MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Solutions #4

1. For $0 \leq r \leq R$ and $\epsilon \in (0,1)$, determine whether or not the set

$$\left\{(x,y,z) \in \mathbb{R}^3 : r^2 \leq x^2 + z^2 \leq R^2,\ |y| \in [\epsilon, 1]\right\}$$

is (a) open, (b) closed, (c) compact, or (d) connected.

Solution: Let the set under consideration be called S.

Let $((x_n, y_n, z_n))_{n=1}^\infty$ be a sequence in S converging to $(x,y,z) \in \mathbb{R}^3$. It follows that

$$r^2 \leq x_n^2 + z_n^2 \leq R^2 \qquad \text{and} \qquad |y_n| \in [\epsilon, 1]$$

for all $n \in \mathbb{N}$. Since $x_n \to x$, $y_n \to y$, and $z_n \to z$, the properties of the limit in $\mathbb{R}$ and the fact that $[\epsilon, 1]$ is closed in $\mathbb{R}$ yield that

$$r^2 \leq x^2 + z^2 \leq R^2 \qquad \text{and} \qquad |y| \in [\epsilon, 1],$$

so that $(x,y,z) \in S$. Consequently, S is closed.

Note that

$$x^2 + y^2 + z^2 \leq R^2 + 1,$$

for $(x,y,z) \in S$, so that $S \subset B_{\sqrt{R^2+1}}[(0,0,0)]$, i.e., S is bounded. Hence, S is compact by the Heine Borel Theorem.

As $\varnothing \neq S \neq \mathbb{R}^3$, it is clear that S cannot be open.

Finally, S is not connected because $\{U, V\}$ with

$$U := \left\{(x,y,z) \in \mathbb{R}^3 : y < 0\right\} \qquad \text{and} \qquad V := \left\{(x,y,z) \in \mathbb{R}^3 : y > 0\right\}$$

is a disconnection for S as one checks easily.

2. A set $S \subset \mathbb{R}^N$ is called *star shaped* if there is $x_0 \in S$ such that $tx_0 + (1-t)x \in S$ for all $x \in S$ and $t \in [0,1]$. Show that every star shaped set is connected, and give an example of a star shaped set that fails to be convex.

Solution: Let S be star shaped, and let $x_0 \in S$ be as in the definition. Assume that there is a disconnection $\{U, V\}$ of S. Without loss of generality suppose that $x_0 \in U$. Let $x \in V \cap S$, and set

$$\tilde{U} := \{t \in \mathbb{R} : tx_0 + (1-t)x \in U\} \qquad \text{and} \qquad \tilde{V} := \{t \in \mathbb{R} : tx_0 + (1-1)t \in V\}.$$

As in the proof for the connectedness of convex sets, one sees that $\left\{\tilde{U}, \tilde{V}\right\}$ is a disconnection for $[0,1]$, which is impossible.

Set, for instance,

$$S := \left\{(x,y) \in \mathbb{R}^2 : y \leq |x|\right\}.$$

For $(x,y) \in S$, i.e., such that $y \leq |x|$, and $t \in [0,1]$, we have $(1-t)y \leq |(1-t)x|$, so that $((1-t)x, (1-t)y) = t(0,0) + (1-t)(x,y) \in S$. Hence, S is star shaped. Clearly, $(1,1), (-1,1) \in S$ whereas

$$(0,1) = \frac{1}{2}(1,1) + \frac{1}{2}(-1,1) \notin S.$$

Hence, S is not convex.

3. Let $C \subset \mathbb{R}^N$ be connected. Show that $\overline{C}$ is also connected.

   Solution: Assume that there is a disconnection $\{U, V\}$ for $\overline{C}$. It is then obvious that $(C \cap U) \cap (C \cap V) = \varnothing$ and $(C \cap U) \cup (C \cup V) = C$. Assume that $C \cap U = \varnothing$, i.e., $C \subset U^c$. As U is open, U^c is closed, so that $\overline{C} \subset U^c$ as well, i.e., $\overline{C} \cap U = \varnothing$. But this is impossible because $\{U, V\}$ is a disconnection for $\overline{C}$. Similarly, one sees that $C \cap V \neq \varnothing$.

