3. Sequences and limits

In this chapter we will lay the foundation for everything else that is to come. Limits are *the* fundamental theoretical concept of calculus. It expresses the intuitive notion of "approximation."

When we discussed the completeness of $\mathbb{R}$, we saw that the supremum s of $S = \{r \in \mathbb{R} \mid r^2 < 2\}$ must satisfy $s^2 = 2$. How can we describe s? It is not a rational number.

One way to do it would be to find reasonably close rational numbers. This is what your calculator/smartphone does if you ask it to compute $\sqrt{2}$. How could we do that?

Start with $r_1 = 1$, say. Then $r_1^2 \leq 2$. But $(r_1 + 1)^2 > 2$. Then there is $0 \leq d_1 \leq 9 \in \mathbb{N}_0$ such that

$r_2 \coloneqq r_1 + \frac{d_1}{10} \in S$ but $r_1 + \frac{d_1+1}{10} \notin S$.

We then repeat this and construct $r_3 = r_2 + \frac{d_2}{100} \in S$ such that $r_2 + \frac{d_2+1}{100} \notin S$ (why does this work)?

Recursively, we obtain a *sequence* $r_1, r_2, \dots, r_n, \dots$

Such that:

1. $r_{n+1} = r_n + \frac{d_n}{10^n} \in S$ with $0 \leq d_n \leq 9 \in \mathbb{N}_0$
2. $r_{n+1} + \frac{d_n+1}{10^n} \notin S$

Note that 2. Implies that $s - r_{n+1} \leq \frac{1}{10^n}$.

While we will not be able to "compute" $\sqrt{2}$ exactly, given any precision requirement $\varepsilon > 0$, we can find a rational number r_n such that $r_n \leq \sqrt{2}$ and $\sqrt{2} - r_n < \varepsilon$. For that choose n above such that $\frac{1}{10^{n-1}} < \varepsilon$.

Moreover, for any $m > n$ we then also have $\sqrt{2} - r_m < \varepsilon$. The sequence $r_1, r_2, \dots$ therefore "approximates" $\sqrt{2}$ to arbitrary precision. We will express this fact as saying that *the limit of r_n is equal to $\sqrt{2}$.*

3.1 Sequences

Definition

A **sequence** is a function $a: \mathbb{N} \to \mathbb{R}$. Sometimes a sequence may "start" at 0, or even a negative integer. Thus, more generally, a sequence is a function $a: \mathbb{Z}_{\geq k} \to \mathbb{R}$ where k is some integer.
We write simply a_n for the sequence (that is, $a_1 = a(1), a_2 = a(2), \dots$). If we want to emphasize that a_n is a sequence and not just a single number labelled by n, we also write (a_n), or $(a_1, a_2, \dots)$ to denote the sequence. EOD.

The numbers a_n are often referred to (imprecisely) as members or elements of the sequence.

Example

1. The natural numbers themselves form a sequence: $a_1, a_2, \dots$ with $a_n = n$.

2. If one wants to formalize the concept of polynomials, one way to do this is to define a polynomial as a sequence $a_0, a_1, \ldots$ such that there exists k such that $a_n = 0$ for all $n > k$. In other words, only finitely many members of the sequence are allowed to be nonzero.
   The sequence $a_0, a_1, \ldots$ is thought of as the polynomial $a_0 + a_1 x + a_2 x^2 + \cdots$. In particular, x itself corresponds to the sequence $0,1,0,0, \ldots$

Recursively defined sequences
A recursive definition for a sequence is a "rule" that specifies how a_{n+1} is computed when a_n or, more generally, $a_1, a_2, \ldots, a_n$ are known.
Formally, a recursive sequence is a template of the form:
1. $a_1 = x$.
2. $a_{n+1} = f_{n+1}(a_1, a_2, \ldots, a_n)$

Here f_n is some function defined on n natural numbers. The Principle of Induction can be used to show that this results in a well-defined sequence.

Example
The **Fibonacci Numbers** form the recursive sequence (f_n) with
1. $f_0 = 0$
2. $f_1 = 1$
3. $f_{n+1} = f_n + f_{n-1}$

So, the sequence is $0,1,1,2,3,5,8, \ldots$ EOE.

3.1.1 Improper sequences
We occasionally encounter situations where it might be useful (simply for convenience) to allow $\pm\infty$ as member of the sequence. We sometimes call such sequences **improper sequences**.

As an example, consider any sequence (a_n), and define the sequence (s_n) as

$$s_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

For many sequences this would not be an allowed definition. For example, if $a_n = n$, then $s_n = \infty$ for all n.

There is an obvious analogue definition for the infimum: $i_n = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$.

We sometimes put $\sup(a_n) \coloneqq s_n$ and $\inf(a_n) \coloneqq i_n$. This is essentially the only time where we might be interested in improper sequences. But note that $\sup(a_n)$ is a sequence, whereas $\sup\{a_1, a_2, \ldots\}$ is a number (or possibly ∞).

Exercise
Show that (s_n) as defined above is always **monotone decreasing** that is $s_m \leq s_n$ whenever $m \leq n$.
Show that (i_n) is always **monotone increasing**. EOE.

3.2 New sequences from old sequences
Given any sequence (a_n) and any real number b we can form a new sequence (c_n) defined by $c_n = ba_n$.

We can generalize this as follows: if $(a_n), (b_n)$ are two sequences, then we can form the **product sequence** $(c_n) = (a_n b_n)$.

Similarly, we can add sequences, and form the sum $(d_n) = (a_n + b_n)$.

In particular, if a_n, b_n are sequences, and c, d real numbers, then $ca_n + db_n$ again is a well defined sequence. This shows that the set of all sequences (all starting at 1, say) form what is called a **vector space**.

We can also do more esoteric things: if a_n, b_n are two sequences, then we can define $\max\{a_n, b_n\}$ as the sequence $\max\{a_1, b_1\}, \max\{a_2, b_2\}, \ldots$. It should be clear how the sequence $\min\{a_n, b_n\}$ is defined.

3.3 Subsequences

Another way to obtain a new sequence from a given sequence is to restrict indices. In other words, we may "leave out" some sequence members. The following definition makes this precise.

Definition
Let (a_n) be a sequence. A **subsequence** of (a_n) is a sequence (b_k) where $b_k = a_{n_k}$ for some strictly increasing sequence n_k of natural numbers. EOD.

To understand this definition let us consider an example:

Let `$a_n = (-1)^n 2^n$. We could decide to focus only on the sequence members for *even* n. So we would put $n_k = 2k$. Consequently $b_k = a_{n_k} = (-1)^{2k} 2^{2k}$.

The only condition is that we must have $n_1 < n_2 < n_3 < \cdots$. Therefore, the notion of a subsequence is not a very deep concept. For further use we record:

Lemma (All there is to say)
Let b_k be a subsequence of a_n. Then for all k we have $n_k \geq k$. EOL.

Proof. We proceed by induction on k and for simplicity assume that both a_n and the subsequence b_k start at 1. Then $n_1 \geq 1$ is obviously true. Now suppose that $n_k \geq k$ for a given $k \in \mathbb{N}$. Then $n_{k+1} > n_k \geq k$. Therefore $n_{k+1} \geq k + 1$ (this uses the fact that there is no natural number between m and $m + 1$ for all $m \in \mathbb{N}$). QED.

The lemma says, informally speaking, that the kth element of (b_k) is appears at least at position k in the sequence (a_n).

3.4 Limits of sequences

In this section we will rigorously define one of the most fundamental concepts of analysis, namely that of a limit. We already discussed one example in the beginning of this chapter, but we will now look at another one.

Example
(This example is a modification of the one found in Heuser, *Lehrbuch der Analysis, Teil 1*, 10th edition, 1993 (German), p 143.)
Many growth processes are described using a quantity $Q(t)$ where t stands for "time" obeying an equation of the form

$$Q(t + \Delta t) \approx Q(t) + cQ(t)\Delta t$$

where c is a fixed constant. That is, Q changes in the interval $[t, t + \Delta t]$ by an amount proportional to the value of Q on this interval. Examples occur in population dynamics (in biology, say) and many other areas of science, finance, …

We will consider the following hypothetical scenario: your bank agrees to pay you an annual interest rate of c. That is, if you deposit $Q(0)$ at the beginning of the year (and don't touch your account during the year), $Q(1) = Q(0) + cQ(0)$. For simplicity, we will assume that $c = 1$ (highly unrealistic).

Now, a competing bank offers you the following deal: you get the same "annual" interest rate ($c = 1$), but interest is paid monthly and compounds. That is, the monthly interest rate is $\frac{1}{12}$.

So for this bank, your account would develop as $Q\left(\frac{1}{12}\right) = Q(0) + \frac{1}{12}Q(0) = \left(1 + \frac{1}{12}\right)Q(0)$. After two months, your statement would show

$$Q\left(\frac{2}{12}\right) = \left(1 + \frac{1}{12}\right)Q\left(\frac{1}{12}\right) = \left(1 + \frac{1}{12}\right)^2 Q(0)$$

and at year's end

$$Q(1) = \left(1 + \frac{1}{12}\right)^{12} Q(0)$$

Which deal is better? For you, clearly the second one. The reason is that in the second scenario the bank agrees to pay interest on interest ("compound interest"). If the second bank paid your interest into a different account, you would end up with exactly the same amount as in the first case. However, since your interest is deposited at month's end into your account, in the next month you receive interest on this interest.

This difference can be significant: a calculator (or if you do not know what that is, a smartphone) shows that $\left(1 + \frac{1}{12}\right)^{12} \approx 2.61$. So, in the second deal you receive roughly 61% of your initial capital extra.

Now a third bank offers you the following deal: you pick a natural number n, and the bank will partition the year in equal intervals of size $\frac{1}{n}$ (so a partition into months would correspond to $n = 12$, a partition into days would mean $n = 365$ (let's ignore leap years), hours would mean $n = 8{,}760$, minutes $n = 525{,}600$, and seconds $n = 31{,}536{,}000$).

They agree to pay interest at each end of the interval and compound it. That is, your account will show

$$Q(1) = \left(1 + \frac{1}{n}\right)^n Q(0)$$

at year's end.

Is this bank endangering its own existence? Which n would you choose?

It is not hard to see that if $m < n$ then $\left(1 + \frac{1}{m}\right)^m < \left(1 + \frac{1}{n}\right)^n$. So to answer the second question: you pick the largest natural number you can imagine (or write down, or transmit to your bank; what is the largest natural number that you can specify in a message with 260 characters using only commonly accepted notation?). It turns out that the answer is not that interesting because of n is large enough, the additional return is relatively small. For fixed $Q(0)$, eventually, the additional return will be below 1 cent. So you only need to pick n *large enough*.

The answer to the first question is no, but that involves some computation. One can show (and that is not too difficult) that always

$$\left(1 + \frac{1}{n}\right)^n < 3$$

This already answers the question: no matter which n you choose, the payout is limited above (which might be surprising at first sight, given the hat $n = 12$ already gives a significant bonus).

Returning to mathematics: let $a_n = \left(1 + \frac{1}{n}\right)^n$. Above we stated that a_n is increasing (that is, that a as a function is monotone increasing), and that $\{a_1, a_2, \dots\}$ is bounded above by 3.
Thus, $e \coloneqq \sup\{a_1, a_2, \dots\}$ exists and is proper (not ∞). On the other hand, since e is the supremum, for any $\varepsilon > 0$ there must be at least one n_0 such that $\left|e - a_{n_0}\right| < \varepsilon$. As a_n is increasing it follows that for *all* $n \geq n_0$ we have $a_{n_0} \leq a_n \leq e$ and hence $|e - a_n| < \varepsilon \ \ \forall n \geq n_0$. Therefore, given n large enough, we can approximate e to arbitrary precision. We will express this situation as

$$\lim_{n\to\infty} a_n = e$$

If we only choose n large enough, a_n will be close to e (closer than any predetermined margin of error). We will devote significant time to make these thoughts precise.
To conclude the example, note that e is the famous **Euler constant**, and Euler wrote in his famous "*Introductio in analysin infintorum*" (1748) that

$$e = 2.71828182845904523536028\dots$$

(And Euler presumably didn't have a smartphone.)
One can show (and we will see) that e is not a rational number. In fact it is a **transcendental** number, that is, it does not satisfy any polynomial equation with rational coefficients (in contrast, $\sqrt{2}$ is not rational but satisfies the equation $x^2 - 2 = 0$). π is another such number.
EOE.

