MATH 371
Homework Assignment #2

Due date/time: Jan, 31, 2022/23:59

a. Write your solutions on paper or pads, and try to keep solutions for different questions on separate pages.

b. Upload scans/photos/pdfs of your solutions to **Assign2** before the due date and time. Make sure to upload solutions to the right slot for each question.

c. Submissions after the deadline will not be graded and will result in a 0 mark.

1. (70 points) Exercises 2.4.1, 2.4.2, 2.4.6, 2.4.10. (Please submit your solutions in separate files for different questions)

Solution: Question 2.4.1. We can assume a simple linear model for the population growth:

$$P_{n+1} - P_n = aP_n, \quad P_0 = b,$$

where P_n is the population size in year $n \in \mathbb{N}$, parameter a is the rate constant for the net increase, and $b > 0$ is the population size at year 0. Year 0 can be any year we begin to count the population size. T he solution to the model is given by

$$P_n = b(1+a)^n, \quad , n \geq 0.$$

To make future projections, we need to determine the values of the parameters r, a from the given data. We note that the solution contains two parameters a, b and we are given two data points: populations size in year 1998 and the births and deaths in 1999. For simplicity, we can choose $n = 0$ as year 1998, so

$$P_0 = b = 82,037,000,$$

and the net increase in year 1999 ($n = 1$) is

$$aP_0 = P_1 - P_0 = 770,744 - 846,330 = -75,556,$$

and thus

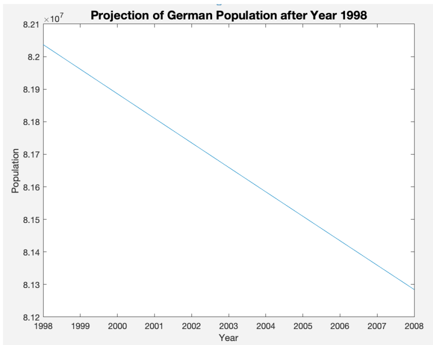
$$a = \frac{-75,556}{P_0} = \frac{-75,556}{82,037,000} = -0.000921.$$

We have calibrated our model using data and obtained

$$P_n = 82,037,000(1 - 0.000921)^n = 82,037,000(0.999079)^n,$$

which gives the future projection for the population, see Figure.

Observation: the model projects that German population will have a steady decline from year 1998.




```
% Plot function P_n = b*(1+a)^n.

clear all;
close all;

a = -0.000921; % growth rate

b = 82037000;    % initial value

x=0:1:10; % define plotting range (x column)

y=b*(1+a).^x; % define function to plot (y column)
%Note: it is important to use ".^" to perform power
%operation in Matlab

plot(x,y) % plot the function (x and y columns)
```

Generalizations. The linear population decline model may not be realistic. It may be modified using either logistic, Ricker, or Beverton-Holt forms of nonlinear decline (choosing $r = 1 + a$ with $a < 0$), described in Sections 2.2.3 and 2.2.4. Which form is more realistic? We need more population data to decide.

Question 2.4.2. **Drug prescriptions.** Consider the following model for a drug prescription:

$$a_{n+1} = a_n - ka_n + b,$$

where an is the amount of a drug (in mg, say) in the bloodstream after administration of n dosages at regular intervals (hourly, say).

(a) Discuss the meaning of the model parameters k and b. What can you say about their size and sign?

Since the drug concentration will generally decline between doses due to body's metabolism (which clears the drug particles), we can see that parameter k represent the rate change in concentration. Because of the negative sign in front of k, we know $k > 0$. It is also quite apparent that $b > 0$ represents the dosage given at regular intervals.

(b) Find the fixed points of the model and their stability via linearization.

Let $f(x) = (1-k)x + b$. Solving the equation $f(x) = x$ we obtain a unique fixed point $\bar{a} = b/k$.

We see that the steady state of the drug concentration is determined by both the dosage b and clearance rate k. The clearance rate should be as small as possible. It is usually depends on the pharmacokinetic properties of the drug. The size of dosage b depends on k and should be chosen in a range such that the concentration is effective and also tolerable to the body.

Since $f'(x) = 1 - k$, and $f'(\bar{a}) = 1 - k$, we know that $\bar{a}$ is stable if $|1 - k| < 1$ and unstable if $|1 - k| > 1$. Solving the inequalities in the range $k > 0$, we obtain the following:

(1) $\bar{a}$ is stable if $0 < k < 2$.

(2) $\bar{a}$ is unstable if $k > 2$.

