Math 127
Suggested solutions to Homework Set 5

Problem 1. We have that

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 6 & 4 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 5 & 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 2 & 5 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \underbrace{\xrightarrow{\substack{R_2-2R_1\to R_2',\, R_3-2R_1\to R_3' \\ R_4-3R_1\to R_4'}}} \left(\begin{array}{cccc|cccc} 1 & 1 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 4 & -2 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\underbrace{\xrightarrow{R_4+R_2\to R_4'}} \left(\begin{array}{cccc|cccc} 1 & 1 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 4 & -2 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -2 & 3 & -5 & 1 & 0 & 1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 1 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 4 & 5 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 5 & 3 & 2 & 1 & 0 & 1 \end{array}\right)$$

$$\underbrace{\xrightarrow{R_4+3R_3\to R_4'}} \left(\begin{array}{cccc|cccc} 1 & 1 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 4 & 5 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 4 & 1 \end{array}\right),$$

therefore A_1 is not invertible because it has a Row Echelon Form with fewer than 4 pivots.

On the other hand,

$$\left(\begin{array}{cccc|cccc} 0.5 & -0.5 & 3 & 2 & 1 & 0 & 0 & 0 \\ -1 & 1.5 & -3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0.5 & 3 & 1 & 0 & 0 & 1 & 0 \\ -3 & 1 & 0 & -4 & 0 & 0 & 0 & 1 \end{array}\right) \underbrace{\xrightarrow{\substack{R_2+2R_1\to R_2' \\ R_4+6R_1\to R_4'}}} \left(\begin{array}{cccc|cccc} 0.5 & -0.5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0.5 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 18 & 8 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\underbrace{\xrightarrow{\substack{R_3-R_2\to R_3' \\ R_4+4R_2\to R_4'}}} \left(\begin{array}{cccc|cccc} 0.5 & -0.5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 0 & 30 & 20 & 14 & 4 & 0 & 1 \end{array}\right)$$

$$\underbrace{\xrightarrow{R_3\leftrightarrow R_4}} \left(\begin{array}{cccc|cccc} 0.5 & -0.5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 30 & 20 & 14 & 4 & 0 & 1 \\ 0 & 0 & 0 & -2 & -2 & -1 & 1 & 0 \end{array}\right) \underbrace{\xrightarrow{0.1R_3\to R_3'}} \left(\begin{array}{cccc|cccc} 0.5 & -0.5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1.4 & 0.4 & 0 & 0.1 \\ 0 & 0 & 0 & -2 & -2 & -1 & 1 & 0 \end{array}\right),$$

thus A_2 is invertible.

Moreover, to find its inverse we continue to do elementary row operations until we get to its RREF, which should be I_4:

$$\left(\begin{array}{cccc|cccc} 0.5 & -0.5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1.4 & 0.4 & 0 & 0.1 \\ 0 & 0 & 0 & -2 & -2 & -1 & 1 & 0 \end{array}\right) \underbrace{\xrightarrow{}}_{\substack{R_3+R_4\to R_3',\, R_2+1.5R_4\to R_2' \\ R_1+R_4\to R_1'}} \left(\begin{array}{cccc|cccc} 0.5 & -0.5 & 3 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0.5 & 3 & 0 & -1 & -0.5 & 1.5 & 0 \\ 0 & 0 & 3 & 0 & -0.6 & -0.6 & 1 & 0.1 \\ 0 & 0 & 0 & -2 & -2 & -1 & 1 & 0 \end{array}\right)$$

$$\underbrace{\xrightarrow{}}_{\substack{R_2-R_3\to R_2' \\ R_1-R_3\to R_1'}} \left(\begin{array}{cccc|cccc} 0.5 & -0.5 & 0 & 0 & -0.4 & -0.4 & 0 & -0.1 \\ 0 & 0.5 & 0 & 0 & -0.4 & 0.1 & 0.5 & -0.1 \\ 0 & 0 & 3 & 0 & -0.6 & -0.6 & 1 & 0.1 \\ 0 & 0 & 0 & -2 & -2 & -1 & 1 & 0 \end{array}\right)$$

$$\underbrace{R_1+R_2\to R_1'} \left(\begin{array}{cccc|cccc} 0.5 & 0 & 0 & 0 & -0.8 & -0.3 & 0.5 & -0.2 \\ 0 & 0.5 & 0 & 0 & -0.4 & 0.1 & 0.5 & -0.1 \\ 0 & 0 & 3 & 0 & -0.6 & -0.6 & 1 & 0.1 \\ 0 & 0 & 0 & -2 & -2 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1.6 & -0.6 & 1 & -0.4 \\ 0 & 1 & 0 & 0 & -0.8 & 0.2 & 1 & -0.2 \\ 0 & 0 & 1 & 0 & -0.2 & -0.2 & \frac{1}{3} & \frac{1}{30} \\ 0 & 0 & 0 & 1 & 1 & 0.5 & -0.5 & 0 \end{array}\right).$$

