Induction

Warm up Problem.
A group of people with assorted eye colors live on an island. They are all perfect logicians -- if a conclusion can be logically deduced, they will do it instantly. No one knows the color of their eyes. Every night at midnight, a ferry stops at the island. Any islanders who have figured out the color of their own eyes then leave the island, and the rest stay. Everyone can see everyone else at all times and keeps a count of the number of people they see with each eye color (excluding themselves), but they cannot otherwise communicate. Everyone on the island knows all the rules in this paragraph.

On this island there are 3 blue-eyed people, 3 brown-eyed people, and the Guru (she happens to have green eyes). So any given blue-eyed person can see 3 people with brown eyes and 2 people with blue eyes (and one with green), but that does not tell him his own eye color; as far as (s)he knows the totals could be 4 brown and 2 blue. Or 3 brown, 2 blue, and (s)he could have red eyes.

The Guru is allowed to speak once (let's say at noon), on one day in all their endless years on the island. Standing before the islanders, she says the following:

"I can see someone who has blue eyes."

Who leaves the island, and on what night?

There are no mirrors or reflecting surfaces, nothing dumb. It is not a trick question, and the answer is logical. It doesn't depend on tricky wording or anyone lying or guessing, and it doesn't involve people doing something silly like creating a sign language or doing genetics. The Guru is not making eye contact with anyone in particular; she's simply saying "I count at least one blue-eyed person on this island who isn't me."

And lastly, the answer is not "no one leaves."

no one leaves the 1st night

$\Rightarrow$ A blue-eyed person saw another blue-eyed person (since otherwise they would have left)

$\therefore$ Everyone knows there are at least 2 blue-eyed people

no one leaves the 2nd night

$\Rightarrow$ The blue-eyed people see two others with blue eyes. (since if they only saw 1 other blue eyed person they would have left)

$\therefore$ Everyone knows there are at least 3 blue-eyed people

$\therefore$ All the blue-eyed people can leave on night 3 (since they only see two others with blue eyes)

The **Principle of Mathematical Induction**: to show a statement P_n is true for all positive integers greater or equal to b there are two steps:

- *Base Case (BC):* Show P_b is true
- *Inductive Step (IS):* Show: $P_n \Rightarrow P_{n+1}$ (for all $n \geq b$)

Example 1: Prove:

$$P_n : 1 + 2 + 3 + \cdots + n = \frac{(n+1)\cdot n}{2}$$

for $n \geq 1$ by using mathematical induction.

BC Show: P_1 is true

$$1 = \frac{(1+1)\cdot 1}{2}$$

IS Show: $P_n \Rightarrow P_{n+1}$ for $n \geq 1$.

$$1 + 2 + \cdots + n + (n+1)$$

$$\overset{P_n}{=} \frac{n(n+1)}{2} + (n+1)$$

$$= (n+1)\left(\frac{n}{2} + 1\right)$$

$$= \frac{(n+1)(n+2)}{2}$$

$\therefore$ P_{n+1} is true.

Example 2: Let m be an integer and prove

$$P_n:\quad m + (m+1) + \cdots + n = \frac{(n+m)(n-m+1)}{2}$$

for $n \geq m$ by using mathematical induction.

$\underline{BC}$ $n = m$ $\quad m = \frac{(m+m)(\cancel{m}-\cancel{m}+1)}{2} = \frac{2m}{2}$ ✓

$\underline{IS}$ Show: $P_n \Rightarrow P_{n+1}$ for $n \geq m$.

$$m + (m+1) + \cdots + n + (n+1)$$

$$\overset{P_n}{=} \frac{(n+m)(n-m+1)}{2} + (n+1)$$

$$= \frac{(n+m)(n-m+1) + 2n + 2}{2}$$

$$= \frac{(n+m)(n-m+1) + (n+\cancel{m}) + (n-\cancel{m}+1) + 1}{2}$$

$\therefore P_{n+1}$ is true ✓

$$= \frac{((n+m)+1)((n-m+1)+1)}{2}$$

Example 3: Find a closed form for the following expression.