3.4.1 The definition of limits

Loosely speaking, the *limit* of a sequence (a_n) (if it exists) is a number L such that the elements of (a_n) *approximate* L arbitrarily well. That does not mean that it is enough that *one* sequence element a_k say is "close" to L. It means that *almost all* of them are. Here is the formal definition.

Definition (Limit)
Let (a_n) be a sequence. We say $L \in \mathbb{R}$ is the **(proper) limit** of (a_n), written $L = \lim\limits_{n\to\infty} a_n$ if for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $|a_n - L| < \varepsilon$.
A sequence where such a limit exists is called **convergent** (and we say the sequence **converges**), and **divergent** (and we say the sequence **diverges**) otherwise. EOD.

Remark
A few remarks are in order.

1. In "normal" English, the definition says that a real number L is the limit of the sequence (a_n) if the sequence members eventually (for large enough n) are closer to L than any prescribed margin (commonly called ε).
2. It is important to realize that n_0 in the definition usually **depends** on ε. For each $\varepsilon > 0$ we possibly have a different n_0.
3. We often also use the notation $a_n \to L$ for $n \to \infty$, or simply $a_n \to L$ to express that $\lim\limits_{n\to\infty} a_n = L$.
4. Likewise, we often simply write $\lim a_n = L$ (with $n \to \infty$ understood).
5. Convince yourself that the definition is equivalent to the following statement: $\lim a_n = L$ if and only if for every $\varepsilon > 0$ *for all but finitely many* n *we have* $|a_n - L| < \varepsilon$.

EOR.

We will discuss a lot of examples below. But without further tools, we can discuss only a very limited number of examples (no pun intended).

Example

1. Let $a_n = \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 0$.
2. Let $a \in \mathbb{R}$. Then $\lim_{k\to\infty} \left(a + \frac{1}{k}\right)^n = a^n$.
3. Let $a_n = (-1)^n$, then (a_n) does not converge.
   Indeed suppose $a_n \to L$ for some $L \in \mathbb{R}$. Then for $\varepsilon = \frac{1}{2}$, there is n_0 such that $|a_n - L| < \frac{1}{2}$ for all $n > n_0$. But then $2 = |(-1)^n - (-1)^{n+1}| = |(a_n - L) + (a_{n+1} - L)| \le \frac{1}{2} + \frac{1}{2} = 1$, a contradiction.

EOE.

While the following result may seem obvious, it nevertheless needs a proof.

Proposition
The limit of a convergent sequence is unique. EOP.

Proof. Suppose (a_n) is a convergent sequence with limit L. Suppose also $a_n \to M$. We must show that $M = L$. Suppose $L \neq M$. Let $\varepsilon = \frac{|L-M|}{2} > 0$. Then there are $n_0, n_1 \in \mathbb{N}$ such that for all $n > n_0$ we have $|a_n - L| < \varepsilon$ and for all $n > n_1$ we have $|a_n - M| < \varepsilon$. Let $n > \max\{n_0, n_1\}$, then $|a_n - L| < \varepsilon$ and $|a_n - M| < \varepsilon$.

But $2\varepsilon = |L - M| \le |L - a_n| + |a_n - M| < 2\varepsilon$. This is a contradiction. QED.

Theorem
Any subsequence of a convergent sequence is convergent with the same limit. EOT.

Proof. Let (a_n) be a convergent sequence with limit L, and let $n_1 < n_2 < \cdots$ be a strictly increasing sequence of natural numbers. Let $b_k = a_{n_k}$. We must show that for every $\varepsilon > 0$ there is n_0 such that $|b_k - L| < \varepsilon$ whenever $k > n_0$. Let n_0 be such that for all $n > n_0$ we have $|a_n - L| < \varepsilon$. As $n_k \ge k$, the same n_0 works for (b_k). QED.

There are some divergent sequences where it still makes sense to define a limit.

Definition (Improper limit)
Let (a_n) be a sequence. We say (a_n) **diverges to** ∞, and say that $\lim a_n = \infty$ if for every $B \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $a_n > B$.
Likewise, we say (a_n) **diverges to** $-\infty$, and write $\lim a_n = -\infty$ if for every $B \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $a_n < B$.
In these cases, we call $\pm\infty$ the **improper limit** or simply **limit** of (a_n). EOD.

In this sense, a divergent sequence may still have a limit. It is also clear that that a sequence with an improper limit cannot be bounded. The converse is again not true: there are unbounded sequences without limit. The sequence defined by $(-1)^n n$ does not have a limit, and the sequence $(-1)^n$ does not have a limit either (the first is not bounded, the second is).

Lemma
Let $A \subseteq \mathbb{R}$ be nonempty. There is a sequence (a_n) in A such that $\lim a_n = \sup A$. EOL.

Proof. (The case of A bounded above was also done in a live meeting.)

First assume that A is bounded above. Then $s = \sup A$ is a real number. For $n \in \mathbb{N}$ there is $a_n \in A$ such that $x_n > s - \frac{1}{n}$. (This uses the Axiom of Choice.)

Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. Then for all $n > n_0$, we have $|x_n - s| = s - x_n < \frac{1}{n} < \frac{1}{n_0} < \varepsilon$. This shows $\lim a_n = s$.

If A is not bounded above, we argue as follows: for every $n \in \mathbb{N}$, there exists $a_n \in A$ such that $a_n > n$. Otherwise n would be an upper bound. But then for any $B \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $n_0 > B$. Then for all $n > n_0$ we have $x_n > n > n_0 > B$. By definition, this means $\lim a_n = \infty = \sup A$. QED.

Exercise
Show that if A contains a sequence with limit s such that s is an upper bound for A or $s = \infty$, then $s = \sup A$. EOE.

Convince yourself that similar results as in the lemma and exercise hold if the supremum is replaced everywhere by the infimum.

Lemma
Let a be a real number. Then a is the limit of a sequence of rational numbers. EOL.

Proof. By the previous lemma it suffices to show that $a = \sup A$ were $A = \{r \in \mathbb{Q} \mid r < a\}$.

For this observe that A is never empty. For example, by the fact that $\mathbb{R}$ is Archimedean, we know that $\mathbb{Q}$ is dense, and so the interval $(a-1, a)$ will contain a rational number.

On the other hand, if $b < a$, then (b, a) will contain a rational number for the same reason, and then b cannot be an upper bound for A. Thus $a = \sup A$ (it is clearly an upper bound), and by the previous lemma, we obtain a sequence in $A \subseteq \mathbb{Q}$ with limit a. QED.

3.4.2 Bounded sequences

Definition
A sequence (a_n) is called **bounded above** if there is $M > 0$ such that $a_n < M$ for all $n \in \mathbb{N}$.
(a_n) is called **bounded below** if there is $M > 0$ such that $a_n > -M$ for all n.
The sequence is called **bounded** if it is both bounded above and below, or equivalently, if there is $M > 0$ such that $-M < a_n < M$ for all $n \in \mathbb{N}$. EOD.

Note that if (a_n) is both bounded below and above, then there are $M, N > 0$ such that $-M < a_n < N$ for all n. By replacing both M and N by their maximum, we can assume they are equal. (a_n) is bounded if and only if $(|a_n|)$ is bounded above.

Lemma
(a_n) is bounded above if and only if $\sup(a_n)$ is a proper sequence.
(a_n) is bounded below if and only if $\inf(a_n)$ is a proper sequence.

Proof. Let $(s_n) = \sup(a_n)$. If (a_n) is bounded above, by $A \in \mathbb{R}$, say, then $a_n \leq A$ for all n, and therefore $s_n = \sup\{a_n, a_{n+1}, \dots\} \leq A < \infty$. Conversely, if (s_n) is a proper sequence, then $a_n \leq s_1$ for all n, so a_n is bounded above. QED.

Lemma
A convergent sequence is bounded. EOL.

Proof. Suppose (a_n) is convergent with limit L. Then for $\varepsilon = 1$ there is $n_0 \in \mathbb{N}$ such that $\forall n > n_0$: $|a_n - L| < 1$. But then $L - 1 < a_n < L + 1$ for all $n > n_0$. Thus $A \leq a_n \leq B$ for all $n \in \mathbb{N}$, were $A = \min\{a_1, a_2, \dots, a_{n_0}, L - 1\}$, and $B = \max\{a_1, a_2, \dots, a_{n_0}, L + 1\}$. QED.

The converse is not true: $a_n = (-1)^n$ is bounded but not convergent.

3.5 Arithmetic with limits

We say a statement about a sequence, or sequences is **almost always** true if it is true for all but possibly finitely many n. Similarly, **almost all** a_n refers to all but at most finitely many of the sequence elements.

For example, $a_n \neq 0$ almost always, means there is $m_0 \in \mathbb{N}$ such that for all $m > m_0$, $a_n \neq 0$. We also say almost all $a_n \neq 0$.

If we look at the definition of a limit in 3.4.1 above and the succeeding remarks, then we find that a sequence (a_n) has the proper limit L if and only if for all $\varepsilon > 0$, $|a_n - L| < \varepsilon$ is almost always true.

Note it is not enough that the statement is true for infinitely many natural numbers n. There must be some natural number m_0 such that the statement is true for all $n > m_0$.

Lemma (Comparison Lemma; CL)
Suppose $a_n \to L$ and $b_n \to M$. If almost always $a_n \leq b_n$, then also $L \leq M$. EOL.

Proof. If $b = \infty$ or $a = -\infty$, there is nothing to do.

Suppose first that M is a proper limit. Let $m_0 \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n > m_0$. Then a_n is bounded above, because b_n is: so if B is an upper bound for b_n, then $\max\{a_1, \dots, a_{m_0}, B\}$ is an upper bound for a_n. In particular if $a \neq -\infty$, then a must be a proper limit.

Suppose $L > M$. Then $\varepsilon := \frac{(L-M)}{2} > 0$ and there is $n_0 \in \mathbb{N}$ such that $|L - a_n|, |M - b_n| < \varepsilon$ for all $n > n_0$. We may also assume that $n_0 > m_0$. Then $a_n \leq b_n \leq M + \varepsilon = L - \varepsilon$. But $a_n > L - \varepsilon$. This is a contradiction.

The case that $b = -\infty$ is similar and left as an exercise. QED.

Remark

- The Comparison Lemma immediately implies that if almost always $a \leq a_n \leq b$ for $a, b \in \mathbb{R}$, then $a \leq \lim a_n \leq b$ if the limit exists. This requires to apply the lemma to the case $a \leq a_n$ where a is viewed as the constant sequence, and we conclude $a \leq \lim\ a_n$. Then apply the lemma to $a_n \leq b$ where b is viewed as the constant sequence with value b, and we can conclude $\lim a_n \leq b$
- If $a_n < b_n$ almost always, we cannot conclude that $\lim a_n < \lim\ b_n$, even if both limits exist. For example, consider the sequence $a_n = 0$ for all n, and the sequence $b_n = \frac{1}{n}$. Then $\lim a_n =$

$0 = \lim\ b_n$ even though $b_n > a_n$ for all n. In that sense a converse for the Comparison Lemma fails.

EOR.

