MATH 217 (Fall 2021)

Honors Advanced Calculus, I

Solutions #3

1. Let $S \subset \mathbb{R}^N$. Show that $x \in \mathbb{R}^N$ is a cluster point of S if and only if each neighbourhood of x contains an infinite number of points in S.

   Solution: Let $x \in \mathbb{R}^N$ be a cluster point of S, and assume that there is a neighborhood U of x such that $U \cap S$ contains only finitely many. If $U \cap S = \{x\}$, then x cannot be a cluster point by definition, so suppose that $(U \cap S) \setminus \{x\}$ is a non-empty finite set. Define
   $$\epsilon := \min\{\|x - y\| : y \in (U \cap S) \setminus \{x\}\}.$$
   Then $\epsilon > 0$, and $U \cap B_\epsilon(x)$ is a neighborhood of x of which the intersection with S contains at most x. Hence, x cannot be a cluster point of S.

   For the converse, let U be any neighborhood of x. Then $U \cap S$ is infinite and therefore has to contain at least one point from $S \setminus \{x\}$.

2. Let $S \subset \mathbb{R}^N$ be any set. Show that ∂S is closed.

   Solution: Let $x \in \mathbb{R}^N \setminus \partial S$. Then there is $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \cap S = \varnothing$ or $B_{\epsilon_0}(x) \cap S^c = \varnothing$.

   Suppose that $B_{\epsilon_0}(x) \cap S = \varnothing$, and let $y \in B_{\epsilon_0}(x)$. Since $B_{\epsilon_0}(x)$ is open, there is $\epsilon > 0$ such that $B_\epsilon(y) \subset B_{\epsilon_0}(x)$; it follows that $B_\epsilon(y) \cap S = \varnothing$ as well, so that $y \notin \partial S$.

   The case where $B_{\epsilon_0}(x) \cap S^c = \varnothing$ is treated analogously.

3. Which of the sets below are compact?

   (a) $\{x \in \mathbb{R}^N : r \leq \|x\| \leq R\}$ with $0 < r < R$;

   (b) $\{x \in \mathbb{R}^N : r < \|x\| \leq R\}$ with $0 < r < R$;

   (c) $\overline{\left\{\left(t, \sin\frac{1}{t}\right) : t \in (0, 2021]\right\}}$;

   (d) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$;

   (e) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$.

   Justify your answers.

   Solution: In each case, let the set under consideration be denoted by K.

   (a) As $K = B_R[0] \cap B_r(0)^c \subset B_r[0]$ is closed and bounded, it is compact by the Heine–Borel Theorem.

   (b) As $\{B_\rho(0) : r < \rho\}$ is an open cover for K without a finite subcover, K cannot be compact.

(c) As $\{(t, \sin\frac{1}{t}) : t \in (0, 2021]\}$ is clearly bounded, so is its closure K, which is therefore compact by Heine–Borel.

(d) Assume that K is compact. The K is, in particular, closed, i.e., K^c is open. As $0 \in K^c$, there is $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset K^c$. Choose $n \in \mathbb{N}$ so large that $\frac{1}{n} < \epsilon$. It follows that $\frac{1}{n} \in K \cap K^c$, which is impossible.

(e) Let $\{U_i : i \in \mathbb{I}\}$ be an open cover for K. Choose $i_0 \in \mathbb{I}$ such that $0 \in U_{i_0}$. Let $\epsilon > 0$ be such that $(-\epsilon, \epsilon) \subset U_{i_0}$, and choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. It follows that $\frac{1}{n} \in (-\epsilon, \epsilon) \subset U_{i_0}$ for $n \geq n_0$. For $k = 1, \ldots, n_0 - 1$, chose $i_k \in \mathbb{I}$ such that $\frac{1}{k} \in U_{i_k}$. It follows that

$$K \subset U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_{n_0-1}}.$$

Hence, K is compact.

4. Show that:

(a) if $U_1 \subset \mathbb{R}^N$ and $U_2 \subset \mathbb{R}^M$ are open, then so is $U_1 \times U_2 \subset \mathbb{R}^{N+M}$;

(b) if $F_1 \subset \mathbb{R}^N$ and $F_2 \subset \mathbb{R}^M$ are closed, then so is $F_1 \times F_2 \subset \mathbb{R}^{N+M}$;

(c) if $K_1 \subset \mathbb{R}^N$ and $K_2 \subset \mathbb{R}^M$ are compact, then so is $K_1 \times K_2 \subset \mathbb{R}^{N+M}$.

Solution:

(a) Let $(x_0, y_0) \in U_1 \times U_2$. As U_1 and U_2 are open, there are $\epsilon_1, \epsilon_2 > 0$ such that $B_{\epsilon_1}(x_0) \subset U_1$ and $B_{\epsilon_2}(y_0) \subset U_2$. Set $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Let $(x, y) \in B_\epsilon((x_0, y_0))$. Then we have

$$\|x - x_0\| \leq \|(x, y) - (x_0, y_0)\| < \epsilon_1 \qquad \text{and} \qquad \|y - y_0\| \leq \|(x, y) - (x_0, y_0)\| < \epsilon_2$$

so that $(x, y) \subset B_{\epsilon_1}(x_0) \times B_{\epsilon_2}(y_0) \subset U_1 \times U_2$. Hence, $U_1 \times U_2$ is open.

