THEOREM LIST

MATH 227

You are expected to know the following notations, definitions, and facts (exact statements and proofs).

Notations.

- $\operatorname{span} A$
- $\operatorname{sign} \sigma$
- $\det A$
- $L(X, Y)$, $L(X)$
- $\operatorname{Null} T$, $\operatorname{Range} T$
- $\operatorname{rank} A$, $\operatorname{nullity}(A)$
- E_λ
- $X + Y$, $X \oplus Y$
- X/Y
- $\widetilde{E}_\lambda$

Definitions. There will be a definition problem on the exam: you will be asked to state a definition from the list below. Ideally, you should state is verbatim as it was stated in class, so it is recommended that you memorize these definitions. It is acceptable to rephrase a definition as long as your statement remains equivalent to the definition given in class.

- vector space, subspace
- spanning set
- linear independent set
- linear operators between vector spaces
- linear isomorphism
- the null space (kernel) of a linear operator
- invertible matrix, singular (= non-invertible) matrix, the inverse matrix
- permutation, transposition, sign of a permutation, inverse of a permutation
- determinant
- minor
- eigenvalues, eigenvectors, and eigenspaces of linear operators

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- characteristic equation, characteristic polynomial
- characteristic equation; characteristic polynomial
- basis of a vector space
- finite and infinite dimensional vector spaces
- dimension of a vector space
- rank of a linear operator or a matrix
- nullity or defect of a linear operator or a matrix
- change of basis (for a vector and a matrix)
- similar matrices
- direct sum of vector spaces (external and internal approaches)
- direct sum decomposition
- algebraic complement
- projection
- partition
- equivalence relation
- quotient map
- quotient of a vector space
- nilpotent matrix
- diagonalizable matrix
- block matrix, block-diagonal matrix
- invariant subspace
- reducing subspace
- ultimate null-space and ultimate range
- generalized eigenspace, generalized eigenvector
- Jordan block
- minimal polynomial
- functions of matrices
- inner product, inner product space, norm (arising from an inner product)
- orthogonal set, orthonormal set, orthogonal basis, orthonormal basis
- orthogonal projection, orthogonal direct sum decomposition
- orthogonal complement
- adjoint matrix, adjoint operator, self-adjoint matrix/operator
- singular value
- unitary matrix/operator
- isometry

Facts. You will be asked on the exam to provide the exact statement and a proof of one of the following statements. Ideally, you should state it exactly as it was stated in class. It is acceptable to rephrase a statement as long as your statement remains equivalent to the statement given in class. As for the proof, you may provide the proof that was given in class. Alternatively, you may submit a different proof under two conditions: first, it must be logically correct, and, second, it must be based on facts that were proved in class before the statement you are proving; if you need any other fact in your proof, you must provide its proof as well.

- There is a one-to-one correspondence between $m \times n$ matrices over $\mathbb{F}$ and linear operators from $\mathbb{F}^n$ to $\mathbb{F}^m$.
- $\operatorname{span} A$ is the smallest subspace containing A.
- The kernel and the range of a linear operator are subspaces.
- Equivalent characterizations of surjective matrices (GST)
- Equivalent characterizations of injective matrices (GIT)
- Equivalent characterizations of invertible matrices (GIT)
- If a linear transformation $T\colon \mathbb{F}^n \to \mathbb{F}^m$ is invertible then $m = n$
- A square matrix is a bijection iff it is onto iff it is one-to-one
- Basic properties of determinant
- Effects of row operations on the determinant
- $\det A = 0$ iff A is singular.
- $\det A^T = \det A$, $\det AB = (\det A) \cdot \det B$.
- A is invertible iff A^T is invertible.
- A square matrix is invertible iff it is left or right invertible
- co-factor expansion
- Cramer's Rule for A^{-1} and for $A\bar{x} = \bar{b}$;
- A linear transformation maps lines to lines, planes to planes, straight line segments to straight line segments, polygons to polygons.
- If U is a square in $\mathbb{R}^2$ with sides parallel to the coordinate exes, and $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then the area of $A(U)$ equals the area of U times $|\det A|$.
- A non-zero vector $\bar{x}$ is an eigenvector for A and λ iff $(\lambda I - A)\bar{x} = \bar{0}$.
- Eigenvalues are the roots of the characteristic polynomial
- A set is a basis iff every vector admits a unique expansion.
- Linear injections preserve liners independence
- Linear surjections map spanning sets to spanning sets
- Characterization of linear isomorphisms in terms of their action on bases

