# 1. Prerequisites

In this chapter we briefly review some of the topics you should be familiar[1] with. We will likely don't have time to go over them in class in any detail.

In addition, there are also some topics that are more of a "nice to know" than "need to know" nature, and not strictly speaking part of this course or required knowledge. Any sections headed with a "*" are not required.

Many of the subtle points raised in the first four sections are merely for the interested. They may be hard to swallow and follow. They will not be required material. **In a first reading skip over any section marked with a ***.

But any mathematician should be aware that mathematics is not as "absolute" as it is often made out to be. And certainties (or "truths") are always hard to come by regardless of the field of study, even in mathematics.

What you should **not** do is let these sections distract you and make you feel insecure about what is "true" or "false." Follow your instinct but be aware that nothing is ultimately "obvious."

## 1.1 Logic

Logic is at once a part of mathematics and not part of it. What is meant by that, and is that not a contradiction?

Of course, logic is a fundamental part of the language of mathematics. But on the other hand, if logic were entirely a part of mathematics, it would be subject to the rules of mathematics. What would those rules be, if not logic? This is not a precise statement, of course, but gives you some indication where problems might arise.

### 1.1.1 Basic logic

We will throughout follow the following logical definitions: if $A, B$ are mathematical statements[2] (e.g. $A =$"1 is a real number." and $B =$"1 is a natural number."), we say

1. $A$ and $B$ holds, in short "$A$ and $B$", if both $A$ is true **and** $B$ is true. Symbolically this is often written as
$$A \wedge B$$
2. $A$ or $B$ holds, in short "$A$ or $B$", if **one** of $A$ or $B$ is true (or both). This is often written as
$$A \vee B$$
3. If $A$ is **not** true, then "not $A$" is true. This is written as
$$\neg A$$

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[1] "Should" is not the same as "have to". It is meant as "in an ideal world, your high school preparation would provide you most of this background." You can also use these notes to develop this background. We will also cover some of it in class

[2] It is quite subtle to define what an admissible "statement" is, but that will rarely cause trouble for us.

Regarding 2. observe that in mathematical logic, “or” is always *non-exclusive*, that is $A \vee B$ is not the same as “either $A$ or $B$ holds (but not both)”. This *exclusive* “or” must be specified explicitly if it is meant.

$A$ **implies** $B$ means that if $A$ is true, also $B$ is true. For example: $A$=”It rains”, $B$=”the street is wet.” This is often written as $A \to B$, or $A \Rightarrow B$.

**Remark**
If $A$ is false, any implication $A \to B$ is true. “You can deduce anything from a false premise.”
EOR.

### 1.1.2 Formulas

Some mathematical statements (or sentences that look like statements) involve one or more undeclared symbols.

For example, "$x$ is a real number” is technically not a statement, as its truth value is unknown (unless we know what $x$ is). We will call such a “statement” a (logical) **formula** and should not be confused with usual calculus formulas or formulas in logic (where $x$ would not be a mathematical object but a statement itself, e.g. $A(x) = x \vee \neg x$). We usually mean that the objects here are sets or elements of sets, of functions.

For simplicity, we assume that each formula has a single free “variable” $x$ (but the situation doesn’t change significantly if there are more variables).

A sentence involving $x$ will be abbreviated by $A(x)$, or similar. For example, we could have $A(x)$: “$x$ is a real number”, and $B(x)$: “$x$ is a natural number.”

Then $B(x) \to A(x)$ is (not a statement, but) a formula that is always true, no matter what $x$ is. Any natural number is also a real number.

Some formulas that are always true are often called tautologies (e.g. $A(x)$: $(x = x) \vee (x \neq x)$).

The important thing to remember here is that a formula $A(x)$ is true if and only if it is true for all admissible values (*interpretations*) for $x$.

The truth value of a formula $A(x)$ is the truth value of the statement: “For all admissible $x$, $A(x)$ is true.”

