Math 127
Homework Problem Set 2

The purpose of the first problem is to highlight why it is important to recall that a field is not just a set of elements, but a triple of mathematical objects (a set and two operations (addition and multiplication) which satisfy the required properties), and that we can have more than one field structure on the same set of elements.

Problem 1. Recall that $\mathbb{R}$ with the standard operations of addition '$+$' and multiplication '$\cdot$' is a field. Define now new operations of addition and of multiplication on $\mathbb{R}$ as follows (note that on the two RHS below, the symbols $+$ and $\cdot$ denote the standard operations): for every $x, y \in \mathbb{R}$,

$$x \oplus y := x + y + 1 \qquad \textit{('new' addition)}$$

$$x \odot y := 2 \cdot x \cdot y + 2 \cdot x + 2 \cdot y + 1 \qquad \textit{('new' multiplication)}$$

(note also that on the two RHS above, the symbol 1 stands for the neutral element of the **standard multiplication** on $\mathbb{R}$, and similarly 2 stands for the standard real number we know, that is, $2 = 1 + 1$; based on these, note e.g. that for the new operations we have $1 \oplus 1 = 3$ and $1 \odot 1 = 7$).

It turns out that the triple $(\mathbb{R}, \oplus, \odot)$ is a field (you don't have to verify this here).

Denote by $0'$ the neutral element of the 'new' addition $\oplus$, and by $1'$ the neutral element of the 'new' multiplication $\odot$.

(i) Verify that $0' = -1$ (that is, $0'$ is equal to the (standard) additive inverse of the neutral element of the standard multiplication in $\mathbb{R}$); in other words, check that $0' = -1$ satisfies the axiom for the neutral element of the addition $\oplus$.

(ii) Find what the 'new' multiplicative identity $1'$ should be, that is, find a real number $1'$ that will satisfy the axiom for the neutral element of the multiplication $\odot$ *(hint: it is not going to be equal to* 1*)*.

(iii) Find what the 'new' additive inverse $\ominus 5$ of 5 should be, that is, find what real number x satisfies $x \oplus 5 = 0'$ *(hint: this number will not equal* -5*)*.

(iv) Find what the 'new' multiplicative inverse of 5 should be, that is, find what real number y satisfies $y \odot 5 = 1'$ *(hint: this number will not equal* 0.2*)*.

(v) Is $\mathbb{R}$ with the 'new' operations a subfield of $\mathbb{C}$ (the complex numbers with the field structure that we discussed in class)? Justify your answer.

If you like Sudoku-type puzzles, then the next problem is precisely for you! Otherwise.... well, it's just one more homework problem.

Problem 2. There is essentially only one field $\mathbb{F}_4$ with exactly 4 elements. Note that among these elements we must have the additive identity 0, as well as the multiplicative identity 1. Recall also that these have to be different elements. This shows that $\mathbb{F}_4 = \{0, 1, c, d\}$.

(i) Explain how the operations of addition and multiplication must be defined (so that $\mathbb{F}_4$ becomes a field) by completing their tables, and give reasons for your choices:

$+$	0	1	c	d
0				
1				
c				
d				

$\cdot$	0	1	c	d
0				
1				
c				
d				

(ii) Subsequently, verify that all the axioms of a field are satisfied.

[*Hint.* Recall that, by standard properties of fields (axioms or facts we have already derived from the axioms), one row and one column of the table of addition are predetermined, and so are two rows and two columns of the table of multiplication (which are these rows and columns?).

Make a note also of what the cancellation laws (see HW1, Problem 2) should give you: in the table of addition every element of $\mathbb{F}$ should appear exactly once in each row and each column (why? convince yourselves that this claim is equivalent to the cancellation law for addition); on the other hand, in the table of multiplication something similar holds for the part of the table that corresponds only to non-zero elements. It may also help you to go back to tables of addition and multiplication that we worked out for other examples (or that you perhaps completed as practice or for problems of HW1), and confirm these claims.

Finally, even though this choice can also be justified, it may be helpful to start with the assumption that $\mathbb{Z}_2$ is a subfield of this field (what would this imply for the tables?).]

Problem 3. (a) For each of the following subsets of $\mathbb{R}$, determine which, if any, of the following properties it has (the operations of addition and multiplication mentioned are the standard ones):

(P1) it is closed under addition;

(P2) it is closed under multiplication;

(P3) it is closed under taking additive inverses;

(P4) it is closed under taking multiplicative inverses (whenever possible).

