HOMEWORK ASSIGNMENTS FOR MATH 227
WINTER 2021

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1. Due Tuesday, January 26

Problem 1.1. Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is linear,
$T\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$, and $T\begin{pmatrix}0\\0\\3\end{pmatrix} = \begin{pmatrix}-1\\5\end{pmatrix}$. Compute $T\begin{pmatrix}3\\3\\9\end{pmatrix}$.

Problem 1.2. Find the matrix of the given linear transformation:

(a) $S\colon \mathbb{R}^2 \to \mathbb{R}^2$, the reflection with respect to the straight line $y = 2x$;

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, the rotation around the y-axis by $\frac{\pi}{2}$ followed by the rotation around the z-axis by α.

Problem 1.3. Let $S, T\colon \mathbb{R}^6 \to \mathbb{R}^6$ be the linear transformations defined by

$$S\colon \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{pmatrix} \mapsto \begin{pmatrix}0\\x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} \quad \text{and} \quad T\colon \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{pmatrix} \mapsto \begin{pmatrix}x_6\\x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}.$$

(a) Describe the action of S and T on the standard unit vectors.

(b) Find the matrix representations of S and T.

(c) Describe powers of these matrices: S^n and T^n for $n \in \mathbb{N}$.

(S is called the "right shift", while T is the "cyclic right shift".)

Problem 1.4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1 + 4x_2 - 3x_3\\-2x_1 - 7x_2 + 6x_3\\x_1 + 7x_2 - 2x_3\end{pmatrix}$. Determine whether T is linear, onto, one-to-one, and/or bijection. If T is invertible, find the inverse operator.

Problem 1.5. Let A and B be two matrices. *True or false:*

(a) If AB is invertible then both A and B are invertible;

(b) If both A and B are invertible then AB is invertible;

(c) If both A and B are invertible then $A + B$ is invertible.

Problem 1.6. Let A be a 5×3 matrix, let B be a 3×3 matrix made up of any three rows of A. *True or false:*

(a) If B is invertible then the columns of A are linearly independent;

(b) If the columns of A are linearly independent then B is invetible.

Problem 1.7. Suppose that A is a square matrix such that the equation $A\bar{x} = \bar{b}$ is consistent for every $\bar{b}$. Show that the equation $A\bar{x} = \bar{b}$ has a *unique* solution for every $\bar{b}$.

Problem 1.8. Let $A = \begin{bmatrix} 2 & 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$. Find $\det A$ *using the definition of the determinant.*

Problem 1.9. Consider the following permutation: $\sigma = (2\ 3\ 1\ 5\ 4)$.

(a) Compute σ^{-1};
(b) Write σ as a product of transpositions;
(c) Find $\operatorname{sign}\sigma$;
(d) Compute powers of σ.

Problem 1.10. Find the determinants of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ 1 & 1 & 3 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & n \end{bmatrix}$$

2. Due Tuesday, February 2

Problem 2.1. Compute the determinants of the following matrices:

$$\textbf{(a)}\ \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix} \qquad \textbf{(b)}\ \begin{bmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{bmatrix} \qquad \textbf{(c)}\ \begin{bmatrix} 2 & i & 3 \\ 7 & 3+i & 4 \\ 2 & 0 & 5i \end{bmatrix}.$$

Problem 2.2. Given two 3×3 matrices A and B with $\det A = 2$ and $\det B = 3$, find $\det(2A^3B^2A^TB^{-1})$.

Problem 2.3. Let $A = \left[\begin{smallmatrix} 2 & 0 & 3 \\ 5 & 1 & 0 \\ 21 & 4 & 2 \end{smallmatrix}\right]$ and $\bar{b} = \left[\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}\right]$.

(a) Find A^{-1} using Cramer's rule (no Gaussian elimination).

(b) Solve $A\bar{x} = \bar{b}$ using Cramer's rule (no Gaussian elimination).

Problem 2.4. Find the eigenvalues and the eigenvectors of the following matrices over $\mathbb{C}$:

$$\textbf{(a)}\ \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} \qquad \textbf{(b)}\ \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \qquad \textbf{(c)}\ \begin{bmatrix} 2 & 6 & 0 & -4 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(d)}\ \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} \qquad \textbf{(e)}\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & 4 \end{bmatrix}$$

Problem 2.5. Show that A and A^T have the same characteristic equation and the same eigenvalues.

Problem 2.6. A matrix is *nilpotent* if some power of it equals the zero matrix. Show that if A is nilpotent then $\det A = 0$ and zero is the only eigenvalue of A.

Problem 2.7. Verify whether the given collection $\mathcal{B}$ is a basis of the given vector space X:

(a) $X = \mathbb{R}^3$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ 3 \end{smallmatrix}\right) \right\}$

(b) $X = \mathbb{R}^3$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 7 \\ 2 \\ 0 \end{smallmatrix}\right) \right\}$

(c) $X = \mathbb{R}^4$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 3 \\ 2 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \\ 2 \\ -3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ 3 \\ -1 \end{smallmatrix}\right) \right\}$

(d) $X = \mathbb{R}^4$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 3 \\ 2 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \\ 2 \\ -3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 \\ 1 \\ 3 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 6 \\ 2 \\ 0 \\ 4 \end{smallmatrix}\right) \right\}$

(e) $X = \mathbb{R}^4$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 3 \\ 2 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \\ 2 \\ -3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 \\ 1 \\ 3 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 6 \\ 2 \\ 0 \\ 5 \end{smallmatrix}\right) \right\}$

(f) $X =$ the xy-plane in $\mathbb{R}^3$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 3 \\ 2 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \\ 0 \end{smallmatrix}\right) \right\}$

(g) $X =$ the xy-plane in $\mathbb{R}^3$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 3 \\ 2 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}\right) \right\}$

(h) $X =$ the xy-plane in $\mathbb{R}^3$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 3 \\ 2 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 6 \\ 4 \\ 0 \end{smallmatrix}\right) \right\}$

Problem 2.8. Find the (complex) eigenvalues of the right shift S and the cyclic right shift T from Problem 1.3 ($n = 6$).

Problem 2.9. On the dark side of Linear Algebra: Let $\lambda_1, \ldots, \lambda_n$ be scalars, find the formula for the van der Monde determinant:

$$\det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{n-1} \end{bmatrix}$$

3. Practice assignment; not to be collected

Problem 3.1. Prove that if A satisfies $3A^2 - 5A - I = 0$ then A is invertible.

Problem 3.2. Let $\bar{u}_1, \ldots, \bar{u}_9 \in \mathbb{C}^4$ such that the set $\{\bar{u}_2, \bar{u}_4, \bar{u}_6, \bar{u}_8\}$ is linearly independent. Show that the set $\{\bar{u}_1, \ldots, \bar{u}_9\}$ spans all of $\mathbb{C}^4$.

Problem 3.3. Let $\bar{u}_1, \ldots, \bar{u}_m$ be vectors in $\mathbb{R}^n$ and let $\bar{v}_k = \sum_{i=1}^k \bar{u}_i$ as $k = 1, \ldots, m$. Show that the set $\{\bar{u}_1, \ldots, \bar{u}_m\}$ is linearly independent iff the set $\{\bar{v}_1, \ldots, \bar{v}_m\}$ is linearly independent.

Problem 3.4. Let A be a 5×3 matrix, let B be a 3×3 matrix made up of any three rows of A. *True or false:*

(a) If $\det B \neq 0$ then the columns of A are linearly independent;

(b) If the columns of A are linearly independent then $\det B \neq 0$.

Problem 3.5. Let $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$; then A is invertible and $A^T = -A$. Does there exist a 3×3 invertible matrix A satisfying $A^T = -A$?

Problem 3.6. Compute the determinant of $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{smallmatrix}\right)$ in $\mathbb{Z}_2$.

Problem 3.7. Compute $\det B^5$, where $B = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{smallmatrix}\right)$ (Hint: no need to compute B^5.)

Problem 3.8. Compute the determinants of the following matricex:

$$\textbf{(a)} \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{bmatrix} \qquad \textbf{(b)} \begin{bmatrix} 1 & \lambda & \lambda^2 \\ \lambda^2 & 1 & \lambda \\ \lambda & \lambda^2 & 1 \end{bmatrix}, \text{ where } \lambda = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}$$

$$\textbf{(c)} \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{bmatrix}, \text{ where } \alpha, \beta, \text{ and } \gamma \text{ are the roots of } x^3 + px + q = 0.$$

$$\textbf{(d)} \begin{pmatrix} 5 & -3 \\ -4 & 4 \end{pmatrix} \qquad \textbf{(f)} \begin{pmatrix} 5 & 3 \\ -4 & 4 \end{pmatrix}$$

Problem 3.9. (a) What can you say about $\det A$ if one of the columns of A is the sum (or, more generally, a linear combination) of other columns of A? Same for rows.

(b) How does the determinant change if one multiplies the entire matrix by a common factor?

(c) Find two matrices A and B such that $\det(A + B) \neq \det A + \det B$.

(d) Show that $\det AB = \det BA$ for any two $n \times n$ matrices A and B;

Problem 3.10. Find all the eigenvalues and all the eigenvectors of $\left[\begin{smallmatrix} 2 & 7 \\ 7 & 2 \end{smallmatrix}\right]$.

Problem 3.11. Let $\bar{u}, \bar{v} \in \mathbb{F}^n$ and $A = \bar{u}\bar{v}^T$. Show that $\bar{u}$ is an eigenvector of A and $\bar{v}$ is an eigenvector of A^T. Identify the null space and the range of A.

Problem 3.12. *True or false*: if $\det A = 0$ then A is nilpotent.

Problem 3.13. Suppose that X and Y are vector spaces and $T\colon X \to Y$ is a linear injection. Show that $X \simeq \operatorname{Range} T$.

Problem 3.14. Let

$$Y = \left\{ \begin{pmatrix} 3\alpha+2\beta \\ 2\alpha-\beta \\ -7\beta \\ -\alpha+3\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

Show that Y is a subspace of $\mathbb{R}^4$. Find a basis of Y and determine $\dim Y$. Show that Y is linearly isomorphic to $\mathbb{R}^2$. Find explicitly a linear isomorphism from $\mathbb{R}^2$ onto Y.

Problem 3.15. Verify whether the given collection $\mathcal{B}$ is a basis of the given vector space X:

(a) $X = M_2(\mathbb{F})$, $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$

(b) $X = \mathcal{P}_2(\mathbb{R})$, $\mathcal{B} = \{x^2 + 2, x + 3, x^2 - 5x\}$.

Problem 3.16. Let $f\colon \mathbb{R}^4 \to \mathbb{R}^3$ given by $f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1-x_2+3x_3-x_4 \\ 2x_1-2x_2+6x_3-5x_4 \\ 3x_1-3x_2+9x_3-10x_4 \end{pmatrix}$.

(a) Find a basis of $\ker f$. Show that $\ker f$ has dimension 2.

(b) Find a basis of $\operatorname{Range} f$. Show that $\operatorname{Range} f$ has dimension 2.

Problem 3.17. Given a collection $\mathcal{B}$ in a vector space X and a vector $\bar{u} \in X$; verify that $\mathcal{B}$ is a basis of X and find the expansion of $\bar{u}$ with respect to $\mathcal{B}$.

(a) $X = \mathbb{C}^4$, $\mathcal{B} = \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$, $\bar{u} = \begin{pmatrix} 10 \\ 11 \\ 8+3i \\ -6 \end{pmatrix}$.

(b) $X = \mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, $\bar{u} = \begin{pmatrix} 15 \\ 11 \\ 8 \end{pmatrix}$.

(c) $X =$ the xy-plane in $\mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$, $\bar{u} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}$.

(d) $X = \mathcal{P}_2(\mathbb{R})$, $\mathcal{B} = \{1 + 2x^2, 5x^2, 1 + 4x + 6x^2\}$, $\bar{u} = 2 + 6x + 10x^2$.

Problem 3.18. For each of the following vector spaces X, find a basis of X and determine $\dim X$.

(a) X is the subspace of $\mathbb{F}^6$ consisting of all the vectors $\bar{x}$ such that $x_2 = x_4 = x_6$.

(b) X is the space of all upper-triangular 3×3 matrices.

(c) X is the space of all 5×5 matrices A that are *antisymmetric*, i.e., satisfy $A^T = -A$.

