MATH 371
Homework Assignment #1

Due date/time: Jan, 17, 2022/23:59

a. Write your solutions on paper or pads, and try to keep solutions for different questions on separate pages.

b. Upload scans/photos/pdfs of your solutions to **Assign2** before the due date and time. Make sure to upload solutions to the right slot for each question.

c. Submissions after the deadline will not be graded and will result in a 0 mark.

1. Find the general solution of ordinary differential equations:

   (a) (10 points) $\dfrac{dN}{dt} = rN \quad t > 0.$

   Solution: Using the method of separation of variables:

   $$\frac{dN}{dN} = rdt, \qquad \Longrightarrow \qquad \log|N(t)| = rt + c \qquad \Longrightarrow \qquad |N(t)| = e^c e^{rt},$$

   where c is an arbitrary constant and e^c is an arbitrary positive constant. This implies that

   $$N(t) = e^c e^{rt}, \quad N(t) = -e^c e^{rt}, \quad \text{and} \quad N(t) = 0$$

   are all the solutions to the DE, and this can be combined into a single family of solutions

   $$N(t) = Ce^{rt},$$

   where C is an arbitrary constants.

   Notes:

   (1) It can be checked that $C = N(0)$, the initial condition of $N(t)$.

   (2) It $N(t)$ represents population size (or density), then $N(t) \geq 0$ for all $t \geq 0$. In this case we will restrict $C \geq 0$ in the general solution.

   (3) If we write the solution as $N(t) = N(0)e^{rt}$, then $N(0) \geq 0$ implies $N(t) \geq 0$ for all $t \geq 0$. This implies that the population model $N' = rN$ is well defined.

   (b) (10 points) $u'' + 3u' + 2u = 0.$

   Solution: This is a second order linear DE of constant coefficients. We use the method of characteristic roots to find solutions of the form $u(t) = e^{\lambda t}$, and the characteristic roots λ satisfy the characteristic equation:

   $$r^2 + 3r + 2 = 0,$$

which has two real roots $\lambda_1 = -1$, $\lambda_2 = -2$, and we obtain two solutions

$$u_1(t) = e^{-t}, \quad u_2(t) = e^{-2t}.$$

Using the Wronskian test for linear independence:

$$W(u_1, u_2; t) = \begin{vmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t} \neq 0, \quad \text{for all } t \geq 0.$$

We know that $u_1 = e^{-t}$ and $u_2 = e^{-2t}$ are linearly independent, and they form a fundamental set of solutions (a basis for the vector space of solutions, which has dimension 2). The general solution is

$$u(t) = c_1 e^{-t} + c_2 e^{-2t}, \quad c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

2. Find the unique solution to the initial value problem
   (a) (10 points) $y'' + 4y = 2\sin 2t, \quad y(0) = y'(0) = 0.$

Solution: The general solution has the form (based on the superposition principle for linear systems)

$$y(t) = y_h(t) + y_p(t),$$

where $y_h(t)$ is the general solution for the homogeneous equation

$$y'' + 4y = 0$$

which has two solutions (by inspection) $y_1(t) = \cos 2t$, $y_2 = \sin 2t$, and the general solution is

$$y_h(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

(Note, the linear independence of y_1 and y_2 can be checked using the Wronskian as in the previous question.)

Also, $y_p(t)$ is a particular solution of the nonhomogeneous DE. We can use the method of undetermined coefficients to find y_p, using a trial solution of form

$$y_p(t) = t(A\cos(2t) + B\sin(2t)).$$

Substitute y_p into the DE, we can find the two constants $A = -1/2, B = 0$, and $y_p = -\frac{1}{2}t\cos(2t)$. Therefore, the general solution is

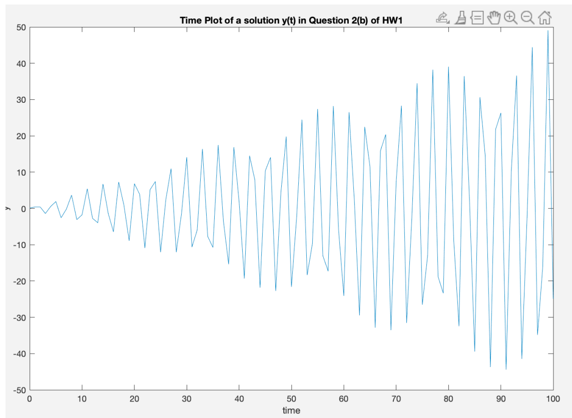
$$y = y_h + y_p = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{2}t\cos(2t).$$

To find the solution to the initial value problem (IVP), we apply the initial conditions $y(0) = 0$ and $y'(0)$ to the general solution $y(t)$ and its derivative $y'(t)$, respective, and we can determine the arbitrary constants $c_1 = 0$, $c_2 = 1/4$. The solution to the IVP is

$$y(t) = \frac{1}{4}\sin(2t) - \frac{1}{2}t\cos(2t).$$

(b) (10 points) Plot the trajectory of the solution in (a). (You can use any software package. Only submit your plot.)

