MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Solutions #7

1. Determine and classify the stationary points of

$$f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, \quad (x,y) \mapsto \frac{1}{y} - \frac{1}{x} - 4x + y.$$

If f attains a local minimum or maximum at a stationary point, evaluate the function there.

Solution: We have

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x^2} - 4 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{1}{y^2} + 1.$$

Hence, the set of stationary points of f is

$$\left\{ \left(\frac{1}{2}, 1\right), \left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, -1\right) \right\}$$

Since

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\frac{2}{x^3}, \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2}{y^3},$$

and

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = 0,$$

we have

$$(\text{Hess } f)(x,y) = \begin{bmatrix} -\frac{2}{x^3} & 0 \\ 0 & \frac{2}{y^3} \end{bmatrix}.$$

It follows that $(\text{Hess } f)(x,y)$ is indefinite at $\left(\frac{1}{2}, 1\right)$ and $\left(-\frac{1}{2}, -1\right)$—so that f has saddles at those points—, positive definite at $\left(-\frac{1}{2}, 1\right)$, and negative definite at $\left(\frac{1}{2}, -1\right)$. Hence, f has a local minimum at $\left(-\frac{1}{2}, 1\right)$, namely $f\left(-\frac{1}{2}, 1\right) = 6$, and a local maximum at $\left(\frac{1}{2}, -1\right)$, namely $f\left(\frac{1}{2}, -1\right) = -6$.

2. Determine and classify the stationary points of

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto (x^2 + 2y^2)e^{-(x^2+y^2)}.$$

If f has a local extremum at a stationary point, determine the nature of this extremum and evaluate f there.

Solution: We have

$$\frac{\partial f}{\partial x}(x,y) = 2xe^{-(x^2+y^2)} - 2x(x^2 + 2y^2)e^{-(x^2+y^2)} = 2x(1 - x^2 - 2y^2)e^{-(x^2+y^2)}$$

and

$$\frac{\partial f}{\partial y}(x,y) = 4ye^{-(x^2+y^2)} - 2y(x^2+2y^2)e^{-(x^2+y^2)} = 2y(2-x^2-2y^2)e^{-(x^2+y^2)}.$$

Suppose that $(\nabla f)(x,y) = 0$. As $e^{-(x^2+y^2)} \neq 0$, this means that

$$2x(1-x^2-2y^2) = 2y(2-x^2-2y^2) = 0.$$

If $x = 0$, it follows that $2y = 0$ or $2 - 2y^2 = 0$, i.e., $y = 0$ or $y = -1$ or $y = 1$. If $y = 0$, then $2x = 0$ or $1 - x^2 = 0$, i.e., $x = 0$ or $x = 1$ or $x = -1$. If $x \neq 0 \neq y$, then $1 - x^2 - 2y^2 = 2 - x^2 - 2y^2 = 0$, which is impossible. Hence, $(0,0)$, $(0,1)$, $(0,-1)$, $(1,0)$, and $(-1,0)$ are the only stationary points of f.

We have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(x,y) &= (2-4y^2-6x^2)e^{-(x^2+y^2)} - 2x\frac{\partial f}{\partial x}(x,y),\\
\frac{\partial^2 f}{\partial y^2}(x,y) &= (4-2x^2-12y^2)e^{-(x^2+y^2)} - 2y\frac{\partial f}{\partial y}(x,y),
\end{aligned}$$

and

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = -4xye^{-(x^2+y^2)} - 2y\frac{\partial f}{\partial x}(x,y).$$

It follows that

$$(\text{Hess } f)(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix},$$

$$(\text{Hess } f)(0,1) = (\text{Hess } f)(0,-1) = \begin{bmatrix} -2e^{-1} & 0 \\ 0 & -8e^{-1} \end{bmatrix},$$

and

$$(\text{Hess } f)(1,0) = (\text{Hess } f)(-1,0) = \begin{bmatrix} -4e^{-1} & 0 \\ 0 & 2e^{-1} \end{bmatrix}.$$

Hence, f has a local minimum at $(0,0)$—$f(0,0) = 0$—, local maxima at $(0,1)$ and $(0,-1)$—$f(0,1) = f(0,-1) = 2e^{-1}$—, and saddles at $(1,0)$ and $(-1,0)$.

