Math 127
Homework Problem Set 4

Problem 1. Let $n > 1$ be a natural number. Consider the set $\mathbb{R}^n$ of all n-tuples $\bar{x} = (x_1, x_2, \ldots, x_n)$ all of whose components are real numbers.

Define the operations of addition and of multiplication of two n-tuples by doing them coordinate-wise. In other words,

$$\begin{aligned}
(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^n \quad &\mapsto \quad \bar{x} + \bar{y} \stackrel{\text{def}}{=} (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n, \\
(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^n \quad &\mapsto \quad \bar{x} \cdot \bar{y} \stackrel{\text{def}}{=} (x_1 y_1, x_2 y_2, \ldots, x_n y_n) \in \mathbb{R}^n.
\end{aligned}$$

Show that $\mathbb{R}^n$ with these operations is a commutative ring. Show also that it is **not** a field.

In mathematics, a *relation* $\mathcal{R}$ on a non-empty set S is a mathematical concept allowing us to "link" certain elements of S (possibly motivated by a certain interesting property we want to study, and which is supposed to be shared by "linked" elements, and not shared by elements that have not been "linked").

Formally, a (binary) relation $\mathcal{R}$ on S is a subset of $S \times S$, that is, it is a set of ordered pairs with components from S. For instance, $\mathcal{R}_1 = \{(m, n) : m, n \in \mathbb{N},\ m \text{ divides } n\}$ is a relation on $\mathbb{N}$, while $\mathcal{R}_2 = \{(0, 1), (1, 2), (2, 4), (3, 3)\}$ is a relation on $\mathbb{Z}_5$ (of course the latter relation may not be particularly meaningful).

Notation. If two elements a, b in S are "linked" in $\mathcal{R}$, the most common ways of denoting this are

$$(a, b) \in \mathcal{R} \qquad \text{or} \qquad a\,\mathcal{R}\,b \qquad \text{or} \qquad a \sim b.$$

There are some types of binary relations that can be really useful.

Definition. A binary relation $\mathcal{R}$ on a non-empty set S is called an *equivalence relation* if:

- for every $a \in S$, we have that $a \,\mathcal{R}\, a$ (we say $\mathcal{R}$ is *reflexive*);
- for every $a, b \in S$, if $a \,\mathcal{R}\, b$, then $b \,\mathcal{R}\, a$ as well (we say $\mathcal{R}$ is *symmetric*);
- for every $a, b, c \in S$, if $a \,\mathcal{R}\, b$ and $b \,\mathcal{R}\, c$, then $a \,\mathcal{R}\, c$ as well (we say $\mathcal{R}$ is *transitive*).

Side Remark. Not every interesting relation in Mathematics has to be an equivalence relation. For example, the relation $\mathcal{R}_1$ above, which is reflexive and transitive, but not symmetric, or the relation

$$\mathcal{R}_3 = \big\{(x, y) : x, y \in \mathbb{R}, x < y\big\} = \big\{(x, y) : x, y \in \mathbb{R}, y - x > 0\big\},$$

which only satisfies the transitive property out of the above three, but fully encodes the standard ordering in $\mathbb{R}$, offer us a lot of useful information.

Problem 2. (a) Let $\mathbb{F}$ be a field, and m, n be positive integers. Show that *row equivalence* of matrices in $\mathbb{F}^{m\times n}$ is an equivalence relation on $\mathbb{F}^{m\times n}$. Recall that we say that $A \in \mathbb{F}^{m\times n}$ is row equivalent to $B \in \mathbb{F}^{m\times n}$ (and we write $A \sim B$) if there are some $k \geqslant 1$ and some elementary matrices $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k \in \mathbb{F}^{m\times m}$ so that

$$B = \mathcal{E}_k \cdots \mathcal{E}_2 \mathcal{E}_1 A.$$

(b) Let $m > 1$ be a positive integer. Recall that congruence modulo m is a relation on $\mathbb{Z}$ defined as follows: $k \equiv l (\operatorname{mod} m)$ if and only if m divides $l - k$. Show that this is an equivalence relation.

