Coin Weighing

Warm up problem:

a) There are 9 coins, all identical except that one is slightly heavier than the others. Find the heavy coin using two weighings on a pan balance.

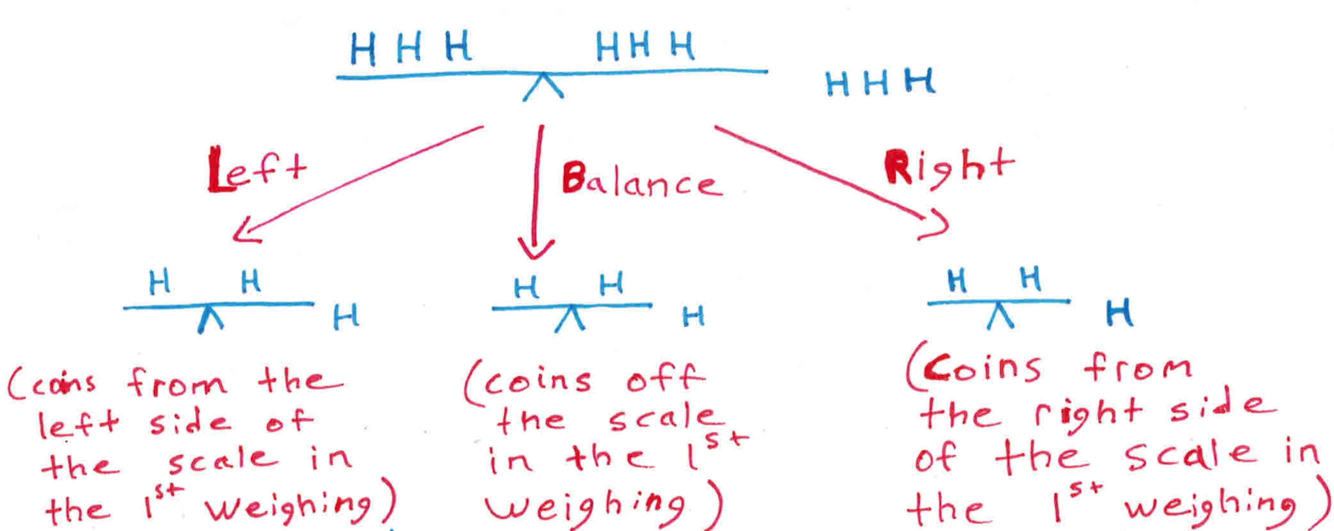
b) There are 8 coins, all identical except that one is slightly heavier or slightly lighter than the others. Find the counterfeit using three weighings on a pan balance. You do not have to identify the counterfeit as being too heavy or too light.

Definition 1: An *adaptive solution* is a step by step solution where at each step we adapt depending upon the outcome of previous steps.

Note: When finding an adaptive solution to the warm up problem the following notation is very helpful. Let:

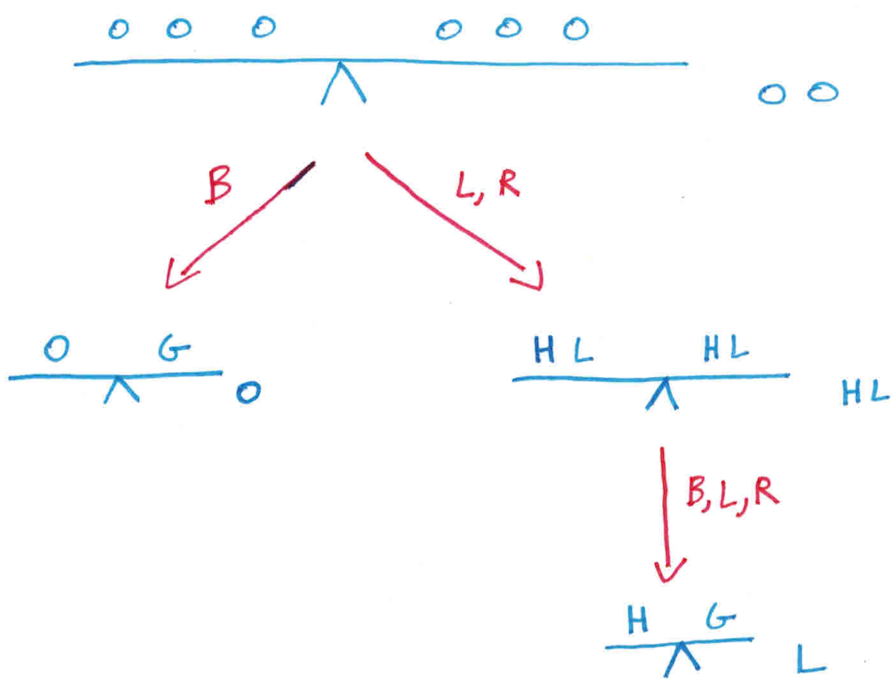
"G" be a coin with the correct weight
"o" be a coin with an unknown weight
"H" be a coin that might be too heavy but is not too light
"L" be a coin that might be too ~~heavy~~ light but is not too ~~light~~ heavy
$\xrightarrow{L}$ represent the case when the scale tips to the left.
$\xrightarrow{R}$ represent the case when the scale tips to the right.
$\xrightarrow{B}$ represent the case when the scale balances.