In conclusion,

$$A_2^{-1} = \begin{pmatrix} -1.6 & -0.6 & 1 & -0.4 \\ -0.8 & 0.2 & 1 & -0.2 \\ -0.2 & -0.2 & \frac{1}{3} & \frac{1}{30} \\ 1 & 0.5 & -0.5 & 0 \end{pmatrix}.$$

Problem 2. (a) Assume that we have an upper triangular matrix $U_1 \in \mathbb{F}^{n\times n}$ all of whose diagonal entries are non-zero.

In that case, all rows of U_1 are non-zero. Moreover,

- the first non-zero entry of the first row of U_1 is the entry $u_{1,1}$ (given that we have assumed that all diagonal entries are non-zero).
- The first non-zero entry of the second row of U_1 is the entry $u_{2,2}$ (given that U_1 is upper triangular, and thus $u_{2,1} = 0$, while the diagonal entry $u_{2,2}$ is non-zero). Thus the first non-zero entry of the second row is to the right of the first non-zero entry of the first row.
- Continuing like this, we see that the first non-zero entry of the i-th row of U_1 is the diagonal entry $u_{i,i}$, and hence it is to the right of the first non-zero entry of any of the previous rows.

The above show that the matrix U_1 is in Row Echelon Form.

(b) Suppose first that U_2 is such that all its diagonal entries are non-zero. Then, as we saw in part (a), U_2 is in REF, and its pivots are its diagonal entries. Thus U_2 has n pivots, and hence it is invertible.

Conversely, suppose that the upper triangular matrix U_2 is invertible. Assume towards a contradiction that there exists $i \in \{1, 2, \ldots, n\}$ such that $u_{i,i} = 0$. In that case, we find the smallest such index i_0; in other words, $u_{i_0,i_0} = 0$, while $u_{s,s} \neq 0$ for $s < i_0$.

We will now try to estimate how many pivots a REF of U_2 would have. As in part (a), we can see that the first non-zero entry of each of the first $i_0 - 1$ rows of U_2 is the diagonal entry in that row, and hence each such entry is to the right of previous such entries.

At the same time, because U_2 is upper triangular, for each such first non-zero entry we have that the entries right below it are all zero.

In other words, we could find a REF of U_2 whose first $i_0 - 1$ rows and first $i_0 - 1$ columns are precisely the first $i_0 - 1$ rows and first $i_0 - 1$ columns of U_2 respectively (that is, we can start replacing U_2 by a matrix in REF without having to apply any elementary row operations to U_2 initially, that is, when we are examining the first $i_0 - 1$ columns of U_2, which will all end up being pivot columns).

We will now try to determine where the next pivot of a REF of U_2 should be: we note that $\mathrm{Col}_{i_0}(U_2)$ satisfies the property that

$$u_{j,i_0} = 0 \quad \text{for all } j \geqslant i_0,$$

and hence none of these entries could be the next pivot. At the same time, none of the first $i_0 - 1$ entries of $\mathrm{Col}_{i_0}(U_2)$ is the first non-zero entry in its corresponding row, and therefore none of these entries could be a pivot either.

Summarising the above, we see that none of the entries of $\mathrm{Col}_{i_0}(U_2)$ can be a pivot of a REF of U_2, and hence the i_0-th column won't be a pivot column.

But this now implies that we can find a REF of U_2 which has fewer than n pivot columns, or equivalently which has fewer than n pivots. From this it would follow that U_2 is **not** invertible, which contradicts the assumption we start from.

Therefore, the assumption that we made, that not all diagonal entries of U_2 are non-zero, was incorrect.

