Math 127
Homework Problem Set 3

Problem 1. Let $\mathcal{S}$ be a ring that is not necessarily commutative (that is, multiplication in $\mathcal{S}$ may or may not be commutative). Let $x, y \in \mathcal{S}$, and suppose that both x and y are invertible. Write x^{-1} and y^{-1} for their respective inverses.

Show that xy is also invertible, and its inverse is a product of the inverses of x and y (which of the two products should you consider here?).

Remark. We said in class that the multiplication of matrices, whenever it makes sense, is not necessarily commutative. That is, we don't always have that the products AB and BA are equal, even when they are both defined.

By this, it shouldn't be understood that we always have $AB \neq BA$. There are 'trivial' nice examples we could consider here, but also examples such as the following: let $A, B \in \mathbb{R}^{3\times 3}$ be the following matrices:

$$A = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -5 & 3 \\ 0 & 1 & -5 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix};$$

then we do have

$$AB = \begin{pmatrix} -6 & -2 & 0 \\ 0 & 3 & -15 \\ 0 & -5 & 3 \end{pmatrix} = BA\,.$$

Terminology. Recall that in such a case we say that the matrices A and B **commute**.

On the other hand, the following problem asks you to essentially check that in general multiplication of matrices is not commutative: no matter what field $\mathbb{F}$ we work with, as long as $n \geqslant 2$, the set $\mathbb{F}^{n\times n}$, together with the addition and multiplication of matrices that we have defined, is a noncommutative ring.

Problem 2. (i) Let $\mathbb{F}$ be an arbitrary field. Give a general proof that multiplication of matrices in $\mathbb{F}^{2\times 2}$ is not commutative.

That is, give an example (that would work in any field $\mathbb{F}$) of two matrices $A, B \in \mathbb{F}^{2\times 2}$ such that $AB \neq BA$, and verify this.

(ii) Could you use (and suitably adjust) the example you gave in part (i) to show that multiplication in $\mathbb{F}^{n\times n}$ is not commutative for any $n \geqslant 3$ as well?

Problem 3. Prove the Distributive Law for matrices. That is, given a field $\mathbb{F}$ and positive integers m, n, k, l, prove that, for any matrices $A, B \in \mathbb{F}^{m\times n}$, $C \in \mathbb{F}^{n\times l}$ and $E \in \mathbb{F}^{k\times m}$, we have that

$$\big(A + B\big)C = AC + BC, \qquad E\big(A + B\big) = EA + EB.$$

[*Hint/Clarification.* Here you will have to rely on (some of) the axioms of the field to prove the desired conclusions.]

Problem 4. (i) Let $\mathbb{F}$ be a field, and let $U, U' \in \mathbb{F}^{n\times n}$ be two upper triangular matrices. Prove that $U + U'$ and UU' are also upper triangular.

(ii) Similarly, let $L, L' \in \mathbb{F}^{n\times n}$ be two lower triangular matrices. Prove that $L + L'$ and LL' are also lower triangular.

Problem 5. Give an example of an <u>inconsistent</u> and <u>underdetermined</u> system of linear equations in at least 3 unknowns, where none of the equations is a multiple of another equation in the system, and also all equations have at least two non-zero coefficients (you can choose to work over any field you want, but specify what field you chose, and also explain why your example has the required properties).

Problem 6. Assume that the following matrices are augmented matrices of certain systems of linear equations. In each case, use the matrix to determine the general form of the solutions: that is, to which $\mathbb{F}^n$ the solutions belong (namely what is $\mathbb{F}$ and what is n).

Furthermore, determine which of these matrices correspond to a staircase system (equivalently, which of these matrices are in row echelon form), and use each such matrix to find the size of the set of solutions of the system: that is, determine whether it is the empty set, or an infinite set, or a nonempty finite set (and if it is a nonempty finite set, determine its cardinality exactly). Justify your answers.

Do not try to find any of these solutions.

$$A_1 = \begin{pmatrix} 2 & -3.5 & 17 & 0 & 9 & 1 & 2 & 0 \\ 0 & 2 & 3 & -4 & 100 & 20 & 5 & 6 \\ 0 & 0 & 35 & 4 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 & 7 & -25 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 113 \end{pmatrix} \in \mathbb{R}^{5\times 8},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 2 & 0 & 9 \\ 0 & -11 & 8 & -4 & 0 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 4 & 12 \end{pmatrix} \in \mathbb{Z}_{13}^{4\times 5}, \qquad A_3 = \begin{pmatrix} 4 & 1 & 0 & 6 & 3 \\ 0 & 0 & -3 & 4 & 2 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \in \mathbb{Z}_7^{4\times 5},$$

$$A_4 = \begin{pmatrix} 3 & 1 & 4 & 0 & 5 & 0 \\ 0 & 2 & -3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_7^{4\times 6}, \qquad A_5 = \begin{pmatrix} 2 & -0.75 & 8 & 0 & 7 & 0 \\ 0 & 13 & 4 & -4 & 7 & 2 \\ 0 & 0 & -99 & 0.25 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 7 & -12 \end{pmatrix} \in \mathbb{Q}^{5\times 6}.$$

Problem 7. This problem has two parts involving a square matrix A. In each part, find $A^2 = A \cdot A$. Moreover, use Gaussian elimination to determine whether A^2 is invertible (you do not have to find its inverse).

[*Hint.* Does it suffice to determine whether A is invertible? See also Problem 1 of this homework set.]

(i) $$A = \begin{pmatrix} 1 & -1 & 1 \\ 11 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

(ii) $$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_5^{4\times 4}.$$