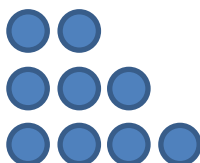


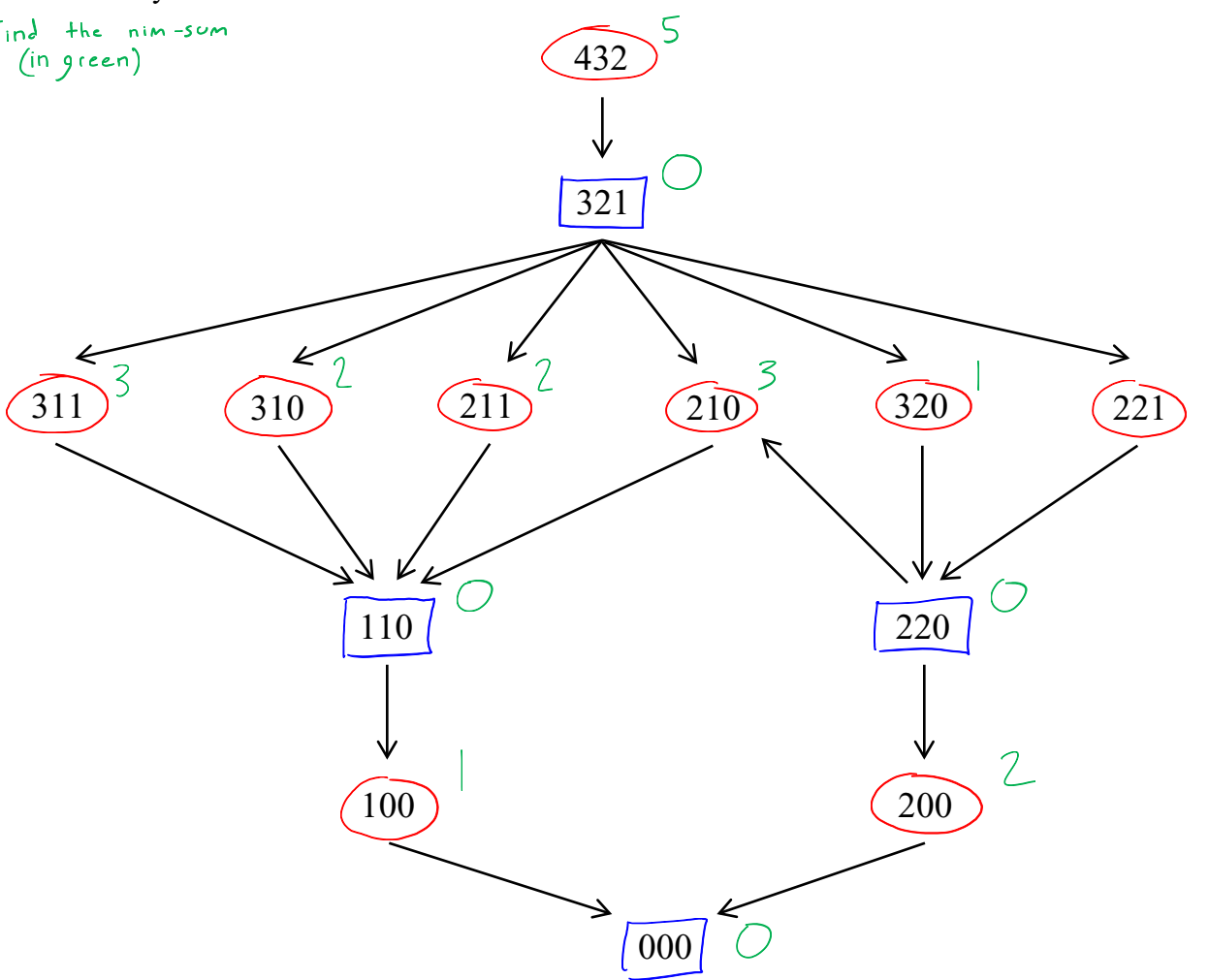
## Nim Games

How to play: Place 9 counters as shown. Players take turns removing counters, provided they are all in the same (horizontal) row. The player that picks up the last counter wins the game. State the player that can always win.