3. Determine the minimum and the maximum of

$$f\colon D \to \mathbb{R}, \quad (x,y) \mapsto \sin x + \sin y + \sin(x+y),$$

where $D := \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x, y \leq \frac{\pi}{2}\right\}$, and all points of D where they are attained.

Solution: By the compactness of D and the continuity of f, the function attains both a minimum and a maximum on D.

Note that $\operatorname{int} D = \left\{(x,y) \in \mathbb{R}^2 : 0 < x, y < \frac{\pi}{2}\right\}$. We start with classifying the stationary points of f on $\operatorname{int} D$.

First, determine the gradient of f:

$$\frac{\partial f}{\partial x}(x,y) = \cos x + \cos(x+y) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = \cos y + \cos(x+y).$$

For $(x,y) \in \operatorname{int} D$ to be a stationary point, it is thus necessary and sufficient that

$$\cos x + \cos(x+y) = 0 = \cos y + \cos(x+y)$$

or, equivalently, that

$$\cos x = \cos y = -\cos(x+y).$$

Since cos in injective on $\left(0, \frac{\pi}{2}\right)$, this means that $x = y$ and thus $\cos x = -\cos(2x)$. For $x \in \left(0, \frac{\pi}{2}\right)$, this is possible only if $x = \frac{\pi}{3}$. Hence, $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is the only stationary point of f.

Next, we calculate the Hessian:

$$(\text{Hess } f)(x,y) = \begin{bmatrix} -\sin x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & -\sin y - \sin(x+y) \end{bmatrix}.$$

Since

$$-\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

and

$$\left(\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\right)^2 - \left(\sin\left(\frac{2\pi}{3}\right)\right)^2 = 3 - \frac{3}{4} > 0$$

the Hessian matrix is negative definite at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$. Hence, f has a local maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, namely $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3}{2}\sqrt{3}$.

Therefore, we know that f attains a local maximum in $\operatorname{int} D$, which is the only local extremum there. We thus have to check the behaviour of f on ∂D.

Let

$$\begin{aligned}
f_1 &: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, \quad x \mapsto f(x,0) = 2\sin x; \\
f_2 &: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, \quad y \mapsto f\left(\frac{\pi}{2}, y\right) = 1 + \sin y + \cos y; \\
f_3 &: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, \quad y \mapsto f\left(x, \frac{\pi}{2}\right) = 1 + \sin x + \cos x; \\
f_4 &: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, \quad x \mapsto f(0,y) = 2\sin y.
\end{aligned}$$

It is immediate that f_1 and f_4 attain their respective minimum—0—at 0 and their respective maximum—2—at $\frac{\pi}{2}$.

Since

$$f_2'(y) = \cos y - \sin y,$$

there is only one candidate for a local extremum of f_2 on $\left(0, \frac{\pi}{2}\right)$, namely $y = \frac{\pi}{4}$. We have

$$f_2(0) = f_3(0) = f_2\left(\frac{\pi}{4}\right) = f_3\left(\frac{\pi}{4}\right) = 2 \quad \text{and} \qquad f_2\left(\frac{\pi}{4}\right) = f_3\left(\frac{\pi}{4}\right) = 1 + \sqrt{2}.$$

Any extremal point of f which is not in int D, must lie on the boundary and hence be either one of $\left\{(0,0), \left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$ or a local extremal point of f_1, f_2, f_3, or f_4. Comparing the values of f at those possible values, we obtain that

- f attains its minimum—0—at $(0,0)$;
- f attains its maximum—$\frac{3}{2}\sqrt{3}$—at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

4. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^N$ with limit x. Show that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ has content zero.

   Solution: Let $\epsilon > 0$, and choose $a_1, b_1, \ldots, a_N, b_N$ with $a_j < b_j$ for $j = 1, \ldots, N$ such that

   $$x \in (a_1, b_1) \times \cdots (a_N, b_N) =: J_0 \qquad \text{and} \qquad \prod_{j=1}^{N} b_j - a_j < \frac{\epsilon}{2}.$$

   As $\lim_{n\to\infty} x_n = x$, and since J_0 is a neighborhood of x, there is $n_0 \in \mathbb{N}$ such that $x_n \in J_0$ for all $n \geq n_0$. Set

   $$I_0 := [a_1, b_1] \times \cdots [a_N, b_N]$$

   Then I_0 is a compact interval in $\mathbb{R}^N$ with

   $$\{x_n : n \geq n_0\} \cup \{x\} \subset I_0 \qquad \text{and} \qquad \mu(I_0) < \frac{\epsilon}{2}.$$

