# Fall 2021, Math 328, Homework 4

Due: End of day on 2021-10-09

## 1 10 points

Let $G$ be an abelian group, and let $a_1, \ldots, a_n \in G$ be elements satisfying

$$G = \langle a_1, \ldots, a_n \rangle$$

and put $k_i := \operatorname{ord}(a_i)$ for $i = 1, \ldots, n$. Assume that $k_i$ is finite for all $i$. Prove that $\#G \leq k_1 \cdots k_n$. Give an example showing that this can fail if $G$ is nonabelian.

## 2 10 points

Let $G$ be an abelian group. Prove that the set

$$\{g \in G \mid \operatorname{ord}(g) < \infty\}$$

is a subgroup of $G$. Give an example showing that this can fail if $G$ is nonabelian.

## 3 10 points

Let $H$ be a subgroup of a group $G$. Show that $H$ is contained in its normalizer $N_G(H)$. Show that $H$ is contained in $C_G(H)$ if and only if $H$ is abelian.

## 4 10 points

Let $H$ be a subgroup of order 2 in a group $G$. Show that $N_G(H) = C_G(H)$. Show $H \leq Z(G)$ provided $N_G(H) = G$.

# 5 10 points

Let $H(\mathbb{R})$ be the set of real-valued matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$.

1. Prove that $H(\mathbb{R})$ is a group with respect to matrix multiplication.
2. Prove that $Z(H(\mathbb{R}))$ is isomorphic to $(\mathbb{R}, +)$.
3. Prove that the map $H(\mathbb{R}) \to \mathbb{R} \times \mathbb{R}$ defined by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (a, b)$$

   is a surjective group homomorphism. What is the kernel of this homomorphism?

# 6 10 points

Prove that $(\mathbb{Q}, +)$ is noncyclic. Prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic. Find an explicit *proper* subgroup of $\mathbb{Q}$ which is noncyclic.

*Remark:* A group $G$ is *finitely generated* if it is generated by a *finite* subset $S \subset G$.