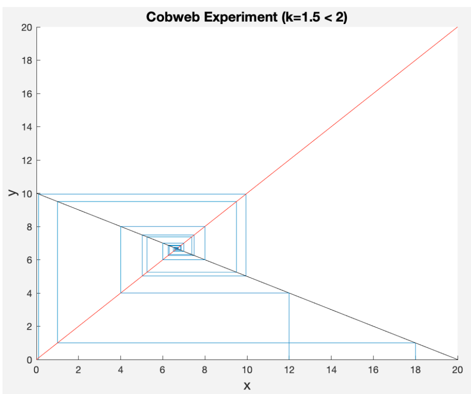
We observe that dosage b does not impact the stability of $\bar{a}$. It influences the level of drug concentration.

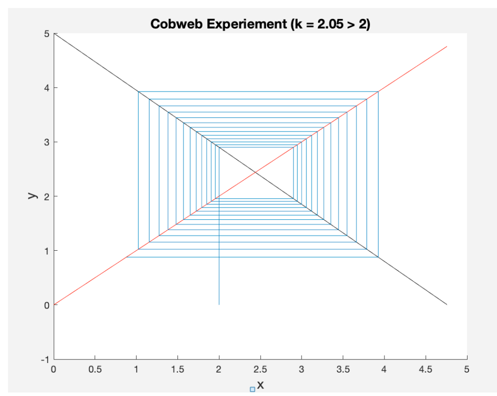
(c) Perform a cobwebbing analysis for this model. What happens to the amount of drug in the bloodstream in the long run? How does the result depend on the model parameters?

Results of two Cobweb experiments are attached: $k = 1$ in (a) and $k = 3$ in (b). The dosage $b = 10$ in both figures. We observe the following:

(1) All trajectories converge to he fixed point $\bar{a}$ when $k = 1.5 < 2$. This should hold for all $0 < k < 2$.

(2) Trajectories move away from $\bar{a}$ for $k = 2.05 > 1$. This should hold for all $k > 2$.





(d) How should b be chosen to ensure that the drug is effective, but not toxic?

When the drug clearance rate k, the minimum drag concentration $a_m > 0$ for the drug to be efficient, and the maximum drug concentration $a_M > 0$ that is safe for human body, the dosage b be can be adjusted so that

$$a_m < \frac{b}{k} < a_M \quad \text{or equivalently} \quad ka_m < b < ka_M.$$

Question 2.4.6. Second-iterate map. This exercise deals with the second-iterate map, $f^2(x)$, for the logistic map, $f(x) = rx(1 - x)$.

(a) Compute $f^2(x)$.

$$\begin{aligned} f^2(x) = f(f(x)) = rf(x)(1 - f(x)) &= r(rx(1 - x))(1 - rx(1 - x)) \\ &= r^2x(1 - x)(1 - rx + rx^2) \end{aligned}$$

(b) Find the fixed points of $f^2(x)$. Verify that a nontrivial 2-cycle exists only for $r > 3$.

Solve $f^2(x) = x$ for fixed points of f^2.

$$x = r^2 x(1-x)(1-rx+rx^2).$$

We know that $x_1 = 0$ is a fixed point, which is also a fixed point of $f(x)$. The rest of fixed points satisfy

$$1 = r^2(1-x)(1-rx+rx^2),$$

or

$$1 - r^2(1-x)(1-rx+rx^2) = r^3x^3 - 2r^3x^2 + r^3x - r^2 + 1 = 0. \tag{1}$$

Note that the logistic map $f(x) = rx(1-x)$ has a second fixed point $x_2 = (r-1)/r$. Since fixed points of f are also fixed points of f^2, we know that $x_2 = (r-1)/r$ is also a solution of the cubic equation (1), and thus $(rx - r + 1)$ is a factor of the cubic polynomial.

By long division, the left hand side of the equation (1) can be factored as

$$(rx - r + 1)(r^2x^2 - r(r+1)x + r + 1) = 0,$$

and thus x_3, x_4 satisfies

$$r^2x^2 - r(r+1)x + r + 1 = 0.$$

Therefore,

$$\begin{aligned} x_{3,4} &= \frac{r(r+1) \pm \sqrt{r^2(r+1)^2 - 4r^2(r+1)}}{2r^2} = \frac{r(r+1) \pm r\sqrt{(r+1)^2 - 4(r+1)}}{2r^2} \\ &= \frac{r(r+1) \pm r\sqrt{(r+1)(r-3)}}{2r^2} \end{aligned}$$

We see that x_3 and x_4 are fixed points of f^2 if and only if $r \geq 3$, and $x_3 = x_4$ when $r = 3$.

