MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Midterm Practice Problems

1. Compute Δf for

$$f\colon \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}, \quad (x,y,z) \mapsto \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

Solution: For $(x,y,z) \neq (0,0,0)$, we have

$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{x^2+y^2+z^2}^3}, \qquad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{x^2+y^2+z^2}^3}, \qquad \text{and} \qquad \frac{\partial f}{\partial z} = -\frac{z}{\sqrt{x^2+y^2+z^2}^3},$$

so that

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{\sqrt{x^2+y^2+z^2}^3} + \frac{3x^2}{\sqrt{x^2+y^2+z^2}^5},$$
$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{\sqrt{x^2+y^2+z^2}^3} + \frac{3y^2}{\sqrt{x^2+y^2+z^2}^5},$$
$$\text{and} \qquad \frac{\partial^2 f}{\partial z^2} = -\frac{1}{\sqrt{x^2+y^2+z^2}^3} + \frac{3z^2}{\sqrt{x^2+y^2+z^2}^5}.$$

It follows that

$$\begin{aligned}\Delta f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= -\frac{3}{\sqrt{x^2+y^2+z^2}^3} + 3\frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}^5} \\ &= -\frac{3}{\sqrt{x^2+y^2+z^2}^3} + \frac{3}{\sqrt{x^2+y^2+z^2}^3} \\ &= 0.\end{aligned}$$

2. Let $\varnothing \neq D \subset \mathbb{R}^N$, let $f : D \to \mathbb{R}^M$ be continuous, and let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in D. Show that $(f(x_n))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^M$ if D is closed or if f is uniformly continuous.

Does this remain true without any additional requirements for D or f?

Solution: Suppose that D is closed. As $(x_n)_{n=1}^\infty$ is a Cauchy sequence, it converges to a limit $x_0 \in \mathbb{R}^N$ that—due to the closedness of D—must lie in D. As f is continuous, it follows that $\lim_{n\to\infty} f(x_n) = f(x_0)$, so that $(f(x_n))_{n=1}^\infty$ is also a Cauchy sequence.

Suppose that f is uniformly continuous. Let $\epsilon > 0$. By the uniform continuity of f, there is $\delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ for all $x, y \in D$ such that $\|x - y\| < \delta$. As $(x_n)_{n=1}^\infty$ is a Cauchy sequence, there is $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < \delta$ for all $n, m \geq n_0$. From the choice of $\delta > 0$ it thus follows that $\|f(x_n) - f(x_m)\| < \epsilon$ for all $n, m \geq n_0$. Hence, $(f(x_n))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^M$.

Let $D = (0, 1]$, and let

$$f\colon D \to \mathbb{R}, \quad x \mapsto \frac{1}{x}.$$

Then D is not closed, and f is continuous, but not uniformly continuous. For $n \in \mathbb{N}$, set $x_n := \frac{1}{n}$. Then $(x_n)_{n=1}^\infty$ is a Cauchy sequence in D, but as $f(x_n) = n$ for $n \in \mathbb{N}$, the sequence $(f(x_n))_{n=1}^\infty$ is definitely not a Cauchy sequence.

3. Show that:

   (a) if $\mathcal{C}$ is a family of connected subsets of $\mathbb{R}^N$ such that $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$, then $\bigcup_{C \in \mathcal{C}} C$ is connected;

   (b) if $C_1 \subset \mathbb{R}^{N_1}$ and $C_2 \subset \mathbb{R}^{N_2}$ are connected, then so is $C_1 \times C_2 \subset \mathbb{R}^{N_1 + N_2}$ (*Hint*: Argue that we can suppose that C_1 and C_2 are not empty, and fix $x_2 \in C_2$; then apply (a) to $\mathcal{C} := \{(C_1 \times \{x_2\}) \cup (\{x_1\} \times C_2) : x_1 \in C_1\}$.);

   (c) if $C_1, C_2 \subset \mathbb{R}^N$ are connected, then so is $C_1 + C_2 \subset \mathbb{R}^N$.

Solution:

(a) Assume that there is a disconnection $\{U, V\}$ for $\bigcup_{C \in \mathcal{C}} C$. For any $C \in \mathcal{C}$, we then have $(U \cap C) \cup (V \cap C) = C$ and $(U \cap C) \cap (V \cap C) = \varnothing$, and as C is connected, this means that $C \subset U$ or $C \subset V$. It follows that

$$\varnothing = \left(U \cap \bigcup_{C \in \mathcal{C}} C\right) \cap \left(V \cap \bigcup_{C \in \mathcal{C}} C\right) = \bigcap_{\substack{C \in \mathcal{C} \\ C \subset U}} C \cap \bigcap_{\substack{C \in \mathcal{C} \\ C \subset V}} C = \bigcap_{C \in \mathcal{C}} C,$$

which contradicts $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$.

