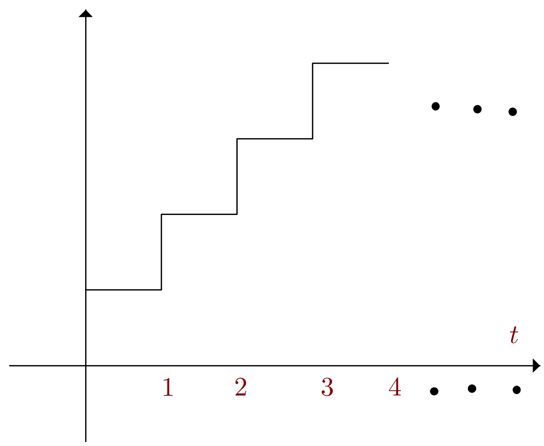
MATH 336- Winter 2022

Assignment 4

Problem 1. Consider the following initial value problem

$$\begin{cases} y'' + y = f(t) \\ y(0) = y'(0) = 0 \end{cases},$$

where $f(t)$ is the function shown below



a) Show

$$\hat{f}(s) = \sum_{n=0}^{\infty} \frac{e^{-ns}}{s}.$$

b) Solve the initial value problem and show

$$y(t) = \sum_{n=0}^{\infty} (1 - \cos(t-n))\, u(t-n)$$

Problem 2.

a) Remember that

$$\mathcal{L}\{t \sin(t)\} = \frac{2s}{(s^2+1)^2}.$$

Find the following Laplace inverse

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}.$$

Hint: You can write

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{2s}{(s^2+1)^2}\right\},$$

and use the formula

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\,\hat{f}(s)\right\} = \int_0^t f(\tau)\, d\tau.$$

b) Now solve the following initial value problem by the LAPLACE transform method

$$\begin{cases} y'' + y = \sin(t)\, u(t) \\ y(0) = 1,\, y'(0) = 0 \end{cases}.$$

Problem 3. Let $p_{1,2}(t)$ be the pulse $p_{1,2}(t) = u(t) - u(t-2)$. Solve the following initial value problem

$$\begin{cases} y'' + 2y' + 2y = e^t\, p_{1,2}(t) \\ y(0) = y'(0) = 0 \end{cases}.$$

Problem 4. Solve the following initial value problem by the LAPLACE transform method

$$\begin{cases} y'' + 3y' + 2y = 0 \\ y(1) = 1,\, y'(1) = -1 \end{cases}$$

Problem 5. Use the LAPLACE transform method and solve the following problems

a) Show the relation

$$\mathcal{L}\left\{\frac{d}{dt}u(t-\tau)\right\} = e^{-\tau s} = \mathcal{L}\{\delta(t-\tau)\}, \tau > 0.$$

In general, we accept the formula for $\tau = 0$ and write *symbolically*

$$\frac{d}{dt}u(t) = \delta(t).$$

b) Show that the solution of the following initial value problem is continuous at $t = \tau > 0$

$$\begin{cases} y' + y = u(t-\tau) \\ y(0) = 0 \end{cases}.$$

c) Show that the solution of the following initial value problem has a finite jump discontinuity at $t = \tau > 0$

$$\begin{cases} y' + y = \delta(t-\tau) \\ y(0) = 0 \end{cases}.$$

d) What do you think about the continuity or discontinuity of the solution of the following initial value problem at $t = \tau > 0$?

$$\begin{cases} y'' + y = \delta(t-\tau) \\ y(0) = 0,\, y'(0) = 0 \end{cases}.$$

Is $y'(\tau)$ continuous at τ?

Problem 6. Consider the following problem

$$\begin{cases} y'' + y = \delta(t) \\ y(0) = y'(0) = 0 \end{cases}.$$

The solution is defined for $t>0$ and is assume to be zero for $t\leq 0$. This solution is called the *impulse response* or the response of the system to the *impulse* $\delta(t)$ and is denoted by $h(t)$.

a) Find the impulse response $h(t)$ of the above system.

b) Now consider the following initial value problem

$$\begin{cases} y''+y=r(t) \\ y(0)=y'(0)=0 \end{cases},$$

where $r(t)$ is an arbitrary function with Laplace transform $\hat{r}(s)$. Show the following relation

$$y(t)=h(t)*r(t).$$

Problem 7. Consider the following *integro-differential* equation

$$y(t)+e^t\int_0^t e^{-\tau}y'(\tau)d\tau=\delta(t-1).$$

a) Find $y(0)$ directly from the equation.

b) Find $y(t)$ for $t>0$.

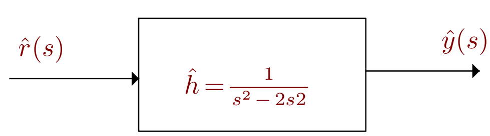
Problem 8. Find the solution of the following system

$$\begin{cases} x'=y+\frac{t}{1+t^2}\delta(t-1) \\ y'=-x \end{cases},$$

with the initial conditions $x(0)=1$, $y(0)=0$.

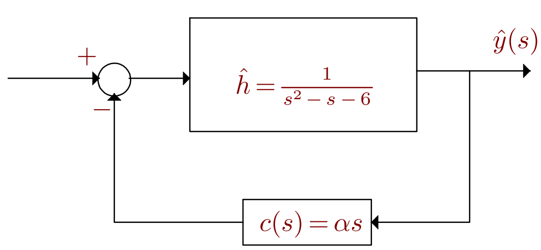
Bonus

Problem 9. Consider the following system



a) Show that the impluse response is unbounded.

b) Now, consider the following *control system*



Determine values of α such that the system is bounded, that is,

$$\lim_{t\to\infty} h(t)<\infty.$$