$$\frac{1 + 2 + \cdots + n}{n + (n+1) + \cdots + 2n}$$

$\underline{\underline{Ex\,2}}$

$$\frac{\frac{n(n+1)}{\cancel{2}}}{\frac{(n+2n)(2n-n+1)}{\cancel{2}}} = \frac{\cancel{n(n+1)}}{3\cancel{n(n+1)}} = \boxed{\frac{1}{3}}$$

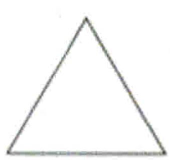
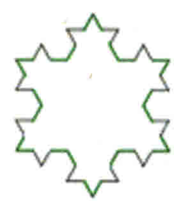
Example 4: Define a sequence of shapes as follows:

- K_1 is an equilateral triangle
- for $n > 1, K_n$ is formed by replacing each line segment

of K_{n-1} with the shape

according to the following three steps
1) The line segment was divided into three segments of equal length.
2) An equilateral triangle was drawn pointing outward that has its middle segment from step 1 as its base.
3) The line segment that is the base of the triangle from step 2 was removed.

The limit of this sequence of shapes is known as the Koch snowflake:

a) Write down a recurrence relation for a_n the number of line segments in K_n.

$$a_n = 4a_{n-1} \qquad a_1 = 3$$

b) Show by induction that the number of line segments in K_n is:

$$3 \cdot 4^{n-1}$$

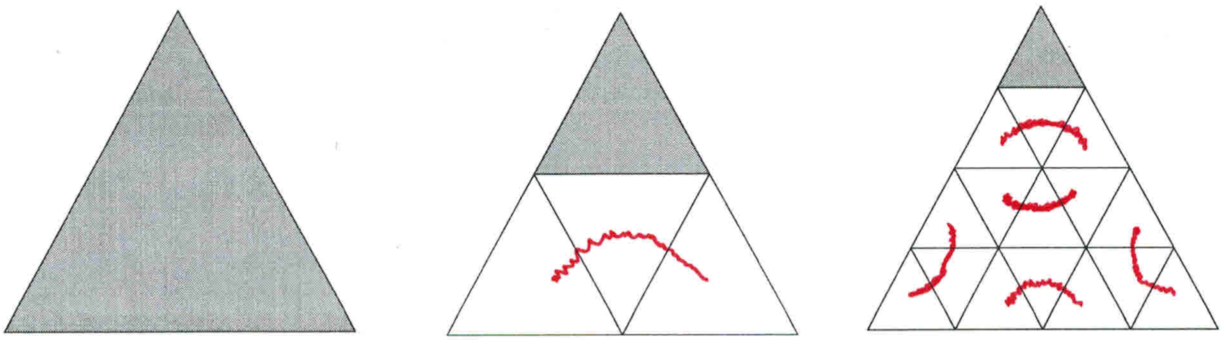
BC $\quad a_1 = 3 \cdot 4^{1-1} = 3$ ✓

IS $\quad$ show: $a_n \overset{P_n}{=} 3 \cdot 4^{n-1} \Rightarrow a_{n+1} = 3 \cdot 4^n$ for $n \geq 1$.

$$a_{n+1} \overset{a)}{=} 4 \, a_{n-1} \overset{P_n}{=} 4 \cdot 3 \cdot 4^{n-1} = 3 \cdot 4^n$$ ✓

Example 5: A corner deficient grid of 4^n triangles is made but cutting an equilateral triangle into 4^n congruent equilateral triangles and removing one of the corners. The deficient grids of 1 and 2 triangles are shown below:

$n = 0$ $\qquad$ $n = 1$ $\qquad$ $n = 2$



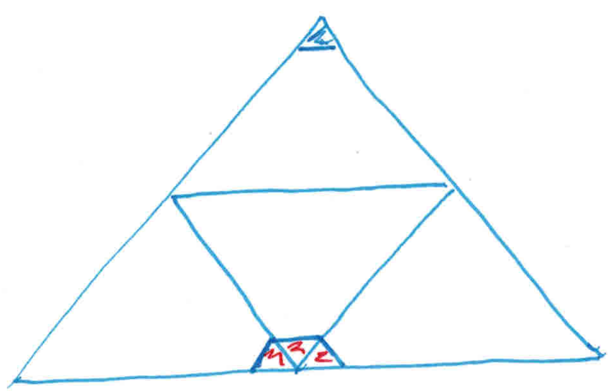
Show that the following statement is true for $n \geq 0$. — no overlaps, loose ends.