(b) Let $C_1 \subset \mathbb{R}^{N_1}$ and $C_2 \subset \mathbb{R}^{N_2}$ be connected. If $C_1 = \varnothing$ or $C_2 = \varnothing$, nothing needs to be shown. Hence, suppose that $C_1 \neq \varnothing \neq C_2$. Fix $x_2 \in C_2$. As $C_1 \times \{x_2\}$ is the image of C_1 under the continuous map

$$\mathbb{R}^{N_1} \to \mathbb{R}^{N_1 + N_2}, \quad x \mapsto (x, x_2),$$

it follows that $C_1 \times \{x_2\}$ is connected. Analogously, one sees that $\{x_1\} \times C_2$ is connected for each $x_1 \in C_1$. As $(x_1, x_2) \in (C_1 \times \{x_2\}) \cap (\{x_1\} \times C_2)$, it follows that $(C_1 \times \{x_2\}) \cup (\{x_1\} \times C_2)$ is connected for each $x_1 \in C_1$. As

$$\varnothing \neq C_1 \times \{x_2\} \subset \bigcap_{x_1 \in C_1} ((C_1 \times \{x_2\}) \cup (\{x_1\} \times C_2)),$$

we conclude that

$$C_1 \times C_2 = \bigcup_{x_1 \in C_1} ((C_1 \times \{x_2\}) \cup (\{x_1\} \times C_2))$$

is connected.

(c) By (b), $C_1 \times C_2$ is connected. As

$$f \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, \quad (x, y) \mapsto x + y$$

is continuous, $C_1 + C_2 = f(C_1 \times C_2)$ is connected as well.

4. Show that the Mean Value Theorem becomes false for vector valued functions: Let

$$f \colon [0, 2\pi] \to \mathbb{R}^2, \quad x \mapsto (\cos(x), \sin(x)).$$

Show that there is *no* $\xi \in (0, 2\pi)$ such that

$$f'(\xi) = \frac{f(2\pi) - f(0)}{2\pi}.$$

Solution: Since f is 2π-periodic, we have $f(2\pi) - f(0) = 0$. Since

$$f'(x) = (-\sin(x), \cos(x))$$

for $x \in [0, 2\pi]$, and since $\sin(x)$ and $\cos(x)$ have no common zero, there is no $\xi \in (0, 2\pi)$ such that $f'(\xi) = 0$.

5. Let

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is twice partially differentiable everywhere, but that

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) \neq \frac{\partial^2 f}{\partial x \partial y}(0, 0).$$

Is f continuous at $(0, 0)$?

Solution: It is clear that f is twice partially differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$. In order to calculate the second partial derivatives at $(0, 0)$, we first need to determine the first partial derivatives of f.

For $(x, y) \neq (0, 0)$, we obtain

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= y\frac{x^2 - y^2}{x^2 + y^2} + xy\frac{2x(x^2 + y^2) - 2x(x^2 - y^2)}{(x^2 + y^2)^2} \\ &= y\frac{x^2 - y^2}{x^2 + y^2} + \frac{4x^2y^3}{(x^2 + y^2)^2} \end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= x\frac{x^2-y^2}{x^2+y^2} + xy\frac{2y(x^2+y^2)+2y(x^2-y^2)}{(x^2+y^2)^2} \\ &= x\frac{x^2-y^2}{x^2+y^2} + \frac{4x^3y^2}{(x^2+y^2)^2}.\end{aligned}$$

From the definition of a partial derivative, we obtain furthermore that

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{f(h,0)-f(0,0)}{h} = \lim_{\substack{h\to 0\\ h\neq 0}} 0 = 0,$$

and, similarly, $\frac{\partial f}{\partial y}(0,0) = 0$.

Consequently, we have

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{1}{h}\left(\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)\right) = 0$$

and similarly $\frac{\partial^2 f}{\partial y^2}(0,0) = 0$.

Moreover, we have

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(0,0) &= \lim_{\substack{h\to 0\\ h\neq 0}} \frac{1}{h}\left(\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)\right) \\ &= \lim_{\substack{h\to 0\\ h\neq 0}} \frac{1}{h} h \frac{h^2}{h^2} \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x}(0,0) &= \lim_{\substack{h\to 0\\ h\neq 0}} \frac{1}{h}\left(\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial y}(0,0)\right) \\ &= \lim_{\substack{h\to 0\\ h\neq 0}} \frac{1}{h} h \frac{-h^2}{h^2} \\ &= -1.\end{aligned}$$

Hence, f is twice partially differentiable at $(0,0)$, but

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1 \neq -1 = \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

As

$$|f(x,y)| = |xy|\frac{|x^2-y^2|}{x^2+y^2} \leq |xy|\frac{x^2+y^2}{x^2+y^2} = |xy|$$

for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, it clear that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, so that f is continuous at $(0,0)$.

6. Show that
$$\mathbb{Q}\left[\sqrt{13}\right] := \left\{p + q\sqrt{13} : p, q \in \mathbb{Q}\right\},$$
with $+$ and $\cdot$ inherited from $\mathbb{R}$, is a field.