Exercise
Show that if $a_n \to L$ and $b_n \to M$, and $L > M$, then $a_n > b_n$ almost always. (This provides a partial converse to the Comparison Lemma, as quality of limits is the "worst" that can happen if we have strict inequalities almost always.) EOE.

Lemma (Squeeze Principle; SP)
Let a_n, b_n be sequences with $a_n \to L$ and $b_n \to L$. Let c_n be a sequence such that almost always
$$a_n \leq c_n \leq b_n$$
Then also $c_n \to L$. EOL.

Proof. First, let $L \in \mathbb{R}$. For $\varepsilon > 0$, there is n_0 such that $|a_n - L| < \varepsilon$ and $|b_n - L| < \varepsilon$ for all $n > n_0$. We may assume that for all $n > n_0$ also $a_n \leq c_n \leq b_n$ (by increasing n_0 if necessary). Then $L - \varepsilon < a_n \leq c_n \leq b_n < L + \varepsilon$, and so $|L - c_n| < \varepsilon$.

If $L = \infty$, then for every $B \in \mathbb{R}$ and almost all n, we have $a_n, b_n > B$. For almost all n we then also have $c_n \geq a_n > B$. If $L = -\infty$, then for ebery B almost all $b_n < B$. And almost always $c_n \leq b_n < B$.

In either case $\lim c_n = L$ follows. QED.

The Squeeze Principle is fundamentally important and allows to compute many limits.

Corollary
If (a_n) is a sequence with limit 0, and if b_n is any sequence such that for some $L \in \mathbb{R}$ and almost always $|b_n - L| \leq a_n$, then $b_n \to L$. EOC.

Proof. We almost always have $0 \leq |b_n - L| \leq a_n$. By the squeeze principle, we conclude that $|b_n - L| \to 0$. But that is equivalent to $b_n \to L$. QED.

Definition
A **zero sequence** is a sequence with limit 0. EOD.

For example, $a_n \to L$, with $L \in \mathbb{R}$, if and only if $(a_n - L)$ or $|a_n - L|$ is a zero sequence.

Occasionally, the following helper result will be useful:

Lemma (Bounded times zero sequence is zero sequence; BTZ)
Let (a_n) be a zero sequence and let (b_n) be a bounded sequence. Then $(a_n b_n)$ is a zero sequence. EOL.

Proof. We use SP. Let $B > 0$ be such that $|b_n| < B$ for all n. Then $|a_n b_n| < |a_n| B$ for all n.

For a given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n| < \frac{\varepsilon}{B}$ for all $n > n_0$. Then $|a_n| B < \varepsilon$ for all $n > n_0$. Thus$B|a_n| \to 0$. By SP, $a_n b_n \to 0$. QED.

Example

1. Let a_n be a convergent sequence with limit L. Then $|a_n|$ converges with limit $|L|$.
   Indeed, we always have $0 \leq \big||a_n| - |L|\big| \leq |a_n - L|$ by the properties of the absolute value.
   But then SP says that $\big||a_n| - |L|\big|$ is a zero sequence. Equivalently, $|a_n| \to |L|$.
2. If $a_n \geq 0$ for all n, and a_n converges to $L \in \mathbb{R}$, then $\sqrt{a_n}$ converges to $\sqrt{L}$.
   First assume $L > 0$.
   Let $\varepsilon > 0$, and let $x_n = \sqrt{a_n} - \sqrt{L}$. a given $\varepsilon > 0$. Note that $a_n - L = x_n(\sqrt{a_n} + \sqrt{L})$.
   We conclude that
   $$|x_n| = \frac{|a_n - L|}{\left|\sqrt{a_n} + \sqrt{L}\right|}$$
   Note that $\sqrt{a_n} + \sqrt{L} \geq \sqrt{L} > 0$.
   Therefore $|x_n| \leq \frac{|a_n - L|}{\sqrt{L}}$. The right hand side is a zero sequence by BTZ, and by SP so is x_n.

   It remains to treat the case that $L = 0$. But then a_n is a zero sequence. For almost all n, $0 \leq a_n < \varepsilon^2$, where $\varepsilon > 0$ is some given real positive number.
   Then $\sqrt{a_n} < \varepsilon$ for almost all n, by some UFO.

EOE.

Theorem (Useful Limit Theorem; ULT)
Let $(a_n), (b_n)$ be sequences with limits L and M respectively.

1. $\lim_{n\to\infty}(a_n + b_n) = L + M$, unless $L = -M$ and $L = \pm\infty$.
2. $\lim_{n\to\infty} a_n b_n = LM$, unless $L = \pm\infty$ and $M = 0$ or vice versa.
3. If $c, d \in \mathbb{R}$ then $\lim_{n\to\infty}(ca_n + db_n) = cL + dM$ unless $cL = -dM$ and $cL = \pm\infty$.
4. If $M \neq 0$ and L or M is a proper limit, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$.

EOT.

For 4. we must observe that $\frac{a_n}{b_n}$ is not necessarily defined for all n. However, if $M \neq 0$, then $b_n \neq 0$ almost always. We therefore view $\frac{a_n}{b_n}$ as the sequence starting at a large enough natural number m_0 such that $b_n \neq 0$ for all $n \geq m_0$.

Proof. We first assume that all limits are proper limits.

1. Let $\varepsilon > 0$. Then there is $n_1 \in \mathbb{N}$ such that for all$n > n_1$, $|a_n - L| < \frac{\varepsilon}{2}$. Likewise, there is $n_2 \in \mathbb{N}$ such that $|b_n - M| < \frac{\varepsilon}{2}$. Let $n_0 = \max\{n_1, n_2\}$. Then for all $n > n_0$ we have
   $$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
   Here we used the triangle inequality. Thus, $a_n + b_n \to L + M$.
   Note the proof would could be stated more concisely (but equivalently) as follows:
   Let $\varepsilon > 0$. Almost always $|a_n - L| < \frac{\varepsilon}{2}$, and almost always $|b_n - M| < \frac{\varepsilon}{2}$. Therefore, almost always $|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \varepsilon$.
2. We must show that $a_n b_n - LM$ is a zero sequence.
   $$a_n b_n - LM = (a_n - L)b_n + (b_n - M)L$$

By BTZ $(a_n - L)b_n$ is a zero sequence, as is $(b_n - M)L$ (where we identify the real number L with the constant sequence $c_n = L$). By 1. the sum of two zero sequences is again a zero sequence

3. This is a combination of 1. and 2. where c is identified with the constant sequence $c_n = c$, and d is identified with the sequence $d_n = d$. So $c_n a_n \to cL$ and$d_n b_n \to dM$, which means $c_n a_n + d_n b_n = ca_n + db_n \to cL + dM$.
4. We first treat the case that $a_n = 1$ for all n. In other words, we show that $b_n^{-1} \to M^{-1}$ (keeping in mind that the sequence b_n^{-1} starts at a large enough natural number m_0 such that $b_n \neq 0$ for all $n > m_0$).
   Now let
   $$x_n = \frac{1}{b_n} - \frac{1}{M} = \frac{M - b_n}{b_n M}$$
   There is $m_1 \in \mathbb{N}$ such hat $|b_n| > \left|\frac{M}{2}\right|$ for all $n > m_1$.
   We must show that $x_n \to 0$. Let $\varepsilon > 0$.
   Then $|x_n| = \frac{|M-b_n|}{|b_n M|} < \frac{2|M-b_n|}{|M|^2}$ for all $n > m_0, m_1$.
   There is $n_0 > m_0, m_1 \in \mathbb{N}$ such that for all $n > n_0$, $|M - b_n| < \frac{\varepsilon |M|^2}{2}$ for all $n > n_0$.
   Then $|x_n| < \varepsilon$.
   The general case then follows by 2.: a_n and $\frac{1}{b_n}$ are convergent sequences, and the limit of $a_n \cdot \frac{1}{b_n}$ is $\frac{\lim a_n}{\lim b_n}$.

This finishes the proof for proper limits.

Now suppose one of the limits is improper. With the exception of 4. The situation is symmetric in $(a_n), (b_n)$.

1. If $L = \infty$, and if $M \neq -\infty$ then for any $B \in \mathbb{R}$, we almost always have $a_n + b_n > B$. This is clear if $M = \infty$. If $M \in \mathbb{R}$, then almost always $b_n > M - 1$, and almost always $a_n > B - (M - 1)$, so almost always $a_n + b_n > \big(B - (M - 1)\big) + (M - 1) = B$. The case $L = -\infty$ is similar.
2. If $L = \infty$, and $B > 0 \in \mathbb{R}$, then almost always $a_n b_n > B$, unless $M = 0$ or $M = -\infty$.
   Indeed, if $M = \infty$, then almost always $a_n, b_n > \sqrt{B}$ and therefore almost always $a_n b_n > B$. If $M \in \mathbb{R}$, but $M \neq 0$, then if $M > 0$, almost always $b_n > \frac{M}{2}$, almost always $a_n > \frac{2B}{M}$, and therefore almost always $a_n b_n > B$, resulting in $\lim a_n b_n = \infty = LM$. If $M < 0$, then almost always $b_n < \frac{M}{2} < 0$, and almost always $a_n > 2B/|M|$, and thus $a_n b_n < -B$. In this case $\lim a_n b_n = -\infty = LM$.
3. This is again a combination of 1. and 2.
4. First suppose $M = \infty$. Then $\frac{1}{b_n} \to 0$. Indeed, for any given $\varepsilon > 0$, almost always $b_n > \frac{1}{\varepsilon} > 0$ and therefore $0 < \frac{1}{b_n} < \varepsilon$.
   If $M = -\infty$, then applying 2. with the constant sequence $a_n = -1$, and to $\frac{1}{-b_n}$, we get that $\frac{1}{b_n} = (-1)\frac{1}{-b_n} \to 0$.
   Now the result follows by applying 2. and observing that $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$.

QED.

Corollary
Two **convergent** sequences (a_n), (b_n) have the same limit if and only if $a_n - b_n$ is a zero sequence. EOC.

Proof. The sequence $a_n - b_n$ converges to $\lim a_n + (-1)\lim b_n$ by ULT 3. This is 0 if and only if $\lim a_n = \lim b_n$. QED.

Note the mere fact that $a_n - b_n \to 0$ does not guarantee that a_n or b_n converge. However look at the following exercise:

Exercise
Suppose a_n converges and b_n is any sequence such that $a_n - b_n \to 0$. Show that b_n converges (to the same limit as a_n. EOE.

***Remark**
ULT3 is important for the following reason: Consider the set S of all sequences starting at 1. That is, S is the set of all *functions* $\mathbb{N} \to \mathbb{R}$. S forms what is called a *vector space*, as we have defined an addition of sequences, and we can scale them by real numbers, and certain natural properties are satisfied (you will recognize them, once you take Linear Algebra).
ULT3 then says that the subset C of *convergent sequences* in S is a linear subspace, and moreover, the function $f: C \to \mathbb{R}$, defined by $f\big((a_n)\big) \coloneqq \lim a_n$ is a so called linear transformation. EOR.

Warning
The exceptions listed in the theorem are all necessary.
If $a_n \to \infty$ and $b_n \to -\infty$, no general statement can be made about $a_n + b_n$

1. $a_n = n$, $b_n = -n + c$, here the sum converges to c.
2. $a_n = n^2, b_n = -n$, here the sum diverges to ∞.
3. $a_n = n$, $b_n = -n + (-1)^n$, here the sum has no limit.
4. $a_n = n, b_n = -n^2$, here the sum diverges to $-\infty$.

If $a_n \to \pm\infty$ and $b_n \to 0$, then no general statement can be made about $a_n b_n$.