Since $x_1 = 0$ and $x_2 = (r-1)/r$ are fixed points of $f(x)$, they won't give rise to nontrivial 2-cycles. The pair of fixed points x_3, x_4 of f^2 give rise to nontrivial 2-cycles of f.

We can verify that $f(x_3) = x_4$ and $f(x_4) = x_3$, the two fixed points x_3, x_4 of f^2 gives rise the orbit $\{x_3, x_4\}$ of single 2-cycle, when $r > 3$.

(c) Compute $\frac{d}{dx} f^2(x)$. Using Mathematica (Wolfram Alpha)

$$\frac{d}{dx} f^2(x) = -r^2(2x-1)(2rx^2 - 2rx + 1).$$

(d) Verify that the nontrivial 2-cycle is stable for $3 < r < 1 + \sqrt{6}$ and unstable for $r > 1 + \sqrt{6}$.

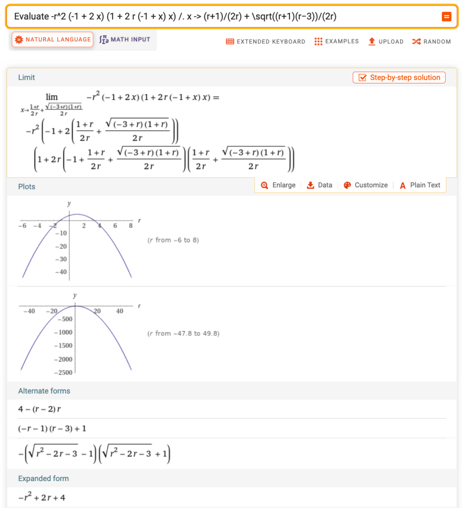
Evaluating the derivative of f^2 at x_3 in Wolfram Alpha, we obtain (see Figure)

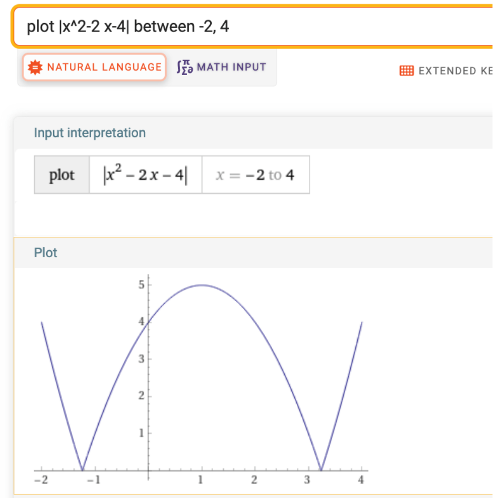
$$\frac{d}{dx} f^2(x_3) = [-r^2(-1+2x)(1+2r(-1+x)x)]\Big|_{x=x_3} = -r^2 + 2r + 4.$$

Solving $-r^2 + 2r + 4 = 1$, we get a positive root $r_1 = 3$. Solving $-r^2 + 2r + 4 = -1$, we get a positive root $r_1 = 1 + \sqrt{6}$. In the plot of $y = |-r^2 + 2r + 4|$, we see that

$$\left|\frac{d}{dx} f^2(x_3)\right| = |-r^2 + 2r + 4| < 1 \qquad \Longleftrightarrow \qquad 3 < r < 1 + \sqrt{6}.$$

Therefore, the nontrivial 2-cycle is stable if and only if when $3 < r < 1 + \sqrt{6}$.

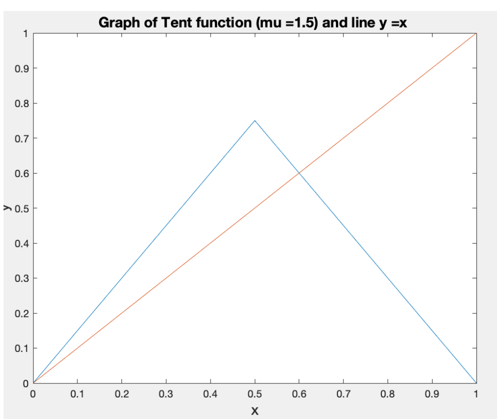




Question 2.4.10. The tent map. The tent map is an approximation to the discrete logistic equation: $x_n + 1 = f(x_n)$ with

$$f(x) = \begin{cases} \mu x & \text{for } 0 \le x < 0.5, \\ \mu(1-x) & \text{for } 0.5 < x \le 1. \end{cases}$$

(a) Sketch the graph of f for $\mu > 0$.




```
%Plot a piecewisely defined function
clear all;
close all;

%Plot the function y = mu x when 0<x<0.5, mu(1-x) when 0.5<x<1.

mu = 1.5;

syms x
y1 = piecewise(0 < x < 0.5, mu.*x, 0.5 < x < 1, mu.*(1-x));

y2 = @(x) x;

fplot(y1,[0,1]);

hold on;

fplot(y2,[0,1]);

hold off;
```

(b) Find the steady states and their stability.

