MATH 217 (Fall 2021)

Honors Advanced Calculus, I

Assignment #5

1. Let $D := \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$, and let

$$f\colon D \to \mathbb{R}, \quad (x,y) \mapsto \frac{x^2}{y}$$

Show that:

(a) $\lim_{\substack{t\to 0\\ t\neq 0}} f(tx_0, ty_0) = 0$ for all $(x_0, y_0) \in D$;

(b) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. A set $D \subset \mathbb{R}^N$ is called *discrete* if, for each $x \in D$, there is $\epsilon > 0$ such that $B_\epsilon(x) \cap D = \{x\}$.

Show that the following are equivalent for $\varnothing \neq D \subset \mathbb{R}^N$:

(i) $D \subset \mathbb{R}^N$ is discrete;

(ii) every sequence $(x_n)_{n=1}^\infty$ in D converging to a point in D is eventually constant, i.e., there is $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ for $n \geq n_0$;

(iii) every function $f\colon D \to \mathbb{R}$ is continuous.

Which are the subsets of $\mathbb{R}^N$ that are both compact and discrete?

3. Let $K, L \subset \mathbb{R}^N$ be compact and non-empty. Show that

$$K + L := \{x + y : x \in K,\, y \in L\}$$

is compact in $\mathbb{R}^N$.

4. Let $\varnothing \neq D \subset \mathbb{R}^N$. A function $f\colon D \to \mathbb{R}^M$ is called *Lipschitz continuous* if there is $C \geq 0$ such that

$$\|f(x) - f(y)\| \leq C\|x - y\|$$

for all $x, y \in D$.

Show that:

(a) each Lipschitz continuous function is uniformly continuous;

(b) if $f : [a,b] \to \mathbb{R}$ is continuous such that f is differentiable on (a,b) with f' bounded on (a,b), then f is Lipschitz continuous;

(c) the function

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto \sqrt{x}$$

is uniformly continuous, but not Lipschitz continuous.

5. Let $C \subset \mathbb{R}^N$. We say that $x_0, x_1 \in C$ can be *joined by a path* if there is a continuous function $\gamma\colon [0,1] \to \mathbb{R}^N$ with $\gamma([0,1]) \subset C$, $\gamma(0) = x_0$, and $\gamma(1) = x_1$. We call C *path connected* if any two points in C can be joined by a path.

   Show that any path connected set is connected.

6*. Let

$$C := \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0 \right\} \subset \mathbb{R}^2.$$

   Show that $\overline{C}$ is connected, but not path connected. (*Hint*: Show that $\{0\} \times [-1,1] \subset \overline{C}$ and that any point in $\{0\} \times [-1,1]$ cannot be joined by a path with any point of the form $\left(x, \sin\left(\frac{1}{x}\right)\right)$ with $x > 0$.)

Due Thursday, October 14, 2020, at 5:00 p.m.; no late assignments.