**Example**
The sentence "$x = 1$" is a formula that can be true or false depending on what $x$ is. As a statement, it is therefore considered false (as there is at least one value for $x$ where it fails: e.g. if $x$ is equal to 2).
The formula “If $x \in \mathbb{R}$, then $x = 1$” is not a true implication, as there are examples of real numbers $x$, which are not equal to 1.
More importantly, consider the following example: $A$ is the statement “Let $x \in \mathbb{R}$.” and $B$ is “$x^2 = 1$.”
Then $A \to B$ is false: While it is true that for *some* real numbers $x$ that $x^2 = 1$, it is not true for all of them. A good way to think about an implication $A \to B$ is to think “Whenever $A$ holds $B$ holds.”
If there is at least one instance where $A$ holds and $B$ does not, then $A \to B$ is false (in the context of thse notes). Logically, the statement “If $x \in \mathbb{R}$, then $x^2 = 1$” should be read as the statement “for all $x \in \mathbb{R}$: $x^2 = 1$". EOR.

The term “tautology” is often reserved for logical statements about statements that are always true (e.g. $A \vee \neg A$, or $\neg(A \wedge \neg A)$). But we also say a sentence of the form “$x \in S$ or $x \notin S$” is a tautology (it is an immediate consequence of $A \vee \neg A$.

### 1.1.3 *Some background

Think about the following question:

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What does it mean for a statement to be true?

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“Truth” is a concept that is ridiculously hard if not impossible to define[3].

Of course, we all have intuitive notions of truth. For example, few would argue that the statement “If $x$ is real number then $x$ is not a real number” is true. (As an aside, this **would** be a true statement if there are no real numbers.)

However, to prove logically a statement of the form “If $A$ is true then its negation is not also true,” one must make assumptions. Usually the assumption is that for all “statements” $A$ one has that

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“$A$ and $\neg A$ is false”

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Symbolically, $\neg(A \wedge \neg A)$ is always true. Here $\neg A$ is shorthand for “not $A$”.

This is obvious (so please prove it[4]). This is commonly referred to as the Law of No Contradiction.

There is ultimately no way around the fact that we need some ground rules that we must treat as obvious and established, and which we cannot prove. The above example is one of them.

In the early 20th century, there was a deep and long discussion about what these rules should be, and to this day there is no universally accepted “rule set.”

One of the main controversies was the question whether it is “true” that for any statement $A$ the statement

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*“$A$ is true or $\neg A$ is true”*

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is true. Symbolically, is $A \vee \neg A$ always true?

[3] I would propose it is impossible to define as any sort of definition would require some sort of logical rules, which in themselves probably are meaningless without a concept of “true” or “false.” But this is a philosophical and not a mathematical argument.

[4] Unfortunately, you highly likely will not be able to prove it. This is usually taken as one of the basic “truths” which cannot be proven.

Think about that: the discussions revolved around whether a statement of the form "$x$ is a real number or not ($x$ is a real number)" is always true.

The postulate (or "axiom") that for all (mathematical or logical) statements $A$ we always have $A$ holds or $\neg A$ holds is commonly known as the **Law of Excluded Middle** (there is nothing "between" true or false, so there is no "middle") or **Tertium non Datur** (which is Latin for "There is no third [option]").

The question revolved around whether if $\neg\neg A$ holds, does that always force $A$ to hold? To show this one must assume the Law of Excluded Middle (LEM).

So called "Intuitionists" reject the LEM. The objections are often grounded in the fact that LEM allows non-constructive proofs (that is, proofs where the existence of an object can be established without giving any concrete recipe of how to obtain the object from given data).

These are profound logical questions, and way above our head (mine included) in this course.

Most of modern mathematics has accepted LEM.

Most proofs by contradiction would be invalid without it: A proof by contradiction rests on the fact that, if we show that the negative of a statement $A$ is false (that is, we show that $\neg\neg A$ is true), then $A$ is true. To show this, one needs LEM.