It is clear that when $0 < \mu < 1$, the graph of $f(x)$ only intersects the line $y = x$ at $x^* = 0$. When $\mu = 1$, there are infinitely many fixed points (we ignore this case). When $\mu > 1$, in addition to $x^* = 0$, a positive intersection with $y = x$ occurs in the interval $0.5 < x < 1$. Solving equation

$$\mu(1-x) = x,$$

we obtain $x^{**} = \mu/(1+\mu)$. It exists only for $\mu > 1$.

Stability of $x^* = 0$. Since $x = 1$ falls on the first branch of the tent function, $f'(0) = \mu$. Therefore, $x^* = 0$ is stable if $0 < \mu < 1$ and unstable if $\mu > 1$.

Stability of $x^{**} = \mu/(1+\mu)$. The positive fixed point x^{**} falls on the second branch. $f'(x^{**}) = -\mu$. Therefore, x^{**} is unstable when $\mu > 1$, i.e. as long as it exists. We see that, when $\mu > 1$, both fixed points $x^* = 0$ and x^{**} are unstable.

(c) Find orbits of period 2. Assuming $\mu > 1$.

$$f^2(x) = f(f(x)) = \begin{cases} \mu f(x), & 0 < f(x) < 0.5, \\ \mu(1 - f(x)), & 0.5 < x < 1. \end{cases}$$

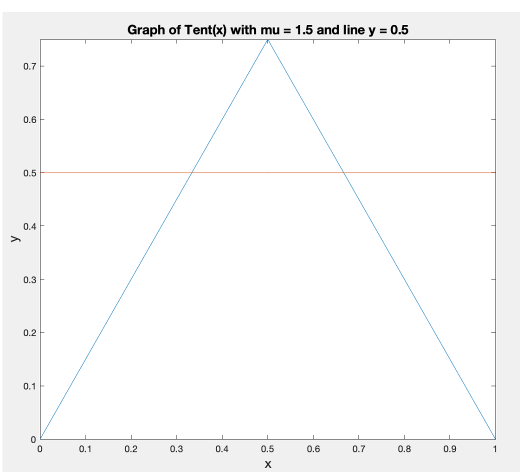
The range of $f(x)$ is $[0, 0.5\mu]$, and the line $y = 0.5$ intersects graph of $f(x)$ in three points, whose x coordinates, together with $x = 0.5$, divide the interval $[0, 1]$ into four subintervals, $[0, 0.5/\mu)$, $(0.5/\mu, 0.5)$, $(0.5, 1 - 0.5/\mu)$, $(1 - 0.5/\mu, 1]$. See Figure. Therefore, $f^2(x)$ can be defined in these subintervals accordingly:

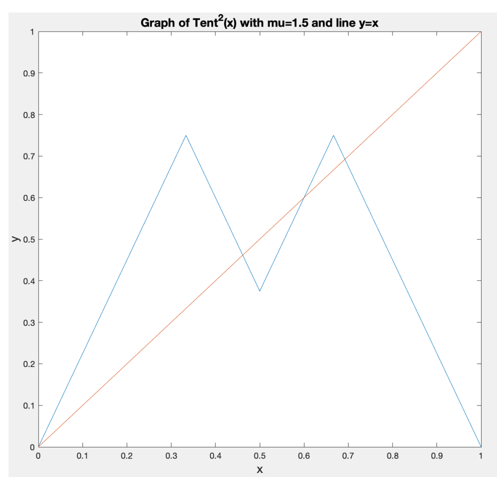
$$f^2(x) = \begin{cases} \mu(\mu x), & 0 \le x < 0.5/\mu, \\ \mu(1 - \mu x), & 0.5/\mu < x < 0.5, \\ \mu(1 - \mu(1 - x)), & 0.5 < x < 1 - 0.5/\mu, \\ \mu(\mu(1 - x)), & 1 - 0.5/\mu < x \le 1 \end{cases}$$

