MATH 217 (Fall 2021)

Honors Advanced Calculus, I

Assignment #9

1. Define

$$f\colon [0,1]^3 \to \mathbb{R}, \quad (x,y,z) \mapsto \begin{cases} xy, & z \le xy, \\ z, & z \ge xy. \end{cases}$$

Evaluate $\int_{[0,1]^3} f$.

2. Let

$$D := \{(x,y) \in \mathbb{R} : x, y \ge 0,\ x^2 + y^2 \le 1\},$$

and let

$$f\colon D \to \mathbb{R}, \quad (x,y) \mapsto \frac{4y^3}{(x+1)^2}$$

Evaluate $\int_D f$.

3. Let $I \subset \mathbb{R}^N$ and $J \subset \mathbb{R}^M$ be compact intervals, let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be continuous, and define

$$f \otimes g\colon I \times J \to \mathbb{R}, \quad (x,y) \mapsto f(x)g(y).$$

Then $f \otimes g$ is continuous and thus Riemann integrable. Show that

$$\int_{I \times J} f \otimes g = \left(\int_I f \right) \left(\int_J g \right).$$

4. Let $a < b$, let $f\colon [a,b] \to [0,\infty)$ be continuous, and let

$$D := \{(x,y) : x \in [a,b],\ y \in [0, f(x)]\}.$$

Show that D has content and that

$$\mu(D) = \int_a^b f(x)\,dx.$$

5. Let $a, b > 0$. Determine the area of the ellipse

$$E := \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}.$$

6*. Define $f\colon [0,1] \times [0,1] \to \mathbb{R}$ by letting

$$f(x,y) = \begin{cases} 2^{2n}, & \text{if } (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}) \text{ for some } n \in \mathbb{N}, \\ -2^{2n+1}, & \text{if } (x,y) \in [2^{-n-1}, 2^{-n}) \times [2^{-n}, 2^{-n+1}) \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the iterated integrals

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx \qquad \text{and} \qquad \int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy$$

both exist, but that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx \neq \int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy.$$

Why doesn't this contradict Fubini's Theorem?

Due Thursday, November 25, 2020, at 5:00 p.m.; no late assignments.