Math 328 – Midterm Exam
University of Alberta, Fall 2021

Due Date: 2021-11-04, 12:00 noon

Instructions – READ CAREFULLY

This midterm exam has five problems of equal weight but varying difficulty. A complete, coherent and correct argument must be given in all cases for full credit. Solutions must be written *legibly*, with *complete sentences* where appropriate. You may use all written resources (lecture notes, books, wikipedia, etc.) when writing this exam. **However, collaboration (both in-person and electronic) is not allowed.** Please submit your solutions using Assign2 before the deadline, which is **Thursday, 2020-11-04 at 12:00 noon**. You may wish to submit your solutions a few hours before the actual deadline in case you encounter any technical difficulties. **This deadline cannot be extended!**

Important: The content of this midterm should not be discussed with any person until the Math 328 lecture on Friday 2021-11-05. This policy is in place to account for any potential academic accommodations that may have been granted. Solutions to this midterm will be discussed during lecture on 2021-11-05.

Problem 1

For each of the following, either give an explicit example, or prove that no example exists.

1. A group G, an element $g \in G$, a set of integers S, and a non-cyclic subgroup of $\langle \{g^a \mid a \in S\} \rangle$, considered as a subgroup of G.

2. A finite group G of odd order, a homomorphism $\phi : D_{16} \to G$, and an element $g \in D_{16}$ such that $\phi(g) \neq 1$.

3. A non-abelian group G and a normal subgroup $N \trianglelefteq G$ such that N and G/N are both abelian.

Problem 2

Let G denote the set

$$G := \mathbb{Z} \times \{\pm 1\}.$$

For (a, b), $(a', b') \in G$, define

$$(a,\ b) \star (a',\ b') := (a + b \cdot a',\ b \cdot b').$$

(The symbols $+$ and $\cdot$ denote the usual addition and multiplication of integers.)

1. Prove that $(G, \star)$ is a group.
2. Prove that $N := \{(a, 1) \mid a \in \mathbb{Z}\}$ is a normal subgroup of G, which is isomorphic to $\mathbb{Z}$.
3. Prove that G/N is cyclic.

Problem 3

Let G be a group. Consider the subset of G defined as follows:

$$S := \{g \cdot h \cdot g^{-1} \cdot h^{-1} \mid g, h \in G\}.$$

Suppose that H is a subgroup of G such that $S \subset H$. Prove that H is a normal subgroup of G and that G/H is abelian.

Problem 4

Let G be a group of order 30625, and H a subgroup of G such that $[G : H] = 5$.

1. Let $g \in G$ be given. Prove that $K := g \cdot H \cdot g^{-1}$ is a subgroup of G, that $[G : K] = 5$, and that $K \cdot H \neq G$.
2. Let $g \in G$ be given and put $K := g \cdot H \cdot g^{-1}$. Prove that $K \cap H = K$.
   Hint: Consider $\#(K \cdot H)$ and use part (1).
3. Prove that H is a normal subgroup of G and that G/H is cyclic.

Note: $30625 = 5^4 \cdot 7^2$.

Problem 5

Let G be a group and H a subgroup of G.

1. Prove that the map
$$G \times (G/H) \to G/H,$$
given by $(g,\ g' \cdot H) \mapsto (g \cdot g') \cdot H$ is well-defined, and that it defines an action of the group G on G/H.

2. Let $\rho : G \to \mathrm{Per}(G/H)$ denote the permutation representation associated to the action from part (1). Prove that $\ker(\rho)$ is contained in H.

3. Assume that $[G : H] = n$ with n a positive integer. Prove that there exists a normal subgroup $K \trianglelefteq G$ such that $K \leq H$ and $[G : K] \leq n!$.