Example 1: Find an adaptive solution to part "a" of the warm up problem.

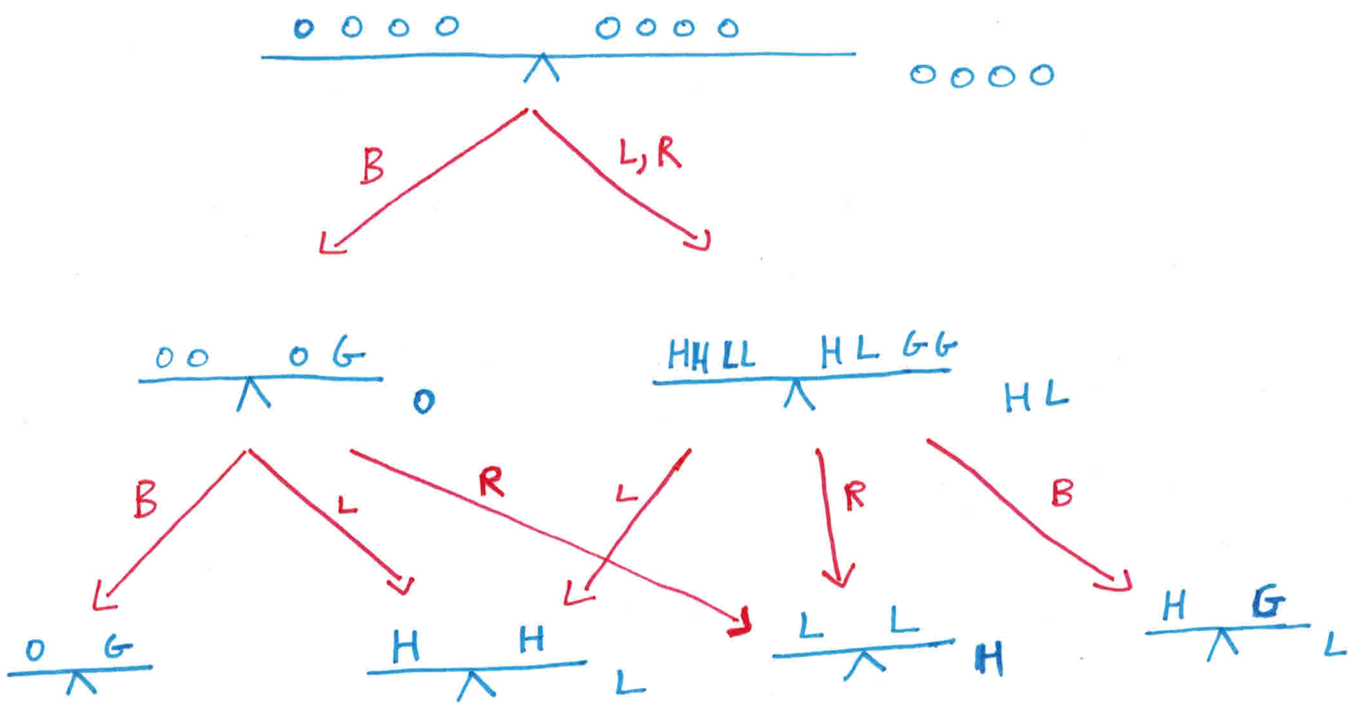


Note: The 2nd weighing has 3 cases that are all theoretically the same, as a shorthand we can combine such cases into one.

Example 2: Find an adaptive solution to part "b" of the warm up problem.

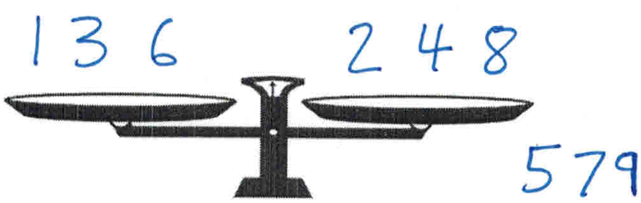


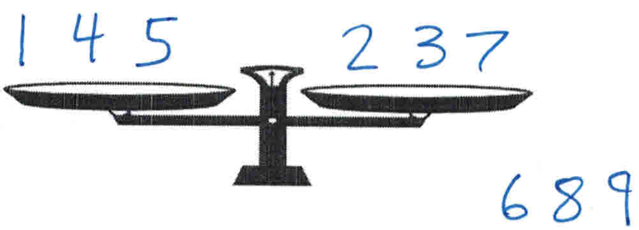
Example 3: There are 12 coins, all identical except that one is slightly heavier or slightly lighter than the others. Find the counterfeit using three weighings on a pan balance. You solution must be able to identify the counterfeit as being too heavy or too light.