Problem 3.19. Show that the space $C[0,1]$ of all real-valued continuous functions on $[0,1]$ is an infinite-dimensional vector space.

Problem 3.20. Find the rank of $\begin{bmatrix} 6 & 2 & 8 & 3 & 8 \\ 3 & 0 & 7 & 1 & 3 \\ 0 & 0 & 5 & 0 & 0 \\ 3 & 1 & 4 & 1 & 4 \end{bmatrix}$.

Problem 3.21. Let A and B be two matrices; let S be an invertible matrix. Assuming that the appropriate sums and products are defined, prove that

(a) $\operatorname{rank}(A+B) \leqslant \operatorname{rank} A + \operatorname{rank} B$;
(b) $\operatorname{rank}(AB) \leqslant \min\{\operatorname{rank} A, \operatorname{rank} B\}$;
(c) $\operatorname{rank} SA = \operatorname{rank} A$, $\operatorname{rank} AS = \operatorname{rank} A$;

4. Due Wednesday, March 3

Problem 4.1. Given two matrices A and B, we say that B is *similar* to A if $B = S^{-1}AS$ for some invertible matrix S. Prove the following statements.

(a) Every matrix is similar to itself;
(b) If A is similar to B then A^n is similar to B^n for every n
(c) If B is similar to A then A is similar to B;
(d) If A is similar to B and B is similar to C then A is similar to C;
(e) If A and B are similar then $\operatorname{rank} A = \operatorname{rank} B$.
(f) If A and B are similar then they share the same determinant, characteristic polynomial, and eigenvalues.
(g) Suppose that B is similar to A. What is the relationship between eigenvectors of A and those of B?

Problem 4.2. Let D be a diagonal matrix with non-zero diagonal entries $d_1, \dots, d_n$. Let $A = (a_{ij})$ be an $n \times n$ matrix. Compute DA, AD, and DAD^{-1}.

Problem 4.3. Determine whether the following two matrices are similar:

(**a**) $\begin{pmatrix} 7 & 2 & 0 & 0 & 0 \\ 0 & 7 & 3 & 0 & 0 \\ 0 & 0 & 7 & 4 & 0 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$ and $\begin{pmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$ (**b**) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

Problem 4.4. Let $A = \begin{bmatrix} 20 & 0 & -10 \\ 0 & 40 & 10 \\ 10 & 0 & 50 \end{bmatrix}$. Find the matrix of A with respect to the basis $\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Problem 4.5. Show that the functions $\sin x$, $\cos x$, and the constant one function are linearly independent, when viewed as vectors in the space $C(\mathbb{R})$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Problem 4.6. Let $T\colon C(\mathbb{R}) \to C(\mathbb{R})$ defined as follows: for $f \in C(\mathbb{R})$, Tf is the function defined by $(Tf)(x) = f(0) + f(1)\sin x + f(2)\cos x$. Show that T is linear. Find $\operatorname{rank} T$.

Problem 4.7. Let c_{00} be the subset $\mathbb{R}^{\mathbb{N}}$ consisting of the sequences that have only finitely many non-zero terms (i.e., eventually zero sequences).

(a) Show that c_{00} is a subspace of $\mathbb{R}^{\mathbb{N}}$.
(b) Show that $\{\bar{e}_n\}_{n=1}^{\infty}$ is a Hamel basis of c_{00}.
(c) Show that c_{00} is linearly isomorphic to $\mathcal{P}(\mathbb{R})$.

Problem 4.8. Let $X = \mathcal{P}_4(\mathbb{R})$, all polynomials of degree at most 4 with real coefficients. Let $\mathcal{B} = \{x^0, x^1, x^2, x^3, x^4\}$ (note that $\mathcal{B}$ is a basis of X). Let $D\colon X \to X$ be the differentiation operator defined by $Dp = p'$. Find the matrix of D with respect to $\mathcal{B}$.

Problem 4.9. Let $n \in \mathbb{N}$; fix n distinct points $x_1, \ldots, x_n$ in $\mathbb{R}$. Let $X = \mathbb{P}_{n-1}(\mathbb{R})$.

(a) Define $q_1, \ldots, q_n$ in X as follows:
$$q_i(x) = \frac{x - x_1}{x_i - x_1} \cdot \frac{x - x_2}{x_i - x_2} \cdots \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdots \frac{x - x_n}{x_i - x_n}.$$
Compute $q_i(x_j)$ for all pairs $i, j \in \{1, \ldots, n\}$. Show that $\{q_1, \ldots, q_n\}$ is a basis of X.

(b) Given $d_1, \ldots, d_n \in \mathbb{R}$, find $p \in \mathbb{P}_{n-1}(\mathbb{R})$ such that $p(x_1) = d_1$, $p(x_2) = d_2$, $\ldots$, $p(x_n) = d_n$.

Problem 4.10. In $\mathbb{R}^6$, consider $X = \operatorname{span}\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and $Y = \operatorname{span}\{\bar{e}_2, \bar{e}_3, \bar{e}_4\}$.

(a) Identify $X \cap Y$ and $X + Y$.

(b) Are $X + Y$ and $X \oplus Y$ linearly isomorphic?

(c) Find $\dim X + Y$ and $\dim X \oplus Y$.

5. Due Monday, March 15

Problem 5.1. Let $X_1, \dots, X_m$ be subspaces of X. Show that TFAE:

(a) Every vector $\bar{x} \in X_1 + \cdots + X_m$ has a unique representation $\bar{x} = \bar{x}_1 + \cdots + \bar{x}_m$ for some $\bar{x}_1 \in X_1, \dots, \bar{x}_m \in X_m$.

(b) If $\bar{x}_1 + \cdots + \bar{x}_m = 0$ for some $\bar{x}_1 \in X_1, \dots, \bar{x}_m \in X_m$ then $\bar{x}_1 = \bar{0}, \dots, \bar{x}_m = \bar{0}$.

Note: if, in addition, $X = X_1 + \cdots + X_m$, we write $X = X_1 \oplus \cdots \oplus X_m$ and say that this is a direct sum decomposition of X.

Problem 5.2. Let $\mathcal{U}$ and $\mathcal{V}$ be two disjoint subsets of a vector space Z such that $\mathcal{U} \cup \mathcal{V}$ is a basis of Z. Show that $Z = \operatorname{span} \mathcal{U} \oplus \operatorname{span} \mathcal{V}$.

Problem 5.3. Let X be a subspace of $\mathbb{R}^4$ spanned by $\left(\begin{smallmatrix}1\\1\\1\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1\\1\\1\\2\end{smallmatrix}\right)$. Find an algebraic complement of X.

Problem 5.4. Let $P = \frac{1}{3}\left(\begin{smallmatrix}1&1&1\\1&1&1\\1&1&1\end{smallmatrix}\right)$. Show that P is a projection. Identify its range.

Problem 5.5. Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 0 \\ 0 & 2 & 4 & 6 & 8 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } Q = I - P.$$

(a) Show that P and Q are projections. Find their ranges and kernels.

(b) Compute PAP, QAQ, PAQ, and QAP.

Problem 5.6. Let X be a finite-dimensional vector space and $P\colon X \to X$ a projection. Show that there is a basis of X in which the matrix of P is diagonal and the diagonal entries are zeros and ones. Conclude that the only possible eigenvalues of P are 0 and 1.

Problem 5.7. Show that $C[0,1] = X \oplus Y$, where X consists of those functions f in $C[0,1]$ such that $f(0) = f(1) = 0$ and Y consists of all the functions of the form $f(x) = ax + b$, where $a, b \in \mathbb{R}$.

Problem 5.8. Let $\mathbb{R}^{\mathbb{R}}$ be the vector space space of all functions $f\colon \mathbb{R} \to \mathbb{R}$. Let $P\colon \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ defined via $(Pf)(x) = \frac{f(x)+f(-x)}{2}$.

(a) Show that P is a projection.

(b) Identify $\operatorname{Range} P$ and $\operatorname{Null} P$.

(c) Show that every function $f\colon \mathbb{R} \to \mathbb{R}$ may be written as a sum of an even and an odd function in a unique way.

Problem 5.9. Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$.

(a) Find the eigenvectors and the eigenvalues of A;
(b) Find a basis in which A is diagonal.
(c) Show that A is similar to a diagonal matrix.

Problem 5.10. Let A be an $n \times n$ matrix. Show that

(a) $\operatorname{Null} A \subseteq \operatorname{Null} A^2 \subseteq \operatorname{Null} A^3 \subseteq \dots$ and $\operatorname{Range} A \supseteq \operatorname{Range} A^2 \supseteq \operatorname{Range} A^3 \supseteq \dots$;
(b) If $\operatorname{Null} A^k = \operatorname{Null} A^{k+1}$ for some k then $\operatorname{Range} A^k = \operatorname{Range} A^{k+1}$ and $\operatorname{Null} A^{k+1} = \operatorname{Null} A^{k+2}$.
(c) There exists $k \leqslant n$ such that $\operatorname{Null} A^i = \operatorname{Null} A^k$ and $\operatorname{Range} A^i = \operatorname{Range} A^k$ for all $i \geqslant k$.

Problem 5.11. Let $A, B \in M_n(\mathbb{F})$ with $n > 1$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Prove the following statements.

(a) If AB is nilpotent then BA is nilpotent.
(b) If A is nilpotent then $A^n = 0$.
(c) A^2 is not equal to the $n \times n$ left shift.

Problem 5.12. Let $A \in M_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

(a) Suppose that $A^k \bar{x} = \bar{0}$ but $A^{k-1}\bar{x} \neq \bar{0}$ for some $\bar{x} \in \mathbb{F}^n$ and $k \in \mathbb{N}$. Show that the set $\{\bar{x}, A\bar{x}, \dots, A^{k-1}\bar{x}\}$ is linearly independent.
(b) Suppose that $A^n = 0$ but $A^{n-1} \neq 0$. Prove that A is similar to the left shift.
(c) Show that the left and the right shift matrices in $M_n(\mathbb{F})$ are similar.

Problem 5.13. Let Y be a subspace of a finite-dimensional vector space X. Show that $\dim X/Y = \dim X - \dim Y$.

6. Practice assignment

Problem 6.1. (a) Suppose that $3, 5, 7 \in \sigma(A)$, u_1, u_2, u_3 are eigenvectors for 2, v_1, v_2, v_3 are eigenvectors for 5, and w_1, w_2, w_3 are eigenvectors for 7. Suppose that each of the three sets $\{u_1, u_2, u_3\}$, $\{v_1, v_2, v_3\}$, $\{w_1, w_2, w_3\}$ is linearly independent. Prove that the set $\{u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3\}$ is linearly independent.

(b) Suppose that $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of A. For each $i = 1, \dots, m$. Let $\mathcal{U}_i$ be a linear independent set in E_{λ_i}. Show that the union $\bigcup_{i=1}^m \mathcal{U}_i$ is linearly independent.

Problem 6.2. Let Y be a subspace of a finite-dimensional vector space X. If $\dim Y = \dim X$ then $Y = X$.

Problem 6.3. Let Y and Z be two subspaces of a finite-dimensional vector space X, such that $\dim Y + \dim Z > \dim X$. Show that Y and Z have a common non-zero vector.

Problem 6.4. Let $\mathcal{B}$ and $\mathcal{C}$ be bases of vector spaces X and Y, respectively. Show that $\mathcal{B} \cup \mathcal{C}$ is a basis of $X \oplus Y$.

Problem 6.5. Let $P\colon Z \to Z$ be a projection onto X along Y.

(a) Show that $I - P$ is a projection onto Y along X;

(b) Find the eigenvalues and the eigenspaces of P.

Problem 6.6. * Let $X = C_\infty(0,1)$, the space of functions on $(0,1)$ that have derivatives of all orders. let Y be the set of all functions f in X such that f vanishes on $(0, \varepsilon)$ for some positive ε (depending on f).

(a) Show that Y is a subspace of X;

(b) Describe elements of X/Y (they are called *germs*); characterize when $f \sim g$;

(c) Show that each equivalence class contains at most one polynomial;

(d) Does the equivalence class of cos contain a polynomial?

Problem 6.7. Find an example of a matrix A over $\mathbb{R}$ such that $\sigma(A) = \{0\}$ yet A is not nilpotent.

Problem 6.8. Show that the following two matrices are similar:

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 6.9. Let X be the set of all polynomials p in $\mathcal{P}_5(\mathbb{R})$ such that $p(7) = 0$. Prove that X is a subspace of $\mathcal{P}_5(\mathbb{R})$; find a basis of X.

