

MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Assignment #1

1. Let $+$ and \cdot be defined on $\{\spadesuit, \dagger, \circ, A\}$ through:

$+$	\spadesuit	\dagger	\circ	A
\spadesuit	\spadesuit	\dagger	\circ	A
\dagger	\dagger	\circ	A	\spadesuit
\circ	\circ	A	\spadesuit	\dagger
A	A	\spadesuit	\dagger	\circ

\cdot	\spadesuit	\dagger	\circ	A
\spadesuit	\spadesuit	\spadesuit	\spadesuit	\spadesuit
\dagger	\spadesuit	\dagger	\circ	A
\circ	\spadesuit	\circ	\spadesuit	\circ
A	\spadesuit	A	\circ	\dagger

Do these turn $\{\spadesuit, \dagger, \circ, A\}$ into a field?

2. Show that

$$\mathbb{Q}[i] := \{p + iq : p, q \in \mathbb{Q}\} \subset \mathbb{C}$$

with $+$ and \cdot inherited from \mathbb{C} , is a field. Is there a way to turn $\mathbb{Q}[i]$ into an ordered field?

(*Hint:* Many of the field axioms are true for $\mathbb{Q}[i]$ simply because they are true for \mathbb{C} ; in this case, just point it out and don't verify the axiom in detail.)

3. Let $\emptyset \neq S \subset \mathbb{R}$ be bounded below, and let $-S := \{-x : x \in S\}$. Show that:

- (a) $-S$ is bounded above;
- (b) S has an infimum, namely $\inf S = -\sup(-S)$.

4. Find $\sup S$ and $\inf S$ in \mathbb{R} for

$$S := \left\{ (-1)^n \left(1 - \frac{1}{n} \right) : n \in \mathbb{N} \right\}.$$

Justify, i.e., *prove*, your findings.

5. Let $S, T \subset \mathbb{R}$ be non-empty and bounded above. Show that

$$S + T := \{x + y : x \in S, y \in T\}$$

is also bounded above with

$$\sup(S + T) = \sup S + \sup T.$$

6*. An ordered field \mathbb{O} is said to have the *nested interval property* if $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ for each decreasing sequence $I_1 \supset I_2 \supset I_3 \supset \cdots$ of closed intervals in \mathbb{O} .

Show that an Archimedean ordered field with the nested interval property is complete.

Due Thursday, September 16, 2020, at 5:00 p.m.; no late assignments.

!!! IMPORTANT !!!

1. The completed assignment has to be placed into the marked mailbox on the third floor of CAB.
2. You are allowed to collaborate on homework assignments—in fact, I encourage you to do so. Still, every student must hand in their own homework assignment.
3. All problems have equal weight.
4. Problems marked with an * are bonus problems: they allow you to earn extra marks on an assignment. On this assignment, for instance, you can thus get a mark of 120%.