Problem 3. (a) We recall that, in HW4 Problem 6(b), we were asked to show that, if a matrix $B \in \mathbb{F}^{n\times n}$ satisfies the property

for every $\bar{u} \in \mathbb{F}^n$, the linear system $B\bar{y} = \bar{u}$ is consistent,

then it will also satisfy that

for every $\bar{u} \in \mathbb{F}^n$, the linear system $B\bar{y} = \bar{u}$ has a unique solution;

then the latter, combined with HW4 Problem 6(a), would imply that this matrix B is invertible.

Based on these, we observe that, in order to show the desired conclusion in this problem, it suffices to prove that

(1) for every $\bar{u} \in \mathbb{F}^n$, the linear system $C\bar{y} = \bar{u}$ is consistent.

Fix now some $\bar{u} \in \mathbb{F}^n$. We have that $CD = I_n$, and therefore if we set $\bar{w} = D\bar{u}$, we will have

$$C\bar{w} = C(D\bar{u}) = (CD)\bar{u} = I_n\bar{u} = \bar{u}.$$

In other words, $\bar{w}D\bar{u}$ is a solution to the system $C\bar{y} = \bar{u}$.

Since $\bar{u}$ was arbitrary, we have shown (1), which then implies the desired conclusion as explained at the beginning.

(b) We will use induction in the number m of matrices whose product we are considering.

Base Case: $m = 2$. Then we have the assumption that the product A_1A_2 is invertible, and need to show that A_1 and A_2 are also invertible.

Given that A_1A_2 is invertible, we can find a matrix $D \in \mathbb{F}^{n\times n}$ such that

$$(A_1A_2)\cdot D = I_n \qquad \text{and} \qquad D\cdot(A_1A_2) = I_n.$$

By associativity of matrix multiplication, we can rewrite the first equality as follows:

$$A_1\cdot(A_2D) = I_n,$$

which then, combined with part (a) of this problem, gives us that A_1 is invertible.

Similarly we can rewrite the second equality as

$$(DA_1)A_2 = I_n.$$

Again, by part (a) this gives us that DA_1 is invertible, which further shows that A_2 is its inverse. Thus we have that A_2 is invertible as well.

Induction Step: Assume now that, for some $m \geqslant 2$, we have that, if the product $A_1 A_2 \cdots A_{m-1} A_m$ of m matrices from $\mathbb{F}^{n\times n}$ is invertible, then each of the matrices is invertible. We need to show that the analogous statement is true when we take the product of $m+1$ matrices.

Suppose that $B_1, B_2, \ldots, B_{m-1}, B_m, B_{m+1} \in \mathbb{F}^{n\times n}$ are matrices such that the product

$$B_1 \cdot B_2 \cdots B_{m-1} \cdot B_m \cdot B_{m+1}$$

is invertible.

Then we can find a matrix $E \in \mathbb{F}^{n\times n}$ such that

$$(B_1 \cdot B_2 \cdots B_{m-1} \cdot B_m \cdot B_{m+1}) \cdot E = I_n \quad \text{and} \quad E \cdot (B_1 \cdot B_2 \cdots B_{m-1} \cdot B_m \cdot B_{m+1}) = I_n.$$

By associativity we can rewrite the first equality as

$$(B_1 \cdot B_2 \cdots B_{m-1} \cdot B_m) \cdot (B_{m+1} \cdot E) = I_n,$$

which combined with part (a) gives us that the product

$$B_1 \cdot B_2 \cdots B_{m-1} \cdot B_m$$

is invertible. We can then apply the Inductive Hypothesis to conclude that, for each $i \in \{1, 2, \ldots, m-1, m\}$, the matrix B_i is invertible.

It remains to explain why the matrix B_{m+1} is invertible. We can rewrite the second equality above as

$$(E \cdot B_1 \cdot B_2 \cdots B_{m-1} \cdot B_m) \cdot B_{m+1} = I_n.$$

By part (a) we now get that both the product $E \cdot B_1 \cdot B_2 \cdots B_{m-1} \cdot B_m$ and the matrix B_{m+1} are invertible (one is the inverse of the other).

This concludes the proof of the Induction Step too.