1. $a_n = n$, $b_n = \frac{c}{n}$, then $a_n b_n \to c$.
2. $a_n = n^2$, $b_n = \frac{c}{n}\,(c \neq 0)$, then $a_n b_n \to c \cdot \infty = \pm\infty$.
3. $a_n = n$, $b_n = \frac{(-1)^n}{n}$, then $a_n b_n = (-1)^n$ has no limit.

Similarly, if $b_n \to 0$, and almost always $b_n \neq 0$, then no general statement can be made about b_n^{-1}. But always $|b_n^{-1}| \to \infty$. But for example, if $b_n = \frac{(-1)^n}{n}$, then b_n^{-1} has no limit.
Note that two sequences can have the same improper limit even though their difference is not a zero sequence. EOW.

For a **generalization** of ULT consider m **convergent** sequences $(a_{1n}), (a_{2n}), \dots (a_{mn})$, with limits $L_1, L_2, \dots, L_m$, respectively.

An immediate induction proof then shows that

$$a_{1n} + a_{2n} + \cdots + a_{mn} \to L_1 + L_2 + \cdots + L_m$$

is also convergent with the obvious limit.

Similarly

$$a_{1n}a_{2n}\cdots a_{mn} \to L_1L_2\cdots L_m$$

One can then generalize ULT3 and ULT4 also to a case of more than two sequences.

For now, the following special case of this generalization is important:

If we take m times the same convergent sequence (a_n), then ULT2 says

$$\lim_{n\to\infty}(a_n)^m = L^m$$

where $L = \lim a_n$.

Likewise, if c is any real number, then $c(a_n)^m$ converges to cL^m. Applying the generalized ULT1 we get:

For any $c_0, c_1, \dots, c_m \in \mathbb{R}$ and any convergent sequence (a_n) with limit L we have

$$\lim_{n\to\infty}(c_0 + c_1a_n + \cdots + c_ma_n^m) = c_0 + c_1L + \cdots + c_mL^m$$

This amounts to the following result:

Proposition (Polynomials are continuous; PC)
Let $f: \mathbb{R} \to \mathbb{R}$ be any polynomial function. Let (a_n) be any convergent sequence with limit $L \in \mathbb{R}$. Then the sequence $f(a_n)$ is convergent with limit $f(L)$. EOP.

3.6 Monotone sequences

As sequences are functions, we know what it means for them to be monotone increasing or decreasing. For convenience, we recall the definition:

A sequence (a_n) is **monotone increasing** if $a_m \le a_n$ whenever $m < n$. (a_n) is **monotone decreasing** if $a_m \ge a_n$ whenever $m < n$. (a_n) is **strictly** monotone increasing (resp. decreasing) if the respective inequalities are strict for all $m < n$.

Exercise
Show that (a_n) is (strictly) monotone increasing if and only if for all n, $a_n \le a_{n+1}$ (respectively $a_n < a_{n+1}$).
Show that (a_n) is (strictly) monotone decreasing if and only if for all n, $a_n \ge a_{n+1}$ (respectively $a_n > a_{n+1}$). EOE.

Convince yourself that a monotone increasing sequence that is also bounded above is bounded. Likewise, a monotone decreasing sequence that is bounded below is bounded.

The ideas we discussed at the beginning of this section come very close to proving the following theorem:

Theorem (Monotone and Bounded is Convergent; MBC)
Let (a_n) be a bounded monotone sequence. Then (a_n) is convergent. EOT.

Proof. We treat the case of a monotone increasing sequence that is bounded above. The case of a bounded below decreasing sequence is similar (or follows, by multiplication with -1).

Let $S = \sup\{a_1, a_2, \dots, a_n, \dots\}$. As (a_n) is monotone increasing, $a_n \leq S$ for all n. As (a_n) is bounded above, $S < \infty$ is a real number. Let $\varepsilon > 0$. By the definition of supremum, there is n_0 such that $a_{n_0} > S - \varepsilon$. As (a_n) is increasing, for all $n > n_0$ we also must have $a_n > S - \varepsilon$. Thus, for all $n > n_0$ we have

$$S - \varepsilon < a_n \leq S$$

Therefore $|a_n - S| < \varepsilon$ for all $n > n_0$, and by definition, $\lim a_n = S$ follows. QED.

If (a_n) is a bounded sequence, then both $\sup(a_n)$ and $\inf(a_n)$ are proper sequences. They are also bounded, but also monotone. They are therefore convergent!

Example
Let a be a positive real number. We want to "compute" $\sqrt{a}$. The following is known as the *Babylonian method*. It was apparently known to the ancient Babylonians (1500 BC).
We construct a sequence (a_n) as follows.

1. Pick any positive number a_0.
2. Put $a_1 = \frac{1}{2}\left(a_0 + \frac{a}{a_0}\right)$
3. Assume a_n is defined. Then $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$.

First note that if $a_n > 0$, then also $a_{n+1} > 0$, so the sequence is actually well defined (and we can make the division to compute a_{n+1} at each step).If by accident we happen to pick $a_0 = \sqrt{a}$, then $a_1 = \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right) = \frac{\frac{1}{2}2a}{\sqrt{a}} = \sqrt{a}$, and then $a_n = \sqrt{a}$ for all n. For all $n > 0$, $a_n^2 \geq a$:
For this consider $\left(a_n + \frac{a}{a_n}\right)^2 = a_n^2 + 2a + \frac{a^2}{a_n^2}$.
Now $a_n^2 + \frac{a^2}{a_n^2} \geq 2a$ because $\left(a_n - \frac{a}{a_n}\right)^2 \geq 0$. Therefore

$$\left(a_n + \frac{a}{a_n}\right)^2 = a_n^2 + 2a + \frac{a^2}{a_n^2} \geq 4a$$

It follows that for any $n \geq 0$, $a_{n+1} \geq \sqrt{a}$. But then $\frac{a}{a_{n+1}} \leq \sqrt{a}$. For $n \geq 2$, a_n is the average of two numbers one greater, one smaller than $\sqrt{a}$. That means $a_1 \geq a_2 \geq \cdots$, and (a_n) is a monotone decreasing function (for $n \geq 2$). It is bounded below by $\sqrt{a}$, and hence there is a limit $L \geq \sqrt{a} > 0$. It follows that $\lim \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) = \frac{1}{2}\left(L + \frac{a}{L}\right) = \lim a_{n+1} = L$.
Solving for L we get $L^2 = a$, and $L = \sqrt{a}$ (since $L > 0$). EOE.

One of the fundamental results on sequences is the following theorem:

Theorem (Bolzano[1]-Weierstrass[2]; BW)
Any sequence contains a monotone subsequence. In particular, any bounded sequence has a convergent subsequence. EOT.

This theorem has profound consequences (which we cannot yet discuss all; but see the Cauchy Criterion below). We will use it repeatedly.

Proof. Let (a_n) be a sequence. Consider the set $S = \{n \in \mathbb{N} \mid a_n \geq a_m \forall m \geq n\}$. The set may be empty, finite, or infinite. (One could think of S as the set of "peaks" of the sequence (a_n), in the sense, that it is always "downhill" from a given $n \in S$.)

If S is infinite, then there exists a strictly monotone increasing sequence $n_1 < n_2 < \cdots$ in S.

(Formally, one could define this sequence recursively as $n_1 = \min S$, and $n_{k+1} = \min\{m \in S \mid m > n_k\}$.)

But then the sequence $b_k = a_{n_k}$ is a monotone decreasing subsequence.

Thus, we may assume S is finite (which includes $S = \emptyset$). Then there is $n_0 \in \mathbb{N}$ such that $n_0 > n$ for all $n \in S$. For any $n \geq n_0$, it is true that there exists at least one $m > n$ such that $a_m > a_n$ because if no such m exists then $a_m \leq a_n$ for all $m > n$, and $n \in S$, which is a contradiction.

We can now construct a subsequence recursively as follows:

$$\begin{aligned} n_1 &= n_0 \\ n_{k+1} &= \min\{m > n_k \mid a_m > a_{n_k}\} \end{aligned}$$

As $n_k \geq n_0$, $n_k \notin S$ and n_{k+1} is always defined. The resulting sequence $b_k = a_{n_k}$ is strictly monotone increasing.

This shows the first part, that every sequence contains a monotone subsequence.

If (a_n) is bounded, then so is this subsequence. It therefore converges by MBC. QED.

3.7 Limit superior and inferior

Let (a_n) be a bounded sequence. Then both $\sup(a_n)$ and $\inf(a_n)$ are proper sequence. They are also bounded by the same upper and lower bounds.

We write $\sup\limits_{k \geq n} a_k$ for $\sup\{a_k \mid k \geq n\}$, and $\inf\limits_{k \geq n} a_k$ for $\inf\{a_k \mid k \geq n\}$. These are the nth elements of the sequences $\sup(a_n)$ and $\inf(a_n)$ respectively.

Exercise
Suppose (a_n) is bounded. Show that $\sup(a_n)$ is a monotone decreasing sequence, and $\inf(a_n)$ is monotone increasing. EOE.

By MBC, both $\sup(a_n)$ and $\inf(a_n)$ converge.

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[1] Bernardus Placidus Johann Nepomuk Bolzano (1781 – 1848)
[2] Karl Theodor Wilhelm Weierstraß (1815 – 1897)

Definition.
Let (a_n) be sequence. We define
$$\limsup_{n\to\infty} a_n := \lim\sup a_n := \lim(\sup(a_n))$$
and call this the **limit superior** of a_n. Similarly, we put
$$\liminf_{n\to\infty} a_n := \lim\inf a_n := \lim(\inf(a_n))$$
and call this the **limit inferior** of a_n. EOD.

Note that if (a_n) is bounded these limits are always finite by the remarks above. If a_n is not bounded at least one of the limits is improper.

Exercise
Let a_n be a bounded sequence. Let S, I be real numbers.

1. Show that $S = \lim\sup a_n$ if and only if: for all $\varepsilon > 0$ there are infinitely many $n \in \mathbb{N}$ such that $a_n > S - \varepsilon$. There are only finitely many n such that $a_n > S + \varepsilon$.
2. Show that $I = \lim\inf a_n$ if and only if: for all $\varepsilon > 0$ there are infinitely many $n \in \mathbb{N}$ such that $a_n < I + \varepsilon$. There are only finitely many n such that $a_n < I - \varepsilon$.
3. Now let $S = \lim\sup a_n$, and $I = \lim\inf a_n$. Conclude that for any $\varepsilon > 0$ we almost always have
$$I - \varepsilon < a_n < S + \varepsilon$$
4. Show that a_n is convergent if and only if $\lim\inf a_n = \lim\sup\ a_n$.

Note that 4. still needs that a_n is bounded. For example, if $a_n = n$, then $\lim\inf a_n = \lim\sup a_n = \infty$, but a_n is not convergent. EOE.

3.8 Cauchy sequences

So far, we have focused on determining convergence of a sequence by SP, the Comparison Lemma, or ULT, or directly by applying the definition. We know that monotone bounded sequences converge. But we have no general criterion for determining whether a given sequence converges (possibly without knowing what it converges to).

However, we haven't yet seen a general criterion that characterizes convergence without specifying a limit. MBC comes close but only applies to monotone bounded sequences.

To determine whether an arbitrary sequence (a_n) is convergent, we so far relied on having a candidate L for a limit, and then show that $a_n \to L$. Of course, we cannot possible check that for all real numbers L.

The Cauchy Criterion below gives the most general characterization of what convergence means.

Definition
A sequence (a_n) is called a **Cauchy sequence** if for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ we have
$$|a_n - a_m| < \varepsilon$$
EOD.

Lemma
A convergent sequence is a Cauchy sequence. EOL.

Proof. Let (a_n) be a convergent sequence with limit L, and let $\varepsilon > 0$. Then there is n_0 such that for all $n > n_0$ we have $|a_n - L| < \frac{\varepsilon}{2}$. But then if both $n, m > n_0$ we have

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < 2\frac{\varepsilon}{2} = \varepsilon$$

QED.