In fact, a proof by contradiction of a statement $A$ usually establishes $\neg A \to A$, which is a contradiction if $\neg A$ is true (in other words, the proof establishes $(\neg A \to A) \to \neg\neg A$). Thus, $\neg A$ must be false, or more precisely $\neg\neg A$ is true. The leap of faith is then that this means $A$ is true because "obviously" $\neg\neg A = A$. To prove this latter fact, one needs LEM. Adherents of intuitionist logic assert that $\neg\neg A \to A$ is not a valid "inference" (logical step). They usually do not object to the inference $A \to \neg\neg A$ (which is essentially the Law of No Contradiction mentioned above).

If that confuses you, you are not alone, and don't worry. We will use "common sense logic" where LEM certainly holds.

To illustrate the issue somewhat in intuitive terms: Suppose someone would like to prove that aliens exist. Which "proof" would convince you more: a theoretical discussion that the non-existence of aliens leads to a logical contradiction, and hence aliens don't "not exist"; or someone showing you an actual alien?

This general problem (to prove existence of something by showing that its non-existence leads to a contradiction) was very controversial at the time.

We also haven't touched upon the question what a "statement" is. This again seems to be something undefinable as any definition of a "statement" seems to be itself a "statement." Set theory and logic offer a way out of that, but we will not have the time to go into that.

## 1.2 Proofs

Mathematics at its essence is about proofs. That is, mathematics is about reasoning why certain statements should or should not be true. Necessarily these statements are mathematical in nature, and do not belong to the "real world".

A mathematical statement is accepted as true, if there is a universally accepted proof of it. Otherwise it is called a “conjecture” (if it is suspected to be true), unclear (if there is no known proof, but also no known counterexample), or false (if there is a counterexample).

The precise nature of what a proof is, is rather involved. “You know it when you see it.”

We will not spend much time on the theoretical foundations of proofs, but rather hope that by doing and seeing it, that you will understand what is meant.

There are a few rules of logic that are important to proving things.

### 1.2.1 *What is a proof?

What is a proof? Again, this is both a mathematical and philosophical question.

Logically, a proof of a statement $A = A_n$is a sequence of logical statements of the form $A_1 \to A_2 \to A_3 \to \cdots \to A_n$ such that if $A_i$ is true this forces $A_{i+1}$ to be true, and $A_1$ is true.

So $A_i \to A_{i+1}$ must be what is called a logical implication or “inference” (that is, if $A_i$ is true then also $A_{i+1}$ is true).

Again, there are problems with this: what are $i$ and $i + 1$? Can we meaningfully talk about these notions without needing “proofs”? Is this definition circular?

There are (philosophical) solutions (or not, depending on where you stand) to these conundrums, and in mathematical day to day business this rarely causes any issues. We all usually know what is meant.

One could regard proofs (and therefore any statements) simply as a trial before a “court of (mathematical) law” where the proponent (prosecutor) tries to establish the fact, and the opponent (defence) tries to argue the particular statement is false. Both parties are subject to certain rules, and a statement is true if the proponent always wins.

Similarly, there is a school of thought that regards all of logic (and by extension all of mathematics) as merely an exercise in formally manipulating symbols on a piece of paper according to some predefined rules without any intrinsic meaning (so, both, the rules and the symbols are devoid of any meaning). This is, of course, a safe choice, in the sense that if you do not claim to “mean” anything, you also cannot be proven false. While I personally subscribe to this notion of mathematics (on philosophical grounds) on *some* level, it is also true that this fundamentally turns mathematics into a useless intellectual exercise without any content or meaning. In a sense, if we want to solve problems (be they mathematical or “real world” problems) using mathematics, we must make assumptions on how mathematics or its rules apply (to mathematics or the “real world”). Nevertheless, I believe we must be aware that there are serious assumptions being made whenever we use any mathematical reasoning to solve “real world” problems (there is no universally accepted “law”, for example, that nature must obey human logic). On a pragmatical note, however, we also do not have any system of thought that works better. All this is intended to encourage you to think about these things (and certainly not to encourage you to doubt everything!). Make up your own mind!