Problem 4. We suppose $Q_3 \in \mathbb{R}^{3\times 3}$ is a stochastic matrix, and we write

$$Q_3 = \big(q_{ij}\big)_{1\leqslant i,j\leqslant 3}.$$

Consider $1 \leqslant i_0 \leqslant 3$. For each $1 \leqslant j \leqslant 3$, the (i_0, j)-entry of $Q_3^2 = Q_3 \cdot Q_3$ is by definition equal to

$$\big\langle \mathrm{Row}_{i_0}(Q_3), \mathrm{Col}_j(Q_3)\big\rangle = \sum_{s=1}^{3} q_{i_0,s} q_{s,j}.$$

Therefore, to verify that Q_3^2 is stochastic, we need to check that

$$\sum_{j=1}^{3} \left(\sum_{s=1}^{3} q_{i_0,s} q_{s,j}\right) = 1.$$

But because of generalised commutativity and associativity in $\mathbb{R}$, we could sum over the index j first:

$$\begin{aligned}\sum_{j=1}^{3} \left(\sum_{s=1}^{3} q_{i_0,s} q_{s,j}\right) &= \sum_{j=1}^{3}\sum_{s=1}^{3} q_{i_0,s} q_{s,j} \\ &= \sum_{s=1}^{3}\sum_{j=1}^{3} q_{i_0,s} q_{s,j} = \sum_{s=1}^{3} q_{i_0,s} \cdot \left(\sum_{j=1}^{3} q_{s,j}\right),\end{aligned}$$

where the latter equality follows by distributivity because $q_{i_0,s}$ is a common factor in the inner sum.

We now use the assumption that Q_3 is stochastic: for every $1 \leqslant s \leqslant 3$, this gives that $\sum_{j=1}^{3} q_{s,j} = 1$.

We can thus continue rewriting the double sum we started with as follows:

$$\sum_{j=1}^{3} \left(\sum_{s=1}^{3} q_{i_0,s} q_{s,j}\right) = \sum_{s=1}^{3} q_{i_0,s} \cdot \left(\sum_{j=1}^{3} q_{s,j}\right) = \sum_{s=1}^{3} q_{i_0,s} \cdot 1 = \sum_{s=1}^{3} q_{i_0,s} = 1,$$

where we use the assumption that Q_3 is a stochastic matrix one more time.

The proof of the corresponding fact for a stochastic matrix $Q_4 \in \mathbb{R}^{4\times 4}$ is completely analogous: write

$$Q_4 = \big(\tilde{q}_{ij}\big)_{1\leqslant i,j\leqslant 4}.$$

Consider $1 \leqslant i_0 \leqslant 4$. For each $1 \leqslant j \leqslant 4$, the (i_0, j)-entry of $Q_4^2 = Q_4 \cdot Q_4$ is by definition equal to

$$\big\langle \mathrm{Row}_{i_0}(Q_4), \mathrm{Col}_j(Q_4)\big\rangle = \sum_{s=1}^{4} \tilde{q}_{i_0,s}\tilde{q}_{s,j}.$$

Therefore, to verify that Q_4^2 is stochastic, we need to check that

$$\sum_{j=1}^{4}\left(\sum_{s=1}^{4} \tilde{q}_{i_0,s}\tilde{q}_{s,j}\right) = 1.$$

But because of generalised commutativity and associativity in $\mathbb{R}$, we could sum over the index j first:

$$\begin{aligned}\sum_{j=1}^{4}\left(\sum_{s=1}^{4} \tilde{q}_{i_0,s}\tilde{q}_{s,j}\right) &= \sum_{j=1}^{4}\sum_{s=1}^{4} \tilde{q}_{i_0,s}\tilde{q}_{s,j}\\ &= \sum_{s=1}^{4}\sum_{j=1}^{4} \tilde{q}_{i_0,s}\tilde{q}_{s,j} = \sum_{s=1}^{4} \tilde{q}_{i_0,s}\cdot\left(\sum_{j=1}^{4} \tilde{q}_{s,j}\right),\end{aligned}$$

where the latter equality follows by distributivity because $\tilde{q}_{i_0,s}$ is a common factor in the inner sum.

We now use the assumption that Q_4 is stochastic: for every $1 \leqslant s \leqslant 4$, this gives that $\sum_{j=1}^{4} \tilde{q}_{s,j} = 1$.

We can thus continue rewriting the double sum we started with as follows:

$$\sum_{j=1}^{4}\left(\sum_{s=1}^{4} \tilde{q}_{i_0,s}\tilde{q}_{s,j}\right) = \sum_{s=1}^{4} \tilde{q}_{i_0,s}\cdot\left(\sum_{j=1}^{4} \tilde{q}_{s,j}\right) = \sum_{s=1}^{4} \tilde{q}_{i_0,s}\cdot 1 = \sum_{s=1}^{4} \tilde{q}_{i_0,s} = 1,$$

where we use the assumption that Q_4 is a stochastic matrix one more time.

