MATH 217 (Fall 2021)

Honors Advanced Calculus, I

Assignment #3

1. Let $S \subset \mathbb{R}^N$. Show that $x \in \mathbb{R}^N$ is a cluster point of S if and only if each neighbourhood of x contains an infinite number of points in S.

2. Let $S \subset \mathbb{R}^N$ be any set. Show that ∂S is closed.

3. Which of the sets below are compact?

   (a) $\{x \in \mathbb{R}^N : r \leq \|x\| \leq R\}$ with $0 < r < R$;

   (b) $\{x \in \mathbb{R}^N : r < \|x\| \leq R\}$ with $0 < r < R$;

   (c) $\overline{\left\{\left(t, \sin\frac{1}{t}\right) : t \in (0, 2021]\right\}}$;

   (d) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$;

   (e) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$.

   Justify your answers.

4. Show that:

   (a) if $U_1 \subset \mathbb{R}^N$ and $U_2 \subset \mathbb{R}^M$ are open, then so is $U_1 \times U_2 \subset \mathbb{R}^{N+M}$;

   (b) if $F_1 \subset \mathbb{R}^N$ and $F_2 \subset \mathbb{R}^M$ are closed, then so is $F_1 \times F_2 \subset \mathbb{R}^{N+M}$;

   (c) if $K_1 \subset \mathbb{R}^N$ and $K_2 \subset \mathbb{R}^M$ are compact, then so is $K_1 \times K_2 \subset \mathbb{R}^{N+M}$.

5. Show that a subset K of $\mathbb{R}^N$ is compact if and only if it has the *finite intersection property*, i.e., if $\{F_i : i \in \mathbb{I}\}$ is a family of closed sets in $\mathbb{R}^N$ such that $K \cap \bigcap_{i \in \mathbb{I}} F_i = \varnothing$, then there are $i_1, \ldots, i_n \in \mathbb{I}$ such that $K \cap F_{i_1} \cap \cdots \cap F_{i_n} = \varnothing$.

6*. For $j = 1, \ldots, N$, let $I_j = [a_j, b_j]$ with $a_j < b_j$, and let $I := I_1 \times \cdots \times I_N$. Determine ∂I. (*Hint*: Draw a sketch for $N = 2$ or $N = 3$.)

Due Thursday, September 30, 2020, at 5:00 p.m.; no late assignments.