The graph of $f^2(x)$ is shown in the Figure below. Solving $f^2(x) = x$ and considering all branches, we find three fixed points:

$$x^* = \frac{\mu}{1+\mu}, \quad x^{**} = \frac{\mu}{1+\mu^2}, \quad x^{***} = \frac{\mu^2}{1+\mu^2}.$$

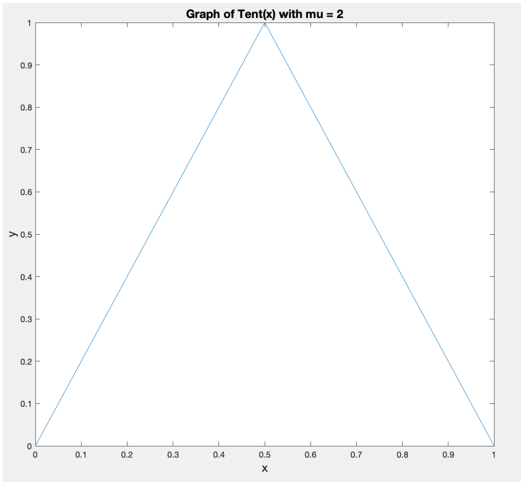
We see that x^* is the fixed point of f, so it is a trivial 2-cycle. We can also see that $x^{**} = \dfrac{\mu}{1+\mu^2} < 0.5$ when $\mu > 1$, and hence $f(x^{**}) = \mu x^{**} = \dfrac{\mu^2}{1+\mu^2} = x^{***}$. Therefore, $\{x^{**}, x^{***}\}$ is the orbit of a single 2-cycle.

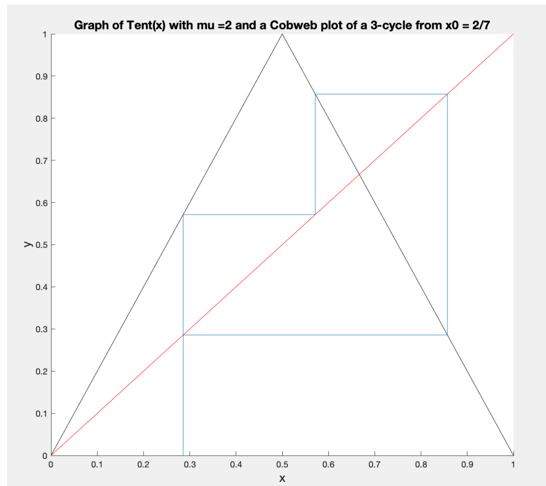




(d) Plot f for $\mu = 2$. Carefully try to find an orbit of period 3.

The graph of f for $\mu = 2$ and Cobweb plot with a period-3 orbit starting at $x_0 = 2/7$ are shown in the Figure below.





2. (30 points) Computation exercises 8.2.1, 8.2.2 and 8.2.3. Please hand in your code with an output (e.g. generated by Word or LaTex) for each question (You may use any software package to do the exercises).

Solution: Question 8.2.1. (a) For the simplified Ricker model:

$$x_{n+1} = ax_n e^{-x_n} = f(x_n), \quad a > 0, \; n \in \mathbb{N},$$

where $f(x) = axe^{-x}$, we can find the equilibria by solving $f(x) = x$, and we obtain two equilibria:

$$x*_1 = 0, \quad x_2^* = \ln a.$$

To determine the stability, we compute the derivative of f: $f'(x) = ae^{-x}(1-x)$.

(a). Stability of x_1^*:

$$f'(x_1^*) = f'(0) = a > 0.$$

Therefore,

(1) x_1^* is stable if $0 < a < 1$;

(2) x_1^* is unstable if $a > 1$.

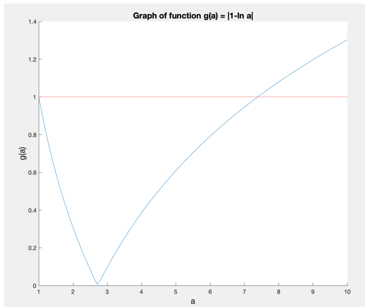
(b). Stability of x_2^*:

$$f'(x_2^*) = f'(\ln a) = 1 - \ln a.$$

We want to determine the ranges of a for which $|1 - \ln a| < 1$ and $|1 - \ln a| < 1$. Note that, equation $|1 - \ln a| = 1$ has two solutions: $a_1 = 1$ and $a_2 = e^2$, these are where the graph of $|1 - \ln a|$ and line $y = 1$ intersect. Using the graph of function $g(a) = |1 - \ln a|$ below, we see that