The main importance of the concept of a Cauchy sequence is that the converse is also true:

Theorem (Cauchy Criterion; GC)
Any Cauchy sequence in $\mathbb{R}$ converges. EOT.

Proof. Let (a_n) be a Cauchy sequence. Then (a_n) is bounded. Indeed let $\varepsilon = 1$. Then there is n_0 such that for all $m, n > n_0$ we have $|a_m - a_n| < 1$. In particular, $a_{n_0+1} - 1 < a_n < a_{n_0+1} + 1$ for all $n > n_0$. Thus $|a_n| \leq \max\{|a_1|, |a_2|, \dots, |a_{n_0+1}| + 1\}$, which means (a_n) is bounded.

By BW, (a_n) contains a convergent subsequence (b_k), say: $b_k = a_{n(k)}$. Let L be the limit. We will show that $\lim a_n = L$. To this end, let $\varepsilon > 0$. Then there exists k_0 such that for all $k > k_0$ we have $|b_k - L| < \frac{\varepsilon}{2}$. Since (a_n) is a Cauchy sequence there is also n_0 such that for all $m, n > n_0$ we have $|a_n - a_m| < \frac{\varepsilon}{2}$.

Let now $n > n_0$. Then there is $k > k_0$ such that $n_k > n_0$. Then

$$|a_n - L| = |a_n - a_{n_k} + a_{n_k} - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

QED.

Remark
The Cauchy Criterion is not helpful in general for determining the limit of a sequence. EOR.

Exercise
Show that an Archimedean ordered field is complete, if and only if every Cauchy sequence converges. (We haven't technically defined the notion of convergence in an arbitrary ordered field. However, the definition is exactly as we stated it for $\mathbb{R}$. Since $\mathbb{Q}$ is contained in any ordered field, we have "enough" small $\varepsilon > 0$ to have a meaningful convergence.)
(*Hint*: First show that if S is a nonempty bounded above set in F, then there is a Cauchy sequence in S whose limit is an upper bound of S.) EOE.

3.9 Examples of limits

Absolute value
Suppose (a_n) is a sequence with limit L, then the sequence $(|a_n|)$ has limit $|L|$ (with the understanding $|\pm\infty| = \infty$).
For finite L this is also referred to as "*the absolute value function is continuous*" (meaning it takes limits to limits).
Let L be finite and $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n > n_0$. Then $\big||a_n| - |L|\big| \leq |a_n - L| < \varepsilon$, and $\lim|a_n| = |L|$ as needed.
If L is not finite, and $B > 0$, there is n_0 such that for all $n > n_0$, $a_n > B$ or $a_n < -B$ (depending on whether $L = \infty$ or $L = -\infty$). In either case $|a_n| > B$, which shows $\lim_{n\to\infty} |a_n| = \infty$.

Note the converse is not true. If $|a_n|$ has a limit, it does not follow that a_n does. Simple examples are given by $a_n = (-1)^n$ or $a_n = (-1)^n n$.

Zero sequences

Recall, a sequence is called a **zero sequence** if it converges to 0.

1. Consider $a_n = \frac{1}{n}$. Then $\lim a_n = 0$.
   Indeed, let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\varepsilon}$ (AF). Then for all $n > n_0$ we have $0 < \frac{1}{n} < \frac{1}{n_0} < \varepsilon$ (this uses some UFOs), and hence $\left|\frac{1}{n}\right| < \varepsilon$.
2. (a_n) is a zero sequence if and only if $(|a_n|)$ is.

Roots and powers

3. Let $x \in \mathbb{R}$ and let $a_n = x^n$. Then (a_n) converges to 0 if and only if $|x| < 1$. It converges to 1 if $x = 1$, diverges to ∞ if $x > 1$, and does not have a limit if $x \leq -1$.
   Note that by 2. above x^n is a zero sequence iff $|x|^n$ is. For the first part therefore, we may assume that $x \in [0,1)$. Then a_n is strictly monotone descending. It is bounded below by 0 so that $a_n \to L$ for some $0 \leq L < 1$. Now $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = L$. On the other hand, $a_{n+1} = xa_n$ has limit $L = xL$ by the Useful Limit Theorem. Now $L = xL$ means $(1-x)L = 0$. Since $x < 1$, this forces $L = 0$.
   If $x > 1$, by what we just saw, we have $\frac{1}{x^n} = \left(\frac{1}{x}\right)^n \to 0$, as $0 < \frac{1}{x} < 1$. Then for any $B > 0$, there is n_0, such that $\frac{1}{x^n} < \frac{1}{B}$ for all $n > n_0$, and therefore $x^n > B$. This shows $\lim x^n = \infty$.
   If $x \leq -1$, then x^n alternates between positive and negative numbers and $|x^n| \geq 1$. Therefore it cannot have a limit.
4. Let (a_n) be a sequence with proper limit L (or $L = +\infty$) and $p \in \mathbb{N}$. Then $\left(a_n^p\right)$ has limit L^p.
   This is a consequence of the Useful Limit Theorem and induction on p.
5. Let $a_n \geq 0$ be a sequence with proper limit $L \geq 0$. Let $p \in \mathbb{N}$. Then $\sqrt[p]{a_n} \to \sqrt[p]{L}$.
   Suppose first $L > 0$. Let $x_n = \sqrt[p]{a_n} - \sqrt[p]{L}$.
   By a homework exercise we have
   $$a^p - b^p = (a-b)\sum_{k=0}^{p-1} a^k b^{p-1-k}$$
   Substituting $a = \sqrt[p]{a}$ and $b = \sqrt[p]{L}$, we get
   $$a_n - L = \left(\sqrt[p]{a} - \sqrt[p]{L}\right)\sum_{k=0}^{p-1} \sqrt[p]{a}^k \sqrt[p]{L}^{p-1-k}$$
   We almost always have $\sqrt[p]{a} > \frac{\sqrt[p]{L}}{2}$. Thus
   $$|a_n - L| = \left|\sqrt[p]{a} - \sqrt[p]{L}\right| \sum_{k=0}^{p-1} \sqrt[p]{a}^k \sqrt[p]{L}^{p-1-k} > \frac{p}{2}\sqrt[p]{L}^{p-1}\left|\sqrt[p]{a} - \sqrt[p]{L}\right|$$
   where we used that $\sqrt[p]{L}^k \sqrt[p]{L}^{p-k} = \sqrt[p]{L}^{p-1}$.
   We get $|x_n| = \left|\sqrt[p]{a} - \sqrt[p]{L}\right| \leq \frac{2|a_n - L|}{p\sqrt[p]{L}^{p-1}}$. The right-hand side is a zero sequence by BTZ. By SP, $x_n \to 0$.

If $L = 0$, then $\sqrt[p]{a_n} \to 0$: for $\varepsilon > 0$, we almost always have $a_n < \varepsilon^p$. By some UFO, then almost always $\sqrt[p]{a_n} < \varepsilon$.

6. As a consequence of 5. $\sqrt[p]{\frac{1}{n^k}} \to 0$ for all $p, k \in \mathbb{N}$.
7. $\sqrt[n]{n} \to 1$. To see why, let $x_n = \sqrt[n]{n} - 1$. Note that $x_n \geq 0$ for all n. Then
$$n = (1 + x_n)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k = 1 + nx_n + \frac{n(n-1)}{2} x_n^2 + \cdots \geq 1 + \frac{n(n-1)}{2} x_n^2$$
Therefore
$$x_n^2 \leq \frac{2(n-1)}{n(n-1)} = \frac{2}{n}$$
and we conclude that $x_n \leq \sqrt{\frac{2}{n}}$.
Since $\frac{1}{\sqrt{n}} \to 0$ (see 6.), and therefore $\frac{\sqrt{2}}{\sqrt{n}} \to 0$, SP tells us that $x_n \to 0$.
8. If $a \geq 1$, then $\sqrt[n]{a} \to 1$.
9. If $0 < a < 1$, then still $\sqrt[n]{a} \to 1$. Indeed, by 8., $\sqrt[n]{\frac{1}{a}} \to 1$, and by ULT4, then $\sqrt[n]{a} = \frac{1}{\sqrt[n]{\frac{1}{a}}} \to 1$.

The Euler number e

Let $a_n = \left(1 + \frac{1}{n}\right)^n$. Then a_n converges. We have remarked that it is strictly monotone increasing:
$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$
To see this, we proceed by induction on n.
If $n = 1$, the statement is $(1 + 1)^1 < \left(1 + \frac{1}{2}\right)^2$, which is (barely) true.
Now suppose $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$ for a given natural number n.
We must show that also $\left(1 + \frac{1}{n+1}\right)^{n+1} < \left(1 + \frac{1}{n+2}\right)^{n+2}$.

All there is to show is that a_n is bounded. Now
$$a_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$
by the Binomial Theorem.
Now $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \leq \frac{n^k}{k!}$. Therefore $a_n \leq \sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}}$ since for $k > 0$, $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$.
But $q_n \coloneqq \sum_{k=1}^{n} \frac{1}{2^{k-1}} = \sum_{k=0}^{n-1} \frac{1}{2^k} \leq 2$ is bounded above. We conclude that (a_n) converges and that its limit is ≤ 3.
The limit of this sequence is called the **Euler number** and usually denoted by e.

Improper limits

10. Let $a_n = n$. Then $\lim a_n = \infty$.
    Let $B \in \mathbb{R}$. There is $n_0 \in \mathbb{N}$ with $n_0 > B$ (AF). But then $\forall n > n_0$ also $n > n_0 > B$.
11. Let $a > 1$, then $a_n = a^n$ has limit ∞. (See 3. above).
12. If $r > 0$ is a rational number, then $\lim n^r = \infty$.
    Write $r = \frac{p}{q}$ with $p, q \in \mathbb{N}$. Then $n^r = \sqrt[q]{n}^p$.
    $\sqrt[q]{n} \to \infty$ by 6.: It is monotone increasing, and $\frac{1}{\sqrt[q]{n}} \to 0$. It can therefore not be bounded.
    Then 4. says $\left(\sqrt[q]{n}\right)^p \to \infty$.

3.10 The Cauchy Limit Theorem

We discuss this theorem at this point mainly because of its clever proof.

Theorem (Cauchy Limit Theorem)

Let (a_n) be a convergent sequence with limit L. Then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = L$$

EOT.

Proof. Let $\varepsilon > 0$. There is $n_0 \in \mathbb{N}$ such that $|L - a_n| < \varepsilon'$ if $n > n_0$. For $n > n_0$ we then have

$$b_n := \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{a_1 + a_2 + \cdots + a_{n_0}}{n} + \frac{\left(a_{n_0+1} + a_{n_0+2} + \cdots + a_n\right)}{n}$$

Applying the triangle inequality (AV3) to

$$|L - b_n| = \left|\frac{(n - n_0)L - \left(a_{n_0+1} + a_{n_0+2} + \cdots + a_n\right)}{n} + \frac{n_0 L - (a_1 + a_2 + \cdots + a_{n_0})}{n}\right|$$

we get

$$|L - b_n| \le \left|\frac{(n - n_0)L - \left(a_{n_0+1} + a_{n_0+2} + \cdots + a_n\right)}{n}\right| + \left|\frac{n_0 L - (a_1 + a_2 + \cdots + a_{n_0})}{n}\right|$$

The right hand summand is bounded above as follows:

$$\left|\frac{n_0 L - (a_1 + a_2 + \cdots + a_{n_0})}{n}\right| = \left|\frac{(L - a_1) + (L - a_2) + \cdots + \left(L - a_{n_0}\right)}{n}\right| \le \frac{\varepsilon}{n}$$

The left hand summand is similarly bounded above by

$$\left|\frac{(n - n_0)L - \left(a_{n_0+1} + a_{n_0+2} + \cdots + a_n\right)}{n}\right| \le \frac{n - n_0}{n}\varepsilon$$

Note that this is true *for all* $n > n_0$.