Problem 3. Let k, λ, μ, u, v, w be unknown constants/parameters allowed to take values in $\mathbb{Z}_5$. Consider the following linear system with coefficients from $\mathbb{Z}_5$:

$$\left\{\begin{array}{ccccccccccccccc}
\kappa x_1 & + & \lambda x_2 & + & 3x_3 & - & 4x_4 & + & 3x_5 & + & x_6 & - & x_7 & = & 1 \\
 & & \mu x_2 & + & x_3 & & & + & 2x_5 & - & x_6 & + & 2x_7 & = & 2 \\
 & & & & 2x_3 & - & 4x_4 & + & x_5 & + & 4x_6 & & & = & 1 \\
 & & & & & & x_4 & - & 2x_5 & & & + & ux_7 & = & 0 \\
 & & & & & & & & x_5 & + & x_6 & + & x_7 & = & 3 \\
 & & & & & & & & vx_5 & & & + & 2x_7 & = & 0 \\
 & & & & & & & & & & & & (u^2-1)x_7 & = & w
\end{array}\right\}.$$

(a) For which combinations of $\kappa, \lambda, \mu, u, v$ and w, if any, do we get a staircase system?
(b) For which of the "good" combinations, which you are asked to find in part (a), does the corresponding (staircase) system have

(i) no solution?

(ii) a unique solution?

(iii) or more than one solutions? (For each of the combinations in this case determine also how many solutions the corresponding system has.)

[*Clarification.* In part (b) here, you don't have to find any solutions to the different systems you will consider, just how many solutions each system has.]

Problem 4. Let $k \in \mathbb{Z}_{17}$ be some (unknown) constant. Consider the system

$$\left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 4 \\ 3x_1 & - & x_2 & + & 5x_3 & = & -2 \\ 2x_1 & - & 3x_2 & + & (k^2-8)x_3 & = & k-2 \end{array} \right\}$$

where the coefficients come from $\mathbb{Z}_{17}$.
(a) For which value(s) of k, if any, does the system have

(i) a unique solution?

(ii) no solution?

(iii) more than one solutions? (And specify how many solutions the system has in each subcase here.)

(b) Solve the system where possible. That is, for those k in case (i) above, find the unique solution $(x_1, x_2, x_3) \in \mathbb{Z}_{17}^3$ (this solution will depend on k), and for each of the k in case (iii) above, parametrise the solutions (x_1, x_2, x_3) appropriately.

[*Clarification.* In part (b), when you try to find the unique solution of the system for each value k belonging to case (i) above, do not plug in different values of k from (i), but rather try to find a formula for the unique solution in terms of k.]

Problem 5. (a) Let $n > 1$ and $\mathbb{F}$ a field. Show that any two diagonal matrices in $\mathbb{F}^{n\times n}$ commute.

(b) <u>TRUE OR FALSE?</u> Decide whether the following statement is correct or not, and give a proof of it or of its negation accordingly.

If D, E are matrices in $\mathbb{F}^{n\times n}$ and D is diagonal, then D and E commute.

The following problem suggests two more criteria for determining when a square matrix is invertible (compare these with Theorem 2 from Lectures 28-35).

Problem 6. (a) Let $\mathbb{F}$ be a field, and suppose that A is a square matrix in $\mathbb{F}^{n\times n}$ with the property that, for <u>at least one</u> $\bar{b} \in \mathbb{F}^n$, the vector equation (or equivalently, the system of linear equations) $A\bar{x} = \bar{b}$ has a unique solution.

Prove that A is invertible.

(b) Suppose that B is a square matrix in $\mathbb{F}^{n\times n}$ with the property that, <u>for every</u> $\bar{c} \in \mathbb{F}^n$, the vector equation $B\bar{y} = \bar{c}$ is consistent (that is, it has at least one solution).

Prove that, for every $\bar{c}$, the vector equation $B\bar{y} = \bar{c}$ has a *unique* solution. (Note that by part (a) this also implies that B is invertible.)