \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$, and let

$$f \colon D \to \mathbb{R}, \quad (x,y) \mapsto \frac{x^2}{y}$$

Show that:

(a) $\lim_{\substack{t \to 0 \\ t \neq 0}} f(tx_0, ty_0) = 0$ for all $(x_0, y_0) \in D$;

(b) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution:

(a) Let $(x_0, y_0) \in D$. For $t \in \mathbb{R} \setminus \{0\}$, we then have that $(tx_0, ty_0) \in D$ as well such that

$$f(tx_0, ty_0) = \frac{t^2\, x_0^2}{t\, y_0} = t\frac{x_0^2}{y_0}$$

It follows that $\lim_{\substack{t \to 0 \\ t \neq 0}} f(tx_0, ty_0) = 0$.

(b) For $n \in \mathbb{N}$, set $(x_n, y_n) := \left(\frac{1}{n}, \frac{1}{n^2}\right)$, so that

$$f(x_n, y_n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1.$$

It follows that $\lim_{n\to\infty} f(x_n, y_n) = 1$. Since by (a), $\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = 0$, we conclude that that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. A set $D \subset \mathbb{R}^N$ is called *discrete* if, for each $x \in D$, there is $\epsilon > 0$ such that $B_\epsilon(x) \cap D = \{x\}$.

Show that the following are equivalent for $\varnothing \neq D \subset \mathbb{R}^N$:

(i) $D \subset \mathbb{R}^N$ is discrete;

(ii) every sequence $(x_n)_{n=1}^\infty$ in D converging to a point in D is eventually constant, i.e., there is $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ for $n \geq n_0$;

(iii) every function $f \colon D \to \mathbb{R}$ is continuous.

Which are the subsets of $\mathbb{R}^N$ that are both compact and discrete?

Solution: (i) $\Longrightarrow$ (ii): Let $(x_n)_{n=1}^\infty$ be a sequence in D converging to $x \in D$. Let $\epsilon > 0$ be such that $B_\epsilon(x) \cap D = \{x\}$. Since $x = \lim_{n\to\infty} x_n$, there is $n_\epsilon \in \mathbb{N}$ such that $\|x_n - x\| < \epsilon$, i.e., $x_n \in B_\epsilon(x) \cap D = \{x\}$, for $n \geq n_\epsilon$. This means that $x_n = x$ for $n \geq n_\epsilon$.

(ii) $\Longrightarrow$ (iii): Let $f : D \to \mathbb{R}$, let $(x_n)_{n=1}^\infty$ be a sequence in D converging to $x \in D$, and let $\epsilon > 0$. As $(x_n)_{n=1}^\infty$ is eventually constant, there is $n_\epsilon \in \mathbb{N}$ such that $x_n = x_{n_\epsilon}$ for $n \geq n_\epsilon$ and thus $x_n = x$ for $n \geq n_\epsilon$. It thus follows for $n \geq n_\epsilon$ that $|f(x_n) - f(x)| = 0 < \epsilon$, i.e., $\lim_{n\to\infty} f(x_n) = f(x)$.

(iii) $\Longrightarrow$ (i): Fix $x \in D$, and define

$$f : D \to \mathbb{R}, \quad y \mapsto \begin{cases} 1, & y = x, \\ 0, & y \neq x. \end{cases}$$

As f is continuous at x, there is $\epsilon > 0$ such that $|f(x) - f(y)| < \frac{1}{2}$ for all $y \in B_\epsilon(x) \cap D$. For any $y \in D \setminus \{x\}$, however, $|f(x) - f(y)| = 1$ holds, so that $B_\epsilon(x) \cap D = \{x\}$.

Clearly, every finite subset of $\mathbb{R}^N$ is both compact and discrete. Conversely, let $\varnothing \neq S \subset \mathbb{R}^N$ be both compact and discrete. Assume that S is infinite. Then there are $x_1, x_2, x_3, \ldots \in S$ with $x_n \neq x_m$ for $n \neq m$. As S is compact, the sequence $(x_n)_{n=1}^\infty$ must have a convergent subsequence $(x_{n_k})_{k=1}^\infty$ with limit in D. By (ii), this means that $(x_{n_k})_{k=1}^\infty$ is eventually constant, which is impossible because $x_n \neq x_m$ for $n \neq m$. All in all, a subset of $\mathbb{R}^N$ is both compact and discrete if it is finite.

3. Let $K, L \subset \mathbb{R}^N$ be compact and non-empty. Show that

$$K + L := \{x + y : x \in K,\, y \in L\}$$

is compact in $\mathbb{R}^N$.

Solution: As you saw in the homework, $K \times L \subset \mathbb{R}^{2N}$ is compact.

As

$$f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, \quad (x, y) \mapsto x + y.$$

is obviously continuous, $K + L = f(K \times L)$ is compact as well.

4. Let $\varnothing \neq D \subset \mathbb{R}^N$. A function $f : D \to \mathbb{R}^M$ is called *Lipschitz continuous* if there is $C \geq 0$ such that

$$\|f(x) - f(y)\| \leq C\|x - y\|$$

for all $x, y \in D$.

Show that:

(a) each Lipschitz continuous function is uniformly continuous;

(b) if $f : [a, b] \to \mathbb{R}$ is continuous such that f is differentiable on (a, b) with f' bounded on (a, b), then f is Lipschitz continuous;

(c) the function

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto \sqrt{x}$$

is uniformly continuous, but not Lipschitz continuous.