Then $|L - b_n| \le \varepsilon\left(\frac{n-n_0+1}{n}\right)$. We may assume that $n_0 > 1$, then $n - n_0 + 1 < n$.

It follows $\lim b_n = L$. QED.

3.11 Quotient Criterion

In the following let (a_n) be any sequence such that almost always $a_n \neq 0$.

For such a sequence, it makes sense to discuss the sequence $\frac{a_{n+1}}{a_n}$ (which is defined almost always).

Propositon (Quotient Criterion; QC)
Let (a_n) be a sequence such that almost always $a_n \neq 0$.

1. If there is q such that $0 \leq q < 1$ and almost always $\left|\frac{a_{n+1}}{a_n}\right| \leq q$, then a_n is a zero sequence.
2. If there is $q > 1$ such that almost always $\left|\frac{a_{n+1}}{a_n}\right| \geq q$, then a_n is not convergent.

EOP.

Proof.

1. There is n_0 such that for all $n > n_0$, $a_n \neq 0$ and also $\left|\frac{a_{n+1}}{a_n}\right| \leq q$. Then for all $k \in \mathbb{N}$ we have
$$\frac{|a_{n_0+k}|}{|a_{n_0}|} = \frac{|a_{n_0+k}|}{|a_{n_0+k-1}|}\frac{|a_{n_0+k-1}|}{|a_{n_0+k-2}|}\cdots\frac{|a_{n_0+2}|}{|a_{n_0+1}|}\frac{|a_{n_0+1}|}{|a_{n_0}|} \leq q^k$$
Then $|a_{n_0+k}| \leq |a_{n_0}|q^k$. Since $0 \leq q < 1$ this means $\lim_{k\to\infty}|a_{n_0+k}| = 0$ by SP. But from this, we also have $\lim_{n\to\infty} a_n = 0$.
2. Similar reasoning as in 1. shows that there is n_0 such that for all $k \in \mathbb{N}$, we have
$$\frac{|a_{n_0+k}|}{|a_{n_0}|} \geq q\hat{}\ell$$
and therefore $|a_{n_0+k}| \geq |a_{n_0}|q^\ell$. Since $a_{n_0} \neq 0$, this means $|a_{n_0+k}| \to \infty$ for $k \to \infty$. As $(a_{n_0+k})_k$ is a subsequence of (a_n), a_n cannot converge.

QED.

Remark.
It is important that $q < 1$, or $q > 1$ in the above.
For example, assuming $a_n \neq 0$ for all n, if $\frac{|a_{n+1}|}{|a_n|} < 1$ for all n, no general statement about the convergence of a_n can be made (it just means the sequence $|a_n|$ is decreasing). Similarly, if $\frac{|a_{n+1}|}{|a_n|} > 1$ for all n, then $|a_n|$ is an increasing sequence.
If a_n is convergent, with a limit $L \neq 0$, then $\frac{a_{n+1}}{a_n} \to 1$ by ULT4 and the fact that a_{n+1} is a subsequence with necessarily the same limit. EOR.

Exercise

1. Show that for a sequence a_n, the statement $a_n \leq q$ for almost all n is equivalent to $\limsup a_n \leq q$.
2. Show that for a sequence a_n, the statement $a_n \geq q$ for almost all n is equivalent to $\liminf a_n \geq q$.
3. Conclude that if a_n is a sequence for which almost always $a_n \neq 0$, then a_n is a zero sequence if $\limsup \frac{|a_{n+1}|}{|a_n|} < 1$. Conclude a_n is divergent, if $\liminf \frac{|a_{n+1}|}{|a_n|} > 1$.

4. Prove that either statement in 3. is "sharp", that is, if the limit superior or limit inferior is equal to 1, the assertion is wrong in general.

EOE.

Examples

1. $a_n = \frac{n!}{n^n} \to 0$.
   $a_n > 0$ for all n, so we can ignore the absolute value. Then
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = (n+1) \frac{1}{n+1} \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n \to e^{-1}$$
   Indeed, $\left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e$, so by ULT4, $\left(\frac{n}{n+1}\right)^n \to e^{-1}$.
   Since the sequence $\left(1 + \frac{1}{n}\right)^n$ is monotone increasing, $e \geq \left(1 + \frac{1}{2}\right)^2 \geq 2$. Thus, almost always $\left(\frac{n}{n+1}\right)^n \leq q < 1$ with $q = \frac{3}{4}$.The Quotient Criterion then

3.12 Series

Occasionally a sequence is formed by forming successive sums. Consider $x \in \mathbb{R}$, then we can form the sequence

$$s_n = 1 + x + x^2 + \cdots + x^n$$

Such a sequence is called a series.

Definition
Let (a_n) be a sequence starting at 0 (or any other integer). A **series** is a sequence (S_n) of the form $S_n = a_0 + a_1 + \cdots + a_n$. The members S_n of the series are called its **partial sums**, and the elements of the original sequence (a_n) its **coefficients**. Occasionally, a series is simply written as $\sum_{k=0}^{\infty} a_k$.
A series may start at any integer not just 0. EOD.

Since a series is just a sequence, all notions of sequences (such as convergence, monotone increasing/decreasing, etc.) carry over unchanged.

No intrinsic meaning should be attributed to the symbol $\sum_{k=0}^{\infty} a_k$ other than a stand-in for the sequence of partial sums it represents. However, if the series *converges* or more generally has any limit L, say, we call L the **value** (or limit) of the series and write $\sum_{k=0}^{\infty} a_k = L$.

Cauchy Criterion
Since a series is a sequence of partial sums, it converges if and only if that sequence is a Cauchy sequence. A series $\sum_{n=0}^{\infty} a_n$ converges if and only if for each $\varepsilon > 0$, there is $N_0 \in \mathbb{N}$ such that for all $N \geq M > N_0$ we have $\left|\sum_{n=M+1}^{N} a_n\right| < \varepsilon$.

EOL.

Proof. Let $\sum_{n=0}^{\infty} a_n$ be a series. For any $N \in \mathbb{N}_0$ let $S_N = a_0 + a_1 + \cdots + a_N = \sum_{n=0}^{N} a_n$.

(S_N) is convergent if and only if for each $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$ such that for all $N, M > N_0$ we have

$$|S_N - S_M| < \varepsilon$$

Since the roles of N, M are interchangeable here, we may assume $N \geq M$, and the series converges if and only if for all $N \geq M$ we have

$$|S_N - S_M| = \left|\sum_{n=M+1}^{N} a_n\right| < \varepsilon$$

QED.

Corollary
If a series $\sum_{n=0}^{\infty} a_n$ is convergent, then a_n must be a zero sequence. EOC.

Proof. This is immediate from the Cauchy Criterion. If $\sum_{n=0}^{\infty} a_n$ converges, for every $\varepsilon > 0$, there exists N_0 such that for all $N \geq M > N_0$, $\left|\sum_{n=M+1}^{N} a_n\right| < \varepsilon$. This means that for all $N = M + 1 > N_0$ we have $|a_N| < \varepsilon$. Therefore, with $n_0 = N_0 + 1$, we conclude that $|a_n| < \varepsilon$ for all $n > n_0$. QED.

The converse is not true as the example of the harmonic series below (see Example 2) shows.

Note by the triangle inequality, for any sequence (a_n)

$$\left|\sum_{n=M+1}^{N} a_n\right| \leq \sum_{n=M+1}^{N} |a_n|$$

If the right hand side is $< \varepsilon$ (for a given $\varepsilon > 0$), the left hand side is as well, and therefore we can conclude the series $\sum_{n=0}^{\infty} a_n$ converges if the series $\sum_{n=0}^{\infty} |a_n|$ does. This fact is so important that it gets its own definition.

Definition
A series $\sum_{n=0}^{\infty} a_n$ is called **absolutely convergent**, if $\sum_{n=0}^{\infty}|a_n|$ converges. EOD.

By the remarks preceding the definition, any absolutely convergent series also converges.

The converse is generally false, as 2. and 3. in the example below show.

Remark
One reason why absolute convergence is important is the fact that the sequence (S_n) of partial sums of the series $\sum_{n=0}^{\infty} |a_n|$ is *monotone increasing*. Indeed, $S_{n+1} = S_n + |a_{n+1}| \geq S_n$. To show convergence of (S_n) it therefore suffices to show it is bounded (above). This is often easier than showing directly that $\sum_{n=0}^{\infty} a_n$ converges. EOR.

Convention
For a series $\sum_{n=0}^{\infty} a_n$ where all $a_n \geq 0$, we write $\sum_{n=0}^{\infty} a_n < \infty$ to indicate that it is bounded above and hence converges. EOC.

Example

1. The **geometric series** is the series $\sum_{k=0}^{\infty} x^k$ where $x \in \mathbb{R}$.
   If S_n denotes its partial sum up to n, then

$$S_n = \frac{1 - x^{n+1}}{1 - x}$$

If $|x| < 1$, (S_n) converges to $\frac{1}{1-x}$.
If $x \geq 1$, (S_n) diverges to ∞.
If $x \leq -1$, the series diverges but does not have a limit.

2. The **harmonic series** is the series $\sum_{k=1}^{\infty} \frac{1}{k}$. It diverges to ∞. To see this let $S_n = \sum_{k=1}^{n} \frac{1}{k}$, and let s_n be the subsequence $s_n = S_{2^n}$. S_n is strictly monotone increasing. It is therefore bounded (and convergent) if and only if s_n is.
   But $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^{n-1}+2^{n-1}} \geq s_{n-1} + \frac{2^{n-1}}{2^n} = s_{n-1} + \frac{1}{2}$. With $s_1 = \frac{3}{2}$, we conclude that $s_n \geq \frac{n+1}{2}$.
3. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges:
   We use the Cauchy Criterion: For any $N \geq M$,
   $$\sum_{n=M+1}^{N} \frac{(-1)^{n+1}}{n} = (-1)^M \left(\frac{1}{M+1} - \frac{1}{M+2} \pm \cdots + \frac{(-1)^{N+1-M}}{N} \right)$$
   If $N - M$ is even, this sum has an even number of summands (namely $N - M$), and they can be paired as
   $$\frac{1}{n} - \frac{1}{n+1} > 0$$
   If $N - M$ is odd, they can still be paired as such except the last summand which is then $\frac{(-1)^{N-M+1}}{N} = \frac{1}{N} > 0$.
   Together, it follows that $\left(\frac{1}{M+1} - \frac{1}{M+2} \pm \cdots + \frac{(-1)^{N+1-M}}{N} \right) > 0$ as long as $N \geq M + 1$.
   On the other hand, we can write
   $$\left(\frac{1}{M+1} - \frac{1}{M+2} \pm \cdots + \frac{(-1)^{N+1-M}}{N} \right) = \frac{1}{M+1} - \left(\frac{1}{M+2} - \frac{1}{M+3} \pm \cdots - \frac{(-1)^{N+1-M}}{N} \right)$$
   The same argument as just discussed shows that $\left(\frac{1}{M+2} - \frac{1}{M+3} \pm \cdots - \frac{(-1)^{N+1-M}}{N} \right)$ Is positive or at least nonnegative. Together
   $$\left| (-1)^M \left(\frac{1}{M+1} - \frac{1}{M+2} \pm \cdots + \frac{(-1)^{N+1-M}}{N} \right) \right| = \left(\frac{1}{M+1} - \frac{1}{M+2} \pm \cdots + \frac{(-1)^{N+1-M}}{N} \right) \geq 0$$
   And
   $$\left(\frac{1}{M+1} - \frac{1}{M+2} \pm \cdots + \frac{(-1)^{N+1-M}}{N} \right) < \frac{1}{M+1}$$
   This means for every $\varepsilon > 0$, there is $N_0 \in \mathbb{N}$ such that for all $N \geq M > N_0$, $\frac{1}{M+1} < \varepsilon$ and therefore $\left| ` \sum_{n=M+1}^{N} \frac{(-1)^{n+1}}{n} \right| < \varepsilon$.

