MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Solutions #2

1. For any set S, its *power set* $\mathfrak{P}(S)$ is defined to be the set consisting of all subsets of S. Show that there is *no* surjective map from S to $\mathfrak{P}(S)$. (*Hint*: Assume that there is a surjective map $f\colon S \to \mathfrak{P}(S)$ and consider the set $\{x \in S : x \notin f(x)\}$.)

   Solution: Assume there is a surjective map $f\colon S \to \mathfrak{P}(S)$, and let

   $$T := \{s \in S : s \notin f(s)\} \in \mathfrak{P}(S).$$

   Since f is surjective, there must be $s \in S$ such that $T = f(s)$. By the definition of T, we have

   $$s \in T \quad \Longleftrightarrow \quad s \notin f(s) = T,$$

   which is nonsense. Hence, there can be no surjective map $f\colon S \to \mathfrak{P}(S)$.

2. Which of the following sets are convex:

   (i) $\{(x, y) \in \mathbb{R}^2 : x > y\}$;

   (ii) $\{x \in \mathbb{R}^N : \|x\| > 2\}$;

   (iii) $\mathbb{R} \setminus \mathbb{Q}$;

   (iv) $\{(x, y, z) \in \mathbb{R}^3 : x + y + z \geq 2021\}$?

   Justify your answers.

   Solution: In each of the following, let C be the set under consideration.

   (a) Let $(x_1, y_1), (x_2, y_2) \in C$, and let $t \in [0, 1]$. It is clear that $t(x_1, y_1) + (1 - t)(x_2, y_2) \in C$ if $t = 0$ or $t = 1$. We may thus suppose without loss of generality that $t \in (0, 1)$. We have

   $$x_1 > y_1 \qquad \text{and} \qquad x_2 > y_2.$$

   Multiplying these inequalities with t and $1 - t$, respectively, we obtain

   $$tx_1 > ty_2 \qquad \text{and} \qquad (1 - t)x_2 > (1 - t)y_2.$$

   Adding these two inequalities, eventually yields

   $$tx_1 + (1 - t)x_2 > ty_1 + (1 - t)y_2,$$

   so that $t(x_1, y_1) + (1 - t)(x_2, y_2) \in C$. Hence, C is convex.

(b) Let $x \in C$. Then $\|-x\| = \|x\| > 2$, so that $-x \in C$ as well. Since

$$0 = \frac{1}{2}x + \frac{1}{2}(-x) \notin C,$$

the set C cannot be convex.

(c) Let $x, y \in C$, and suppose, without loss of generality, that $x < y$. As we have seen in class, there is $q \in (x, y) \cap \mathbb{Q}$. Set $t := \frac{y-q}{y-x}$, so that $t \in [0,1]$ and $q = tx + (1-t)y$. Hence, C is not convex.

(d) Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in C$, and let $t \in [0,1]$. Then

$$x_j + y_j + z_j \geq 2020$$

folds for $j = 1, 2$ and therefore

$$t(x_1 + y_1 + z_1) \geq t\,2020 \qquad \text{and} \qquad (1-t)(x_2 + y_2 + z_2) \geq (1-t)2020.$$

Adding these two inequalities yields

$$t(x_1 + y_1 + z_1) + (1-t)(x_2 + y_2 + z_2) \geq 2020.$$

Hence, C is convex.

3. Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^N$. Show that $\bigcap_{C \in \mathcal{C}} C$ is again convex. Is $\bigcup_{C \in \mathcal{C}} C$ necessarily convex?

Solution: Let $x, y \in \bigcap_{C \in \mathcal{C}} C$, i.e., $x, y \in C$ for each $C \in \mathcal{C}$. Let $t \in [0,1]$. Since each $C \in \mathcal{C}$ is convex, we have $tx + (1-t)y \in C$ for each $C \in \mathcal{C}$. Hence, $tx + (1-t)y \in \bigcap_{C \in \mathcal{C}} C$. Consequently, $\bigcap_{C \in \mathcal{C}} C$ is convex.

Let $x, y \in \mathbb{R}^N$ be such that $x \neq y$, and set $\mathcal{C} = \{\{x\}, \{y\}\}$. Then $\{x\}$ and $\{y\}$ are convex, but $\frac{1}{2}x + \frac{1}{2}y \notin \{x\} \cup \{y\}$.

4. Show that $\mathbb{Z}$ is closed in $\mathbb{R}$, but not open, and that $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

Solution: Let $x \in \mathbb{R} \setminus \mathbb{Z}$, and let $\lfloor x \rfloor$ be the largest integer less than or equal to x, e.g., $\lfloor 2 \rfloor = 2$, $\lfloor \pi \rfloor = 3$, or $\lfloor -\frac{9}{5} \rfloor = -5$. It follows that $\lfloor x \rfloor < x < \lfloor x \rfloor + 1$ (as $x \notin \mathbb{Z}$, the equalities must be strict). Set

$$\epsilon := \min\{x - \lfloor x \rfloor, \lfloor x \rfloor + 1 - x\},$$

so that

$$(x - \epsilon, x + \epsilon) \subset (\lfloor x \rfloor, \lfloor x \rfloor + 1).$$

It follows that $(x - \epsilon, x + \epsilon) \cap \mathbb{Z} = \varnothing$. Hence, $\mathbb{R} \setminus \mathbb{Z}$ is open, and $\mathbb{Z}$ is closed.