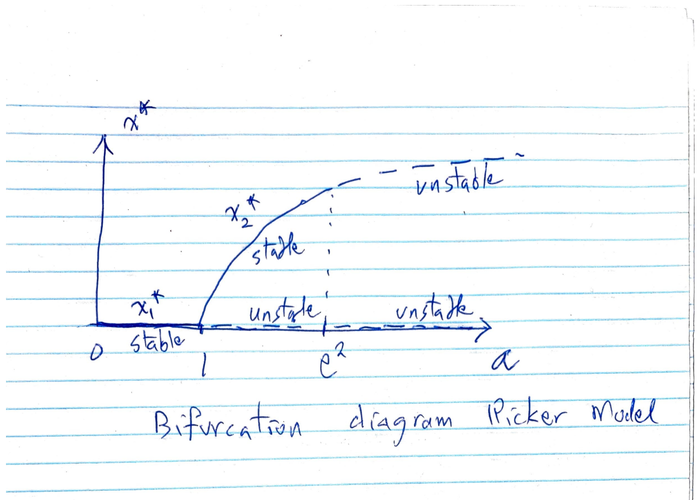
(1) x_2^* is stable if $1 < a < e^2$;

(2) x_1^* is unstable if $a > e^2$.

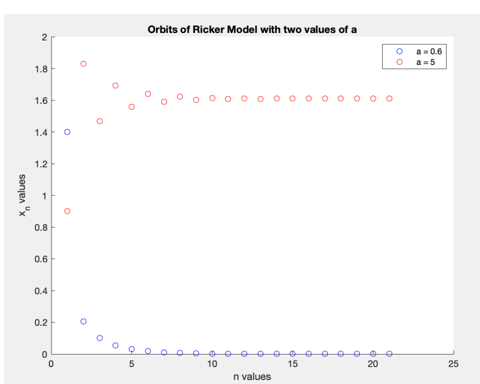



```
%Plot the grapha of g(a) = |1 - ln a|, for 1<a<10, together with line y=1.

clear all;
close all;

x = 1:0.1:10;
y = abs(1 - log(x));


hold on;

plot(x,y);

yline(1, 'r');

hold off;
```

(c). A hand sketch of bifurcation diagram:



Question 8.2.2. Plot a trajectory of Ricker model for different values of a. The plot is shown below, together with the Matlab codes.



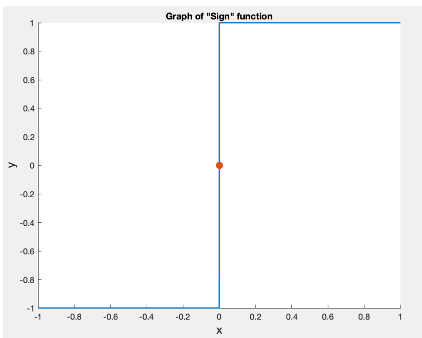

```
%Model analysis of Simplified Ricker Model, and plotting its trajectories (Section 8.2)
% x(n+1) = a*x(n).*exp(-x(n));
% for different values of a and 0<x(0)<1.

close all; %Close all previous plots
clear all; %Clear aall previous variables an values

a1=0.6;
a2=5;

x(1)=1.4;    % initial value
y(1)=0.9;

for n=1:20 % n is the iteration variable
x(n+1)=a1*x(n).*exp(-x(n));
end

for n=1:20 % n is the iteration variable
y(n+1)=a2*y(n).*exp(-y(n));
end

hold on
plot(x, 'ob')
plot(y, 'or')
hold off

xlabel('n values');
ylabel('x_n values');
title('Orbits of Ricker Model with two values of a');
legend('a = 0.6', 'a = 5')
```

Exercise 8.2.3. The sign function is defined as:

$$\begin{cases} -1 & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

Write a Matlab function to define the sign function.

The Matlab codes and plot of the graph of "sign" function is given below. The code can be adapted to define any piece-wise functions.




```
%Plot a piecewisely defined function
clear all;
close all;

%Plot the function y = 1 when x< 0, 0 when $x=0, and -1 when x>0.

syms x

y1 = piecewise(x<0, -1, x>0, 1, 0);


hold on
fplot(y1,[-1,1]);
plot(0,0, '.', 'MarkerSize', 30)
hold off;



```