Problem 6.10. Let $\mathcal{B}$ be a Hamel basis of a vector space X; let $\mathcal{U}$ and $\mathcal{V}$ be two subsets of X such that $\mathcal{U} \subsetneq \mathcal{B} \subsetneq \mathcal{V}$. Show that both $\mathcal{U}$ and $\mathcal{V}$ fail to be Hamel bases of X.

Problem 6.11. Show that the set $\{\bar{e}_1, 2\bar{e}_2, 3\bar{e}_3, 4\bar{e}_4, \dots\}$ is a Hamel basis of c_{00}.

Problem 6.12. Let X be the subset of $\mathbb{R}^{\mathbb{N}}$ consisting of those sequences that are eventually constant. That is, for $\bar{x} \in \mathbb{R}^{\mathbb{N}}$, we have $\bar{x} \in X$ when there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = x_{n+2} = \dots$. (Note that n depends on $\bar{x}$.) Show that X is a vector space. Find a Hamel basis of X.

Problem 6.13. Let $P\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the xy-plane along the vector $\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$. Find the matrix if P.

Problem 6.14. Let X be a subspace of a vector space Z. Show that there exist linear operators S and T in $L(Z)$ with $\operatorname{Range} T = \operatorname{Null} S = X$.

Problem 6.15. Find an algebraic complement of $\operatorname{span}\left\{ \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 2 \\ 3 \end{smallmatrix}\right] \right\}$ in $\mathbb{R}^4$.

Problem 6.16. Let Z be as follows, let

$$X = \bigl\{ f \in Z \,:\, f \text{ vanishes on } (-\infty, 0) \bigr\} \text{ and } Y = \bigl\{ f \in Z \,:\, f \text{ vanishes on } [0, \infty) \bigr\}.$$

Verify whether $Z = X \oplus Y$.

(**a**) $Z = \mathbb{R}^{\mathbb{R}}$ (**b**) $Z = C(\mathbb{R})$.

Problem 6.17. Let $T, S \in L(X)$ be two linear operator on a vector space X such that $TS = ST$. Show that $\ker T$ and $\operatorname{Range} T$ are invariant under S. Show that every eigenspace of T is invariant under S. Show that for every $m \in \mathbb{N}$, $\operatorname{Null} T^m$ and $\operatorname{Range} T^m$ are invariant under T.

Problem 6.18. Let $T \in L(X)$ and Y be a subspace of X. Show that

(a) Y is invariant under T iff $(I - P)TP = 0$ for a projection P with $\operatorname{Range} P = Y$;
(b) T has a reducing decomposition iff T commutes with a projection.

Problem 6.19. Let A be a block diagonal matrix, $A = \left[\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right]$. Show that $\operatorname{Range} A = \operatorname{Range} B \oplus \operatorname{Range} C$ and $\operatorname{rank} A = \operatorname{rank} B + \operatorname{rank} C$.

7. Due Wednesday, March 31

Problem 7.1. We write $J_{\lambda,k}$ for the $k \times k$ Jordan block with diagonal entries equal to λ. By $J_{0,1}$ we mean the 1×1 zero matrix. Suppose that $A \in M_{28}(\mathbb{C})$ such that its Jordan canonical form consists of the following blocks: $J_{2,3}$, $J_{3,1}$, $J_{3,5}$, $J_{1,4}$, $J_{0,2}$, $J_{0,3}$ repeated twice, $J_{0,4}$, and $J_{0,1}$ repeated thrice. Compute rank A^m as $m = 1, 2, 3, \ldots$.

Problem 7.2. Let $A \in M_9(\mathbb{C})$ such that $a_{2,3} = 2$, $a_{4,1} = 3$, $a_{5,4} = 4$, $a_{7,9} = 5$, $a_{9,2} = 6$, $a_{8,8} = 7$, and the rest of the entries are zero. Find the Jordan canonical form of A.

Problem 7.3. Let A be a 7×7 matrix. Use the following data to find the Jordan form of A: $\operatorname{rank}(A - 2I) = 6$, $\operatorname{rank}(A - 3I) = 5$, $\operatorname{rank}(A - 3I)^2 = 3$, $\operatorname{rank}(A - 3I)^3 = 2$, $\operatorname{rank}(A - 3I)^4 = 1$. (This problem illustrates that that the Jordan Canonical form of a matrix is uniquely determined by the matrix up to the order of the blocks.)

Problem 7.4. Let R be the 5×5 right shift matrix. Find the Jordan canonical forms of R and of R^2.

Problem 7.5. Let $A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix}$. Find the eigenvalues of A and their algebraic and geometric multiplicities. Find a basis that brings A to its Jordan form. Find the Jordan form of A. Find the eigenspaces and the generalized eigenspaces of A.

Problem 7.6. Let A be a complex matrix and p a polynomial with complex coefficients. Show that if $\sigma(A) = \{\lambda_1, \ldots, \lambda_m\}$ then $\sigma\big(p(A)\big) = \big\{p(\lambda_1), \ldots, p(\lambda_m)\big\}$, without using the Spectral Mapping Theorem.

8. Due Wednesday, April 14

Problem 8.1. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Fix any invertible matrix A in $M_n(\mathbb{F})$. For $\bar{x}, \bar{y} \in \mathbb{F}^n$, define $[\bar{x}, \bar{y}] = \langle A\bar{x}, A\bar{y} \rangle$. Show that $[\cdot, \cdot]$ is an inner product. Find a basis which is orthonormal under this inner product.

Problem 8.2. *When are Cauchy-Schwartz and triangle inequalities equalities?* Let $\bar{x}$ and $\bar{y}$ be two vectors in an inner product space. Show that $\langle \bar{x}, \bar{y} \rangle = \|\bar{x}\|\|\bar{y}\|$ iff $\|\bar{x}+\bar{y}\| = \|\bar{x}\|+\|\bar{y}\|$ iff one of the vectors is a positive scalar multiple of the other.

Problem 8.3. Let $X = C[0, 2\pi]$ viewed as a vector space over $\mathbb{R}$. For $f, g \in X$, define $\langle f, g \rangle = \int_0^{2\pi} f(t)g(t)\,dt$.

(a) Show that this is an inner product on X;
(b) Consider the set of all functions of the form $\sin nt$ and $\cos nt$, where $n \in \mathbb{N}$. Show that this is an orthogonal set. Is it orthonormal?

Problem 8.4. Let $\mathcal{B}$ be an orthonormal basis in an inner product space X. Let $\bar{x}, \bar{y} \in X$ with $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $[\bar{y}]_{\mathcal{B}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Show that $\langle \bar{x}, \bar{y} \rangle = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n$.

Problem 8.5. Let Y be a subspace of an inner product space X. For $x \in X$, we write $x \perp Y$ if $x \perp y$ for every $y \in Y$. Let $Y^\perp = \{x \in X \,:\, x \perp Y\}$. Show that $Y^\perp$ is again a subspace of X. Show that if there exists an orthogonal projection onto Y then $(Y^\perp)^\perp = Y$.

Problem 8.6. A linear functional is a linear operator from a vector space to $\mathbb{F}$. Let X be an inner product space.

(a) Fix $\bar{y} \in X$ and define a function $f \colon X \to \mathbb{F}$ via $f(\bar{x}) = \langle \bar{x}, \bar{y} \rangle$. Show that f is a linear functional.
(b) Show that every linear functional on a finite-dimensional inner product space arises in this way.

Problem 8.7. Given a subspace Y, find orthonormal bases of Y and $Y^\perp$. Find the orthogonal projection onto Y.

(a) Y is the plane $4x_1 - x_2 + x_3 = 0$ in $\mathbb{R}^3$.
(b) $Y = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Problem 8.8. Let $T \in L(X)$, where X is an inner product space over $\mathbb{C}$. Let $\bar{x}, \bar{y} \in X$. Show that

$$\langle T\bar{x}, \bar{y} \rangle = \tfrac{1}{4}\Big[\big\langle T(\bar{x}+\bar{y}), \bar{x}+\bar{y} \big\rangle - \big\langle T(\bar{x}-\bar{y}), \bar{x}-\bar{y} \big\rangle\Big] + \tfrac{i}{4}\Big[\big\langle T(\bar{x}+i\bar{y}), \bar{x}+i\bar{y} \big\rangle - \big\langle T(\bar{x}-i\bar{y}), \bar{x}-i\bar{y} \big\rangle\Big].$$

The problem shows that $\langle T\bar{x}, \bar{y}\rangle$ may be written as a combination of expressions of the form $\langle T\bar{z}, \bar{z}\rangle$.

Problem 8.9. Let X be an inner product space over $\mathbb{C}$. Let $\bar{x}, \bar{y} \in X$. Show that

$$\langle \bar{x}, \bar{y}\rangle = \frac{1}{4}\Big[\|\bar{x}+\bar{y}\|^2 - \|\bar{x}-\bar{y}\|^2\Big] + \frac{i}{4}\Big[\|\bar{x}+i\bar{y}\|^2 - \|\bar{x}-i\bar{y}\|^2\Big].$$

This problem shows that inner product is uniquely determined by the norm.

Problem 8.10. Let $T, S \in L(X)$, where X is an inner product space over $\mathbb{C}$. Show that

(a) If $\langle T\bar{x}, \bar{y}\rangle = 0$ for all $\bar{x}, \bar{y} \in X$ then $T = 0$;
(b) If $\langle T\bar{x}, \bar{y}\rangle = \langle S\bar{x}, \bar{y}\rangle$ for all $\bar{x}, \bar{y} \in X$ then $T = S$;
(c) If $\langle T\bar{x}, \bar{x}\rangle = \langle S\bar{x}, \bar{x}\rangle$ for all $\bar{x} \in X$ then $T = S$.

9. Extra practice problems

Problem 9.1. Let X and Y be vector spaces over $\mathbb{F}$; let $\mathcal{B}$ be a Hamel basis of X. Let $f\colon \mathcal{B} \to Y$ be an arbitrary function. Prove that f extends to a unique linear operator from X to Y. That is, prove that

(a) there is a linear operator $T\colon X \to Y$ such that $Tx = f(x)$ for every $x \in \mathcal{B}$, and
(b) that if $S\colon X \to Y$ is a linear operator such that $Sx = f(x)$ for every $x \in \mathcal{B}$ then $S = T$.

Problem 9.2. Let $A \in M_n(\mathbb{F})$, $\lambda_1, \dots, \lambda_n$ *distinct* eigenvalues of A, and $\bar{u}_i$ an eigenvector for λ_i as $i = 1, \dots, n$. Show that $\operatorname{span}\{\bar{u}_1, \dots, \bar{u}_n\} = \mathbb{F}^n$.

Problem 9.3. The goal of this problem is to understand whether invertibility of a matrix is related to the number of non-zero entries in it.

(a) Find an $n \times n$ non-invertible matrix with no zero entries;
(b) What is the minimal possible number of non-zero entries in an invertible matrix?
(c) Describe all invertible matrices with the minimal number of non-zero entries.

Problem 9.4. Let X be a vector space, Y and Z subspaces of X. *True or false:*

(a) $Y \cap Z$ is a subspace of X;
(b) $Y \cup Z$ is a subspace of X.

Problem 9.5. Let $\bar{u}, \bar{v} \in \mathbb{R}^n$ and $A = \bar{u}\bar{v}^T$. Identify the kernel (the null space) and the range of A.

Problem 9.6. Let Y be the subspace of $\mathbb{P}$ spanned by the polynomials $p(x) = 1 + x^2$, $q(x) = x^2 - x^3$, and $r(x) = x^5$. Does the following polynomial belong to Y?

(a) $3 + x + 2x^3$ **(b)** $3 + x^2 + 2x^3 + 7x^5$ **(c)** $3 + x^2 + 4x^3 + 7x^5$

Problem 9.7. Let $f, g, h \in C(\mathbb{R})$ given by $g(t) = \sin t$, $h(t) = 1 + t^2$, and $f(t) = 1$, the constant 1 function. Prove that the set $\{f, g, h\}$ is linearly independent.

Problem 9.8. Let $T\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the "left shift":

$$T\colon (x_1, x_2, x_3, x_4, \dots) = (x_2, x_3, x_4, x_5, \dots).$$

Show that T is linear. Find $\ker T$ and $\operatorname{Range} T$. Is T onto? One-to-one? Show that every real number λ is an eigenvalue of T and find an eigenvector for T and λ.

