Fall 2021, Math 328, Homework 7

Due: End of day on 2021-11-29

1 10 points

Let X and Y be two sets and G a group acting on X and Y. A function $f : X \to Y$ is called *G-equivariant* provided that for all $x \in X$ and $g \in G$, one has $f(g \cdot x) = g \cdot f(x)$. A G-equivariant *isomorphism* between X and Y is a G-equivariant function $f : X \to Y$ such that there exists a G-equivariant function $g : Y \to X$ satisfying $f \circ g = \mathbf{1}$ and $g \circ f = \mathbf{1}$.

Now suppose that X is a set with an action of G, and let $x \in X$ be given.

1. Prove that there is a *unique* action of G on $\mathrm{Orb}_G(x)$ such that the inclusion function
$$\mathrm{Orb}_G(x) \hookrightarrow X$$
is G-equivariant.

2. Prove that the function
$$\mathrm{Orb}_G(x) \to G/\operatorname{Stab}_G(x)$$
defined by $g \cdot x \mapsto g \cdot \operatorname{Stab}_G(x)$ is well-defined and G-equivariant. Here G acts on $G/\operatorname{Stab}_G(x)$ by left multiplication.

3. Prove that the function from item (2) above is a G-equivariant isomorphism.

4. Suppose that G acts transitively on X and that X is nonempty. Prove that there is a G-equivariant isomorphism between X and G/H for some subgroup H of G, where G acts on G/H by left multiplication.

Optional: Suppose that G acts on X. Prove that there is a G-equivariant isomorphism between X and
$$\coprod_i G/H_i,$$
where H_i is a (possibly empty) collection of subgroups of G, and $\coprod$ denotes the disjoint union. *Note:* You should first think about how G acts on a disjoint union of sets each endowed with an action of G.

2 10 points

Let G be a group, and let A and B be two normal subgroups of G with $A \cdot B = G$. Prove that $G/(A \cap B) \cong G/A \times G/B$.

3 10 points

Consider the *unit circle group* $S := \{z \in \mathbb{C} \mid |z| = 1\}$, which is a group with respect to multiplication of complex numbers. Let n be a positive integer. Consider the map $S \to S$ sending z to z^n. Prove that this is a surjective homomorphism with finite kernel. Deduce that S has a normal subgroup N such that $N \neq \{1\}$ and such that $S \cong S/N$. Can such a subgroup exist in a finite group?

4 10 points

Let G be a group acting transitively on a nonempty finite set X, and let H be a normal subgroup of G. Let H act on X via the inclusion $H \hookrightarrow G$, and let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the distinct orbits of H acting on X.

1. Prove that for all $g \in G$ and all $i = 1, \ldots, r$, there is a j such that $g \cdot \mathcal{O}_i = \mathcal{O}_j$. Prove that this induces an action of G on $\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$, and that this action is transitive. Prove that $\mathcal{O}_1, \ldots, \mathcal{O}_r$ all have the same size.

2. Suppose $a \in \mathcal{O}_1$. Prove that one has $\#\mathcal{O}_1 = [H : H \cap \mathrm{Stab}_G(a)]$. Prove that one has $r = [G : \mathrm{Stab}_G(a) \cdot H]$.

5 10 points

1. Let G be a group, and N a normal subgroup of order 2. Show that N is contained in the center of G. (See homework 4).

2. Prove that every nonabelian group of order 6 has a nonnormal subgroup of order 2 (See homework 1).

3. Classify all groups of order 6, up-to isomorphisms.

6 10 points

1. Find all finite groups which have exactly two conjugacy classes.

2. Find all finite groups which have exactly three conjugacy classes.