Represent a state of this game by three numbers. Each of the three numbers will correspond to one of the three rows. Each number will track how many counters remain in its corresponding row. By symmetry, write the larger numbers on the left. This means the game starts in the state 432. With this representation the following portion of the state diagram will reveal the player that can always win.

Find the nim-sum  
(in green)



**Remark:** When analyzing a Nim game with a small number of rows and counters, the partial state diagram is a reasonable way of determining the winner. However, for a large number of rows and counters, a more sophisticated method is needed. For this reason we introduce the following definition.

**Definition 1:** Given a list of whole numbers, the *nim-sum* is found by representing the positive numbers as sums of distinct powers of 2, and then canceling all pairs of equal powers, and finally adding what is left. The nim-sum of a list of whole numbers  $a_1, a_2, \dots, a_n$  is denoted by:

$$a_1 \oplus a_2 \oplus \dots \oplus a_n$$

**Example 1:** Find the following nim-sums.

a)  $2 \oplus 3$

$$= \cancel{2} \oplus (\cancel{2} + 1) = \boxed{1}$$

b)  $2 \oplus 3 \oplus 4$

$$\cancel{2} \oplus (\cancel{2} + 1) \oplus 4 = \boxed{5}$$

c)  $15 \oplus 15$

$$\begin{aligned} & (\cancel{1} + \cancel{2} + \cancel{4} + \cancel{8}) \\ & \oplus (\cancel{1} + \cancel{2} + \cancel{4} + \cancel{8}) = \boxed{0} \end{aligned}$$

d)  $25 \oplus 31$

$$\begin{aligned} & (\cancel{16} + \cancel{8} + \cancel{1}) \oplus (\cancel{1} + \cancel{2} + \cancel{4} + \cancel{8} + \cancel{16}) \\ & = \boxed{6} \end{aligned}$$

e)  $4 \oplus 7 \oplus 12$

$$4 \oplus (\cancel{1} + \cancel{2} + \cancel{4}) \oplus (\cancel{4} + \cancel{8}) = \boxed{15}$$

f)  $4 \oplus 7 \oplus \cancel{12} \oplus \cancel{7} \oplus \cancel{12} \oplus \cancel{7}$

$$\cancel{4} \oplus (\cancel{1} + \cancel{2} + \cancel{4}) = \boxed{3}$$

**Lemma 1:** Given whole numbers  $a, b, c$  the following nim-sum properties hold:

1.  $a \oplus b$  is a whole number
2.  $a \oplus b = b \oplus a$
3.  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
4.  $a \oplus a = 0$
5.  $a \oplus b \neq 0$  whenever  $a \neq b$

**Note:** Nim is also played in a more general fashion than the version at the beginning of this lecture. Start by picking the number of rows to play with; any amount is fine. In each row place a random amount of counters. Players take turns removing counters provided they are all in the same row. The player that picks up the last counter wins the game.

**Definition 2:** Similar to the 4, 3, 2 Nim game (at the beginning of this lecture) represent a state of the general Nim game by a list of numbers, one number for each row. Each number will track how many counters remain in its corresponding row. Calculating the nim-sum of these numbers gives the *nim-sum* of the game.

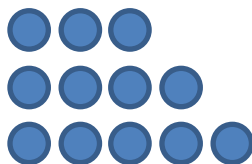
**Lemma 2:** Every move in the game of Nim will change its nim-sum.

*Proof.* Consider a game of Nim in the state:  $a_1, a_2, \dots, a_n$  where  $s = a_1 \oplus a_2 \oplus \dots \oplus a_n$  and suppose decreasing  $a_1$  to  $a'_1$  does not change the nim-sum. Then by lemma 1 we have:

$$0 = s \oplus s = (a_1 \oplus a_2 \oplus \dots \oplus a_n) \oplus (a'_1 \oplus a_2 \oplus \dots \oplus a_n) = a_1 \oplus a'_1 \neq 0$$

which is a contradiction. Therefore the nim-sum must change.

**Example 2:** Consider the following game of Nim:



a) Using definition 2 represent the state this game of Nim starts in.

5 4 3

b) Using definition 2 find the nim-sum of the game.

$$\begin{aligned} & 5 \oplus 4 \oplus 3 \\ &= (\cancel{1+4}) \oplus \cancel{4} \oplus (\cancel{2+1}) \\ &= \boxed{2} \end{aligned}$$

c) Make any move you like and notice that the nim-sum of the game changes.