Problem 9.9. Let $S\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the "right shift":

$$S\colon (x_1, x_2, x_3, x_4, \dots) = (0, x_1, x_2, x_3, \dots).$$

Show that S is linear. Find $\ker S$ and $\operatorname{Range} S$. Is S onto? One-to-one? Show that S has no eigenvalues.

Problem 9.10. Let X be the set of all upper-triangular 5×5 real matrices. Show that X is a vector space over $\mathbb{R}$. Show that X is linearly isomorphic to $\mathbb{R}^{15}$.

Problem 9.11. Let X be the set of all antisymmetric 5×5 real matrices, that is, matrices satisfying $A^T = -A$. Show that X is a vector space over $\mathbb{R}$. Show that X is linearly isomorphic to $\mathbb{R}^{10}$.

Problem 9.12. Show that the space of all polynomials with real coefficients is linearly isomorphic to c_{00}.

Problem 9.13. Let X be the vector space of all upper triangular 3×3 matrices with zero trace. Find a basis of X, and determine $\dim X$.

Problem 9.14. Let $T\colon X \to Y$ be a linear operator between vector spaces with $\operatorname{rank} T = n$. Show that there exist linear operators $U\colon X \to \mathbb{F}^n$ and $V\colon \mathbb{F}^n \to Y$ such that $T = VU$ (we say that T *factors through* $\mathbb{F}^n$).

Problem 9.15. Let X be a vector space, $x \in X$, and $T \in L(X)$. Put $Y = \operatorname{span}\{x, Tx, T^2x, \dots\}$. Show that Y is a subspace of X. Show that Y is *invariant under* T, that is, $T(Y) \subseteq Y$. Find an example where $T(Y) \neq Y$.

Problem 9.16. Find two non-similar matrices whose characteristic polynomial is $\lambda^2(\lambda - 2)^4(\lambda - 5)^6$ and whose minimal polynomial is $\lambda^2(\lambda - 2)^3(\lambda - 5)^4$.

Problem 9.17. Let $A = J_{0,3} \oplus J_{0,4}$. Compute e^A.

Problem 9.18. Let A be as in Problem 7.5. Find $\sin(\frac{\pi}{4}A)$.

Problem 9.19. Let $A \in M_n(\mathbb{F})$. Suppose that every $\lambda \in \sigma(A)$ satisfies $|\lambda| = 1$ and $\arg \lambda$ is a rational multiple of π. Show that there exists $k \in \mathbb{N}$ such that $(A^k - I)^n = 0$.

Problem 9.20. Prove that $(A + B)^* = A^* + B^*$, $(\alpha A)^* = \bar{\alpha} A^*$, and $(AB)^* = B^* A^*$.

Problem 9.21. Prove that $\langle A\bar{x}, \bar{y} \rangle = \langle \bar{x}, A^*\bar{y} \rangle$ for every $A \in M_n(\mathbb{C})$ and every $\bar{x}, \bar{y} \in \mathbb{C}^n$.

Problem 9.22. Let $A \in M_n(\mathbb{C})$. Show that the matrix A^*A is self-adjoint.

Problem 9.23. Let $A = \begin{bmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{bmatrix}$. Find an orthonormal basis which diagonalizes A.

Problem 9.24. Let T be a linear operator between two finite-dimensional inner product spaces. Show that

(a) $\ker T^* = (\operatorname{Range} T)^{\perp}$;
(b) $\operatorname{Range} T^* = (\ker T)^{\perp}$;
(c) $\ker T = (\operatorname{Range} T^*) \perp$;
(d) $\operatorname{Range} T = (\ker T^*)^{\perp}$;
(e) T is injective iff T^* is surjective;
(f) T is surjective iff T^* is injective;
(g) $\operatorname{rank} T^* = \operatorname{rank} T$.

Problem 9.25. Let $\theta \in [0, 2\pi]$ and $U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Show that U is unitary. Show that every unitary matrix in $M_2(\mathbb{R})$ is of this form.

Problem 9.26. Show that the product of two unitary matrices is unitary.

Problem 9.27. A self-adjoint operator T is said to be *positive semi-definite* (or just *positive*) if $\langle T\bar{x}, \bar{x}\rangle \geqslant 0$ for all $\bar{x}$.

(a) Let T be self-adjoint; then is positive semi-definite iff the eigenvalues of T are non-negative.
(b) Show that T^*T is positive semi-definite for every operator T.
(c) Show that if T is positive semi-definite then there is a positive semi-definite operator S such that $S^2 = T$.

Problem 9.28. Let A be a 3×3 matrix with real entries, such that $4 + i$ is an eigenvalue of A. Show that A is diagonalizable.

Problem 9.29. Let Y be the subspace of $\mathbb{R}^4$ given by the equation $x_1+2x_2-3x_3+4x_4 = 0$.

(a) Find an algebraic complement of Y in $\mathbb{R}^4$.
(b) Find a projection $P\colon \mathbb{R}^4 \to \mathbb{R}^4$ with $\operatorname{Range} P = Y$.

Problem 9.30. Let $A \in M_n(\mathbb{C})$. Prove that there is a unique pair of self-adjoint operators T and S such that $A = T + iS$. (Hint: put $T = \frac{A+A^*}{2}$.)

Problem 9.31. Let X be an inner product space and $\bar{x} \in X$. Show that if $\bar{x} \perp \bar{y}$ for all $\bar{y} \in X$ then $\bar{x} = \bar{0}$. (Hint: put $\bar{y} = \bar{x}$.)

Problem 9.32. Let S be the set of all upper-triangular matrices in $M_n(\mathbb{F})$; let D be the set of all diagonal matrices in $M_n(\mathbb{F})$, let N be the set of matrices in S with only zero entries on the diagonal. Show that $S = D \oplus N$ as vector spaces. Show that S/N is linearly isomorphic to D.

Problem 9.33. Let X and Y are two vector spaces, let $T \in L(X, Y)$, and let Z be a subspace of $\ker T$. Show that $\widetilde{T}\colon X/Z \to Y$ defined via $\widetilde{T}\tilde{x} = Tx$ is well defined.

Problem 9.34. * Let X be a k-dimensional subspace of $\mathbb{R}^n$ for some $1 \leqslant k \leqslant n$.

(a) Show that X contains a vector $\bar{x}$ such that $-1 \leqslant x_i \leqslant 1$ for all $i = 1, \ldots, n$ and there exist k distinct indices $i_1, \ldots, i_k$ such that $|x_{i_1}| = \cdots = |x_{i_k}| = 1$.

(b) Show that X contains a vector $\bar{x}$ such that $-1 \leqslant x_i \leqslant 1$ for all $i = 1, \ldots, n$ and there exist k distinct indices $i_1 < \cdots < i_k$ such that $x_{i_1} = -1$, $x_{i_2} = 1$, $x_{i_3} = -1$, $x_{i_4} = 1$, $\ldots$

Solutions

1.1

$$T\begin{pmatrix}3\\3\\9\end{pmatrix}=T\left(3\begin{pmatrix}1\\1\\2\end{pmatrix}+\begin{pmatrix}0\\0\\3\end{pmatrix}\right)=3T\begin{pmatrix}1\\1\\2\end{pmatrix}+T\begin{pmatrix}0\\0\\3\end{pmatrix}=3\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}-1\\5\end{pmatrix}=\begin{pmatrix}3\\11\end{pmatrix}$$

1.2(a) Hint: find $S\bar{e}_1$ and $S\bar{e}_2$ and use them as the columns of the matrix. Alternatively, observe that $S = A^{-1}BA$, where A is the rotation by θ clockwise and B is the reflection with respect to the x-axis, where θ is the slope of the line $y = 2x$.

1.2(b) Find $T\bar{e}_1$, $T\bar{e}_2$, and $T\bar{e}_3$ and use them as the columns of the matrix. Alternatively, write $T = AB$ where A is the rotation around the z-axis by α and B is the rotation around the y-axis by $\frac{\pi}{2}$ and find separately the matrices of A and B. Note: your answer may depend on how you orient the axis!

1.3 $S\bar{e}_1 = \bar{e}_2$, $S\bar{e}_2 = \bar{e}_3$, $\ldots$, $S\bar{e}_5 = \bar{e}_6$, and $S\bar{e}_6 = 0$. This allows us to write down the matrix of S. Then, using matrix product, we compute $S^2 = SS$, $S^3 = SSS$, etc:

$$S=\begin{pmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{pmatrix},S^2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\end{pmatrix},S^3=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix},$$

$$S^4=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix},S^5=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\end{pmatrix},S^6=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}.$$

It is clear that S^n is the zero matrix whenever $n \geqslant 6$.

Observe that $T\bar{e}_i = e_{i+1}$ when $i < 6$ and $T\bar{e}_6 = e_1$. That is, $T\bar{e}_i = e_{i+1 \mod 6}$. Iterating, we get $T^n\bar{e}_i = e_{i+n \mod 6}$. In particular, $T^6 = I$ and $T^7 = T$, $T^8 = T^2$, etc. So it suffices to identify T^2 through T^5. Using $T^n\bar{e}_i = e_{i+n \mod 6}$, we can now write the matrix of T^n:

$$T=\begin{pmatrix}0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{pmatrix},T^2=\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\end{pmatrix},T^3=\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix},$$

$$T^4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, T^5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T^6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

1.4 The operator T may be given by a matrix-vector product: for every $\bar{x}$, we have $T\bar{x} = A\bar{x}$ where $A = \begin{pmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{pmatrix}$. It follows that T is linear. Now use row reduction on A to answer the questions and to find the inverse.

1.5(a) This may fail for non-square matrices. For example, $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, yet the two matrices in the left hand side are non-square, hence not invertible.

The statement is true for square matrices, say $n \times n$. Indeed, in this case, it is easy to see that $\operatorname{Null} B \subseteq \operatorname{Null} AB = \{0\}$, which implies that B is one-to-one and, therefore invertible. On the other hand, it follows from $\operatorname{Range} A \supseteq \operatorname{Range} AB = \mathbb{F}^n$ that A is onto and, therefore, invertible.

1.5(b) True: direct verification shows that $B^{-1}A^{-1}$ is the inverse of AB: $ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$ and $B^{-1}A^{-1}AB = B^{-1}B = I$.

1.5(c) False, for a counterexample, take $A = I$ and $B = -I$.

1.7 Since $A\bar{x} = \bar{b}$ is consistent for every $\bar{b}$, this means that for every $\bar{b}$ there is an $\bar{x}$ such that $A\bar{x} = \bar{b}$; hence $\bar{b}$ is in $\operatorname{Range} A$, hence A is onto. Being a square matrix, A is also one-to-one. It follows that there may be only one $\bar{x}$ satisfying $A\bar{x} = \bar{b}$. Actually, we can also say that A is invertible and the unique solution of $A\bar{x} = \bar{b}$ is $\bar{x} = A^{-1}\bar{b}$.

1.8 There is only only way to select one non-zero entry from every row and one non-zero entry from every column: we have to use 4 in row 2 (and in column 2), 5 in row 4 (and column 5), and 2 in row 5 (and column 3). This means that we cannot use 3 in column 3 (and row 1), hence we have to use 2 in row 1 (and column 1). Hence we cannot use 1 in column 1, so we have to use 2 in row 3. Hence $\det A = \operatorname{sign}(\sigma) a_{11} a_{22} a_{24} a_{35} a_{43} = \operatorname{sign}(\sigma) \cdot 2 \cdot 4 \cdot 2 \cdot 5 \cdot 2$. where $\sigma = (1\ 2\ 4\ 5\ 3)$. Note that σ is obtained from the identity permutations by two swaps (transpositions):

$$(1\ 2\ 3\ 4\ 5) \mapsto (1\ 2\ 5\ 4\ 3) \mapsto (1\ 2\ 4\ 5\ 3)$$

hence σ is even and $\operatorname{sign}(\sigma) = 1$.

1.9(a) We have $\sigma(1) = 2$ so $\sigma^{-1}(2) = 1$. Similarly, $\sigma^{-1}(3) = 2$, $\sigma^{-1}(1) = 3$, $\sigma^{-1}(5) = 4$, $\sigma^{-1}(4) = 5$, so that $\sigma^{-1} = (3\ 1\ 2\ 5\ 4)$.