P_n: A corner deficient grid of 4^n triangles can be tiled with trapezoidal tiles made of 3 triangles:

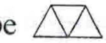
BC $\quad n=0 \quad 4^0 = 1$ triangle with the corner missing is tiled.

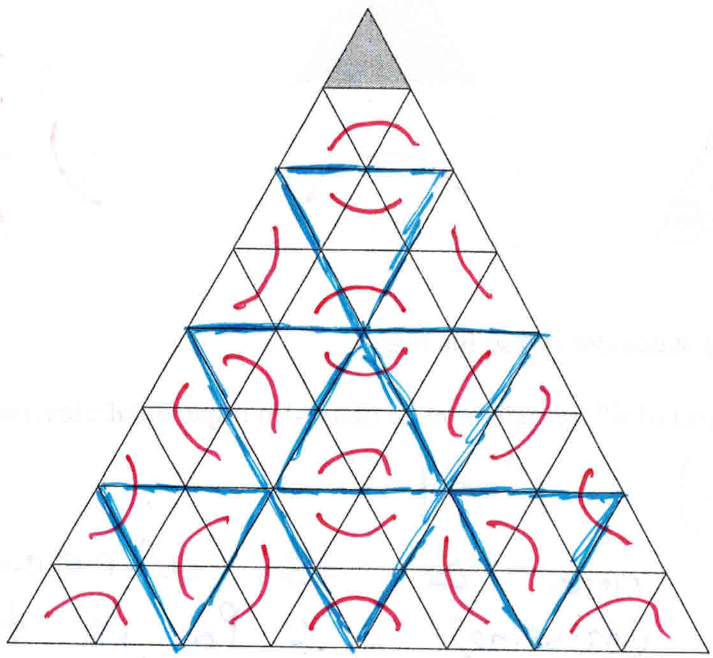
IS Show: $P_n \Rightarrow P_{n+1}$, for $n \geq 0$

Consider a corner deficient grid of 4^{n+1} Δ's



By placing one trapezoid as show each of the 4 sections can be tiled by P_n. $\therefore$ P_{n+1} is true.

Example 6: Use example 5 to tile a corner deficient grid of 4^3 triangles with the shape  :



Example 7: Conjecture a formula for the sum of the first n Fibonacci numbers with even indices and prove your formula works by using mathematical induction. That is, find and prove a formula for $F_2 + F_4 + \cdots + F_{2n}$.

Note:

$$F_1 = 1,$$
$$F_2 = 1,$$

$$F_n = F_{n-1} + F_{n-2}.$$

$F_1 = 1$

$F_2 = 1$

$F_3 = 2 \xrightarrow{-1} 1 = F_2$

$F_4 = 3$

$F_5 = 5 \xrightarrow{-1} 4 = F_2 + F_4$

$F_6 = 8$

$F_7 = 13 \xrightarrow{-1} 12 = F_2 + F_4 + F_6$

Guess:

P_n:

$F_{2n+1} - 1 = F_2 + F_4 + \cdots + F_{2n}$

BC $\quad n = 1 \qquad F_3 - 1 = F_2$ ✓

IS $\quad$ Show: $P_n \Rightarrow P_{n+1}$, for $n \geq 1$.

$$F_2 + F_4 + \cdots + F_{2n} + F_{2n+2}$$

$$\overset{P_n}{=} F_{2n+1} - 1 + F_{2n+2}$$

$$\overset{\text{Def}}{=} F_{2n+3} - 1 \qquad \therefore \; P_{n+1} \text{ is true}$$ ✓

Extra: Solve the warm up problem in general by proving the following statement:

P_n: if there are n blue-eyed people then on night n they all leave

for $n \geq 1$.

BC Show: P_1 is true.

One blue-eyed person will see no others with blue eyes and can conclude (s)he has blue eyes.

IS Show $P_n \Rightarrow P_{n+1}$. If there are $n+1$ blue-eyed people.