Side note: By now, it may have become clear that we only need to slightly adjust the (very similar) arguments above in order to justify the most general statement we can get here, that for every $n \geqslant 2$ and every stochastic matrix $Q \in \mathbb{R}^{n\times n}$, the matrix Q^2 is also stochastic.

Indeed, fix some $n \geqslant 2$ and consider a stochastic matrix $Q \in \mathbb{R}^{n\times n}$. For notational convenience, write

$$Q = \big(q_{ij}\big)_{1\leqslant i,j\leqslant n}.$$

For every $1 \leqslant i_0 \leqslant n$, and every $1 \leqslant j \leqslant n$, the (i_0, j)-entry of Q^2 is

$$\sum_{s=1}^{n} q_{i_0,s} q_{s,j}.$$

Therefore, we need to check that

$$\sum_{j=1}^{n} \left(\sum_{s=1}^{n} q_{i_0,s} q_{s,j} \right) = 1.$$

But

$$\begin{aligned}
\sum_{j=1}^{n} \left(\sum_{s=1}^{n} q_{i_0,s} q_{s,j} \right) &= \sum_{s=1}^{n} \sum_{j=1}^{n} q_{i_0,s} q_{s,j} \\
&= \sum_{s=1}^{n} q_{i_0,s} \cdot \left(\sum_{j=1}^{n} q_{s,j} \right) && \text{(the entries of the s-th row of Q add up to 1)} \\
&= \sum_{s=1}^{n} q_{i_0,s} \cdot 1 = \sum_{s=1}^{n} q_{i_0,s} = 1. && \text{(the entries of the i_0-th row of Q add up to 1)}
\end{aligned}$$

This completes the proof of the general statement.

Problem 5. (a) We have that:

(i) this f is linear, injective and surjective. Indeed, for every $\bar{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right), \bar{y} = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \in \mathbb{R}^2$ and every $r \in \mathbb{R}$ we have

$$f(\bar{x} + \bar{y}) = f\left(\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}\right) = \begin{pmatrix} -x_1 - y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} = f(\bar{x}) + f(\bar{y})$$

$$\text{and} \quad f(r\bar{x}) = f\left(\begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix}\right) = \begin{pmatrix} -rx_1 \\ rx_2 \end{pmatrix} = rf(\bar{x}),$$

thus f is linear.

Also, if $f(\bar{x}) = f(\bar{y})$, then

$$\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} \quad \Rightarrow \quad x_1 = y_1 \text{ and } x_2 = y_2$$

$$\Rightarrow \quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \bar{y},$$

thus f is injective.

Finally, given $\bar{z} = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) \in \mathbb{R}^2$, we have that $\left(\begin{smallmatrix} -z_1 \\ z_2 \end{smallmatrix}\right) \in \mathbb{R}^2$ too and

$$f\left(\begin{pmatrix} -z_1 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} -(-z_1) \\ z_2 \end{pmatrix} = \bar{z},$$

thus f is surjective.

(ii) this f is not linear, it is not injective and it is not surjective. Indeed,

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{while } f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \neq 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right),$$

thus f is not linear.

Also, $f\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = f\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)$, thus f is not injective.

Finally, we show that $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ is not in the range of f. Assume towards a contradiction that there is $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^2$ such that

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Given that we want $x_1 x_2 = 1$, none of the x_1, x_2 can be zero, and moreover either both of them are positive or both of them are negative. But if both of them were negative, then $x_1 + x_2 < 0$ and thus $\neq 1$. So both x_1, x_2 need to be positive. But then $x_1 = x_1 + 0 < x_1 + x_2 = 1$, and similarly $x_2 < x_1 + x_2 = 1$. Thus $x_1, x_2 \in (0,1)$ and they satisfy $x_1 x_2 = 1 \Rightarrow x_2 = \frac{1}{x_1}$. These imply that

$$x_1 + x_2 = x_1 + \frac{1}{x_1} = \frac{x_1^2 + 1}{x_1} > \frac{1}{x_1} > 1 \quad \text{given that} \quad 0 < x_1 < 1,$$

which contradicts our assumption that $x_1 + x_2 = 1$.