1.9(b) One of possible solutions is $\varepsilon_{45}\varepsilon_{13}\varepsilon_{12}$.

1.9(c) Since σ may be written as a product of an odd number of transpositions, $\operatorname{sign}(\sigma) = -1$.

1.9(d) $\sigma^2(1) = \sigma(2) = 3$. Similarly, find $\sigma^2(i)$ as $i = 2, 3, 4, 5$. Find σ^n for $n = 2, 3, 4, 5, 6$ in a similar way. Observe that $\sigma^6 = e$. It follows that $\sigma^n = \sigma^{n \bmod 6}$ for $n > 6$.

1.10 Subtracting the first row from every other row results in an upper-triangular matrix.

2.6 Suppose $A^n = 0$. Then $0 = \det A^n = (\det A)^n$, hence $\det A = 0$, so that A is not invertible and zero is an eigenvalue of A. To show that A has no other eigenvalues, assume, for the sake of contradiction, that some non-zero λ is an eigenvalue of A. Let $\bar{x}$ be a (non-zero) eigenvector for A and λ. It follows from $A^n = 0$ that

$$0 = A^n\bar{x} = A^{n-1}A\bar{x} = A^{n-1}\lambda\bar{x} = \lambda A^{n-1}\bar{x} = \cdots = \lambda^n\bar{x}.$$

Since $\bar{x} \neq 0$, we have $\lambda = 0$.

2.7(a) Yes because the matrix whose columns are the vectors is $\mathcal{B}$ is invertible.

2.7(b) No, 4 vectors in $\mathbb{R}^3$ have to be linearly dependent.

2.7(c) No, 3 vectors cannot span $\mathbb{R}^4$.

2.7(d) No, because the matrix whose columns are the vectors is $\mathcal{B}$ is not invertible.

2.7(e) Yes, because the matrix whose columns are the vectors is $\mathcal{B}$ is invertible.

2.7(f) Yes. The vectors are linearly independent and span X.

2.7(g) No, the second vector is not even in X.

2.7(h) No, the vectors are linearly dependent.

2.8 Since S is nilpotent, its one and only eigenvalue is zero. To find the characteristic polynomial of T, write down the matrix of T, then apply the co-factor expansion over the first row of $\det(\lambda I - T)$. This gives the characteristic polynomial $\lambda^6 - 1$. This polynomial has two real roots, ± 1. Viewed over $\mathbb{C}$, it has six roots; namely, the six 6-th roots of unity. Thus, this matrix has six complex eigenvalues, two of which are real.

3.1 *Solution 1.* If $A\bar{x} = 0$ then

$$\bar{0} = (3A^2 - 5A - I)x = (3A)(A\bar{x}) - 5A\bar{x} - \bar{x} = -\bar{x},$$

so that $\bar{x} = 0$ and, therefore, $A\bar{x} = 0$ has only the trivial solution. Since the problem implies that A is a square matrix, it means that A is invertible.

Solution 2. It follows that $I = 3A^2 - 5A = BA$ where $B = 3A - 5I$. Hence, A is left invertible. Since the problem implies that A is a square matrix, it means that A is invertible.

3.2 Being a set of four linearly independent vectors in $\mathbb{C}^4$, the set $\{\bar{u}_2, \bar{u}_4, \bar{u}_6, \bar{u}_8\}$ spans all of $\mathbb{C}^4$. Since $\{\bar{u}_1, \dots, \bar{u}_9\}$ is even larger, it also spans $\mathbb{C}^4$.

3.3 Suppose $\alpha_1\bar{v}_1 + \cdots + \alpha_m\bar{v}_m = \bar{0}$. Then

$$\alpha_1\bar{u}_1 + \alpha_2(\bar{u}_1 + \bar{u}_2) + \cdots + \alpha_m(\bar{u}_1 + \cdots + \bar{u}_m) = \bar{0}$$

Rearranging, we get

$$(\alpha_1 + \cdots + \alpha_m)\bar{u}_1 + (\alpha_2 + \cdots + \alpha_m)\bar{u}_2 + \cdots + (\alpha_{m-1} + \alpha_m)u_{m-1} + \alpha_m u_m = \bar{0}.$$

Since $\{\bar{u}_1, \dots, \bar{u}_m\}$ is linearly independent, we have

$$\alpha_1 + \cdots + \alpha_m = 0, \quad \alpha_2 + \cdots + \alpha_m = 0, \quad \dots, \quad \alpha_{m-1} + \alpha_m = 0, \quad \alpha_m = 0.$$

Working our way through these equation backwards, starting the last one, we get $\alpha_m = 0$, so that $\alpha_{m-1} = 0$, so that $\alpha_{m-2} = 0$, dots, $\alpha_2 = 0$, $\alpha_1 = 0$. Therefore, the set is $\{\bar{v}_1, \dots, \bar{v}_m\}$ is linearly independent.

3.5 $\det A = \det A^T = \det(-A) = (-1)^3 \det A = -\det A$ implies $\det A = 0$.

3.8 (c) Adding row 2 and row 3 to row 1, we make all the entries in row 1 equal to $\alpha+\beta+\gamma$. Equating the powers at x^2 in the identity $x^3 + px + q = (x-\alpha)(x-\beta)(x-\gamma)$, we conclude that $\alpha + \beta + \gamma = 0$. It follows that the determinant is zero.

3.9(a) The columns are linearly dependent, hence A is not invertible, so that $\det A = 0$.

3.9(b) $\det(\lambda A) = \lambda^n \det A$.

3.9(d) $\det AB = (\det A)(\det B) = \det BA$

3.12 False. Counterexample: $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then $A^n = A$ for every n, so A is not nilpotent. Yet $\det A = 0$.

3.14(3.14) Yes,

$$Y = \left\{ \alpha \begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -1 \\ -7 \\ 3 \end{pmatrix} \ : \ \alpha, \beta \in \mathbb{R} \right\} = \operatorname{span}\{\bar{u}_1, \bar{u}_2\}, \text{ where } \bar{u}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \end{pmatrix} \text{ and } \bar{u}_2 = \begin{pmatrix} 2 \\ -1 \\ -7 \\ 3 \end{pmatrix}.$$

Being a span, Y is a subspace of $\mathbb{R}^4$. Since the two vectors are not proportional, they are linearly independent, hence form a basis of Y. Since Y has a basis consisting of two vectors, $\dim Y = 2$ and, therefore, Y is linearly isomorphic to $\mathbb{R}^2$. The linear operator $T\colon \mathbb{R}^2 \to Y$ defined by $T\bar{e}_1 = \bar{u}_1$ and $T\bar{e}_2 = \bar{u}_2$ is a linear isomorphism from $\mathbb{R}^2$ to Y; note that $T\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = \begin{pmatrix} 3\alpha+2\beta \\ 2\alpha-\beta \\ -7\beta \\ -\alpha+3\beta \end{pmatrix}$.

3.15(a) Yes, every 2×2 matrix admits a unique expansion:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

3.15(b) Hint: we learned in class that there is a linear isomorphism between $\mathcal{P}_2(\mathbb{R})$ and $\mathbb{R}^3$. Using this isomorphism, we may "transfer" the problem to $\mathbb{R}^3$.

3.19 Suppose that $C[0,1]$ is finite-dimensional. Since $\mathbb{P}$ may be viewed as a subspace of $C[0,1]$; hence $\mathbb{P}$ is finite-dimensional; contradiction.

3.21(a) Suppose that A and B have k columns; let $\bar{a}_1,\dots,\bar{a}_k$ be the columns of A and $\bar{b}_1,\dots,\bar{b}_k$ be the columns of B. Let $n=\operatorname{rank}A$ and $m=\operatorname{rank}B$. Find a basis $\{\bar{u}_1,\dots,\bar{u}_n\}$ of Range A and a basis $\{\bar{v}_1,\dots,\bar{v}_m\}$ of Range B. Then for every $i=1,\dots,k$, $\bar{a}_i$ is a linear combination of $\bar{u}_1,\dots,u_n$, and $\bar{b}_i$ is a linear combination of $\bar{v}_1,\dots,\bar{v}_m$, hence $\bar{a}_i+\bar{b}_i$ is a linear combination of $\bar{u}_1,\dots,\bar{u}_n,\bar{v}_1,\dots,\bar{v}_m$, hence

$$\bar{a}_i+\bar{b}_i\in\operatorname{span}\{\bar{u}_1,\dots,\bar{u}_n,\bar{v}_1,\dots,\bar{v}_m\}.$$

It follows that

$$\operatorname{Range}(A+B)=\operatorname{span}\{\bar{a}_1+\bar{b}_1,\dots,\bar{a}_k+\bar{b}_k\}\subseteq\operatorname{span}\{\bar{u}_1,\dots,\bar{u}_n,\bar{v}_1,\dots,\bar{v}_m\},$$

so that

$$\begin{aligned}\operatorname{rank}(A+B)&=\dim\operatorname{Range}(A+B)\\&\leqslant\dim\operatorname{span}\{\bar{u}_1,\dots,\bar{u}_n,\bar{v}_1,\dots,\bar{v}_m\}\leqslant n+m=\operatorname{rank}A+\operatorname{rank}B.\end{aligned}$$

3.21(b) Range $AB\subseteq$ Range A, so that $\operatorname{rank}(AB)\leqslant\operatorname{rank}A$. Furthermore,

$$\operatorname{rank}AB=\operatorname{rank}(AB)^T=\operatorname{rank}B^TA^T\leqslant\operatorname{rank}B^T=\operatorname{rank}B.$$

3.21(c) $\operatorname{rank}SA\leqslant\operatorname{rank}A$ and $\operatorname{rank}A=\operatorname{rank}S^{-1}SA\leqslant\operatorname{rank}SA$, hence $\operatorname{rank}SA=\operatorname{rank}A$. The other identity is similar.

4.1(a) $A=I^{-1}AI$.

4.1(b) If $A=S^{-1}BS$ then, after cancellations, $A^n=(S^{-1}BS)(S^{-1}BS)\dots(S^{-1}BS)=S^{-1}B^nS$.

4.1(c) If $A=S^-BS$ then $B=SAS^{-1}=TAT^{-1}$ where $T=S^{-1}$.

4.1(e) Let $B=S^{-1}AS$. Using Problem 3.21(c), we get $\operatorname{rank}B=\operatorname{rank}S^{-1}AS=\operatorname{rank}S^{-1}A=\operatorname{rank}A$.

4.1(g) Suppose that $B=S^{-1}AS$ and $A\bar{x}=\lambda\bar{x}$. Put $\bar{y}=S^{-1}\bar{x}$. Then

$$B\bar{y}=(S^{-1}AS)(S^{-1}\bar{x})=S^{-1}\lambda\bar{x}=\lambda\bar{y},$$

so that $B\bar{y}=\lambda\bar{y}$.

4.3 Hint: for (a), use Problem 4.2; for (b), consider the powers of the matrices.

4.5 Assign names to the functions: let $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = x$. Suppose $\alpha f + \beta g + \gamma h = 0$ for some scalars α, β, and γ. This means that

$$\alpha \sin x + \beta \cos x + \gamma x = 0$$

for all values of $x \in \mathbb{R}$. In particular, plugging in $x = 0$, $x = \pi/2$, and $x = \pi$, we get a homogeneous linear system of three equations for α, β, and γ. This system has only the trivial solution: $\alpha = \beta = \gamma = 0$.

4.6(4.6) Linearity of T is straightforward. $\operatorname{Range} T = \operatorname{span}\{u_1, u_2, u_3\}$, where u_1 is the constant one function, $u_2(x) = \sin x$, and $u_3(x) = \cos x$. The three functions are linearly independent, hence form a basis of $\operatorname{Range} T$. It follows that $\operatorname{rank} T = 3$.

4.7(a) Verify that it is closed under addition and scalar multiplication.

4.7(b) Every member of c_{00} may be written as a linear combination of finitely many $\bar{e}_i$'s in a unique way:

$$\text{if } \bar{x} = (x_1, \dots, x_m, 0, 0, \dots) \text{ then } \bar{x} = x_1\bar{e}_1 + \cdots + x_n\bar{e}_n.$$

4.7(c) Hint: the map $T\colon c_{00} \to \mathcal{P}(\mathbb{R})$ that sends $\bar{a} = (a_1, \dots, a_m, 0, 0, \dots)$ to the polynomial $a_1 + a_2x + \cdots + a_mx^{m-1}$ is a linear bijection, hence an isomorphism.