Take 2 from row 2

$$\begin{aligned} & 5 \oplus 2 \oplus 3 \\ &= (\cancel{1+4}) \oplus \cancel{2} \oplus (\cancel{2+1}) \\ &= 4 \neq 2 \quad (\text{an example showing the nim-sum has changed}) \end{aligned}$$

**Lemma 3:** There is always a move that changes a Nim game with a non-zero nim-sum to a zero nim-sum.

*Proof.* Consider a game of Nim in the state:  $a_1, a_2, \dots, a_n$  where  $s = a_1 \oplus a_2 \oplus \dots \oplus a_n \neq 0$ . Since  $s$  is not zero it can be written as a (non-empty) sum of distinct powers of 2. One of the powers appearing in that sum will be the largest, let it be  $2^k$ . Since  $2^k$  did not cancel when calculating  $a_1 \oplus a_2 \oplus \dots \oplus a_n$  it must have appeared an odd number of times in the sums of distinct powers of 2 of  $a_1, a_2, \dots, a_n$ . This means we can pick  $a_i$  so that  $2^k$  appears in its sum of distinct powers of 2. The move we want to make is to decrease  $a_i$  to  $a_i \oplus s$  but first we must verify  $a_i \oplus s < a_i$  to show this is a valid move:

The powers larger than  $2^k$  do not cancel

The powers smaller than  $2^k$  might cancel

$$\begin{aligned}
 a_i \oplus s &= (2^{l_a} + \dots + 2^{l_1} + \cancel{2^k} + 2^{s_b} + \dots + 2^{s_1}) \oplus (\cancel{2^k} + 2^{t_c} + \dots + 2^{t_1}) \\
 &\leq 2^{l_a} + \dots + 2^{l_1} + 2^{k-1} + 2^{k-2} + \dots + 2^0 \\
 &= 2^{l_a} + \dots + 2^{l_1} + 2^k - 1 \\
 &< 2^{l_a} + \dots + 2^{l_1} + 2^k \\
 &\leq 2^{l_a} + \dots + 2^{l_1} + 2^k + 2^{s_b} + \dots + 2^{s_1} \\
 &= a_i
 \end{aligned}$$

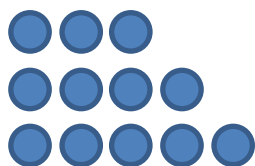
Finally, making the move to decrease  $a_i$  to  $a_i \oplus s$  changes the nim-sum from:

$$a_1 \oplus a_2 \oplus \dots \oplus a_i \oplus \dots \oplus a_n \neq 0$$

to

$$\begin{aligned}
 &a_1 \oplus a_2 \oplus \dots \oplus (a_i \oplus s) \oplus \dots \oplus a_n \\
 &= (a_1 \oplus a_2 \oplus \dots \oplus a_i \oplus \dots \oplus a_n) \oplus s \\
 &= s \oplus s \\
 &= 0
 \end{aligned}$$

**Example 3:** In the following Nim game the nim-sum is not zero. By lemma 3, there should be a move that changes the nim-sum to zero. Find such a move.