EOE.

Remark
Series are not really special sequences. It is just a special way to talk about sequences. Any sequence can be written as a series, and any series is a sequence (the sequence of its partial sums).

Indeed, let (a_n) be any sequence. Let $b_n := a_n - a_{n-1}$ for $n > 0$, and $b_0 = a_0$. Then the sequence of partial sums of the series $\sum_{n=0}^{\infty} b_n$ is equal to the sequence (a_n). EOR.

QC may be used with series as follows:

Lemma (Quotient Criterion for series)
Let (a_n) be a sequence starting at 0 such that almost always $a_n \neq 0$.

1. If there is $0 \le q < 1$ such that almost always $\left|\frac{a_{n+1}}{a_n}\right| \le q$ then $\sum_{n=0}^{\infty} a_n$ converges absolutely.
2. If there is $q > 1$ such that almost always $\left|\frac{a_{n+1}}{a_n}\right| \ge q$, then $\sum_{n=0}^{\infty} a_n$ diverges.

EOL.

Proof.

1. As in QC one shows that there is k_0 such that for all $\ell \ge 0$ $|a_{k_0+\ell}| \le |a_{k_0}| q^{\ell}$.
   Then
$$\sum_{n=0}^{N} |a_n| \le |a_0| + |a_1| + \cdots + |a_{k_0-1}| + \sum_{\ell=0}^{N-k_0} |a_{k_0+\ell}|$$
$$\le |a_0| + |a_1| + \cdots + |a_{k_0-1}| + \sum_{\ell=0}^{N-k_0} |a_{k_0+\ell}|\, q\text{^}\ell$$
$$\le |a_0| + |a_1| + \cdots + |a_{k_0-1}| + \frac{|a_{k_0}|}{1-q}$$
   This (monotone) sequence of partial sums is therefore bounded and thus convergent.
2. In this case $|a_{k_0+\,\ell}| \ge |a_{k_0}| q^{\ell}$ for all $\ell \ge 0$ (and k_0 large enough). But then a_n is not a zero sequence.

QED.

Example
For $x \in \mathbb{R}$ consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. The series converges for all x. Let $E(x)$ denote its limit.
We obtain a function $E: \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto E(x)$.
This is called the exponential function, often also denoted $\exp x$. It is also common to write e^x for $\exp x$, where e is the Euler number (we will justify this later).
To see convergence, observe that
$$\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1} \to 0$$
So, for almost all n, $\frac{|a_{n+1}|}{|a_n|} < \frac{1}{2}$, for example, where $a_n = \frac{x^n}{n!}$.
This shows the series is absolutely convergent, and we obtain a well-defined function E. EOE.

3.13 *Excursion: Recursive definitions

We are finally in a position to formally discuss recursive definitions.

The general idea of recursive definitions is closely related to (complete) induction. If an object A_{n+1} can be unambiguously defined provided $A_1, A_2, \dots, A_n$ have been defined, and if $A_1$is specified, then this defines a unique sequence $A_1, A_2, \ldots$ of objects (one for each natural number). The objects in questions can be numbers, sets, or functions (which, mathematically speaking are not really distinct classes).

To make this precise we will assume that the objects we want to define are all elements of a given set X.

Theorem (Recursive Definition Theorem; RDT)
Let X be a nonempty set, and $f: X \to X$ be a function. For every $x_0 \in X$, there exists a unique function $a: \mathbb{N} \to X$ such that

1. $a(1) = x_0$.
2. $\forall n \in \mathbb{N}: a(n+1) = f(a(n))$

EOT.

To prove this theorem, we will need the following definition: for a natural number n, we put

$$D_n = \{ m \in \mathbb{N} \mid m \leq n \} = \{1, 2, \ldots, n\}$$

Note that the order on $\mathbb{N}$ can be defined in a non-recursive fashion, even if $\mathbb{N}$ is not defined a subset of $\mathbb{R}$. We have constructed the order (in principle) just from the Peano Axioms (see our discussion of G_n in 2.1.2).

Proof (of RDT). We will prove the following statement by induction:

For each $n \in \mathbb{N}$, there exists a unique function $a_n: D_n \to X$ such that

a. $a_1(1) = x_0$; and
b. for all $k < n$, $a_n(k+1) = f\big(a_n(k)\big)$.

The base case $n = 1$ is simple: we define $a_1(1) = x_1$ and are done. It is obviously also the only possible function satisfying the conditions a. and b.

Now suppose the statement holds for a specific n. On D_{n+1} we define a_{n+1} by $a_{n+1}(k) = a_n(k)$ if $k < n$ and $a_{n+1}(n+1) = f\big(a_n(n)\big)$. It is clear that a_{n+1} satisfies the conditions a. and b.

For uniqueness, let b be another function defined on D_{n+1} satisfying a. and b. Then b, restricted to D_n also satisfies a. and b. (with respect to n). Since a_n is unique, we must have that the restriction $b|_{D_n} = a_n$. But then b. says $b(n+1) = f\big(b(n)\big) = f\big(a(n)\big) = a_{n+1}(n+1)$, and so $b = a_{n+1}$.Thus, for each n there is a unique a_n.

To define $a: \mathbb{N} \to X$ we put $a(n) \coloneqq a_n(n)$. Then for each$n \in \mathbb{N}$, $a(n+1) = a_{n+1}(n+1) = f(a_n(n))$ because a_{n+1} satisfies a. and b. Thus a satisfies the conditions 1. and 2. required by the theorem.

Let b be any function satisfying 1. and 2. Then $b|_{D_n}$ satisfies a. and b. and therefore $b|_{D_n} = a_n$ by the uniqueness we proved above. But then $b(n) = a_n(n) = a(n)$ for all n. Hence $b = a$. QED.

For a complete and more general version of this theorem see van der Waerden.

What has that to do with recursive definitions? We will now provide a few examples that should illustrate the general idea. We usually write a_n instead of $a(n)$, and also write (a_n) to indicate that this is a sequence with values a_n.

1. $n!$
   Here, $X = \mathbb{N} \times \mathbb{N}$ and $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is defined as $f(n, m) \coloneqq (n + 1, (n + 1)m)$.
   Then for $x_0 = (1,1)$, we get a function $a: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that
   a. $a(1) = (1,1)$
   b. $a(n + 1) = f\big(a(n)\big)$

   If we write $a(n) = (a_1(n), a_2(n))$, then $f\big(a(n)\big) = (a_1(n) + 1, (a_1(n) + 1)a_2(n))$. A straight-forward induction shows that $a_1(n) = n$. $f\big(a(n)\big) = (n + 1, (n + 1)a_2(n))$.

   Therefore $a_2: \mathbb{N} \to \mathbb{N}$ satisfies $a_2(1) = 1$ and $a_2(n + 1) = (n + 1)a_2(n)$. This is precisely the recursive formula we gave for $n!$, so we can now properly define $n! \coloneqq a_2(n)$.

   Note, for any function $b: \mathbb{N} \to \mathbb{N}$ such that $b(1) = 1$ and $b(n + 1) = (n + 1)b(n)$, we have that $a'(n) \coloneqq (n, b(n))$ must coincide with $a(n)$ by the uniqueness assertion of the RDT. Therefore $b(n) = a_2(n)$ for all n.

2. a^n
   Suppose $a \in \mathbb{R}$. Let $X = \mathbb{R}$, and define $f: X \to X$ by $f(x) = ax$. Put $x_0 = a$. Then the resulting function $a: \mathbb{N} \to \mathbb{R}$ satisfies $a(1) = a$ and $a(n + 1) = a \cdot a(n)$.
3. We can define the *Fibonacci*[3] sequence (F_n) by
   a. $F_1 = F_2 = 1$.
   b. $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$

   Here, we could define $X = \mathbb{N} \times \mathbb{N}$, and $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ as $f(n, m) = (m, n + m)$, and put $x_0 = (1,1)$. The resulting function $a: \mathbb{N} \to X$ can be written as $a(n) = (a_1(n), a_2(n))$.

   Then $a(n + 1) = \big(a_1(n + 1), a_2(n + 1)\big) = f\big(a_1(n), a_2(n)\big) = (a_2(n), a_1(n) + a_2(n))$. In particular his means $a_2(n + 1) = a_1(n) + a_2(n)$, and $a_1(n + 1) = a_2(n)$.

4. In general, if X is a set, and we want to define a sequence $A_1, A_2, \ldots$ of elements in X recursively such that A_{n+1} is determined by $A_1, A_2, \ldots, A_n$ we define X' as the set of all objects of the form $\big(n, (B_1, B_2, \ldots, B_n)\big)$ where $n \in \mathbb{N}$, and $B_1, B_2, \ldots, B_n \in X$. The recursive definition then must be a function $f: X' \to X'$ such that $f(\big(n, (B_1, B_2, \ldots, B_n)\big) = (n + 1, (B'_1, B'_2, \ldots, B'_{n+1}))$ with the condition that $B'_1 = B_1, B'_2 = B_2, \ldots, B'_n = B_n$. Technically, this means that $f(\big(n, (B_1, B_2, \ldots, B_n)\big) = ((n + 1), (B_1, B_2, \ldots, B_n, g_n(B_1, B_2, \ldots, B_n))$, where $g_n: X^n \to X$ is a function. By RFT there is a sequence $a': \mathbb{N} \to X'$ such that $a'(n) = \big(n, (A_1, A_2, \ldots, A_n)\big)$ where $a'(n + 1) = (n + 1, \big(A_1, A_2, \ldots, A_n, g_n(A_1, A_2, \ldots, A_n)\big)$. Then the last component of $a'(n)$ is A_n. This may be confusing: Is it not obvious that if $A_1, A_2, \ldots, A_n \in X$ are given we can define A_{n+1} as $g_n(A_1, A_2, \ldots, A_n)$? Yes, of course. The problem is that we must do so in a set theoretically sound way. The set theory axioms (which we haven't defined precisely) are all admissible rules. "Obvious" is not strictly speaking a rule. Ultimately, the issue is that we make infinitely many definitions (one for each $n \in \mathbb{N}$) simultaneously, and dependent on each other. The above makes this rigorous: the "first" component of $a'(n)$ is well defined as n. Using this we know that

[3] Fibonacci (a.k.a. Leonardo Bonacci (Wikipedia)) (c. 1170 – c. 1240 – 1250)

$a'(n)$ has the form (n, A) where A is an n-tuple of elements in X, and therefore the n-th component of A is well-defined.

Virtually all cases of recursive definitions we ever encounter will be covered by Example 4.

We look at one more example of how to apply the RDT.

Suppose we are given n real numbers $a_1, a_2, \dots, a_n$. We want to define what

$$a_1 + a_2 + \cdots + a_n$$

should mean. Of course, we all know what it should mean: it means "add them up". But in which order? If $n = 4$, there are several ways to add 4 numbers:

$$(a_1 + a_2) + (a_3 + a_4), a_1 + \big(a_2 + (a_3 + a_4)\big), a_1 + \big((a_2 + a_3) + a_4\big),$$

$$\big((a_1 + a_2) + a_3\big) + a_4, \big(a_1 + (a_2 + a_3)\big) + a_4$$

You can imagine that the number of possibilities explodes with the number of summands. By the associative law of the addition, all these expressions result in the same number. But, each of these expressions represents a distinct way of computing the result. That they are all equal *does need* a proof.