   All in all, $\{U, V\}$ is a disconnection for C, which is impossible because C is connected.

4. Let $S \subset \mathbb{R}^N$, and let $x \in \mathbb{R}^N$. Show that $x \in \overline{S}$ if and only if there is a sequence $(x_n)_{n=1}^\infty$ in S such that $x = \lim_{n\to\infty} x_n$.

   Solution: Suppose that there is a sequence $(x_n)_{n=1}^\infty$ in S such that $x = \lim_{n\to\infty} x_n$. As $(x_n)_{n=1}^\infty$ is also contained in $\overline{S}$ and since $\overline{S}$ is closed, it follows that $x \in \overline{S}$.

   Conversely, let $x \in \overline{S}$. If $x \in S$, there certainly is a sequence $(x_n)_{n=1}^\infty$ converging to x: just set $x_n := x$ for $n \in \mathbb{N}$. If $x \notin S$, then x must be a cluster point of S by the definition of $\overline{S}$, i.e., for each $n \in \mathbb{N}$, there is $x_n \in B_{\frac{1}{n}}(x) \cap S$, so that $x_n \to x$.

5. Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}^N$ with limit x. Show that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.

   Solution: Let $\{U_i : i \in \mathbb{I}\}$ be an open cover for $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Choose $i_0 \in \mathbb{I}$ such that $x \in U_{i_0}$. Since U_{i_0} is open, it is a neighborhood of x. Hence, there is $n_0 \in \mathbb{N}$ such that $x_n \in U_{i_0}$ for all $n \geq n_0$. For $j = 1, \ldots, n_0 - 1$, choose $i_j \in \mathbb{I}$ such that $x_j \in U_{i_j}$. It follows that

   $$K \subset U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_{n_0-1}},$$

   so that K is compact as claimed.

6*. Show that $\mathbb{R}^N \setminus \{0\}$ is disconnected if and only if $N = 1$.

Solution: If $N = 1$, then $\{(-\infty, 0), (0, \infty)\}$ is a disconnection for $S := \{x \in \mathbb{R}^N : x \neq 0\}$.

Let $N \geq 2$ and assume that there is a disconnection $\{U, V\}$ for S. Fix $x \in U \cap S$ and $y \in V \cap S$.

Suppose first that $x + t(y - x) \neq 0$ for all $t \in \mathbb{R}$. Define

$$\tilde{U} := \{t \in \mathbb{R} : x + t(y - x) \in U \cap S\}$$

and

$$\tilde{V} := \{t \in \mathbb{R} : x + t(y - x) \in V \cap S\}.$$

As in the proof of the connecteness of convex sets, one sees that $\left\{\tilde{U}, \tilde{V}\right\}$ is a disconnection for $\mathbb{R}$, which is not possible.

Suppose now that there is $t_0 \in \mathbb{R}^N$ such that $x + t_0(y - x) = 0$. Since $y \neq 0$, we have $t_0 \neq 1$ and thus $x = -\frac{t_0}{1-t_0} y$. Let $j \in \{1, \ldots, N\}$ be such that $y_j \neq 0$; then we have $-\frac{t_0}{1-t_0} = \frac{x_j}{y_j}$ and thus $x = \frac{x_j}{y_j} y$. Let $\epsilon > 0$ be such that $B_\epsilon(x) \subset U \cap S$. Fix $k \in \{1, \ldots, N\} \setminus \{j\}$, and define $\tilde{x} \in \mathbb{R}^N$ by letting

$$\tilde{x}_l := \begin{cases} x_l, & l \neq k, \\ x_k + \epsilon, & k = l, \end{cases}$$

for $l = 1, \ldots, N$. It follows that $\tilde{x} \in B_\epsilon(x) \subset U \cap S$. Assume that there is $\tilde{t}_0 \in \mathbb{R}$ such that $\tilde{x} + \tilde{t}_0(y - \tilde{x}) = 0$. Then—as before—it follows that

$$\tilde{x} = \frac{\tilde{x}_j}{y_j} y = \frac{x_j}{y_j} y = x,$$

which is impossible by the definition of $\tilde{x}$. Hence, $\tilde{x} + t(y - \tilde{x}) \neq 0$ must hold for all $t \in \mathbb{R}$, which is impossible as we just saw.