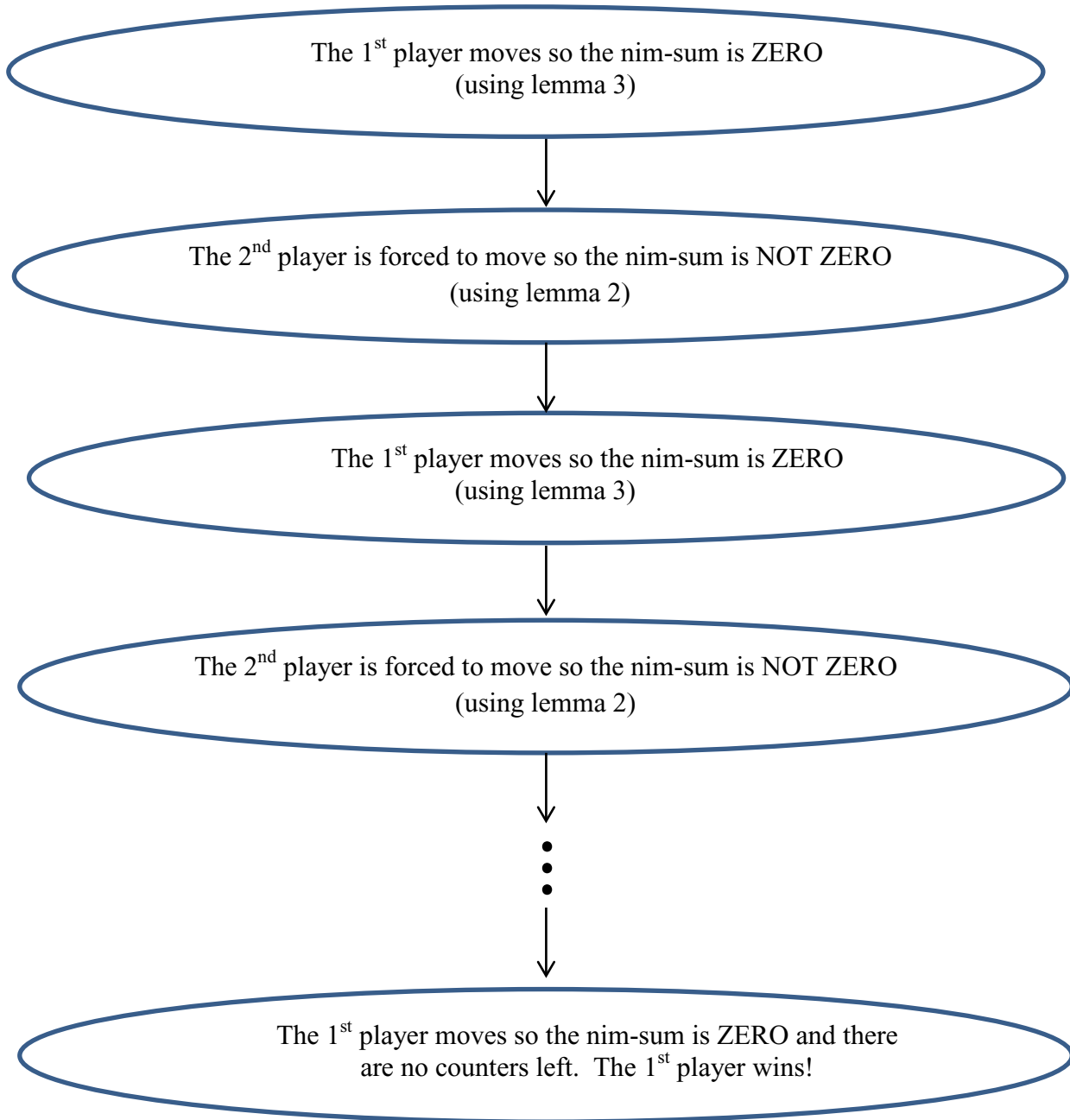
Take this

~~(4+1)~~ ⊕ ~~4~~ ⊕ ~~(1+2)~~

Take 2 from row 1

**Theorem 1:** A Nim game is in a hot position if and only if the nim-sum is non-zero.

*Proof.* Suppose the game starts with a non-zero nim-sum, the first player can use the following winning strategy:

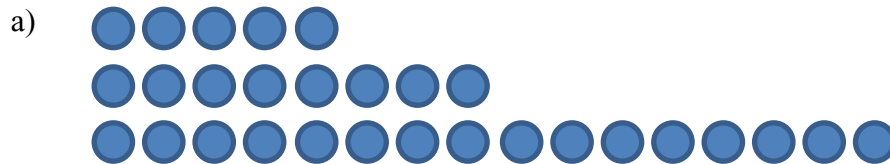


**Note:** the 2<sup>nd</sup> player can steal this strategy if the game starts with a nim-sum of zero.

**Corollary 1:** To change a non-zero nim-sum to zero follow these steps.

1. Calculate the nim-sum  $s = 2^k + 2^{t_c} + \dots + 2^{t_1}$ . ( $2^k$  is the largest power of 2)
2. Find row  $i$  with  $a_i = 2^{l_a} + \dots + 2^{l_1} + 2^k + 2^{s_b} + \dots + 2^{s_1}$  counters. ( $2^k$  appears)
3. Leave  $a_i \oplus s$  counters in row  $i$ .

**Example 4:** In the following Nim games, who can always win, the 1st player or the 2nd player? If the first player can always win, state the first move that he or she needs to make.