4.9(a) Observe that $q_i(x_j)$ equals 1 if $i = j$ and zero otherwise. Deduce from this that the set $\{q_1, \dots, q_n\}$ is linearly independent. Since $\dim X = n$, this set is a basis.

4.9(b) Take $p = d_1q_1 + \cdots + d_nq_n$.

4.10 $X \cap Y = \operatorname{span}\{\bar{e}_2, \bar{e}_3\}$, $X + Y = \operatorname{span}\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$. It follows that $\dim X + Y = 4$. On the other hand, properties of direct sums yield $\dim X \oplus Y = \dim X + \dim Y = 3 + 3 = 6$. Since isomorphic vector spaces must have the same dimensions, this implies that $X + Y$ and $X \oplus Y$ fail to be isomorphic.

Since $X \cap Y$ is non-trivial, we cannot interpret $X \oplus Y$ as an internal direct sum; that is, $X + Y \neq X \oplus Y$. So the only plausible way to interpret $X \oplus Y$ is as an external direct sum. While $X + Y$ is a subset of $\mathbb{R}^6$, $X \oplus Y$ is the Cartesian product $X \times Y$ with the appropriate operations. As a set, $X \oplus Y$ consists of pairs of the form $\left(\begin{bmatrix} a \\ b \\ c \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d \\ e \\ f \\ 0 \\ 0 \end{bmatrix}\right)$

5.1 (a)$\Rightarrow$(b) The zero vector has a unique expansion. But the zero vector has the "trivial" expansion: $\bar{0} = \bar{0} + \cdots + \bar{0}$. Hence, every expansion of zero has to be the trivial one.

(b)$\Rightarrow$(a) By the definition of the sum of subspaces, every vector in $X_1 + \cdots + X_m$ has a decomposition of the required form; we only need to check uniqueness. Suppose that $\bar{x}$ has two decompositions: $\bar{x} = \bar{x}_1 + \cdots + \bar{x}_m = \bar{y}_1 + \cdots + \bar{y}_m$ for some $\bar{x}_1, \bar{y}_1 \in X_1$, ...,

$\bar{x}_m, \bar{y}_m \in X_m$. Subtracting one from the other, we get $(\bar{x}_1 - \bar{y}_1) + \cdots + (\bar{x}_m - \bar{y}_m) = 0$. By (b), we conclude that $\bar{x}_1 - \bar{y}_1 = 0$, $\ldots$, $\bar{x}_m - \bar{y}_m = 0$, so that $\bar{x}_1 = \bar{y}_1$, $\ldots$, $\bar{x}_m = \bar{y}_m$.

5.2 Put $X = \operatorname{span}\mathcal{U}$ and $Y = \operatorname{span}\mathcal{V}$. Since $\mathcal{U}$ is a subset of a basis, it is linearly independent, hence $\mathcal{U}$ is a basis of X. Similarly, $\mathcal{V}$ is a basis of Y.

We claim that $Z = X + Y$. Indeed,

$$X + Y = \operatorname{span} X \cup Y \supseteq \operatorname{span}\mathcal{U} \cup \mathcal{V} = Z$$

because $\mathcal{U} \cup \mathcal{V}$ is a basis of Z. Hence, $X + Y = Z$.

It is left to show that $X \cap Y = \{\bar{0}\}$. Indeed, if $\bar{x} \in X \cap Y$ then $\bar{x}$ can be written as a linear combination of elements of $\mathcal{U}$ and as a linear combination of elements of $\mathcal{V}$:

$$\bar{x} = \alpha_1\bar{u}_1 + \cdots + \alpha_n\bar{u}_n = \beta_1\bar{v}_1 + \cdots + \beta_m\bar{v}_m,$$

which implies that

$$\alpha_1\bar{u}_1 + \cdots + \alpha_n\bar{u}_n - \beta_1\bar{v}_1 - \cdots - \beta_m\bar{v}_m = \bar{0}.$$

Since the set $\{\bar{u}_1, \ldots, \bar{u}_n, \bar{v}_1, \ldots, \bar{v}_m\}$ is linearly independent, it follows that $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_m = 0$, hence $\bar{x} = \bar{0}$.

5.3 Since column operations do not affect the span, we may replace the second vector by the difference of the two, that is, $X = \operatorname{span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$. We can now easily find an invertible matrix whose columns include these two vectors: $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}$. Then the four columns of this matrix form a basis of $\mathbb{R}^4$. The 1st and the 4th columns form a basis of X, so we can use the remaining two columns as a basis of an algebraic complement of X, that is, we may take $Y = \operatorname{span}\left\{\begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}$. Note that the answer is not unique.

5.4 It can be easily verified that $P^2 = P$, hence P is a projection. The range of P is the span of its columns, hence the span of $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

5.6 Let $Y = \operatorname{Range} P$ and $Z = \ker P$, then $X = Y \oplus Z$ and P is the projection onto Y along Z. Being subspaces of a finite-dimensional vector space, Y and Z are finite-dimensional. Let $\{\bar{e}_1, \ldots, \bar{e}_m\}$ be a basis of Y; let $\{\bar{e}_{m+1}, \ldots, \bar{e}_n\}$ be a basis of Z. Then $\mathcal{B} = \{\bar{e}_1, \ldots, \bar{e}_m, \bar{e}_{m+1}, \ldots, \bar{e}_n\}$ is a basis of X. Note that $P\bar{e}_1 = \bar{e}_1$, $\ldots P\bar{e}_m = \bar{e}_m$, $P\bar{e}_{m+1} = \bar{0}$, $\ldots$, $P\bar{e}_n = \bar{0}$. So the matrix of P with respect to the basis $\mathcal{B}$ has columns $\bar{e}_1, \ldots, \bar{e}_n, \bar{0}, \ldots, \bar{0}$, so it has the required form.

In particular, the matrix of P with respect to this basis is diagonal; hence the eigenvalues of P are he diagonal entries, which are just zeros and ones.

5.7 A function of the form $f(x) = ax+b$ is called *affine*. Observe that $X \cap Y = \{0\}$ because the only affine function that vanishes at the endpoints is the zero function. To show that $X + Y = C[0,1]$, take any $f \in C[0,1]$ and observe that there is an affine function h such that h agrees with f at the endpoints. It follows that $g := f - h$ is in X. We have $f = g+h$ with $g \in X$ and $h \in Y$. It follows that $X + Y = C[0,1]$.

5.8(5.8) Hint: P is the projection onto the subspace of all even functions along the subspace of all odd functions.

5.9(a) The characteristic polynomial $\det(\lambda I - A) = \lambda^3 - 12\lambda - 16 = (\lambda-4)(\lambda+2)^2$; the roots = the eigenvalues are $\lambda = 4, -2$. Solving the homogeneous linear system $(4I - A)\bar{x} = 0$, we see that the eigenspace of 4 is spanned by $\bar{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, hence every eigenvector for 4 is a scalar multiple of this vector. Solving the homogeneous linear system $(-2I - A)\bar{x} = 0$, we see that the eigenspace of -2 is the span of $\bar{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\bar{u}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

5.9(b) The set $\mathcal{B} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is a basis. It follows from $A\bar{u}_1 = 4\bar{u}_1$, $A\bar{u}_2 = -2\bar{u}_2$, and $A\bar{u}_3 = -2\bar{u}_3$ that in this basis the matrix of A is $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

5.9(c) Since A is diagonal in some basis, it is similar to a diagonal matrix. More precisely, $S^{-1}AS$ is diagonal, where the columns of S are $\bar{u}_1$, $\bar{u}_2$, and $\bar{u}_3$.

5.10(b) By Rank Theorem,

$$\dim \operatorname{Range} A^k = n - \dim \operatorname{Null} A^k = n - \dim \operatorname{Null} A^{k+1} = \dim \operatorname{Range} A^{k+1}.$$

Combining this with $\operatorname{Range} A^k \supseteq \operatorname{Range} A^{k+1}$, we conclude that the two subspaces are equal.

By (a), we have $\ker A^{k+1} \subseteq \ker A^{k+2}$. To prove the converse inclusion, let $\bar{x} \in \ker A^{k+2}$. Then $\bar{0} = A^{k+2}\bar{x} = A^{k+1}(A\bar{x})$. It follows that $A\bar{x} \in \ker A^{k+1}$ and, by assumption, $A\bar{x} \in \ker A^k$. Then $\bar{0} = A^k(A\bar{x}) = A^{k+1}\bar{x}$. It follows that $\bar{x} \in \ker A^{k+1}$.

5.10(c) By (a), $\dim \operatorname{Null} A^i \leqslant \dim \operatorname{Null} A^{i+1}$ for every i. Also, $\dim \operatorname{Null} A^i \leqslant n$ for every n as $\operatorname{Null} A^i$ is a subspace of $\mathbb{F}^n$. It follows that $\operatorname{Null} A^k = \operatorname{Null} A^{k+1}$ for some $k \leqslant n$. Now apply (b).

5.11(a) Suppose $(AB)^k = 0$ for some k. Then $(BA)^{k+1} = (BA)\cdots(BA) = B(AB)\cdots(AB)A = B(AB)^k B = 0$.

5.11(b) By Problem 5.10, the sequence of the null spaces of A^k grows, and then stabilizes after some $k \leqslant n$. Since A is nilpotent, $A^m = 0$ for some m and, therefore $\operatorname{Null} A^i = \mathbb{F}^n$ for all $i \geqslant m$. It follows that $\operatorname{Null} A^k = \mathbb{F}^n$ and, therefore, $\operatorname{Null} A^n = \mathbb{F}^n$.

5.11(c) Suppose $A^2 = S$. Then $A^{2n} = S^n = 0$, so that A is nilpotent. Then, by (b), $A^n = 0$. $2n - 2 \geqslant n$, we have $A^{2n-2} = 0$. However, $A^{2n-2} = S^{n-1} \neq 0$; contradiction.

5.12(a) Suppose that

$$\alpha_0\bar{x} + \alpha_1 A\bar{x} + \cdots + \alpha_{k-2}A^{k-2}\bar{x} + \alpha_{k-1}A^{k-1}\bar{x} = \bar{0}.$$

Applying A^{k-1} to both sides and using the fact that $A^k = 0$, we get $\alpha_0 A^{k-1}\bar{x} = \bar{0}$ and, therefore, $\alpha_0 = 0$. Therefore, we have

$$\alpha_1 A\bar{x} + \cdots + \alpha_{k-2}A^{k-2}\bar{x} + \alpha_{k-1}A^{k-1}\bar{x} = \bar{0}.$$

Applying A^{k-2} to both sides and using the fact that $A^k = 0$, we get $\alpha_1 A^{k-1}\bar{x} = \bar{0}$ and, therefore, $\alpha_1 = 0$. Proceeding in a similar manner, we conclude that $\alpha_2 = 0, \ldots, \alpha_{k-1} = 0$.

5.12(b) Since $A^{n-1} \neq 0$, we can find a vector $\bar{x} \in \mathbb{F}^n$ such that $A^{n-1}\bar{x} \neq \bar{0}$. Since $A^n = 0$, we have $A^n\bar{x} = \bar{0}$. By the first part of the problem , we conclude that the set $\{\bar{x}, A\bar{x}, \ldots, A^{n-1}\bar{x}\}$ is linearly independent. Since this set consists of n vectors, it forms a basis of $\mathbb{F}^n$. Re-label this basis as $\bar{e}_1 = A^{n-1}\bar{x}$, $\bar{e}_2 = A^{n-2}\bar{x}$, …, $\bar{e}_n = \bar{x}$. Then $A\bar{e}_1 = \bar{0}$, $A\bar{e}_2 = \bar{e}_1$, $A\bar{e}_3 = \bar{e}_2$, etc. Hence, in this basis, A is the left shift.

5.12(c) The right shift satisfies the assumption of (b).

6.1(b) Enumerate each $\mathcal{U}_i$: let $\mathcal{U}_i = \{\bar{u}_{i,1}, \ldots, \bar{u}_{i,k_i}\}$. Note that here $\bar{u}_{i,j}$ is a *vector*, namely, the j-th vector of $\mathcal{U}_i$.