Definition 2: In a *nonadaptive solution* all the steps are fully predetermined.

Example 4: Find a nonadaptive solution to part a) of the warm up problem. In your solution let "L" represent when the scale tips to the left, let "R" represent when the scale tips to the right, and let "B" represent when the scale balances. With this notation "LB" will mean that the first weighing tipped to the left and the second weighing balanced.





Counterfeit Coin	How the Scale Responds
1	LL
2	RR
3	LR
4	RL
5	BL
6	LB
7	BR
8	RB
9	BB

Example 5: Can example 4 be solved with 10 coins and two weighings?

No, since the scale can only respond in 9 ways.

Example 6: How many different ways can the scale respond with 3 weighings?

$$\left.\begin{matrix} LLL \\ RRR \\ \vdots \\ BBB \end{matrix}\right\} \; 3\cdot 3\cdot 3 = 3^3 = 27$$

Example 7: Can example 4 be solved with 27 coins and three weighings?

Yes, assign coin

1 to LLL

2 to RRR

⋮

27 to BBB

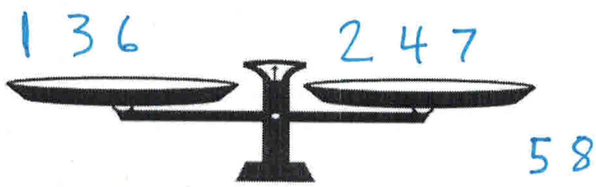
Example 8: Can example 4 be solved with 28 coins and three weighings?

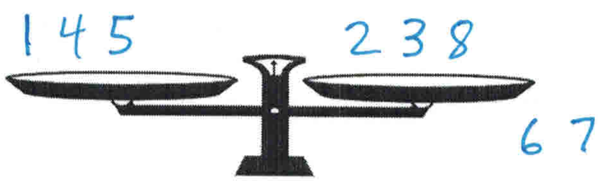
No, since the scale can only respond in 27 ways

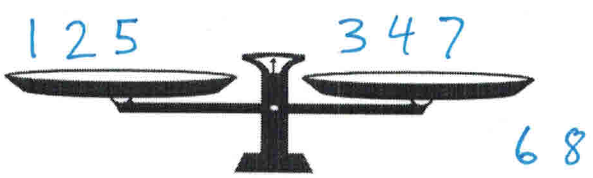
Theorem 1: Let $2 \le c \le 3^w$. Given c identical coins (one which is slightly heavier than the others), the counterfeit can be found in w weighings on a pan balance. However, given $3^w + 1$ identical coins, $w + 1$ weighings are needed.

Weighings	Max. Coins
1	3
2	9
3	27
4	81
⋮	⋮

Example 9: Find a nonadaptive solution to part b) of the warm up problem.







Counterfeit Coin	How the Scale Responds
1	LLL or RRR
2	RRL or LLR
3	LRR or RLL
4	RLR or LRL
5	BLL or BRR
6	LBB or RBB
7	RBR or LBL
8	BRB or BLB

Definition: The *ceiling* function $\lceil x \rceil$ rounds the number x up to the closest integer. If x is already an integer, no rounding is done. For example: $\lceil 5.001 \rceil = 6$, $\lceil 5.0 \rceil = 5$.

Example 10: In example 9 the scale responds in 2 ways for each coin (that was on the scale at least once). What is the largest number of coins that the counterfeit could be found among?

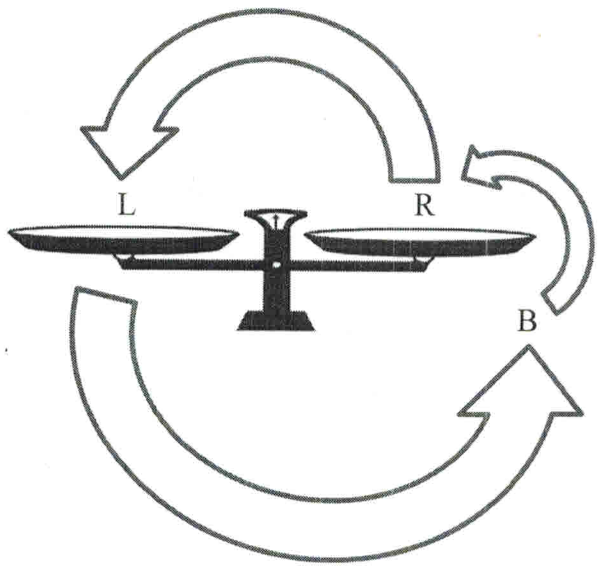
Count the responses:

LLL or RRR
RRL or LLR
⋮
BBR or BBL
BBB

$$\left\lceil \frac{27}{2} \right\rceil = 14$$

The Shifting Trick: This trick gives a non-adaptive solution to some coin weighing problems when the number of coins is multiple of 3. You can also use this trick to add three coins to some existing non-adaptive solutions.

Step 1: Take any way the scale can respond, other than LLL, RRR, BBB. Say for example, we have three weighings, pick LLR or RRL. Shift this response according to the following diagram:

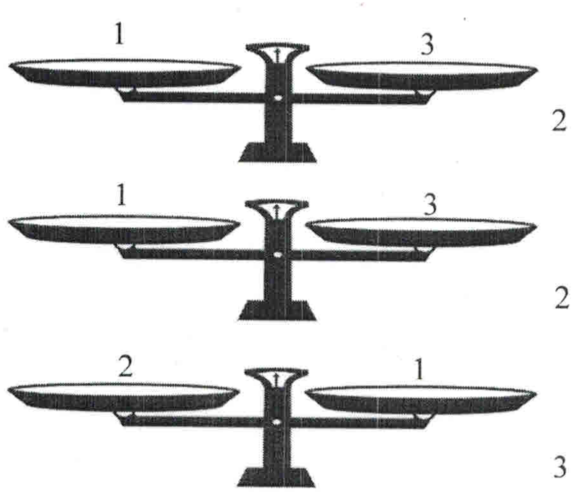


to get a sifting set:

LLR	RRL
BBL	BBR
RRB	LLB

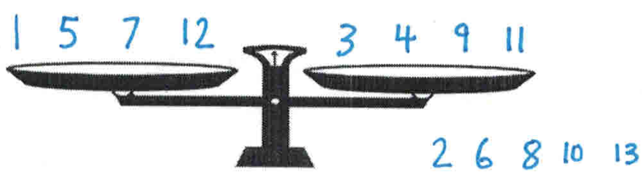
Repeat step one until you find all the shifting sets.

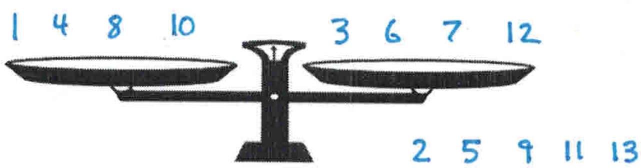
Step 2: Identify a shifting set that does not have any coins on the scale corresponding to it. Place three coins on the scale corresponding to this shifting set. Repeat step 2 until you have placed as the desired amount of coins on the scale. For example, the shifting set above gives a solution with three coins :

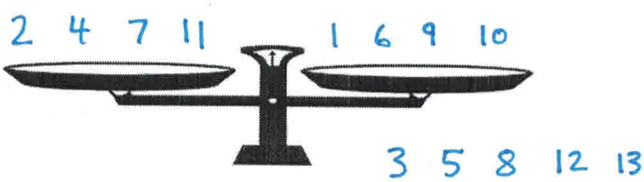


Shift : L R B

Example 11: Solve example 6 with 13 coins?







Counterfeit Coin	How the Scale Responds	
1	LLR or RRL	
2	BBL or BBR	Shifting set
3	RRB or LLB	
4	RLL or LRR	
5	LBB or RBB	Shifting set
6	BRR or BLL	
7	LRL or RLR	
8	BLB or BRB	Shifting set
9	RBR or LBL	
10	BLR or BRL	
11	RBL or LBR	Shifting set
12	LRB or RLB	
	LLL or RRR	
13	BBB	

Example 12: Show that example 9 cannot be solved with 14 coins?

Assume the problem can be solved with 14 coins. Every way the scale responds is used.

$\Rightarrow$ 5 coins are off the scale in each weighing

$\Rightarrow$ 9 coins are on the scale in each weighing ↯ (need an even # on the scale)

Theorem 2: Let $3 \leq c \leq \left\lceil \frac{3^w}{2} \right\rceil - 1$. Given c identical coins (one of which is slightly heavier or lighter than the others) the counterfeit can be found in w weighings on a pan balance. However, given $\left\lceil \frac{3^w}{2} \right\rceil$ identical coins, $w + 1$ weighings are needed.

Weighings	Max. Coins
2	4
3	13
4	40
⋮	⋮