§ Recursion & Induction

Lecture 12

Open and Closed Forms:

Definition 1: A *closed form* expression is an expression which has a fixed number of operations.

Definition 2: An *open form* expression is an expression in which the number of operations grows depending on a variable in the expression.

Definition 3: The sum (in open form) of the first n natural numbers each to the power i is given by:

$$S_i = 1^i + 2^i + \cdots + n^i$$

Example 1: Find a closed form for:

$$S_1 = 1 + 2 + 3 + \cdots + n$$

To find the closed form, use the expansion:

$$(x+1)^2 = x^2 + 2x + 1$$

by plugging in $x = 1, x = 2, x = 3, \dots, x = n$.

Note: (Gauss in grade 1)

$$S_1 = 1 + 2 + \cdots + n$$

$$S_1 = n + (n-1) + \cdots + 1$$

$$2S_1 = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n} = n(n+1)$$

$$\Rightarrow S_1 = \frac{(n+1)n}{2}$$

$$x = 1: \quad (1+1)^2 = 1^2 + 2\cdot 1 + 1$$

$$x = 2: \quad (1+2)^2 = 2^2 + 2\cdot 2 + 1$$

$$\vdots \qquad\qquad \vdots$$

$$x = n: + \quad (1+n)^2 = n^2 + 2\cdot n + 1$$

$$(n+1)^2 + \cdots + 2^2 + 1^2 - 1^2 = S_2 + 2S_1 + n$$

$$\Rightarrow (n+1)^2 + \cancel{S_2} - 1 = \cancel{S_2} + 2S_1 + n$$

$$\Rightarrow 2\cdot S_1 = (n+1)^2 - 1 - n = n^2 + n = (n+1)n$$

$$\Rightarrow \boxed{S_1 = \frac{(n+1)n}{2}}$$

Example 2: Find a closed form for:

$$S_2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

To find the closed form, use S_1 and the expansion:

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

by plugging in $x = 1, x = 2, x = 3, \dots, x = n$.

$x = 1: \quad 2^3 = 1^3 + 3 \cdot 1^2 + 3 \cdot 1 + 1$

$x = 2: \quad 3^3 = 2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1$

$\vdots \qquad\qquad \vdots$

$x = n: \quad + \quad (n+1)^3 = n^3 + 3n^2 + 3 \cdot n + 1$

$$\underbrace{(n+1)^3 + n^3 + \cdots + 2^3 + 1^3}_{S_3} - 1^3 = S_3 + 3S_2 + 3S_1 + n$$

$$\Rightarrow (n+1)^3 + \cancel{S_3} - 1 = \cancel{S_3} + 3S_2 + 3S_1 + n$$

$$\Rightarrow 3S_2 = (n+1)^3 - 3S_1 - (n+1)$$

$$= (n+1)\left((n+1)^2 - \frac{3n}{2} - 1\right)$$

since: $S_1 = \frac{n(n+1)}{2}$

$$= (n+1)\left(n^2 + 2n \cancel{+1} - \frac{3n}{2} \cancel{-1}\right)$$

$$= \frac{(n+1)}{2}\left(2n^2 + 4n - 3n\right)$$

$$= \frac{(n+1)(2n+1)n}{2}$$

$$\Rightarrow \boxed{S_2 = \frac{(2n+1)(n+1)n}{6}}$$

Example 3: Find a closed form expression for

$$a_n = 1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1$$

valid for $n \geq 1$.

$$a_n = \sum_{i=1}^{n} i \cdot (n+1-i)$$

$$= \sum_{i=1}^{n} i\,(n+1) - i^2$$

$$= (n+1)\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2$$

$$= \frac{(n+1)^2 n}{2} - \frac{(2n+1)(n+1)n}{6}$$

Example 4: For $x \neq 1$ find a closed form for:

$$S = x^0 + x^1 + x^2 + \cdots + x^n$$

$$\Rightarrow - xS = x^1 + x^2 + \cdots + x^n + x^{n+1}$$

$$S - xS = x^0 - x^{n+1}$$

$$\Rightarrow S(1-x) = 1 - x^{n+1}$$

$$\Rightarrow S = \frac{1 - x^{n+1}}{1-x}$$

$$\Rightarrow \boxed{S = \frac{x^{n+1} - 1}{x - 1}}$$

(Note! if $0 < x < 1$ then:

$$S = \frac{1}{1-x} \quad \text{as} \quad n \to \infty$$)

Definition 4: The number of ways to chose k objects from a group of n objects is given by:

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

Definition 5: Pascal's Formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Example 5: Find a closed form for:

$$\sum_{i=3}^{n-1} \binom{i}{3}$$

$$= \sum_{i=3}^{n-1} \binom{i+1}{4} - \binom{i}{4} \qquad \text{since } \binom{n-1}{k-1} = \binom{n}{k} - \binom{n-1}{k}$$

$$= \binom{4}{4} + \binom{5}{4} + \cdots + \binom{n-1}{4} + \binom{n}{4}$$

$$- \underbrace{\binom{3}{4}}_{=0} - \binom{4}{4} - \binom{5}{4} - \cdots - \binom{n-1}{4}$$

$$= \binom{n}{4}$$

Example 6: Show that

$$a_n = \frac{1}{n+1} \cdot \binom{2n}{n}$$

is an integer for $n = 1, 2, 3 \ldots$

$$a_n = \frac{(2n)!}{(n+1)\, n!\, n!} = \frac{(2n)!}{(n+1)!\, n!} = \frac{(2n)!\,((n+1)-n)}{(n+1)!\, n!}$$

$$= \frac{(2n)!}{n!\, n!} - \frac{(2n)!}{(n+1)!\,(n-1)!} = \underbrace{\binom{2n}{n}}_{\in \mathbb{Z}} - \underbrace{\binom{2n}{n+1}}_{\in \mathbb{Z}} \in \mathbb{Z}$$