Solution: Using the Matlab codes at the end of this question, we can produce the following simulation results. In the plot, we see that the solution is oscillatory, and the maximum amplitude of the oscillations grows without bound as t increases to ∞. This happens when the external forcing function $2\sin(2t$ has a *forcing frequency* 2, which matches the *natural frequency* of the free oscillation $y_h(t) = c_1 \cos(2t) + c_2 \sin(2t)$. This phenomenon is called *resonance*.



3. (20 points) Find the eigenvalues and eigenvectors of the following matrices.

(a) $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$, (d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution: Answers:

(a). $\lambda_1 = -2, \quad v_1 = (-1, 3)$, and $\lambda_2 = 2, \quad v_2 = (1, 1)$.

(b). $\lambda_1 = \lambda_2 = 2, \quad v_1 = (1, 0)$

(Note: In this case, we have a double root and the algebraic multiplicity of $\lambda = 2$ is 2. In contrast, the eigen-space of $\lambda = 2$ has dimension 1, and thus the geometric multiplicity is 1, smaller than the algebraic multiplicity. In such a case, the matrix is not *diagonalizable*, and we cannot obtain a basis for $\mathbb{R}^2$ that consists of eigenvectors of the matrix. Furthermore, we know that there is alway a basis of $\mathbb{R}^2$ that consists of *generalized eigenvectors*. You can find that a generalized eigenvector of $\lambda = 2$ is $u = (0, 1)$.)

(c). Two complex eigenvalues: $\lambda_1 = -2 + i, \quad v_1 = (-i, 1)$, and $\lambda_2 = -2 - i, \quad v_2 = (i, 1)$.

Note: We can see that when the two complex eigenvalues are complex conjugates, so are the two complex-valued eigevectors. This relation is true all real matrices. Using this relationship, we can choose to use the real and imaginary parts of the complex eigenvectors, $u_1 = (1, 0)$ and $u_2 = (0, 1)$ as a basis to get the Jordan canonical form. In the case of question (c), such a basis is the standard basis, and canonical form is the matrix itself. This tells us that If a 2×2 matrix A has a pair of complex eigenvalues $\lambda = a \pm bi$, then using the the real and imaginary parts of the complex eigenvectors as a basis, the canonical form is

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

4. (20 points) Exercises 1.4.1 (page 8) and 1.4.3 in Textbook.

Solution: Question 1.4.1:

(a). We can choose the time unit as one minute (could also be one day or one hour). Then probability of cell division is $p_1 = 2/10 = 1/5$.

(b) - (d). Let $N(t)$ be the number of cells at n minutes, and let Δt a small time interval. The balance equation is

$$N(t + \Delta t) = N(t) + p\Delta t N(t) = (1 + p\Delta t)N(t)$$

Let $\Delta t = \dfrac{1}{10}$ of a minute, and let $N(n) = N(t + n\Delta t)$, for $n = 1, 2, \cdots$. Then, the above equation becomes

$$N(n + 1) = (1 + \frac{1}{5}\frac{1}{10}N(t) = (1 + \frac{1}{50})N(t).$$

This is out discrete model, whose solution is

$$N(n) = N(0)(1 + 1/50)^n, \quad n = 1, 2, 3, \cdots.$$

To derive a continuous time model, we choose the time unit as one minute, then the growth rate α is

$$\alpha = \frac{p}{\text{time unit}} = p_1 = \frac{1}{5} \ \ 1/\text{day}.$$

From the balance equation

$$N(t + \Delta t) = N(t) + p_1\Delta t N(t),$$

we obtain

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = p_1 N(t).$$

Letting $t \to \infty$ and assume that $N(t)$ is differentiable for all t, we obtain the continuous time model:

$$N'(t) = p_1 N(t) = \frac{1}{5}N(t),$$

whose solution is

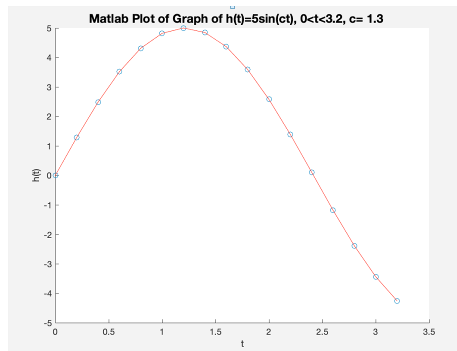
$$N(t) = N(0)e^{\frac{t}{5}}.$$

(e) There is no right and wrong solutions to this question. For instance, one may think discrete models are generally appropriate since life events such as birth or death occur at discrete time points. Data are also collected at discrete time points. When the population size is large, discrete time series can be well approximated by continuous curves, which can be described by a continuous time model.