Solution:

(a) Suppose that, for $f\colon D \to \mathbb{R}^M$, there is $C \geq 0$ such that

$$\|f(x) - f(y)\| \leq C\|x - y\|$$

for all $x, y \in D$. Let $\epsilon > 0$, and choose $\delta := \frac{\epsilon}{C+1}$. For $x, y \in D$ with $\|x-y\| < \delta$, it follows that

$$\|f(x) - f(y)\| \leq C\|x - y\| < C\frac{\epsilon}{C+1} < \epsilon.$$

Hence, f is uniformly continuous.

(b) Set $C := \sup_{\xi \in (a,b)} |f'(\xi)|$. Let $x, y \in [a,b]$, and suppose without loss of generality that $x < y$. By the Mean Value Theorem, there is $\xi \in (x,y)$ such that

$$f'(\xi) = \frac{f(y) - f(x)}{y - x},$$

so that

$$|f(x) - f(y)| = |f'(\xi)||x - y| \leq C|x - y|.$$

(c) As f is continuous and as $[0,1]$ is compact, it follows that f is uniformly continuous. Assume that there is $C \geq 0$ as in the definition of Lipschitz continuity. It then follows that

$$\frac{1}{2\sqrt{x}} = f'(x) \leq C$$

for $x \in (0,1]$, which is impossible.

5. Let $C \subset \mathbb{R}^N$. We say that $x_0, x_1 \in C$ can be *joined by a path* if there is a continuous function $\gamma\colon [0,1] \to \mathbb{R}^N$ with $\gamma([0,1]) \subset C$, $\gamma(0) = x_0$, and $\gamma(1) = x_1$. We call C *path connected* if any two points in C can be joined by a path.

   Show that any path connected set is connected.

   Solution: Assume that C is not connected, i.e., there is a disconnection $\{U, V\}$ for C. Choose $x_0 \in U \cap C$ and $x_1 \in V \cap C$. Since C is path connected, there is a continuous function $\gamma\colon [0,1] \to \mathbb{R}^N$ with $\gamma([0,1]) \subset C$, $\gamma(0) = x_0$, and $\gamma(1) = x_1$. Since γ is continuous, there are open sets $\tilde{U}, \tilde{V} \subset \mathbb{R}$ such that

$$\tilde{U} \cap [0,1] = \gamma^{-1}(U) \qquad \text{and} \qquad \tilde{V} \cap [0,1] = \gamma^{-1}(V).$$

   It is easy to see that $\left\{\tilde{U}, \tilde{V}\right\}$ is a disconnection for $[0,1]$, which is impossible.

6*. Let

$$C := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x > 0 \right\} \subset \mathbb{R}^2.$$

Show that $\overline{C}$ is connected, but not path connected. (*Hint*: Show that $\{0\} \times [-1,1] \subset \overline{C}$ and that any point in $\{0\} \times [-1,1]$ cannot be joined by a path with any point of the form $\left(x, \sin\left(\frac{1}{x}\right)\right)$ with $x > 0$.)

Solution: The map

$$(0,\infty) \to \mathbb{R}^2, \quad t \mapsto \left(t, \sin\left(\frac{1}{t}\right) \right)$$

is continuous and has C as its range. As $(0,\infty)$ is connected, C is connected as is $\overline{C}$ by Solution 3 to Assignment #4.

Let $y \in [-1,1]$, and let $x_y > 0$ be such that $\sin x_y = y$. For $n \in \mathbb{N}$, let $x_n := \frac{1}{2n\pi + x_y}$. It follows that

$$\left(x_n, \sin\left(\frac{1}{x_n}\right) \right) = (x_n, \sin x_y) = (x_n, y) \to (0, y),$$

so that $(0,y) \in \overline{C}$.

Let $y \in [-1,1]$, let $t_0 > 0$, and suppose that there is a continuous function $\gamma = (\gamma_1, \gamma_2) : [0,1] \to \overline{C}$ such that $\gamma(0) = (0,y)$ and $\gamma(1) = \left(t_0, \sin\left(\frac{1}{t_0}\right)\right)$. Let $a := \sup\{t \in [0,1] : \gamma_1(t) = 0\}$. It follows that $\gamma_1(a) = 0$, $a \in [0,1)$, and $\gamma_2(t) = \sin\left(\frac{1}{\gamma_1(t)}\right)$ for $t \in (a,1]$. Consider

$$\tau : [0,1] \to [a,1], \quad t \mapsto a + t(1-a)$$

Then $\gamma \circ \tau$ is a path joining $(0, \gamma_2(a))$ with $\left(t_0, \sin\left(\frac{1}{t_0}\right)\right)$. Replacing γ by $\gamma \circ \tau$, we can thus suppose without loss of generality that $\gamma_1(t) > 0$ for all $t \in (0,1]$.

Let $n \in \mathbb{N}$, and note that $\lim_{t \to 0} \gamma_1(t) = 0 < \gamma_1\left(\frac{1}{n}\right)$. Choose $m_n \in \mathbb{N}$ such that:

- if n is even, then so is m_n, and if n is odd, so is m_n;
- $\frac{1}{m_n \pi + \frac{\pi}{2}} \le \gamma_1\left(\frac{1}{n}\right)$.

Then use the Intermediate Value Theorem to find $t_n \in \left(0, \frac{1}{n}\right]$ such that $\gamma_1(t_n) = \frac{1}{m_n \pi + \frac{\pi}{2}}$

It follows that $t_n \to 0$, so that $\gamma(t_n) \to (0,y)$. However, we have

$$\gamma_2(t_n) = \sin\left(m_n \pi + \frac{\pi}{2}\right) = (-1)^n$$

for $n \in \mathbb{N}$, which does not converge as $n \to \infty$.