Solution: Let $p, q, r, s \in \mathbb{Q}$. Then
$$\left(p + \sqrt{13}q\right) + \left(r + \sqrt{13}s\right) = (p + r) + (q + s)\sqrt{13} \in \mathbb{Q}\left[\sqrt{13}\right]$$
and
$$\left(p + \sqrt{13}q\right)\left(r + \sqrt{13}s\right) = \underbrace{(pr + 13qs)}_{\in\mathbb{Q}} + \underbrace{(qr + ps)}_{\in\mathbb{Q}}\sqrt{13} \in \mathbb{Q}\left[\sqrt{13}\right]$$
hold, so that (F 1) is satisfied.

Since (F 2), (F 3), and (F 4) hold for $\mathbb{R}$, they also hold for $\mathbb{Q}\left[\sqrt{13}\right]$.

Since $0 = 0 + 0\sqrt{13}, 1 = 1 + 0\sqrt{13} \in \mathbb{Q}\left[\sqrt{13}\right]$, (F 5) is satisfied as well.

Let $p, q \in \mathbb{Q}$, and let $x = p + q\sqrt{13}$. Then $-x = -p - q\sqrt{13} \in \mathbb{Q}\left[\sqrt{13}\right]$ as well. Suppose that $x \neq 0$. Assume that $p^2 - 13q^2 = 0$. If $q = 0$, this implies that $p = 0$ as well and thus $x = 0$. Suppose therefore that $q \neq 0$. Then $p^2 - 13q^2 = 0$ implies $\sqrt{13} = \frac{|p|}{|q|} \in \mathbb{Q}$, which is impossible. Hence, $p^2 - 13q^2 \neq 0$ holds. Let
$$y := \frac{p}{p^2 - 13q^2} - \frac{q}{p^2 - 13q^2}\sqrt{13} \in \mathbb{Q}\left[\sqrt{13}\right].$$
Then we have
$$\begin{aligned} xy &= \frac{p - q\sqrt{13}}{p^2 - 13q^2}\left(p + q\sqrt{13}\right) \\ &= \frac{\left(p - q\sqrt{13}\right)\left(p + q\sqrt{13}\right)}{p^2 - 13q^2} \\ &= \frac{p^2 - 13q^2}{p^2 - 13q^2} \\ &= 1. \end{aligned}$$
Hence, (F 6) is also satisfied.

7. Show that the function
$$f \colon \mathbb{R}^N \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}, \quad (x, t) \mapsto \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{\|x\|^2}{4t}\right)$$
solves the *heat equation*
$$\Delta f - \frac{\partial f}{\partial t} = 0,$$
where Δ denotes the *spatial* Laplace operator, i.e.,
$$\Delta f = \sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}.$$

Solution: Note that

$$f(x_1, \dots, x_N, t) = \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right)$$

for $x_1, \dots, x_N, t \in \mathbb{R}$ with $t \neq 0$. It follows for $j = 1, \dots, N$ that

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x,t) &= \frac{1}{t^{\frac{N}{2}}} \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right)\left(-\frac{x_j}{2t}\right) \\
&= -\frac{x_j}{2t^{\frac{N}{2}+1}} \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right)
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_j^2}(x,t) &= -\frac{1}{2t^{\frac{N}{2}+1}} \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right) \\
&\quad - \frac{x_j}{2t^{\frac{N}{2}+1}} \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right)\left(-\frac{x_j}{2t}\right) \\
&= -\frac{1}{2t^{\frac{N}{2}+1}} \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right) \\
&\quad + \frac{x_j^2}{4t^{\frac{N+1}{2}}} \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right) \\
&= \left(\frac{x_j^2}{4t^{\frac{N+1}{2}}} - \frac{1}{2t^{\frac{N}{2}+1}}\right) \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\Delta f(x,t) &= \sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(x,t) \\
&= \sum_{j=1}^{N} \left(\frac{x_j^2}{4t^{\frac{N+1}{2}}} - \frac{1}{2t^{\frac{N}{2}+1}}\right) \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right) \\
&= \left(\frac{\|x\|^2}{4t^{\frac{N+1}{2}}} - \frac{N}{2t^{\frac{N}{2}+1}}\right) \exp\left(-\frac{x_1^2 + \cdots + x_N^2}{4t}\right)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial f}{\partial t}(x,t) &= -\frac{N}{2}\frac{1}{t^{\frac{N}{2}+1}} \exp\left(-\frac{\|x\|^2}{4t}\right) + \frac{1}{t^{\frac{N}{2}}}\frac{\|x\|^2}{4t^2} \exp\left(-\frac{\|x\|^2}{4t}\right) \\
&= \left(\frac{\|x\|^2}{4t^{\frac{N+1}{2}}} - \frac{N}{2t^{\frac{N}{2}+1}}\right) \exp\left(-\frac{\|x\|^2}{4t}\right) \\
&= \Delta f(x,t),
\end{aligned}$$

so that f solves the heat equation.