5. (20 points) Computation exercises 8.1.1 (page 204) and 8.1.3 (page 211). Please submit your numerical output for these questions only.

Solution: Question 8.1.1. The questions can be done using the following Matlab codes, with plot shown to the right.

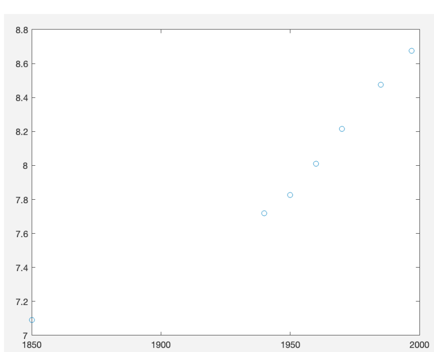
```
% HW1 Question 1.8.1

clear all;
close all;

c = 1.3;
x = 0:0.2:3.2;
y = 5*sin(c*x);

y(1), y(6), y(17)

hold on
plot(x,y,'o')
plot(x,y,'r')
hold off
```

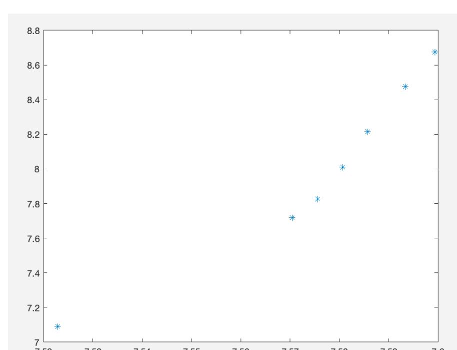


Question 8.1.3. The questions can be answered using the Matlab codes below, with outputs shown in figures.

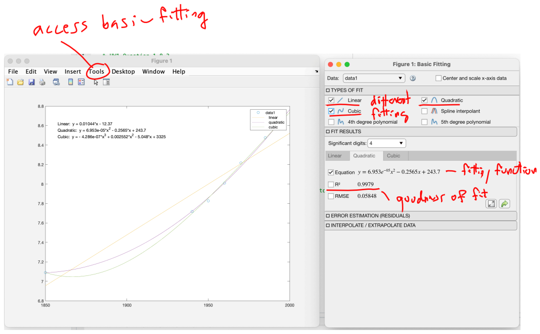
```
% HW1 Question 1.8.3

clear all;
close all;

% a. define a list for the data:
data = [1200,2249,2509,3010,3700,4800,5848.7];

% b. Plot data
linx = [1850, 1940, 1950, 1960, 1970, 1985, 1997];
%plot(year,data,'o');

% c. plot logdata in both linear scale x and log scale x
logdata = log(data);
logx = log(x);

plot(x,logdata,'o');
plot(logx, logdata,'*');

% d, e. Least-squres fit of logdata using Matlab's Basic Fitting tool directly
% from the plot panel (see figures for reqults and discussions)
```

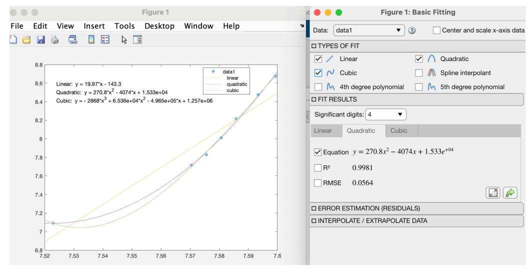
Output1: The plots of logdata in both linear scale and logarithmic scale of time. From the the figures, we see that the plots are quite similar.





Output2: Least-sqaures fit of logdata in both linear and logarithmic scales of time, using linear, quadratic and cubic polynomials:





We see that the fit using quadratic polynomials of both data plots look similar and have similar R^2 values, which measures the goodness of the fits. The R^2 value for the cubic fit is actually higher, but it gives a decline between year 1850 to 1900. If this decline really happened, we need to verify it by finding more data.

Further observations of the logdata show that the data points in the 1900s may fit a linear function better. This can be further explored. If verified, it may suggest that the population growth in the 1900s is closer to an exponential growth. The exponential growth rate can be estimated from the slope of the linear regression of the logdata, as we have done before.

Students can also use the fitting codes given in Section 1.8.3 of the Matlab Chapter of the textbook.