1. Find the nim-sum:

$$s = 5 \oplus 8 \oplus 16 = (1+4) \oplus 8 \oplus 16 = 29$$

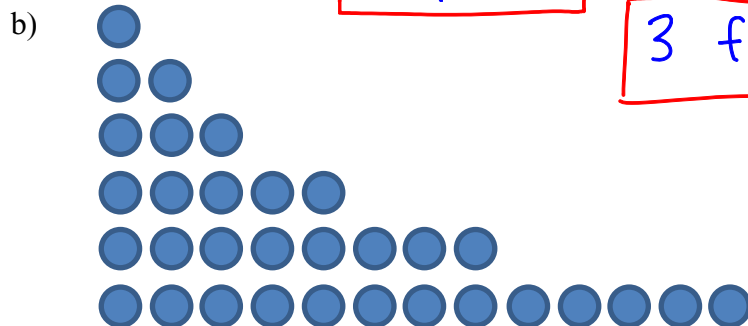
Note the largest power of 2 in the # 29 is 16

2. Find a row with 16 in it.

16 is in row 3. ( $i=3$ )

3. Find  $a_3 \oplus s = 16 \oplus 29 = 13$

∴ The 1<sup>st</sup> player can always win by taking 3 from row 3.



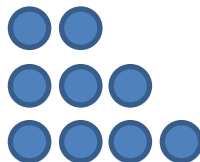
$$1 \oplus 2 \oplus 3 \oplus 5 \oplus 8 \oplus 13$$

$$= \cancel{1} \oplus \cancel{2} \oplus (\cancel{2} + \cancel{1}) \oplus (\cancel{4} + \cancel{1}) \oplus \cancel{8} \oplus (\cancel{8} + \cancel{4} + \cancel{1})$$

$= 0$  ∴ The 2<sup>nd</sup> player can always win

## Classic Nim

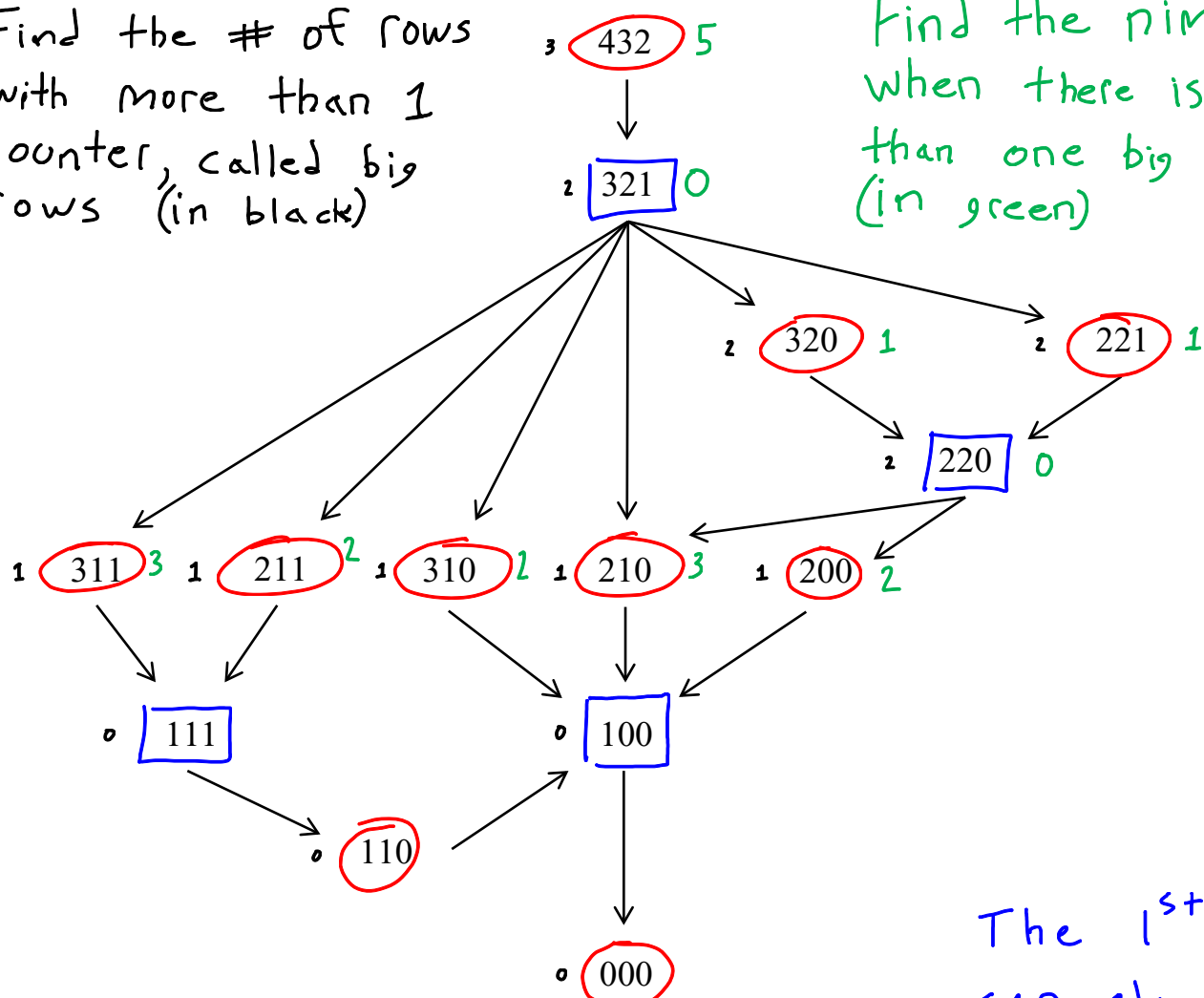
How to play: Place 9 counters as shown. Players take turns removing counters provided they are all in the same row. The player who is forced to pick up the last counter loses.