(iii) this f is not linear, it is not injective and it is not surjective. Indeed,

$$f\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\1\end{pmatrix}, \quad \text{while} \quad f\left(\begin{pmatrix}2\\2\end{pmatrix}\right) = \begin{pmatrix}0\\1\end{pmatrix} \neq \begin{pmatrix}0\\2\end{pmatrix} = 2f\left(\begin{pmatrix}1\\1\end{pmatrix}\right),$$

thus f is not linear.

Also, we just saw that $f\left(\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right) = f\left(\left(\begin{smallmatrix}2\\2\end{smallmatrix}\right)\right)$, thus f is not injective.

Finally, we show that $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ is not in the range of f. Assume towards a contradiction that there is $\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right) \in \mathbb{Z}_3^2$ such that

$$f\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix} \qquad \Leftrightarrow \qquad \begin{pmatrix}x_1 - x_2\\x_1 x_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

Given that we want $x_1 x_2 = 1$, none of the x_1, x_2 can be zero, thus $x_1, x_2 \in \{1, 2\}$. Note moreover that if we had $x_1 \neq x_2$ combined with $x_1, x_2 \in \{1,2\}$, then we would get $x_1 x_2 = 2 \neq 1$. Therefore we must have $x_1 = x_2$. This in turn implies that $x_1 - x_2 = 0$, which contradicts our assumption that $x_1 - x_2 = 1$.

(iv) this f is linear and surjective, but it is not injective. Indeed, for every $\bar{x} = \left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right), \bar{y} = \left(\begin{smallmatrix}y_1\\y_2\end{smallmatrix}\right) \in \mathbb{R}^2$ and every $r \in \mathbb{R}$ we have

$$\begin{aligned} f(\bar{x} + \bar{y}) &= f\left(\begin{pmatrix}x_1 + y_1\\x_2 + y_2\end{pmatrix}\right) = (x_1 + y_1) + (x_2 + y_2) \\ &= (x_1 + x_2) + (y_1 + y_2) = f(\bar{x}) + f(\bar{y}) \\ \text{and} \quad f(r\bar{x}) &= f\left(\begin{pmatrix}rx_1\\rx_2\end{pmatrix}\right) = rx_1 + rx_2 = r(x_1 + x_2) = rf(\bar{x}), \end{aligned}$$

thus f is linear.

Also, given $s \in \mathbb{R}$, we have that $f\left(\left(\begin{smallmatrix}s\\0\end{smallmatrix}\right)\right) = s + 0 = s$, thus f is surjective.

However, we also have $f\left(\left(\begin{smallmatrix}0\\s\end{smallmatrix}\right)\right) = s$, thus f is not injective (given that the vectors $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, which are different, are mapped to the same element).

(v) this f is injective and surjective, but it is not linear. Indeed, $f(\bar{0}) = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right) \neq \bar{0}$, thus f is not linear.

On the other hand, if $f(\bar{x}) = f(\bar{y})$, then

$$\begin{pmatrix} x_1+1 \\ x_2+2 \end{pmatrix} = \begin{pmatrix} y_1+1 \\ y_2+2 \end{pmatrix} \quad \Rightarrow \quad x_1+1 = y_1+1 \text{ and } x_2+2 = y_2+2$$

$$\Rightarrow \quad x_1 = y_1 \text{ and } x_2 = y_2 \Rightarrow \quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \bar{y},$$

thus f is injective.

Finally, given $\bar{z} = \left(\begin{smallmatrix}z_1\\z_2\end{smallmatrix}\right) \in \mathbb{R}^2$, we have that $\left(\begin{smallmatrix}z_1-1\\z_2-2\end{smallmatrix}\right) \in \mathbb{R}^2$ too and

$$f\left(\begin{pmatrix} z_1-1 \\ z_2-2 \end{pmatrix}\right) = \begin{pmatrix} (z_1-1)+1 \\ (z_2-2)+2 \end{pmatrix} = \bar{z},$$

thus f is surjective.