Regardless, in modern mathematics, we usually do not worry too much about such foundational questions. We have come to realize that there is likely no solution that satisfies all, and we must live with the uncertainty that comes with it.

### 1.2.2 Proof by contradiction

A proof by contradiction is a form of reasoning that involves proving a statement $A$ by disproving $\neg A$ ("not $A$"). In other words, $\neg\neg A$ is shown to be true, and then the conclusion is that $A$ must be true. This relies heavily on LEM (discussed above).

**Example**
A typical example of a proof by contradiction is Cantor[5]'s argument that the set of real numbers is uncountably infinite (if these words do not mean anything to you, do not despair; this is just an example). Here "countably infinite" means that you can list the elements of a set in an ordered list and number them using natural numbers. Cantor's proof goes as follows: suppose you could list all real numbers in this way, then there is a real number with a certain decimal expansion that cannot possibly be a member of the list (because its $n$th digit after the decimal point differs from the $n$th digit of the $n$th element in the list). This results in a contradicton, so therefore the real numbers cannot form a countably infinite set. This is an example of a not very controversial proof by contradiction, since we do not assert a positive statement, we only assert that something is false. EOE.

A more approachable example is the following:

**Example**
No real number $x$ satisfies $x^2 = -1$.
Proof: Suppose $x^2 = -1$. Then $x \neq 0$. For if $x = 0$ then $x^2 = 0 \neq -1$, **which is a contradiction**. If $x > 0$, then also $x^2 > 0 > -1$, **a contradiction**. If $x < 0$, then $x = (-1)y$ for some $y > 0$. Then $x^2 = (-1)^2 y^2 = y^2 > 0$, **again a contradiction**. We have exhausted all possibilities, and each lead to a contradiction, so the premise must be false: we cannot have $x^2 = -1$. EOE.

### 1.2.3 Proof by induction

Induction is one of the most powerful proving techniques. You will see examples of this in many areas of mathematics. In its simplest form it is used to prove statements about natural numbers:

Formally if $A(n)$ is a statement about a natural number $n$, then $A(n)$ is true for all $n \in \mathbb{N}$ if

1. $A(1)$ is true.
2. Whenever $A(n)$ is true for a specific $n \in \mathbb{N}$, then also $A(n+1)$ is true.

The naïve idea behind this is to count or enumerate the numbers for which $A(n)$ holds: by the first statement the count starts at 1. By the second statement, the count never ends, and therefore (as every natural number eventually is "reached"), it is true for all of them.

Formally this is called the **Principle of Induction** (POI) which is one of the Peano[6] Axioms which we assume the natural numbers to satisfy. Precisely, it states that:

[5] Georg Cantor (1845 – 1918)
[6] Giuseppe Peano (1858 – 1932)

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*Given a subset $S \subseteq \mathbb{N}$ of natural numbers such that $1 \in S$ and $\forall n \in S: n+1 \in S$, then $S = \mathbb{N}$.*

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Technically, the POI states this in terms of the successor function on $\mathbb{N}$, out which (together with the POI) one constructs the addition, multiplication, and order relation of natural numbers.

The connection to proofs by induction is the definition $S \coloneqq \{n \in \mathbb{N} \mid A(n) \text{ is true}\}$. So, a proof by induction exactly establishes that $S = \mathbb{N}$.

***Remark**
As always there are issues with this: for example, this needs that $S$ is an actual set. The axioms of set theory (see below) guarantee this to be true if $A(n)$ is a well-formed statement in first order logic, for example. Many statements (about sequences $a_n$ for example) are of this form. In fact, we will not encounter any statement in this course where this is a problem. But for the sake of argument imagine you are convinced that any natural number is either "beautiful" or "not beautiful". Is it then true that $\{n \mid n \text{ is beautiful}\}$ is a set? Of course, this depends on your definition of "beautiful," but unless you can define that in logically precise terms, there is no reason why it should be. EOR.