No one leaves on the n^{th} night

$\Rightarrow$ The blue-eyed people see n others with blue-eyes (since if they only saw $n-1$ other blue-eyed people they would have left on night n by P_n)

$\therefore$ Everyone knows there are at least $n+1$ blue-eyed people

$\therefore$ All the blue eyed people leave on the $(n+1)^{th}$ night.

$\therefore$ P_{n+1} is true

Example 8:

P_n: All girls have the same hair colour. We claim that all girls in any group of n girls have the same hair colour, for each $n = 1, 2, \ldots$

BC *Step 1:* When $n = 1$, there is only one girl in the group, so all girls within the group clearly have the same hair colour.

IS *Step 2:* Assume that the case $n = k$ holds. Given a group of $k + 1$ girls, remove one of them from the group. By assumption, each of the remaining k girls have the same hair colour. Now swap one of these girls with the girl we removed. Since every girl in this new group of k girls also have the same hair colour, we know now that all $k + 1$ girls have the same hair colour!

By induction, the claim holds.

Is there an error in the above "proof"? If so, where is the flaw?

Yes there is a flaw:

$$P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow P_4 \Rightarrow$$

true FALSE true true true

Since $P_1 \Rightarrow P_2$ is not true the proof doesn't work.

we need to show $P_n \Rightarrow P_{n+1}$ for all $n \geq 1$.

Strong Induction; to show a statement P_n is true for all positive integers greater or equal to n_0 there are two steps:

- *Base Case*: Show P_{n_0} is true (you may need to do this for other values also)

- *Inductive Step:* Show: $(P_{n_0} \,\&\, P_{n_0+1} \,\&\, P_{n_0+2} \,\&\, \cdots \,\&\, P_n) \Rightarrow P_{n+1}$ (for all $n \geq n_0$)

Example 9: P_n: "Postage of n cents can be formed using 4-cent and 5-cent stamps." Show: P_n is true for $n \geq 12$.

BC

$$\left.\begin{aligned} 3(4) + 0(5) &= 12 \\ 2(4) + 1(5) &= 13 \\ 1(4) + 2(5) &= 14 \\ 0(4) + 3(5) &= 15 \end{aligned}\right\} \Rightarrow P_{12}, P_{13}, P_{14}, P_{15} \text{ are true}$$

IS Show: $P_n \Rightarrow P_{n+4}$ for $n \geq 12$

$$n+4 \overset{P_n}{=} 4x + 5y + 4$$ (since P_n is true $n = 4x+5y$ for some non-negative integers x, y.)

$$= 4(x+1) + 5y$$

$\therefore$ P_{n+4} is true.

Note we have shown:

$$P_{12} \Rightarrow P_{16} \Rightarrow P_{20} \Rightarrow \cdots$$
$$P_{13} \Rightarrow P_{17} \Rightarrow P_{21} \Rightarrow \cdots$$
$$P_{14} \Rightarrow P_{18} \Rightarrow P_{22} \Rightarrow \cdots$$
$$P_{15} \Rightarrow P_{19} \Rightarrow P_{23} \Rightarrow \cdots$$

Example 10: Show that every positive integer can be written as a sum of distinct powers of two.

Let P_n be the statement for $n \geq 1$.

<u>BC</u> $1 = 2^0 \implies P_1$ is true

<u>IS</u> Show: $P_n \implies (P_{2n} \;\&\; P_{2n+1})$ for $n \geq 1$

$$2n \overset{P_n}{=} 2(\underbrace{2^i + \cdots + 2^j}_{\text{distinct powers}}) = \underbrace{2^{i+1} + \cdots + 2^{j+1}}_{\text{distinct powers}}$$

$$\implies 2n+1 = \overbrace{2^{i+1} + \cdots + 2^{j+1}}^{\text{distinct powers}} + 1 = \overbrace{2^{i+1} + \cdots + 2^{j+1} + 2^0}^{\text{distinct powers}}$$

$\therefore$ P_{2n} & P_{2n+1} are true.

Note we have shown:

$$P_1$$

$$\implies P_2 \qquad P_3$$

$$\implies P_4 \quad P_5 \quad P_6 \quad P_7$$

$$\implies P_8 \quad P_9 \quad P_{10} \quad P_{11} \quad P_{12} \quad P_{13} \quad P_{14} \quad P_{15}$$

$$\vdots$$