(b) Note that

$$(F_1 \times F_2)^c = (\mathbb{R}^N \times F_2^c) \cup (F_1^c \times \mathbb{R}^M)$$

is open by (a), so that $F_1 \times F_2$ has to be closed.

(c) By (b), $K_1 \times K_2$ is closed. Let $r_1, r_2 > 0$ be such that $K_j \subset B_{r_j}[0]$ for $j = 1, 2$. For $(x, y) \in K_1 \times K_2$, it follows that

$$\|(x, y)\| = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}} \leq \sqrt{2} \max\{\|x\|, \|y\|\} \leq \sqrt{2} \max\{r_1, r_2\}.$$

so that $K_1 \times K_2 \subset B_{\sqrt{2}\max\{r_1, r_2\}}[0]$. Hence, $K_1 \times K_2$ is also bounded and thus compact by the Heine–Borel Theorem.

5. Show that a subset K of $\mathbb{R}^N$ is compact if and only if it has the *finite intersection property*, i.e., if $\{F_i : i \in \mathbb{I}\}$ is a family of closed sets in $\mathbb{R}^N$ such that $K \cap \bigcap_{i\in\mathbb{I}} F_i = \varnothing$, then there are $i_1, \ldots, i_n \in \mathbb{I}$ such that $K \cap F_{i_1} \cap \cdots \cap F_{i_n} = \varnothing$.

   Solution: Suppose that K is compact and that $\{F_i : i \in \mathbb{I}\}$ is a family of closed sets in $\mathbb{R}^N$ such that $K \cap \bigcap_{i\in\mathbb{I}} F_i = \varnothing$. It follows that

$$K \subset \left(\bigcap_{i\in\mathbb{I}} F_i\right)^c = \bigcup_{i\in\mathbb{I}} F_i^c,$$

   so that $\{F_i^c : i \in \mathbb{I}\}$ is an open cover for K. Since K is compact, there are $i_1, \ldots, i_n \in \mathbb{I}$ such that

$$K \subset F_{i_1}^c \cup \cdots \cup F_{i_n}^c = (F_{i_1} \cap \cdots \cap F_{i_n})^c,$$

   and thus

$$K \cap F_{i_1} \cap \cdots \cap F_{i_n} = \varnothing.$$

   Conversely, suppose that K has the finite intersection property, and let $\{U_i : i \in \mathbb{I}\}$ be an open cover for K, so that

$$K \cap \bigcap_{i\in\mathbb{I}} U_i^c = \varnothing.$$

   It follows that there are $i_1, \ldots, i_n \in \mathbb{I}$ such that

$$K \cap U_{i_1}^c \cap \cdots \cap U_{i_n}^c = \varnothing$$

   and thus

$$K \subset U_{i_1} \cup \cdots \cup U_{i_n}.$$

   Hence, K is compact.

6*. For $j = 1, \ldots, N$, let $I_j = [a_j, b_j]$ with $a_j < b_j$, and let $I := I_1 \times \cdots \times I_N$. Determine ∂I. (*Hint*: Draw a sketch for $N = 2$ or $N = 3$.)

   Solution: Since I is closed by part (b) of Problem 4, it is clear that $\partial I \subset I$.

   For $j = 1, \ldots, N$ let

$$J_j := I_1 \times \cdots \times I_{j-1} \times \{a_j, b_j\} \times I_{j+1} \times \cdots \times I_N.$$

   and let $J := J_1 \cup \cdots \cup J_N$.

   We claim that $\partial I = J$.

   It is immediate from this definition that

$$I \setminus J = (a_1, b_1) \times \cdots \times (a_N, b_N),$$

which is open by part (a) of Problem 4. Hence, for any $x \in I \setminus J$, there is $\epsilon > 0$ such that $B_\epsilon(x) \subset I \setminus J \subset I$. It follows that $B_\epsilon(x) \cap I^c = \varnothing$, so that x cannot be a boundary point. It follows that $\partial I \subset J$.

For the converse inclusion, let $x \in J$. Without loss of generality, suppose that $x \in J_1$, i.e., $x_1 = a_1$ or $x_1 = b_1$. Without loss of generality also suppose that $x_1 = a_1$. Let $\epsilon > 0$, and let $\delta < \min\{\epsilon, b_1 - a_1\}$. Define

$$y := (x_1 + \delta, x_2, \ldots, x_N) \qquad \text{and} \qquad z := (x_1 - \delta, x_2, \ldots, x_N).$$

Then $y, z \in B_\epsilon(x)$, but $y \in I$, whereas $z \notin I$. Hence, x is a boundary point of I.