A secondary issue is that for all this to be non-vacuous, there better be a set $\mathbb{N}$ of natural numbers.

**Remark**
Finally, POI should be distinguished from "inductive reasoning" in some natural sciences. Inductive reasoning makes assumptions about future events based on experience. A primitive example is, that, since we all observed that the sun rises every day in the past, it will also rise tomorrow. This is not a logical conclusion. There may be physical models that predict the sun rising tomorrow, and so far, they had a perfect score in terms of accuracy; but this is not a proof. Experimental sciences rely on the belief that past observations are indicative of future behaviour. A famous quip is that the statement "All odd natural numbers greater than 1 are prime" is true: 3,5,7 are all prime, 9 is a measuring error, 11,13 are prime….". Of course, this does not mean that the inductive method in experimental sciences is invalid. The point is that two different "principles of induction" are at work here. EOR.

## 1.3 *Axioms

Nowadays, in mathematics we usually do not care much about what mathematical objects "are" (as that is often mostly a philosophical question), but rather describe what **properties** they have. Often these properties are referred to as "axioms" and deemed true by default or definition.

For example, the Peano Axioms (see below) are a way to capture the fundamental properties of natural numbers. It is completely irrelevant to know "what natural numbers are" as long as they obey these axioms.

For the "user" there are two classes of axioms. The first kind are statements that are treated as self-evident truths, such as the Law of Excluded Middle discussed above.

The second kind are axioms by definition: a function has certain properties, and its properties do not have to have any intrinsic meaning or be particularly self-evident. A thing is a function if it satisfies these properties.

This is not a logical distinction, by the way.

Below, we will briefly discuss set theory. In the early 20th century, it was also discovered that the "naïve set theory" as proposed and fundamentally furthered by people like Cantor lead to contradictions.

As a result, there was a movement to "axiomatize" mathematics: base everything on a few "first principles."

This applied to the most fundamental mathematical theories of all: set theory. There was hope that one could prove that the axioms of set theory would be consistent (meaning you cannot deduce a logical contradiction from them).

At the heart of this thought is that "truth" is closely related to "provable."

This is a natural thought: if a statement is "true" there should be a way to prove that it actually is.

This approach was supremely successful. However, as Goedel[7] showed in the 1930s, it is also in vain.

Goedel proved in his famous 1st Incompleteness Theorem that any system of axioms that is sufficiently complex (for example complex enough that basic arithmetic of natural numbers can be defined and performed) will allow statements that are neither provable nor disprovable, using only the axioms of the system (and predetermined rules of logic).

Goedel later proved as well that no sufficiently complex system of axioms can prove its own consistency. Here that means that there is no proof using only the axioms of the system (and predetermined axioms of logic) that shows that the axioms are not contradictory (meaning that no contradiction could be deduced from them).

For us, these things make little difference. It is mentioned to show that mathematics is far from being without controversies.

Often the first incompleteness theorem is paraphrased as "there are statements that are true but not provable." That is slightly misleading as there is no way to know a statement is "true" without being able to prove it (in principle of course, this could just mean that we can prove that an element of a set of several statements is true, but we cannot prove which one specifically is true, but that is not what is meant). What is meant is that there are systems that satisfy the axioms where the statement is true, but the axioms are not enough to prove it, and therefore there are other systems where the statement is false.

Think about the Peano Axioms minus the Principle of Induction (POI). There are probably sets with structure[8] that satisfy the modified system, but do not satisfy the POI, and there are other sets that are satisfying all of the axioms including the POI, like $\mathbb{N}$. If that is true, then the POI cannot be proved from the other axioms. It is an "undecidable" statement, but "true" in some instances (like that of $\mathbb{N}$).