Use the same representation that was used for Nim and use the following portion of the state diagram to help find the player that can always win.

Find the # of rows with more than 1 counter, called big rows (in black)

Find the nim-sum when there is more than one big row (in green)



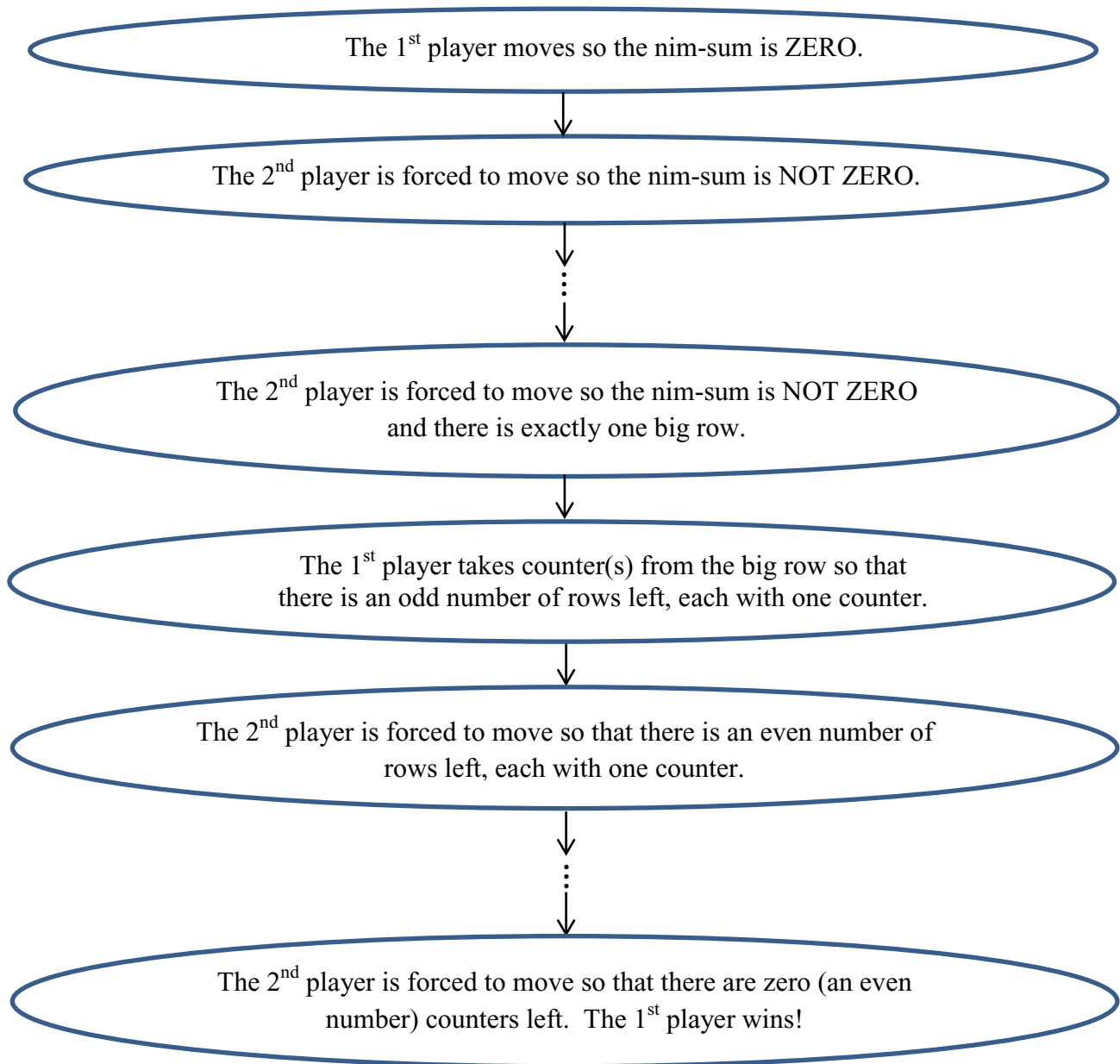
**Note:** Playing Nim under the rules of Classic Nim is called misère play. Classic Nim can also be played in a more general fashion than the version above. It can be set up in all the same ways as the general version of Nim. The two versions are played in the same way, the only difference is, in Classic Nim, the player forced to pick up the last counter loses.