Suppose some linear combination of all these vectors equals zero: $\sum_{i=1}^{m}\sum_{j=1}^{k_i}\alpha_{i,j}\bar{u}_{i,j} = 0$. For each i, put $\bar{v}_i = \sum_{j=1}^{k_i}\alpha_{i,j}\bar{u}_{i,j}$. Then $\sum_{i=1}^{m}\bar{v}_i = 0$. Note that $\bar{v}_i \in E_{\lambda_i}$. We proved in class that eigenvectors corresponding to distinct eigenvalues are linearly independent. It follows that $\bar{v}_i = 0$ for each i. Hence, $\sum_{j=1}^{k_i}\alpha_{i,j}\bar{u}_{i,j} = 0$. Since for every fixed i, the vectors $\bar{u}_{i,1}, \ldots, \bar{u}_{i,k_i}$ come from $\mathcal{U}_i$, they are linearly independent; it follows from $\sum_{j=1}^{k_i}\alpha_{i,j}\bar{u}_{i,j} = 0$ that $\alpha_{i,1} = 1, \ldots, \alpha_{i,k_i} = 0$.

6.6(b) $f \sim g$ iff there exists $\varepsilon > 0$ such that f and g agree on $(0, \varepsilon)$.

6.6(c) If two polynomials p and q are in the same equivalence class then $p \sim q$, so that p and q agree on some open interval, hence $p = q$.

6.6(d) No. Let $f(x) = \cos x$, and suppose that the equivalence class of f contains a polynomial p. Then p and f agree on some open interval. Then $p^{(n)}$ agrees with $f^{(n)}$ on this interval for every $n \in \mathbb{N}$. For a sufficiently large n, $p^{(n)}$ is the zero function, while derivatives of f do not vans on any open interval; a contradiction.

6.7 We know that the matrix $R = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ represents the rotation of $\mathbb{R}^2$ by $90°$, and has no real eigenvectors and no real eigenvalues, so $\sigma(R) = \varnothing$ over $\mathbb{R}$. Let N be the 1×1 zero matrix. Put A to be the block diagonal matrix $A = \left[\begin{smallmatrix} R & 0 \\ 0 & N \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, then $\sigma(A) =$

$\sigma(R) \cup \sigma(N) = \varnothing \cup \{0\} = \{0\}$. On the other hand, R is invertible, hence $R^m \neq 0$ for any m. It follows that $A^m = \left[\begin{smallmatrix} R^m & 0 \\ 0 & N^m \end{smallmatrix}\right] \neq 0$, hence A is not nilpotent.

6.8 Hint: Use Problem 4.2.

6.9 Let Y be the set of all polynomials p in $\mathbb{P}_5(\mathbb{R})$ such that $p(0) = 0$. Clearly, Y consists of all the polynomials with zero constant term, i.e., $\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5$. Hence Y is a subspace and the set $\mathcal{B} = \{x^1, x^2, x^3, x^4, x^5\}$ is a basis of Y. Let $T\colon \mathbb{P}_5(\mathbb{R}) \to \mathbb{P}_5(\mathbb{R})$ given by $(Tf)(x) = f(x-7)$. It is easy to see that T is a linear isomorphism and $T(Y) = X$. Hence X is a subspace of $\mathbb{P}_5(\mathbb{R})$ and $X \simeq Y$. In particular, the set $T(\mathcal{B})$, which equals

$$\left\{x - 7, (x-7)^2, (x-7)^3, (x-7)^4, (x-7)^5\right\}$$

is a basis of X.

6.11 Every linear combination of elements of this set may be also viewed as a linear combination of $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \dots\}$. Since the latter set is a Hamel basis of c_{00}, so is the given set.

6.12 Let $\bar{u}$ be the constant one sequence. Then $X = \operatorname{span} \mathcal{B}$, where $\mathcal{B} = \{\bar{u}, \bar{e}_1, \bar{e}_2, \dots\}$, hence X is a subspace of $\mathbb{R}^{\mathbb{N}}$. Every element of X has a unique expansion with respect to $\mathcal{B}$, hence $\mathcal{B}$ is a Hamel basis of X.

6.14 Let Y be any algebraic complement of X in Z. Let T be the projection onto X along Y and S the projection onto Y along X.

6.17 Let $\bar{x} \in \ker T$. Then $T(S\bar{x}) = S(T\bar{x}) = S(\bar{0}) = \bar{0}$, so that $Sx \in \ker T$. It follows that $\ker T$ is S-invariant.

Let $\bar{y} \in \operatorname{Range} T$. Then $\bar{y} = T\bar{x}$ for some $\bar{x} \in X$. Then $S\bar{y} = S(T\bar{x}) = T(S\bar{x}) \in \operatorname{Range} T$. It follows that $\operatorname{Range} T$ is S-invariant.

Let λ be an eigenvalue of T. Then the corresponding eigenspace is $\ker(T - \lambda I)$. Since $T - \lambda I$ commutes with S, its kernel is S-invariant.

For the last claim, just note that T^m commutes with T.

6.18(a) Suppose that Y is T-invariant. Take any projection $P\colon X \to X$ with $Y = \operatorname{Range} P$. For every $\bar{x} \in X$, we have $P\bar{x} \in Y$ and, since Y is T-invariant, $\bar{y} := TP\bar{x} \in Y$. It follows that $P\bar{y} = \bar{y}$, so that $PTP\bar{x} = TP\bar{x}$ and, therefore, $(I - P)TP\bar{x} = 0$. Since $\bar{x}$ was an arbitrary element of X, we conclude that $(I - P)TP = 0$.

Conversely, suppose that $(I - P)TP = 0$ for some projection P with $\operatorname{Range} P = Y$. It follows that $TP = PTP$. Let $\bar{y} \in Y$, we need to show that $T\bar{y} \in Y$. Indeed, $\bar{y} = P\bar{y}$, so $T\bar{y} = TP\bar{y} = PTP\bar{y} \in Y$.

6.18(b) Suppose $TP = PT$ for some projection P. Then $\operatorname{Range} P$ and $\ker P$ are invariant under T by Problem 6.17. Since P is a projection, we have $X = (\operatorname{Range} P) \oplus (\ker P)$, hence this is a reducing decomposition for T.

Conversely, suppose that $X = Y \oplus Z$, where Y and Z are both invariant under T. Let $P\colon X \to X$ be the projection onto Y along Z. Let $x \in X$, then $x = y + z$ where $y = Px \in Y$ and $z = (I - O)x \in Z$. Then $TPx = Ty = PTy$ because $Ty \in Y$ as Y is invariant. On the other hand, $PTx = PTy + PTz$. Since Z is invariant, we have $Tz \in Z$ and, therefore, $TPz = 0$, so that $PTx = PTy$. Thus, $TPx = PTx$, so that $TP = PT$.

7.1 WLOG, changing to the appropriate basis, we may assume that A equals its Jordan Canonical Form. We will use the following notation:

$$A = J_{2,3} \oplus J_{3,1} \oplus J_{3,5} \oplus J_{1,4} \oplus J_{0,2} \oplus J_{0,3} \oplus J_{0,3} \oplus J_{0,4} \oplus J_{0,1} \oplus J_{0,1} \oplus J_{0,1}.$$

Using block-matrix arithmetic, we conclude that

$$A^k = J_{2,3}^k \oplus J_{3,1}^k \oplus J_{3,5}^k \oplus J_{1,4}^k \oplus J_{0,2}^k \oplus J_{0,3}^k \oplus J_{0,3}^k \oplus J_{0,4}^k \oplus J_{0,1}^k \oplus J_{0,1}^k \oplus J_{0,1}^k$$

for every power k. By Problem 6.19, the rank of A^k is the sum of the ranks of its blocks. The block $J_{2,3}$ is an upper-triangular matrix with non-zero diagonal entries, hence is invertible, and every power of it is invertible. It follows that it has full rank, that is, $\operatorname{rank} J_{2,3} = 3$, and the same is true for every power of $J_{2,3}$. The same reasoning goes for all the other blocks with non-zero entries. On the other hand, $J_{0,4}$ is a 4×4 shift, hence $\operatorname{rank} J_{0,4} = 3$, $\operatorname{rank} J_{0,4}^2 = 2$, $\operatorname{rank} J_{0,4}^3 = 1$, and $\operatorname{rank} J_{0,4}^k = 0$ for all $k \geqslant 4$. The other nilpotent blocks behave in a similar way. Therefore, grouping together the non-nilpotent blocks and grouping together the nilpotent block, we get

$$\begin{aligned}
\operatorname{rank} A &= (3+1+5+4) + (1+2+2+3+0+0+0) = 21,\\
\operatorname{rank} A^2 &= (3+1+5+4) + (0+1+1+2+0+0+0) = 17,\\
\operatorname{rank} A^3 &= (3+1+5+4) + (0+0+0+1+0+0+0) = 14,\\
\operatorname{rank} A^4 &= (3+1+5+4) + (0+0+0+0+0+0+0) = 13,\\
\operatorname{rank} A^5 &= (3+1+5+4) + (0+0+0+1+0+0+0) = 13,\\
&\vdots \qquad \vdots
\end{aligned}$$

7.2 Observe that

$$\bar{e}_1 \mapsto \bar{e}_4 \mapsto \bar{e}_5 \mapsto \bar{0}, \quad \bar{e}_3 \mapsto \bar{e}_2 \mapsto \bar{e}_9 \mapsto \bar{e}_7 \mapsto \bar{0}, \quad \bar{e}_6 \mapsto \bar{0}. \bar{e}_8 \mapsto \bar{e}_8,$$

Here an arrow means that A maps $\bar{e}_i$ into a scalar multiple of $\bar{e}_j$. This includes three chains and one loop. Consider a basis which is a permutation of the original basis (we just enumerate the chains in the reverse order):

$$\bar{u}_1 = \bar{e}_5, \bar{u}_2 = \bar{e}_4, \bar{u}_3 = \bar{e}_1, \bar{u}_4 = \bar{e}_7, \bar{u}_5 = \bar{e}_9, \bar{u}_6 = \bar{e}_2, \bar{u}_7 = \bar{e}_3, \bar{u}_8 = \bar{e}_6, \bar{u}_9 = \bar{e}_8.$$

Then

$A\bar{u}_1 = \bar{0}, A\bar{u}_2 = 4\bar{u}_1, A\bar{u}_3 = 3\bar{u}_2, A\bar{u}_4 = \bar{0}, A\bar{u}_5 = 5\bar{u}_4, A\bar{u}_6 = 6\bar{u}_5, A\bar{u}_7 = 2\bar{u}_6, A\bar{u}_8 = \bar{0}, A\bar{u}_9 = 7\bar{u}_9.$

Hence, in this basis, the matrix of A is

$$\begin{pmatrix} . & 4 & . & . & . & . & . & . & . \\ . & . & 3 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & 5 & . & . & . & . \\ . & . & . & . & . & 6 & . & . & . \\ . & . & . & . & . & . & 2 & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 7 \end{pmatrix}$$

For convenience, we replace zeros with dots. This is a block-diagonal matrix with blocks of size 3×3, 4×4, 1×1, and 1×1. Since it is obtained by re-arranging the order of the basis vectors, it is of the form PAP^{-1} for some permutation matrix P.

Finally, using Problem 4.2, we can find an appropriate diagonal matrix D to convert all the entries in the 2nd diagonal into ones:

$$DPAP^{-1}D^{-1} = \begin{pmatrix} . & 1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 7 \end{pmatrix}$$

Clearly, this is the Jordan Canonical Form of A.

7.3 It is more convenient to convert the data into nullities using the Rank-Nullity Theorem: $d(A-2I) = 1$, $d(A-3I) = 2$, $d(A-3I)^2 = 4$, $d(A-3I)^3 = 5$, $d(A-3I)^4 = 6$. Since $\ker(A-2I)$ and $\ker(A-3I)$ are non-trivial, it follows that $2, 3 \in \sigma(A)$. Consider their generalized eigenspaces. It follows from $\widetilde{E}_2 \supseteq \ker(A-2I)$ that $\dim \widetilde{E}_2 \geqslant \dim(A-2I) = d(A-2I) = 1$. Similarly, $\widetilde{E}_3 \supseteq \ker(A-3I)^4$ implies $\dim \widetilde{E}_3 \geqslant \dim(A-3I)^4 = d(A-2I)^4 = 6$. It follows from $\dim \widetilde{E}_2 + \dim \widetilde{E}_3 = 6 = \dim X$ that $X = \widetilde{E}_2 \oplus \widetilde{E}_3$. Therefore, $\sigma(A) = \{2, 3\}$. It also follows that $\dim \widetilde{E}_2 = 1$ and $\dim \widetilde{E}_3 = 6$, so the corresponding superblocks in the Jordan form are 1×1 and 6×6. It follows from $d(A-3I) = 2$ that there are two Jordan blocks for $\lambda = 3$. It now follows from $d(A-3I)^2 = 4$ and $d(A-3I)^3 = 5$ that these blocks have to be 4×4 and 2×2. Thus, $A = J_{2,1} \oplus J_{3,4} \oplus J_{3,2}$.