   As a finite set, $\{x_1, \ldots, x_{n_0-1}\}$ has content zero, i.e., there are compact intervals $I_1, \ldots, I_m \subset \mathbb{R}^N$ such that

   $$\{x_1, \ldots, x_{n_0-1}\} \subset \bigcup_{j=1}^{m} I_j \qquad \text{and} \qquad \sum_{j=1}^{m} \mu(I_j) < \frac{\epsilon}{2}.$$

   It follows that

   $$\{x_n : n \in \mathbb{N}\} \cup \{x\} \subset \bigcup_{j=0}^{m} I_j \qquad \text{and} \qquad \sum_{j=0}^{m} \mu(I_j) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

   As $\epsilon > 0$ is arbitrary, we conclude that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ has content zero.

5. Let $I \subset \mathbb{R}^N$ be a compact interval. Show that ∂I has content zero.

Solution: Let

$$I = [a_1, b_1] \times \cdots \times [a_N, b_N].$$

For $j = 1, \ldots, N$ and $x \in \mathbb{R}^N$, set

$$S_{j,x} := [a_1, b_1] \times \cdots \times [a_{j-1}, b_{j-1}] \times \{x\} \times [a_{j+1}, b_{j+1}] \times \cdots \times [a_N, b_N]$$

In Problem 6* on Assignment #3, you showed that

$$\partial I = \bigcup_{j=1}^{N} S_{j,a_j} \cup S_{j,b_j}$$

It is therefore sufficient to show that, $\mu(S_{j,x}) = 0$ for any $j = 1, \ldots, N$ and $x \in \mathbb{R}$. Let $\epsilon > 0$, and let

$$J := [a_1, b_1] \times \cdots \times [a_{j-1}, b_{j-1}] \times [x - \delta, x + \delta] \times [a_{j+1}, b_{j+1}] \times \cdots \times [a_N, b_N],$$

where

$$\delta < \frac{1}{2} \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{\epsilon}{b_k - a_k}.$$

We then have

$$S_{j,x} \subset J \qquad \text{and} \qquad \mu(J) = 2\delta \prod_{\substack{k=1 \\ k \neq j}}^{N} (b_k - a_k) < \epsilon,$$

so that $\mu(S_{j,x}) = 0$.

6*. Let $I_1, \ldots, I_n \subset \mathbb{R}$ be compact intervals such that $\mathbb{Q} \cap [0,1] \subset I_1 \cup \cdots \cup I_n$. Show that $\sum_{j=1}^{n} \mu(I_j) \geq 1$.

Solution: Let $\epsilon > 0$. For $j = 1, \ldots, n$ and $I_j = [a_j, b_j]$ with $0 \leq a_j$ and $b_j \leq 1$, set $J_j := (a_j - \epsilon, b_j + \epsilon)$. We claim that $[0,1] \subset J_1 \cup \cdots \cup J_n$. To see this, let $x \in [0,1]$. Then there is $q \in \mathbb{Q} \cap [0,1]$ such that $|x - q| < \epsilon$, i.e., $q - \epsilon < x < q + \epsilon$. Let $j_q \in \{1, \ldots, n\}$ be such that $q \in I_{j_q}$, i.e., $a_{j_q} \leq q \leq b_{j_q}$. It follows that

$$a_{j_q} - \epsilon \leq q - \epsilon < x < q + \epsilon \leq b_{j_q} + \epsilon,$$

i.e., $x \in J_{j_q}$.

Let $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\{t_0, t_1, \ldots, t_m\}$ consists precisely of 0 and 1 and those boundary points of $J_1, \ldots, J_n$ that lie in $[0,1]$. Then we obtain that

$$\begin{aligned} 1 = \sum_{k=1}^{m} t_k - t_{k-1} &\leq \sum_{j=1}^{n} \sum_{(t_{k-1}, t_k) \subset J_j} t_k - t_{k-1} \\ &\leq \sum_{k=1}^{n} (b_n + \epsilon) - (a_n - \epsilon) = 2n\epsilon + \sum_{k=1}^{n} b_k - a_k = 2n\epsilon + \sum_{k=1}^{n} \mu(I_k). \end{aligned}$$

As $\epsilon > 0$ is arbitrary, this yields the claim.