(vi) this f is linear, injective and surjective. Indeed, for every $\bar{x} = \left(\begin{smallmatrix}x_1\\x_2\\x_3\end{smallmatrix}\right), \bar{y} = \left(\begin{smallmatrix}y_1\\y_2\\y_3\end{smallmatrix}\right) \in \mathbb{Z}_5^3$ and every $s \in \mathbb{Z}_5$ we have

$$f(\bar{x}+\bar{y}) = f\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}\right) = \begin{pmatrix} x_2+y_2 \\ x_3+y_3 \\ x_1+y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} + \begin{pmatrix} y_2 \\ y_3 \\ y_1 \end{pmatrix} = f(\bar{x}) + f(\bar{y})$$

$$\text{and } \ f(s\bar{x}) = f\left(\begin{pmatrix} sx_1 \\ sx_2 \\ sx_3 \end{pmatrix}\right) = \begin{pmatrix} sx_2 \\ sx_3 \\ sx_1 \end{pmatrix} = sf(\bar{x}),$$

thus f is linear.

Also, if $f(\bar{x}) = f(\bar{y})$, then

$$\begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_1 \end{pmatrix} \quad \Rightarrow \quad x_2 = y_2, \quad x_3 = y_3 \text{ and } x_1 = y_1$$

$$\Rightarrow \quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \bar{y},$$

thus f is injective.

Finally, given $\bar{z} = \left(\begin{smallmatrix}z_1\\z_2\\z_3\end{smallmatrix}\right) \in \mathbb{Z}_5^3$, we have that $\left(\begin{smallmatrix}z_3\\z_1\\z_2\end{smallmatrix}\right) \in \mathbb{Z}_5^3$ too and

$$f\left(\begin{pmatrix} z_3 \\ z_1 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},$$

as we wanted. Thus f is surjective.

(vii) this f is linear and injective, but it is not surjective. Indeed, for every $\bar{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right), \bar{y} = \left(\begin{smallmatrix} y_1 \\ y_2 \\ y_3 \end{smallmatrix}\right) \in \mathbb{R}^3$ and every $r \in \mathbb{R}$ we have

$$\begin{aligned} f(\bar{x}+\bar{y}) &= f\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}\right) = \begin{pmatrix} x_1+y_1 & x_2+y_2 \\ 0 & x_3+y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix} = f(\bar{x}) + f(\bar{y}) \\ \text{and} \quad f(r\bar{x}) &= f\left(\begin{pmatrix} rx_1 \\ rx_2 \\ rx_3 \end{pmatrix}\right) = \begin{pmatrix} rx_1 & rx_2 \\ 0 & rx_3 \end{pmatrix} = \begin{pmatrix} rx_1 & rx_2 \\ r\cdot 0 & rx_3 \end{pmatrix} = rf(\bar{x}), \end{aligned}$$

thus f is linear.

Also, if $f(\bar{x}) = f(\bar{y})$, then

$$\begin{aligned} \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix} \quad &\Rightarrow \quad x_1 = y_1, \quad x_2 = y_2 \text{ and } x_3 = y_3 \\ &\Rightarrow \quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \bar{y}, \end{aligned}$$

thus f is injective.

Finally, note that $C = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ is not in the range of this f, thus f is not surjective.

(viii) this f is not linear and it is not surjective, but it is injective. Indeed, for every matrix $C = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the range of f, we should have $c+d = 1$. But then the zero matrix cannot be in the range of f, which shows both that f is not linear (why?) and that it is not surjective.

On the other hand this f is injective: consider $\bar{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right), \bar{y} = \left(\begin{smallmatrix} y_1 \\ y_2 \\ y_3 \end{smallmatrix}\right) \in \mathbb{R}^3$ that satisfy

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) \quad \Leftrightarrow \quad \begin{pmatrix} x_1 & x_2 \\ 1-x_3 & x_3 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ 1-y_3 & y_3 \end{pmatrix}.$$

Then we must have $x_1 = y_1, x_2 = y_2$ and $x_3 = y_3$, hence $\bar{x} = \bar{y}$.

(b) Note that out of the first six functions, the only ones that we found are linear are the functions in part (i), part (iv) and part (iv). We give a matrix representation for each one of these:

(i) Let

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Then, for every $\bar{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^2$, we have

$$A_1\bar{x} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = f(\bar{x}).$$

(iv) Let

$$A_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \in \mathbb{R}^{1\times 2}.$$

Then, for every $\bar{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^2$, we have

$$A_2\bar{x} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2 = f(\bar{x}).$$

(vi) Let

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_5^{3\times 3}.$$

Then, for every $\bar{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \in \mathbb{Z}_5^3$, we have

$$A_3\bar{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} = f(\bar{x}).$$

Problem 6. (a) Consider two arbitrary vectors $\bar{x}, \bar{y} \in V_1$, and $r \in \mathbb{F}$.