**Remark:** Amazingly, the same opening move in both 4, 3, 2 Nim and 4, 3, 2 Classic Nim is successful. To that end, the winning strategy for the general version of Classic Nim is very similar to the winning strategy for the general version of Nim.

**Definition 3:** In the game of Classic Nim a *big row* is a row that has more than one counter.

**Theorem 2:** Suppose a Classic Nim game has two or more big rows. The game is in a hot position if and only if the nim-sum is non-zero.

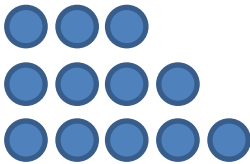
*Proof.* If the game starts with a non-zero nim-sum, the first player can use the following winning strategy. This strategy works the same as the one in theorem 1 until there is exactly one big row.

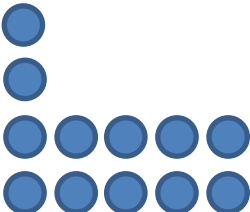


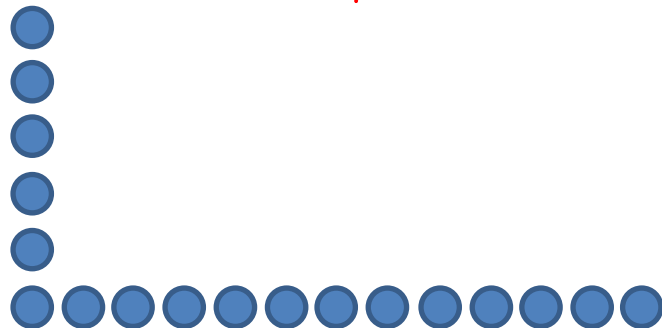
Note: the 2<sup>nd</sup> player can steal this strategy if the game starts with a nim-sum of zero.



**Example 5:** In the following Classic Nim games, who can always win, the 1<sup>st</sup> player or the 2<sup>nd</sup> player? If the 1<sup>st</sup> player can always win, state the first move that he or she needs to make.

- a)  Since there are 3 big rows the strategy is the same as nim. From example 3 the 1<sup>st</sup> player can always win by taking 2 from row 1

- b)  Since there are 2 big rows the strategy is the same as nim. Find the nim-sum:  
 $1 \oplus 1 \oplus 5 \oplus 5 = 0$   
 ∴ the 2<sup>nd</sup> player can always win

- c)  There is one big row, therefore the 1<sup>st</sup> player leaves an odd # of counter

The 1<sup>st</sup> player can always win by taking 13 counters from row 6

## Circle Nim

This is a two-player game. How to play: To begin, place a marker in each circle. Players take turns removing either one or two markers from the circles. If a player removes two markers, they must be beside each other, with no empty circles between them. The player that removes the last marker(s) wins.

State the player that can always win and describe a winning strategy.