7.4 Hint: similar to Problem 7.2.

7.5 $p_A(\lambda) = \det(\lambda I - A) = (\lambda-2)^2(\lambda-3)^2$. It follows that the eigenvalues of A are 2 and 3, both have algebraic multiplicities 2.

To find $E_2 = \ker(A-2I)$, solve $(A-2I)\bar{x} = \bar{0}$. Row reduction of $A - 2I = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, so that x_2 and x_4 are free variables. Taking first $x_2 = 1$ and $x_4 = 0$ and then taking $x_2 = 0$ and $x_4 = 1$, we get two vectors $\bar{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $\bar{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $\{\bar{u}_1, \bar{u}_2\}$ is a basis of E_2. It follows that the geometric multiplicity of $\lambda = 2$ is two, equals its algebraic multiplicity. So the superblock for $\lambda = 2$ is just $\operatorname{diag}(2,2)$.

To find $E_3 = \ker(A-3I)$, solve $(A-3I)\bar{x} = \bar{0}$. Row reduction of $A - 3I = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 4 & 0 & 1 & -1 \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$, so that x_4 is a free variable. Taking first $x_4 = 3$, we get $\bar{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$. $\bar{u}_3$ is an eigenvector for $\lambda = 3$. $E_3 = \operatorname{span}\{\bar{u}_3\}$. Hence, the geometric multiplicity of $\lambda = 3$ is 1. Find a generalized eigenvector $\bar{u}_4$ for $\lambda = 3$ by solving $(A-3I)\bar{u}_4 = \bar{u}_3$. Then $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4\}$ is a required basis, and the Jordan form of A is $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

7.6 WLOG, A is a Jordan Form, with eigenvalues $\lambda_1, \dots, \lambda_m$. Then $p(A)$ is an upper triangular matrix with $p(\lambda_1), \dots, p(\lambda_m)$ on the diagonal, so $\sigma\big(p(A)\big) = \big\{p(\lambda_1), \dots, p(\lambda_m)\big\}$.

8.1 To show that $[\cdot,\cdot]$ is an inner product, verify that it satisfies the axioms of the inner product. The set $\{A^{-1}\bar{e}_1, \dots, A^{-1}\bar{e}_n\}$ is an orthogonal basis for this inner product. Normalizing it, we get a required basis.

8.2 *Hint:* analyze the proofs of the inequalities.

8.5 To show that $Y^\perp$ is a subspace, we need to verify that it is closed under scalar multiplication and addition.

Suppose that $\bar{x} \in Y^\perp$ and $\alpha \in \mathbb{F}$. Then $\bar{x} \perp \bar{y}$ for every $\bar{y} \in Y$, so that $(\alpha\bar{x}) \perp \bar{y}$ for every $\bar{y} \in Y$, hence $\alpha\bar{x} \in Y^\perp$.

Suppose that $\bar{x}_1\bar{x}_2 \in Y^\perp$ and $\alpha \in \mathbb{F}$. Then $\bar{x}_1 \perp \bar{y}$ and $\bar{x}_2 \perp y$ for every $\bar{y} \in Y$, so that $(\bar{x}_1 + \bar{x}_2) \perp \bar{y}$ for every $\bar{y} \in Y$, hence $\bar{x}_1 + \bar{x}_2 \in Y^\perp$.

Let P be the orthogonal projection onto Y. Then $Y = \operatorname{Range} P$ and $Y^\perp = \ker P$. Let $Q = I - P$ be the complementary projection. Then $\ker Q = \operatorname{Range} P = Y$ and $\operatorname{Range} Q = \ker P = Y^\perp$. It follows that Q is again orthogonal and that $Y = (Y^\perp)^\perp$.

8.6(b) We may identify X with $\mathbb{F}^n$ for some n. Let $f\colon X \to \mathbb{F}$ be a linear functional. Then we may view f as a $1 \times n$ matrix. Writing it as a column and taking the complex conjugate of every entry, we get the required vector $\bar{y}$.

9.2 Use the facts that eigenvectors corresponding to distinct eigenvalues are linearly independent, and that any n linearly independent vectors in $\mathbb{F}^n$ span all of $\mathbb{F}^n$.

9.3(a) Take the matrix of all 1's. Its columns are linearly dependent, hence this matrix is not invertible.

9.3(b) We claim that the minimal possible number is n. The columns must be linearly independent, hence no column is zero, hence every column must contain at least one non-zero entry, so there must be at least n non-zero entries in total. On the other hand, I has exactly n non-zero entries.

9.3(c) The matrices that have exactly one non-zero entry in every row and in every column.

9.4(a) Yes. If $\bar{x}, \bar{y} \in Y \cap Z$ then $\bar{x}, \bar{y} \in Y$ and $\bar{x}, \bar{y} \in Z$, so that $\bar{x} + \bar{y} \in Y$ and $\bar{x} + \bar{y} \in Y$, hence $\bar{x} + \bar{y} \in Y \cap Z$. Therefore, $Y \cap Z$ is closed under addition. It can be shown in a similar fashion that $Y \cap Z$ is closed under scalar multiplication.

9.4(b) No. Counterexample: let Y and Z be the x-axis and the y-axis in $\mathbb{R}^2$.

9.5 If $\bar{u} = \bar{0}$ or $\bar{v} = \bar{0}$ then $A = 0$, in which case $\operatorname{Range} A = \{0\}$ and $\ker A = \mathbb{R}^n$. So from now on, assume that both $\bar{u}$ and $\bar{v}$ are non-zero.

For any vector $\bar{x}$, we have

$$A\bar{x} = (\bar{u}\bar{v}^T)\bar{x} = \bar{u}(\bar{v}^T\bar{x}) = \langle \bar{v}, \bar{x} \rangle \bar{u}$$

Note that $\bar{v}^T\bar{x} = \langle \bar{v}, \bar{x} \rangle$ is a number, so $A\bar{x} = \bar{0}$ iff this number is zero, so $\bar{x} \in \ker A$ iff $\langle \bar{v}, \bar{x} \rangle = 0$ iff $\bar{x}$ is orthogonal to $\bar{v}$. Thus,

$$\begin{aligned}\ker A = \big\{\bar{x} \in \mathbb{R}^n \,:\, \langle \bar{v}, \bar{x} \rangle = 0\big\} &= \big\{\bar{x} \in \mathbb{R}^n \,:\, \bar{x} \perp \bar{v}\big\} \\ &= \text{the set of all vectors orthogonal to } \bar{v}.\end{aligned}$$

Since $\bar{v}$ is non-zero, $\langle \bar{v}, \bar{x} \rangle$ could be any number as $\bar{x}$ runs through $\mathbb{R}$, so $A\bar{x}$ could be any scalar multiple of $\bar{u}$, hence $\operatorname{Range} A = \operatorname{span}\{\bar{u}\}$.

9.6 (a) No. The polynomials p, q, and r lack the first power of x, hence every linear combination of them also lacks it, while $3 + x + 2x^3$ has it.

(b)

$$3 + x^2 + 2x^3 + 7x^5 = \alpha p(x) + \beta q(x) + \gamma r(x) = \alpha + (\alpha + \beta)x^2 - \beta x^3 + \gamma x^5,$$

which is satisfied when $\alpha = 3$, $\beta = -2$, and $\gamma = 7$.

9.7 Suppose that $\alpha_1 f + \alpha_2 g + \alpha_3 h = 0$ (the constant zero function). Then $\alpha_1 + \alpha_2 \sin t + \alpha_3(1+t^2) = 0$ for every t. Plugging in $t = 0$ and $t = \pi$, we get $\alpha_1 + \alpha_3 = 0$ and $\alpha_1 + \alpha_3(1+\pi^2) = 0$. This is a linear system for α_1 and α_3; solving it, we get $\alpha_1 = \alpha_3 = 0$. Plugging this into the original equation, we get $\alpha_2 \sin t = 0$ for all t, which yields $\alpha_2 = 0$.

9.8 The proof that T is linear is straightforward. $\ker T = \{(x_1, 0, 0, \dots) : x_1 \in \mathbb{R}\}$. The kernel is non-trivial, hence T is not one-to-one. $\operatorname{Range} T = \mathbb{R}^{\mathbb{N}}$; so T is onto. Let $\bar{x} = (1, \lambda, \lambda^2, \lambda^3, \dots)$, then $T\bar{x} = \lambda\bar{x}$, so λ is an eigenvalue for T, while $\bar{x}$ is an eigenvector for T and λ.

9.9 The proof that S is linear is straightforward. It follows from the definition that S is one-to-one; hence its kernel is trivial. S is not onto because the vector $(1, 0, 0, \dots)$ is not in the range. The range of S consists of all the sequences whose first component is 0. Since $\ker S = \{0\}$, zero is not an eigenvalue of S. Let $\lambda \neq 0$, show that λ is not an eigenvalue of S. Suppose it is, then $S\bar{x} = \lambda x$ for some non-zero vector x. Then

$$(0, x_1, x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \dots).$$

Since $\lambda \neq 0$, we conclude from $0 = \lambda x_1$ that $x_1 = 0$. Then $0 = \lambda x_2$, so that $x_2 = 0$. Proceeding inductively, we see that $x_n = 0$ for all n, hence $\bar{x} = 0$; a contradiction.

9.16

$$J_{2,0} \oplus J_{3,2} \oplus J_{1,2} \oplus J_{4,3} \oplus J_{2,5} \quad \text{and} \quad J_{2,0} \oplus J_{3,2} \oplus J_{1,2} \oplus J_{4,3} \oplus J_{1,5} \oplus J_{1,5}$$

9.19 Suppose that $\sigma(A) = \{\lambda_1, \dots, \lambda_m\}$ and $\arg \lambda_i = \frac{m_i}{k_i}\pi$, where $m_i \in \mathbb{Z}$ and $k_i \in \mathbb{N}$. Put $k = 2k_1 \cdots k_m$. Then $k \cdot \arg \lambda_i$ is an integer multiple of 2π, hence $\lambda_i^k = 1$. It follows from Spectral Mapping Theorem that $\sigma(A^k) = \{1\}$, so that $\sigma(A^k - I) = \{0\}$. Hence $A^k - I$ is an $n \times n$ nilpotent matrix, so that $(A^k - I)^n = 0$.

9.22 $(A^*A)^* = A^*A^{**} = A^*A$.

9.24(a) Let $\bar{x} \in \ker T^*$; we will show that $\bar{x} \in (\operatorname{Range} T)^\perp$ or, equivalently, $\bar{x} \perp \operatorname{Range} T$, that is, $\bar{x} \perp (T\bar{y})$ for every $\bar{y}$. Indeed, take an arbitrary $\bar{y}$, then

$$\langle \bar{x}, T\bar{y} \rangle = \langle T^*\bar{x}, \bar{y} \rangle = \langle \bar{0}, \bar{y} \rangle = 0.$$

Conversely, suppose that $\bar{x} \in (\operatorname{Range} T)^\perp$; we will show that $\bar{x} \in \ker T^*$. For every vector $\bar{y}$, we have

$$0 = \langle \bar{x}, T\bar{y} \rangle = \langle T^*\bar{x}, \bar{y} \rangle.$$

Since this holds true for every vector $\bar{y}$, it follows from Problem 9.31 that $T^*\bar{x} = 0$, hence $\bar{x} \in \ker T^*$.

9.24(b) Take the orthogonal complements of both sides of (a).

9.24(c) Replace T with T^* in (a).

9.28 Let p_A be the characteristic polynomial of A. Then $4+i$ is a root of p_A. Since p_A has real coefficients, $\overline{4+i} = 4-i$ is also a root of p_A. Since p_A is a cubic polynomial, it has a real root. Therefore, p_A has three distinct roots, so that A has three distinct eigenvalues. It follows that A is diagonalizable.