We then have

$$\begin{aligned}
(\mu_1 f_1 + \mu_2 f_2)(\bar{x}+\bar{y}) &= (\mu_1 f_1)(\bar{x}+\bar{y}) + (\mu_2 f_2)(\bar{x}+\bar{y}) && \text{(by definition of sum of functions)}\\
&= \mu_1 \cdot f_1(\bar{x}+\bar{y}) + \mu_2 \cdot f_2(\bar{x}+\bar{y}) && \text{(by definition of scalar multiplication for functions)}\\
&= \mu_1 \cdot (f_1(\bar{x}) + f_1(\bar{y})) + \mu_2 \cdot (f_2(\bar{x}) + f_2(\bar{y})) && \text{(because } f_1, f_2 \text{ are linear)}\\
&= \big(\mu_1 \cdot f_1(\bar{x}) + \mu_1 \cdot f_1(\bar{y})\big) + \big(\mu_2 \cdot f_2(\bar{x}) + \mu_2 \cdot f_2(\bar{y})\big)\\
&= (\mu_1 f_1)(\bar{x}) + (\mu_1 f_1)(\bar{y}) + (\mu_2 f_2)(\bar{x}) + (\mu_2 f_2)(\bar{y})\\
&= \big((\mu_1 f_1)(\bar{x}) + (\mu_2 f_2)(\bar{x})\big) + \big((\mu_1 f_1)(\bar{y}) + (\mu_2 f_2)(\bar{y})\big)\\
&= (\mu_1 f_1 + \mu_2 f_2)(\bar{x}) + (\mu_1 f_1 + \mu_2 f_2)(\bar{y}),
\end{aligned}$$

which shows the additivity of $\mu_1 f_1 + \mu_2 f_2$.

Moreover,

$$\begin{aligned}
(\mu_1 f_1 + \mu_2 f_2)(r\bar{x}) &= (\mu_1 f_1)(r\bar{x}) + (\mu_2 f_2)(r\bar{x}) && \text{(by definition of sum of functions)}\\
&= \mu_1 \cdot f_1(r\bar{x}) + \mu_2 \cdot f_2(r\bar{x}) && \text{(by definition of scalar multiplication for functions)}\\
&= \mu_1 \cdot (r \cdot f_1(\bar{x})) + \mu_2 \cdot (r \cdot f_2(\bar{x})) && \text{(because } f_1, f_2 \text{ are linear)}\\
&= (\mu_1 r) \cdot f_1(\bar{x}) + (\mu_2 r) \cdot f_2(\bar{x})\\
&= (r\mu_1) \cdot f_1(\bar{x}) + (r\mu_2) \cdot f_2(\bar{x})\\
&= r \cdot (\mu_1 \cdot f_1(\bar{x})) + r \cdot (\mu_2 \cdot f_2(\bar{x}))\\
&= r \cdot \big((\mu_1 f_1)(\bar{x}) + (\mu_2 f_2)(\bar{x})\big)\\
&= r \cdot (\mu_1 f_1 + \mu_2 f_2)(\bar{x}).
\end{aligned}$$

Combining the above, we conclude that $\mu_1 f_1 + \mu_2 f_2$ is a linear map from V_1 to V_2.

(b) Consider two arbitrary vectors $\bar{x}, \bar{y} \in V_1$, and $r \in \mathbb{F}$. Then, since f is linear, we have $f(\bar{x}+\bar{y}) = f(\bar{x}) + f(\bar{y})$ and $f(r\bar{x}) = rf(\bar{x})$. But then,

$$\begin{aligned}
(g \circ f)(\bar{x}+\bar{y}) = g\big(f(\bar{x}+\bar{y})\big) &= g\big(f(\bar{x}) + f(\bar{y})\big)\\
&= g(f(\bar{x})) + g(f(\bar{y})) = (g \circ f)(\bar{x}) + (g \circ f)(\bar{y}),
\end{aligned}$$

where we used that g is linear too.

Similarly,

$$(g \circ f)(r\bar{x}) = g\big(f(r\bar{x})\big) = g\big(rf(\bar{x})\big) = rg(f(\bar{x})) = r(g \circ f)(\bar{x}).$$

Since $\bar{x}, \bar{y} \in V_1$ and $r \in \mathbb{F}$ were arbitrary, we conclude that $g \circ f$ is linear.