Historically, axioms appeared in Euclidean[9] geometry: "Two (distinct) points define a line." It was later (much later) shown that these axioms do not rely on any intrinsic meaning of the words "line" or

[7] Kurt Friedrich Gödel (1906 – 1978)
[8] Meaning a set with additional properties like having a successor function such as required by the Peano Axioms.
[9] Euclid, also Euclid of Alexandria (fl. 300 BC)

“point,” but only on their properties and relations as specified in the axioms. It is famously attributed to David Hilbert[10] that one should be able to do Euclidean geometry with the same theorems and results but replacing every occurrence of “point, line, or plane” by “table, chair, beer mug,” respectively (I am not sure about the order nor about the authenticity of the attribution, but it is mentioned for example in the Wikipedia article on Hilbert). The point, however, is that labels are not important.

## 1.4 Sets

Set theory is a foundational topic of modern mathematics. From a purely logical viewpoint **all** mathematical objects (at least those that we encounter in this course) are *sets* (yes, in foundational mathematics, a natural number is a set: one could define for example (after von Neumann[11])

$$0 = \emptyset, \qquad 1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}, \qquad 2 = 1 \cup \{1\} = \big\{\emptyset, \{\emptyset\}\big\}$$

and so on $n + 1 = n \cup \{n\}$. It then follows that $n + 1 = \{0, 1, 2, \dots, n\}$.

We will use so called *Naïve Set Theory* throughout. That is, for us a **set** is a *collection* of mathematical objects[12]. That means if $X$ is a set and $x$ is a mathematical object, then one and only one of the following two statements is true:

- $x \in X$ (“$x$ is an element of $X$, or belongs to the collection $X$”); or
- $x \notin X$ (“$x$ is not an element of $X$”).

For example, ½ ∈ ℝ, but ½ ∉ ℤ.

If $X, Y$ are sets, we can form $X \cup Y$, the **union** of $X, Y$, which is the set precisely containing the elements of both $X$ and $Y$. Similarly, we can form the **intersection** $X \cap Y$ of elements that belong to both.

$$X \cup Y = \{\, z \mid (z \in X) \vee (z \in Y) \,\}$$

$$X \cap Y = \{z \mid (z \in Z) \wedge (z \in Y)\}$$

Going back to what we said about statements or formulas, if $A(x)$ is a formula, then

$$\{x \mid A(x)\}$$

is the set of all mathemetical objects $x$ for which $A(x)$ is true (this is not generally a set, as the example $A(x)\colon x \notin x$ shows; see Russell’s Paradox below). To be on the safe side, one usually must assume that $x$ is restricted to a predetermined set.

Most of the time we will use such a construction as

$$T \coloneqq \{\, x \in S \mid A(x) \,\}$$

the set of all $x \in S$ which satisfy the formula $A(x)$. Then $A(x)$ is a valid formula, if this is actually a set, and vice versa, if this is a set, then $A(x)$ is a valid formula (it is logically equivalent to $x \in T$).

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[10] David Hilbert (1862 – 1943)

[11] John von Neumann (1903 – 1957)

[12] This is obviously a circular definition. We just chose to define “set” by using a language synonym “collection,” which begs the question, what is a “collection?” But neither “set”, nor “collection” need to be defined precisely. We only need to know what properties a “set” or “collection” has.

### 1.4.1 *Naïve set theory

Naïve set theory goes back to Georg Cantor who first established a well-founded theory of cardinality of sets. As mentioned before, he showed that the reals form an uncountable set.

The set theory as used by Cantor works for most practical purposes. However, if taken to the logical extreme, it does lead to contradictions.

As often is true in logic, self-reference is a problem.

So, if sets are collections of arbitrary mathematical objects, then surely there is a set whose elements are exactly all the sets (as these are undeniably mathematical objects). So, there should be a set of all sets.

**Russell[13]'s paradox**
Let $S$ be the set containing all sets as elements. Let $T \subseteq S$ be the subset formed as follows

$$T = \{A \in S \mid A \notin A\}$$

Is $T \in T$?
$T$ is clearly a set. If $T \in T$, then by the definition of $T$, also $T \notin T$, a contradiction.
Thus, we must conclude that $T \notin T$. But then, as $T$ is a set, by definition of $T$, we also find $T \in T$, again a contradiction. EOR.