Assume that $\mathbb{Q}$ is open. Then, for any $q \in \mathbb{Q}$, there is $\epsilon > 0$ such that $(q-\epsilon, q+\epsilon) \subset \mathbb{Q}$. Choose $n \in \mathbb{N}$ so large that $\frac{\sqrt{13}}{n} < \epsilon$; it follows that $q + \frac{\sqrt{13}}{n} \in (q - \epsilon, q + \epsilon)$, but $q + \frac{\sqrt{13}}{n} \notin \mathbb{Q}$, which is a contradiction.

Assume that $\mathbb{Q}$ is closed, i.e., $\mathbb{R} \setminus \mathbb{Q}$ is open. Then, for any $x \in \mathbb{R} \setminus \mathbb{Q}$, there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus \mathbb{Q}$. In class, however, it was shown that there is a rational number between $x - \epsilon$ and $x + \epsilon$. Hence, $\mathbb{R} \setminus \mathbb{Q}$ cannot be open, so that $\mathbb{Q}$ is not closed.

5. Let $\varnothing \neq S \subset \mathbb{R}^N$ be arbitrary, and let $\varnothing \neq U \subset \mathbb{R}^N$ be open. Show that

$$S + U := \{x + y : x \in S,\, y \in U\}$$

is open.

Solution: Let $x \in S$, and define

$$x + U := \{x + y : y \in U\}.$$

We claim that $x + U$ is open. Let $\tilde{x} \in x + U$, so that $\tilde{x} - x \in U$. Let $\epsilon > 0$ be such that $B_\epsilon(\tilde{x} - x) \subset U$, and let $\tilde{y} \in \mathbb{R}^N$ be such that $\|\tilde{x} - \tilde{y}\| < \epsilon$. It follows that

$$\|(\tilde{y} - x) - (\tilde{x} - x)\| = \|\tilde{y} - \tilde{x}\| < \epsilon,$$

i.e., $\tilde{y} - x \in B_\epsilon(\tilde{x} - x) \subset U$ and thus $\tilde{y} \in x + U$. Hence, $x + U$ is open.

Since

$$S + U := \bigcup_{x \in S} (x + U),$$

it is clear that $S + U$ is also open.

6* For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, set

$$\|x\|_1 := |x_1| + \cdots + |x_N| \qquad \text{and} \qquad \|x\|_\infty := \max\{|x_1|, \ldots, |x_N|\}.$$

(a) Show that the following are true for $j = 1, \infty$, $x, y \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$:

(i) $\|x\|_j \geq 0$ and $\|x\|_j = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\|_j = |\lambda| \|x\|_j$;

(iii) $\|x + y\|_j \leq \|x\|_j + \|y\|_j$.

(b) For $N = 2$, sketch the sets of those x for which $\|x\|_1 \leq 1$, $\|x\| \leq 1$, and $\|x\|_\infty \leq 1$.

(c) Show that

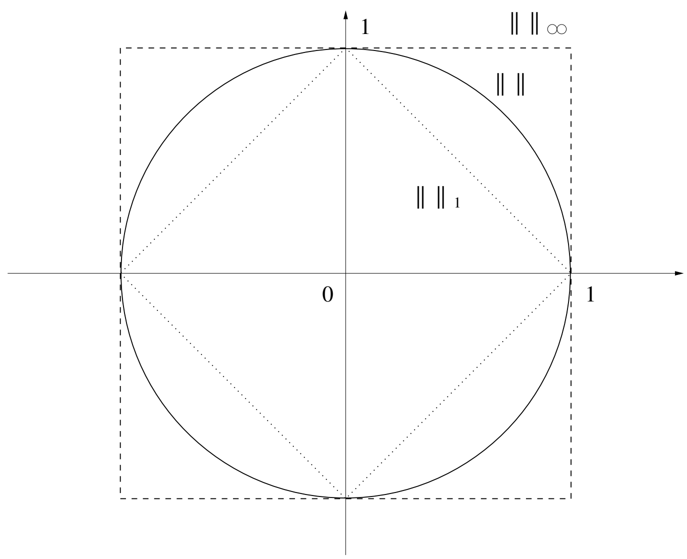
$$\|x\|_1 \leq \sqrt{N} \|x\| \leq N \, \|x\|_\infty$$

for all $x \in \mathbb{R}^N$.

Solution:

(a) The verification of (a) is routine (just use the corresponding properties of the absolute value on $\mathbb{R}$).

(b) Your sketch should look like this:



(c) Let $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, and let $y = (1, \ldots, 1)$. The Cauchy–Schwarz Inequality then yields that

$$\|x\|_1 = \sum_{j=1}^{N} |x_j y_j| \leq \|x\| \|y\| = \sqrt{N}\, \|x\|.$$

Moreover, we have

$$\|x\| = \sqrt{\sum_{j=1}^{N} x_j^2} \leq \sqrt{\sum_{j=1}^{N} \|x\|_\infty^2} = \sqrt{N}\, \|x\|_\infty.$$