- Every finite-dimensional vector space X is linearly isomorphic to $\mathbb{F}^n$, where $n = \dim X$;
- If X is a finite-dimensional vector space then any two bases in X have the same number of vectors.
- Every finite spanning set contains a basis
- Every linearly independent set in a finite-dimensional vector space may be enlarged to a basis
- Every subspace of $\mathbb{F}^n$ is finite-dimensional
- If X and Y are both n-dimensional for some n then $X \simeq Y$.
- Let A be an $m \times n$ matrix with $m \geqslant n$, such that there is a subset of the rows of A that forms an invertible square matrix. Then the columns of A are linearly independent.
- $\text{nullity}(A)$ is the number of free variables in the echelon form of A.
- $\text{rank}(A)$ is the number of leading variables (pivots) in the echelon form of A
- Rank Theorem: if A is an $m \times n$ matrix then $\text{rank}(A) + \text{nullity}(A) = n$.
- $\operatorname{rank} A^T = \operatorname{rank} A$
- A has a $k \times k$ invertible submatrix iff $k \leqslant \operatorname{rank} A$
- $\operatorname{rank} A$ is the size of the largest invertible submatrix
- If X and Y are subspaces of Z and $X \cap Y = \{0\}$ then $X + Y$ is linearly isomorphic to $X \oplus Y$.
- Subspaces X and Y of Z form a direct sum decomposition iff every vector in Z has a unique decomposition $z = x + y$ with $x \in X$ and $y \in Y$.
- The union of bases of summands in a direct sum decomposition is a basis of the entire space.
- Every subspace of a vector space has an algebraic complement.
- Every splitting (partition) of a basis yields a direct sum decomposition.
- If P is a projection onto X along Y then $\operatorname{Range} P = X$ and $\ker P = Y$.
- A linear operator $P\colon X \to X$ is a projection iff $P^2 = P$.
- Relationship between partitions and equivalence relations.
- X/Y is linearly isomorphic to any algebraic complement of Y in X
- $\dim X/Y = \dim X - \dim Y$.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.
- A matrix A is diagonalizable iff it is diagonal with respect to some basis iff there is a basis consisting of eigenvectors of A.
- If an $n \times n$ matrix has n distinct eigenvalues then it is diagonalizable.
- The only diagonalizable nilpotent matrix is the zero matrix.

- If A is block-diagonal with $A = A_1 \oplus \cdots \oplus A_m$ then $\det A = (\det A_1) \cdots (\det A_m)$, $p_A = p_{A_1} \cdots p_{A_m}$, and $\sigma(A) = \sigma(A_1) \cup \cdots \cup \sigma(A_m)$.
- Characterizations of invariant and reducing subspaces in terms of block-matrices.
- Ultimate Decomposition Theorem (nilpotent plus invertible)
- $A \in M_n(\mathbb{C})$ is nilpotent iff $\sigma(A) = \{0\}$.
- Decomposition into generalized eigenspaces
- Characterization of diagonalizable matrices in terms of multiplicities
- Jordan Decomposition Theorem
- Cayley-Hamilton Theorem
- Spectral Mapping Theorem for matrices
- Basic properties of inner product, orthogonality, and norm
- Pythagoras Theorem
- Cauchy-Schwartz Inequality
- Triangle inequality
- If $\{\bar{u}_1, \dots, \bar{u}_m\}$ is an orthogonal set then $\|\bar{u}_1 + \cdots + \bar{u}_m\|^2 = \|\bar{u}_1\|^2 + \cdots + \|\bar{u}_m\|^2$.
- Every orthogonal set is linearly independent.
- If $\{\bar{u}_1, \dots, \bar{u}_m\}$ is an orthonormal set then $\|\alpha_1 \bar{u}_1 + \cdots + \alpha_m \bar{u}_m\| = \sqrt{\sum_{i=1}^m |\alpha_i|^2}$.
- Let $\mathcal{B} = \{\bar{u}_1, \dots, \bar{u}_n\}$ is an orthonormal basis of X and $\bar{x} \in X$. Then the i-th component of $\bar{x}$ with respect to $\mathcal{B}$ is $x_i = \langle \bar{x}, \bar{u}_i \rangle$, and $\|\bar{x}\| = \sqrt{\sum_{i=1}^n |x_i|^2}$. That is, $\bar{x} = \sum_{i=1}^n \langle \bar{x}, \bar{u}_i \rangle \bar{u}_i$ and $\|\bar{x}\| = \sqrt{\sum_{i=1}^n \left| \langle \bar{x}, \bar{u}_i \rangle \right|^2}$.
- A projection P is orthogonal iff $x - Px \perp \operatorname{Range} P$ for every $x \in X$.
- For every subspace Y, there is at most one orthogonal projection onto it.
- If P is an orthogonal projection onto a subspace Y then $\operatorname{Null} P = Y^\perp$.
- If $X = Y \oplus Z$ is an orthogonal decomposition then $Z = Y^\perp$.
- Orthogonal projections minimize distances.
- Let $\{\bar{u}_1, \dots, \bar{u}_m\}$ be an orthonormal basis of a subspace Y of X. For $\bar{x} \in X$, put $P\bar{x} = \sum_{i=1}^m \langle \bar{x}, \bar{u}_i \rangle \bar{u}_i$. Then P is an orthogonal projection with $\operatorname{Range} P = Y$.
- Every finite-dimensional inner product space admits an orthonormal basis (Gram-Schmidt orthogonalization).
- Every finite-dimensional subspace of an inner product space is a range a unique orthogonal projection.
- Every finite dimensional inner product space is isomorphic as an inner product space to $\mathbb{F}^n$ with the canonical inner product.
- For finite dimensional subspace Y there exists an orthogonal projection onto Y.
- $Y^{\perp\perp} = Y$ for finite dimensional subspace Y
- $(A + B)^* = A^* + B^*$, $(\lambda A)^* = \bar{\lambda} A^*$, $(AB)^* = B^* A^*$.

- $B = A^*$ iff $\langle Ax, y\rangle = \langle x, By\rangle$ for all x, y.
- A is self-adjoint iff $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all x, y.
- If A is self-adjoint then $\sigma(A) \subseteq \mathbb{R}$
- A is self-adjoint iff it is real-diagonal with respect to an orthonormal basis.
- A projection is orthogonal iff it is self-adjoint.
- A^*A is self-adjoint and $\sigma(A^*A) \subseteq [0, \infty)$.
- A is invertible iff A^*A is invertible iff all singular values of A are non-zero.
- Equivalent characterizations of unitary operators.
- Singular Value Decomposition Theorem (proof for invertible matrices only)