Proposition (General Associativity; GA)
Let F be a field. Then there is a unique collection of functions $S_n: F^n \to F$ ($n \in \mathbb{N}$), such that

1. $S_1(x) = x$ for all $x \in F$.
2. For any $n > 2$ and any $1 \le k < n$, and all $a_1, a_2, \dots, a_n \in F$,
$$S_n(a_1, a_2, \dots, a_n) = S_k(a_1, a_2, \dots, a_k) + S_{n-k}(a_{k+1}, a_{k+2}, \dots, a_n)$$

EOP.

Proof. Let X be the set of functions $F^n \to F$ for some $n \in \mathbb{N}$, so $X = \{f: F^n \to F \mid n \in \mathbb{N}\}$. This is a well-defined nonempty set. Recall that $F^n = \{(a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in F\}$.

Let $T: X \to X$ be defined as follows: given $f: F^n \to F$, $T(f): F^{n+1} \to F$ is defined as $T(f)(a_1, a_2, \dots, a_{n+1}) = f(a_1, a_2, \dots, a_n) + a_{n+1}$.

By the RDT there exists a unique function $S: \mathbb{N} \to X$, such that $S(1)$ is the identity function on F (that is, $S(1)(x) = x$ for all x) and such that $S(n+1) = T(S(n))$ for all $n \in \mathbb{N}$.

Let $S_n = S(n)$. Then S_1 satisfies 1. Note that any collection S_n' of functions satisfying 1. and 2. must be equal to S_n by the RDT because 2. forces the collection to satisfy $S_{n+1}'(a_1, a_2, \dots, a_{n+1}) = S_n'(a_1, a_2, \dots, a_n) + a_{n+1} = T(S_n')(a_1, a_2, \dots, a_{n+1})$.

To show that S_n satisfies 2., we will proceed by induction. The base case $n = 1$ is trivial as nothing is to show.

Now suppose S_ℓ satisfies 2. for all $1 \le \ell \le n$. And let $1 \le k < n + 1$, and $a_1, a_2, \dots, a_{n+1} \in F$. If $k = n$, then $S_{n+1}(a_1, a_2, \dots, a_{n+1}) = S_n(a_1, a_2, \dots, a_n) + a_{n+1} = S_n(a_1, a_2, \dots, a_n) + S_1(a_{n+1})$ by definition. If $k < n$, then

$$
\begin{aligned}
S_{n+1}(a_1, a_2, \dots, a_{n+1}) = S_n(a_1, a_2, \dots, a_n) + a_{n+1} \\
&= \big(S_k(a_1, a_2, \dots, a_k) + S_{n-k}(a_{k+1}, a_{k+2}, \dots, a_n)\big) + a_{n+1} \\
&= S_k(a_1, a_2, \dots, a_k) + (S_{n-k}(a_{k+1}, a_{k+2}, \dots, a_n) + a_{n+1}) \\
&= S_k(a_1, a_2, \dots, a_k) + S_{n-k+1}(a_{k+1}, a_{k+2}, \dots, a_n, a_{n+1})
\end{aligned}
$$

QED.

There is an analogous result regarding multiplication, where we denote the corresponding function $F^n \to F$ by P_n (playing the role of S_n).

This allows us to completely ignore brackets when only addition or multiplication is concerned. Furthermore, it allows to define two shorthand notations[4]:

If $a_1, a_2, \dots, a_n$ are elements of F we define

$$\sum_{i=1}^{n} a_i \coloneqq a_1 + a_2 + \cdots + a_n = S_n(a_1, a_2, \dots, a_n)$$

and

$$\prod_{i=1}^{n} a_i = a_1 a_2 \cdots a_n = P_n(a_1, a_2, \dots, a_n)$$

The "starting" index is not important here. We can easily extend the definition to adding or multiplying numbers $a_{-5}, a_{-4}, \dots, a_n$ etc.

However, if the starting index k, say, is strictly greater than the "ending" index n, we define

$$\sum_{i=k}^{n} a_i = 0$$

and

$$\prod_{i=k}^{n} a_i = 1$$

(In this case, there is no collection of numbers required.)

3.14 *Excursion: The Axiom of Choice

The Axiom of Choice (AoC) is an axiom of set theory. The Zermelo-Fraenkel system of axioms of set theory, plus the Axiom of Choice is usually referred to as ZFC. It is the system of set theory that we are assuming.

In purely set theoretic terms, the Axiom of Choice can be phrased as

Axiom of Choice

$$\emptyset \notin X \Rightarrow \exists f: X \to \cup_{A \in X} A: f(A) \in A$$

[4] Of course, we have used them many times before. But now we have put their use on a rigorous foundation.

Such a function f is called a *choice function*: it "chooses" an element out of each element of X. For this to be possible, the elements of X must all be nonempty sets.

Note that $\bigcup_{A \in X} A = \{\, x \mid \exists A \in X : x \in A \}$. This set exists by other axioms of ZF. Also note, as mentioned before, in set theory all elements of a set are considered sets themselves, and therefore the statement $\emptyset \notin X$ simply says that every element of X is a nonempty set.

AoC has been (and occasionally still is) controversial in mathematics. The main reason of that is that it allows for non-constructive existence proofs.

For example, AoC implies that *every* set can be well-ordered. That is, every set allows a total o rder $\prec$ such that every nonempty element has a minimal element (with respect to $\prec$). We know the usual order $<$ satisfies that for $\mathbb{N}$. However, by AoC such an order also exists for e.g. $\mathbb{R}$. However, it is impossible, in general, to define such an order for an arbitrary set concretely and explicitly.

But AoC is also used in many more mundane mathematical applications.

Example
Let $x \in \mathbb{R}$. Then there exists a sequence of rational numbers $r_n \in \mathbb{Q}$ such that $\lim r_n = x$. One possible proof is the following: We know that $\mathbb{Q}$ is dense in $\mathbb{R}$ in the sense that every nonempty interval contains a rational number. Thus, for each $n \in \mathbb{N}$, there is $r_n \in \mathbb{Q}$ such that $r_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$. This does not rely on AoC. But now we claim, that there is therefore a sequence r_n. In other words, for each n we pick or ("choose") one such r_n. As there are infinitely many possible choices, and infinitely many n, we must make infinitely many such choices. That this is possible, is a consequence of AoC. Indeed, let X be the set of all such intervals: $X = \left\{ I \;\middle|\; I = I_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \mathbb{Q} \text{ for some } n \in \mathbb{N} \right\}$. This is a well defined set (it is a subset of the power set of $\mathbb{R}$). Every element of X is nonempty. Therefore, there is a choice function $R: X \to \bigcup_{n \in \mathbb{N}} I_n$ such that $R(I_n) \in I_n$. We now put $r_n = R(I_n)$, which no longer involves any choice.
The use of AoC can be avoided by specifying a concrete sequence r_n. For example, $r_n = \frac{[nx]}{n}$, where for any $z \in \mathbb{R}$, $[z] = \max\{\, n \in \mathbb{Z} \mid z \leq x \,\}$. EoE.

Example
Let $A \subset \mathbb{R}$ be bounded and nonempty. Suppose $S = \sup A$ is not an element of A. Then there exists a sequence $a_n \in A$ such that $a_n \to \sup A$.
We have seen this before. The proof we used is the following: for $n \in \mathbb{N}$, let $a_n \in A$ be any element such that $a_n > \sup A - \frac{1}{n}$. Such an element exists by the definition of $\sup A$. However, there may be infinitely many such elements. For each n, we may have infinitely many choices. So, if someone were to read your proof, they might say: "OK, I agree, if there is a sequence a_n such that $a_n > \sup A - \frac{1}{n}$, then $\lim a_n = \sup A$. But please tell me what a_5 is, or a_{99}, or a_n in general. And please don't say 'Some real number'. Please tell me which number."
Here, again, we use AoC: For each $n \in \mathbb{N}$, let $S_n = \left\{ a \in A \;\middle|\; a > \sup A - \frac{1}{n} \right\}$. Then we know that $S_n \neq \emptyset$. Let $X = \{\, S_n \mid n \in \mathbb{N} \}$, and let A be a choice function for X. Then we define $a_n \coloneqq A(S_n)$. Now your critic should be satisfied, because now we precisely define a_n as the value of a function. Of course, you might be in trouble, if they now ask, "OK, please tell me what $A(S_n)$ is for$n = 1,2,99, \ldots$." Your answer

then should be: "I don't have to. AoC says A exists." But you can see that some mathematicians might find that unsatisfactory. EoE.

Example

Another example hails from the realm of linear algebra (if you do not know what a vector space or a basis is, simply skip this example, and maybe return to it at a later stage).
There is a theorem that says that every vector space has a basis. By definition, a *basis* is a maximal linearly independent subset. It heavily relies on AoC; in fact, it is logically equivalent to AoC. Usually, the proof of this theorem relies on what is known as Zorn[5]'s Lemma. Zorn's Lemma is logically equivalent to AoC (provided the other axioms of ZF set theory). It is a bit complicated to state, and we avoid it here. EoE.

Note that the axiom of choice can also be modified as follows: Let Z be a set, where each element of Z is of the form (x, A) where x is an element of some set X, and A is a nonempty set (possibly depending on x), such that for each x there is a unique A such that $(x, A) \in Z$. Then there is a function $f: X \to \bigcup_{(x,A)\in Z} A$ such that $f(x) \in A$ where $(x, A) \in Z$. Indeed, let $S = \{ A \mid \exists x \in X, (x, A) \in Z\}$. This set exists by some axioms of set theory. Then S has a choice function F, and there is a function $g: X \to S$ such that $g(x) = A$, where A is the unique set such that $(x, A) \in Z$.

Sometimes AoC and RDT combine. There are cases where we want to define an object recursively but the object for the $n + 1$ case is not uniquely defined but involves a choice.

Example

One example is the following situation: $X \subset \mathbb{R}$ is an infinite set. Find an injective function $f: \mathbb{N} \to X$.
We would start with, "let $f(1) \in X$ be any element." And then "If $f(n)$ has been defined let $f(n + 1)$ be any element of X that is not equal to $f(1), f(2), \dots, f(n)$."
Recall that for a recursive definition the object for "$n + 1$" must be uniquely determined. This is not the case here, at least not a priori, so this approach does not work directly.

Let S be the set of distinct finite sequences: that is,

$$S = \{(x_1, x_2, \dots, x_n) \mid n \in \mathbb{N}, x_1, x_2, \dots, x_n \in X, x_i \neq x_j \forall i \neq j \leq n\}$$

A simple induction proof on n shows that for each n, there is at least one tuple of length n in S. Here we define the length of $(x_1, x_2, \dots, x_n)$ to be n. This is where we use that X is infinite.
Then let Z be the set of all elements of the form (s, A_s), where $s = (x_1, x_2, \dots, x_n) \in S$ and $A_s = \{x \in X \mid x \neq x_i \ \forall 1 \leq i \leq n\}$. Note that $s \mapsto A_s$ defines a function $S \to P(X)$, the power set of X. By AoC (and the remarks above) there is a function $f: S \to \bigcup_{(s,A_s)\in Z} A_s$ such that $f(s) \in A_s$.
Now define $F: S \to S$ by $F\big((x_1, x_2, \dots, x_n)\big) = \big(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)\big)$. This gives us a recursive definition, and for any $x_1 \in X$ by the RDT we get $a: \mathbb{N} \to S$ such that $a(1) = x_1 \in X$, and $a(n + 1) = F(a(n))$. We can then define $x_n \coloneqq a(n)_n$ (that is, the last component of $a(n)$). EOE.

Exercise (Challenge)

Suppose $A \subset \mathbb{R}$ is a nonempty set with $\sup A \notin A$. Show that there is a strictly monotone increasing sequence $a_n \in A$ such that $\lim a_n = \sup A$.

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[5] Max August Zorn (1906 – 1993)