(i) $[-1,1]$. (ii) $\{-1,0,1\}$. (iii) $\mathbb{R}\setminus\mathbb{Q}$. (iv) $\{0\}$.

(v) $\mathbb{N}_0$ (that is, the non-negative integers). (vi) $\mathbb{Z}$. (vii) $\mathbb{R}\setminus\{0\}$.

$(viii)$ $\left\{r\in\mathbb{R} : \exists\, q_1,q_2\in\mathbb{Q} \text{ such that } r=q_1+q_2\sqrt{5}\,\right\}$.

(ix) $\left\{r\in\mathbb{R} : \exists\, p_1,p_2\in\mathbb{Q} \text{ such that } r=p_1-p_2\sqrt{20}\,\right\}$.

(x) $\left\{r\in\mathbb{R} : \exists\, s_1,s_2\in\mathbb{Q} \text{ such that } r=s_1+es_2\right\}$.

[*Hint.* You may find the following fact useful: the number e is not only irrational, but also does not solve any non-trivial polynomial equation of the form $a_mx^m+a_{m-1}x^{m-1}+\cdots+a_2x^2+a_1x+a_0=0$ where the coefficients $a_0,a_1,\ldots,a_{m-1},a_m$ are **integers** (and not all of them are zero). Such real numbers are called *transcendental*; another such number is π.]

(b) Which of these subsets are subfields of $\mathbb{R}$? Justify your answer.

Problem 4. For each of the following lines or planes, which are defined via a linear equation (or a pair of linear equations), give an equivalent vector equation.

(i) The line $\ell_1=\left\{(x,y)\in\mathbb{R}^2 \mid 3x+4y+2=0\right\}$.

(ii) The line ℓ_2 in $\mathbb{R}^3$ defined by the system of linear equations

$$\left\{\begin{array}{c} x-y+3z=0 \\ x+y-z-2=0 \end{array}\right\}.$$

(iii) The plane $\mathcal{P}_1=\left\{(x,y,z)\in\mathbb{R}^3 \mid 2y-x-z=0\right\}$.

(iv) The plane $\mathcal{P}_2=\left\{(x,y,z)\in\mathbb{R}^3 \mid x-3y+z+6=0\right\}$.

Problem 5. (i) Prove that $\langle\bar{u},\bar{v}+\bar{w}\rangle=\langle\bar{u},\bar{v}\rangle+\langle\bar{u},\bar{w}\rangle$ for every three vectors $\bar{u},\bar{v}$ and $\bar{w}$ in $\mathbb{R}^n$.
(ii) Suppose $\bar{u}_1,\bar{v}_1$ and $\bar{w}_1$ are three vectors in $\mathbb{R}^n$ such that $\langle\bar{u}_1,\bar{v}_1\rangle\cdot\langle\bar{u}_1,\bar{w}_1\rangle\neq 0$. Show that we can find **non-zero** $t,s\in\mathbb{R}$ so that the vector $t\bar{v}_1+s\bar{w}_1$ is orthogonal to $\bar{u}_1$.

Problem 6. (i) Prove that $\langle\bar{x},\bar{x}\rangle\geqslant 0$ for every $\bar{x}\in\mathbb{R}^n$.
(ii) Prove that $\langle\bar{x},\bar{x}\rangle=0\Leftrightarrow\bar{x}=\bar{0}$. [*Note.* The symbol "$\Leftrightarrow$" means "if and only if".]

Problem 7. In $\mathbb{R}^n$ set $\bar{e}_1=\begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}$, $\bar{e}_2=\begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}$, $\ldots$, $\bar{e}_{n-1}=\begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}$, $\bar{e}_n=\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$.
These are called the *standard basis vectors* of $\mathbb{R}^n$.

For instance, in $\mathbb{R}^2$ there are two standard basis vectors: $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\bar{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, while in $\mathbb{R}^3$ there are three: $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\bar{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(i) Show that any vector $\bar{x}$ in $\mathbb{R}^n$ is a linear combination of $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$. In other words, $\mathrm{span}(\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n) = \mathbb{R}^n$.

(ii) Show that, for every $1 \leqslant i \leqslant n$, $\|\bar{e}_i\| = 1$. Moreover, show that, for every $1 \leqslant i, j \leqslant n$ with $i \neq j$, $\langle \bar{e}_i, \bar{e}_j \rangle = 0$.