Math 127
Homework Problem Set 5

Problem 1. For each of the following matrices, use Gauss-Jordan elimination to determine whether it is invertible. Moreover, if it is, find its inverse too.

$$A_1 = \begin{pmatrix} 1 & 1 & 6 & 4 \\ 2 & 3 & 2 & 5 \\ 2 & 2 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{pmatrix} \in \mathbb{Z}_7^{4\times 4}, \qquad A_2 = \begin{pmatrix} 0.5 & -0.5 & 3 & 2 \\ -1 & 1.5 & -3 & -1 \\ 0 & 0.5 & 3 & 1 \\ -3 & 1 & 0 & -4 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

Problem 2. (a) Let $n \geqslant 2$, let $\mathbb{F}$ be a field, and suppose that $U_1 \in \mathbb{F}^{n\times n}$ is an upper triangular matrix.

Justify the following: if all diagonal entries of U_1 are non-zero, then U_1 is in REF.

[**Remark.** Recall that we have mentioned that a square matrix in REF will necessarily be upper triangular. However, as we have seen for instance in Lecture 30, the converse is not always true. This question provides a partial converse.]

(b) Consider again an upper triangular matrix $U_2 \in \mathbb{F}^{n\times n}$, and prove the following:

U_2 is invertible **if and only if** all diagonal entries of U_2 are non-zero.

Problem 3. (a) Let $n \geqslant 2$, let $\mathbb{F}$ be a field, and suppose $C \in \mathbb{F}^{n\times n}$ is a square matrix satisfying the following:

there exists a matrix D in $\mathbb{F}^{n\times n}$ such that $CD = I_n$

(recall that in this case we say that C has a right inverse).

Prove that C is invertible. *(In other words, given a square matrix C, it suffices to know that C has a right inverse in order to conclude that C has an inverse. Similarly, we could say that it suffices to know that D has a left inverse in order to conclude* *that D has an inverse; why?)*

[**Remark / Hint.** How strong this property of multiplication in $\mathbb{F}^{n\times n}$ is, can be appreciated even more once we see examples of other non-commutative rings that fail to have the corresponding property.

It may help to look at Problem 6 from HW4.]

(b) Let $m \geqslant 2$, and let $A_1, A_2, \ldots, A_m$ be matrices in $\mathbb{F}^{n\times n}$. Suppose that we know that the product

$$A_1 \cdot A_2 \cdots A_{m-1} \cdot A_m$$

is an invertible matrix.

Prove that, for every $i \in \{1, 2, \ldots, m\}$, the matrix A_i is invertible too.

[**Hint.** It may help to use induction in m, as well as part (a).]

The next problem is about a special class of square matrices with real entries, called *stochastic* matrices.

These matrices appear in numerous applications and are very useful in several areas of Mathematics and of other disciplines (as for example Probability Theory, Statistics, Mathematical Finance, Communications Theory or Evolutionary Biology). They allow us to encode the possible ways in which certain probabilistic/stochastic phenomena can evolve over time, and how likely each of these ways is: for instance, questions like

“If we have a standard deck of 52 cards, how likely is it
for, say, 5 consecutive cards to be of the same colour or the same suit
after one shuffle of the deck, or two shuffles, or eight shuffles, and so on?”

can be answered with the use of stochastic matrices.

Problem 4. Let $n > 1$. A square matrix $Q = \big(q_{ij}\big)_{1\leqslant i,j\leqslant n} \in \mathbb{R}^{n\times n}$ is called *stochastic* (or sometimes *row stochastic*) if:

- all the entries q_{ij} are non-negative numbers, and
- for each row the total sum of the entries contained in it is 1, that is, for every $1 \leqslant i_0 \leqslant n$,

$$\sum_{j=1}^{n} q_{i_0,j} = 1.$$

Consider a stochastic matrix $Q_3 \in \mathbb{R}^{3\times 3}$, and show that $Q_3^2 = Q_3 \cdot Q_3$ is also stochastic. Moreover, show that the same is true for a stochastic matrix $Q_4 \in \mathbb{R}^{4\times 4}$.

[*Remark.* This problem is a good instance of how we can rearrange a sum using generalised commutativity and associativity, with the purpose of simplifying it, when the summation is over more than one index.

Although the problem doesn't ask for this, is it possible to generalise your approach in order to prove the conclusion for a stochastic matrix of any size?]

Problem 5. (a) For each of the following functions, determine whether it is linear, injective and/or surjective (give brief justifications). The spaces that appear as domains or codomains of these functions should be understood as the standard vector spaces we have seen (over the largest possible scalar field in each case).

(i) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$

(ii) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \end{pmatrix}$

(iii) $f\colon \mathbb{Z}_3^2 \to \mathbb{Z}_3^2$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_1 x_2 \end{pmatrix}$

(iv) $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 + x_2$

(v) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 1 \\ x_2 + 2 \end{pmatrix}$

(vi) $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5^3$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$

(vii) $f\colon \mathbb{R}^3 \to \mathbb{R}^{2\times 2}$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}$

(viii) $f\colon \mathbb{R}^3 \to \mathbb{R}^{2\times 2}$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 & x_2 \\ 1 - x_3 & x_3 \end{pmatrix}$

(b) For any of the functions in parts (i)-(vi) that is linear, find a matrix representation.

Problem 6. (a) Let $\mathbb{F}$ be a field, and consider two vector spaces V_1, V_2 over $\mathbb{F}$. If $g_1, g_2, g : V_1 \to V_2$ are functions (not necessarily linear) and if $\lambda \in \mathbb{F}$, then we can define the functions $g_1 + g_2 : V_1 \to V_2$ and $\lambda g : V_1 \to V_2$ as follows:

$$\text{for every } \bar{x} \in V_1, \quad \big(g_1 + g_2\big)(\bar{x}) := g_1(\bar{x}) + g_2(\bar{x})$$
$$\text{and} \quad (\lambda g)(\bar{x}) := \lambda \cdot g(\bar{x})$$

(note that, for any $\bar{x} \in V_1$, we have that $g_1(\bar{x})$, $g_2(\bar{x})$ and $g(\bar{x})$ are vectors in V_2, and thus

- we can consider the sum $g_1(\bar{x}) + g_2(\bar{x})$ in V_2, and set this to be the image of $\bar{x}$ under $g_1 + g_2$,
- and similarly, we can consider the scalar multiple $\lambda \cdot g(\bar{x})$ of $g(\bar{x})$ in V_2, and set this to be the image of $\bar{x}$ under λg).

Prove the following: if $\mu_1, \mu_2 \in \mathbb{F}$, and if f_1, f_2 are **linear maps** from V_1 to V_2, then

$$\mu_1 f_1 + \mu_2 f_2$$

is a linear map too.

(b) Let V_3 be a vector space over $\mathbb{F}$ too. If $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are linear maps, show that $g \circ f : V_1 \to V_3$ is a linear map too.