This shows that naïve set theory can become problematic when dealing with large or self-referencing sets.

In the case that all sets in question are part or elements of a large (often unmentioned) set, there is usually no (known) problem.

In modern mathematics this problem has been solved by introducing axiomatic set theory (see below). The role of naïve sets is then played by so called "classes." A *class* is a set if and only if it is an element of a class.

### 1.4.2 *Axiomatic set theory

To avoid paradoxes such as above, in modern mathematics, set theory is governed by axioms. They are quite formal and beyond the scope of this class. The usually accepted system of axioms is the Zermelo[14]-Fraenkel[15] system of axioms. It is abbreviated ZF, or ZFC if the Axiom of Choice (see below) is included.

One of these axioms stands out, as it is/was -just like the Law of Excluded Middle- the source of great controversy. As discussed, at the heart of the problem with the Law of Excluded Middle is that existence proofs by contradiction don't tell you much about the object that exists. For example, you might be able to prove that the nonexistence of the solution of an equation $f(x) = 0$ leads to a contradiction. You then know that a solution exists. But that may not help you to find that solution.

In a similar fashion, the Axiom of Choice (AOC) in set theory has led to complaints. The AOC is simple: let $X$ be any set whose elements are *nonempty* sets. Then $X$ admits a **choice function** $C: X \to \bigcup_{A \in X} A$ such

[13] Bertrand Arthur William Russell (1872 – 1970)
[14] Ernst Friedrich Ferdinand Zermelo (1871 – 1953)
[15] Abraham Halevi Fraenkel (1891 – 1965)

that $\forall A \in X: C(A) \in A$. In other words, one can "choose" for each of the sets $A$ belonging to $X$ one element $C(A)$ each.

You will encounter many such applications: For example, when we construct many sequences we argue as follows: "for each $n \in \mathbb{N}$, pick $x_n$ such that …". Whenever you see "pick X" in a mathematical context, there is a significant chance that this choice needs AOC. Here is an example, that will start to make sense once we discuss real numbers in greater detail: Consider a nonempty bounded set $A \subseteq \mathbb{R}$ and we want a sequence $a_n \to \sup A$. We then say, "For $n \in \mathbb{N}$ let $a_n \in A$ be any element with `$|a_n - \sup A| < \frac{1}{n}$." In many cases, such an $a_n$ is not unique. Assuming $\sup A$ is finite, consider $A_n \coloneqq \{ x \in A \mid |x - \sup A| < \frac{1}{n} \}$. Then $A_n$ is never empty. Let $X = \{ A_n \mid n \in \mathbb{N} \}$. Then a choice function $C$ has the property that $x_n \coloneqq C(A_n) \in A_n$ as desired. Unfortunately, the AOC tells us nothing about how to explicitly obtain such a $C$ (and consequently, $x_n$) other than that it exists.

To put it into concrete terms, a proof using the AOC might tell you that an equation of the form $f(x) = 0$ has a solution. But it may not tell you anything about how to find it.

### 1.4.3 Real numbers etc.

Having said all this, we will assume that there is a distinguished set $\mathbb{R}$ together with a multiplication and addition that satisfies the axioms of a field (see below). We also assume that this field is *ordered* and *complete*. One can show that there is at most one such object (up to what is called an *isomorphism*; essentially this says that if $F$ is any other such field, then there is a natural one-to-one correspondence between $F$ and $\mathbb{R}$, respecting the ordering and as a consequence suprema and infima.

You should simply think of $\mathbb{R}$ as the set of real numbers familiar from high school: For example, every real number admits a decimal expansion that uniquely determines it. (The converse is not true. For example, $.9999999999\ldots$ is the same as 1.0.)

We will make this precise below.