6 Power series and Taylor's Theorem

Why are we interested in series? In analysis (or calculus, if you will), they provide a very important and rich class of functions, based on *power series*.

6.1 Taylor's Theorem and Taylor series

One of the most interesting theorems in analysis is the fact that one can *approximate* sufficiently differentiable functions just by knowing their multiple derivatives at a given point.

We already saw a glimpse of that by observing that if f is differentiable at x_0, then $f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x)$ with $r(x)$ "small." How can we generalize that to obtain better approximations?

We have also seen that if f is a power series, it may be approximated by polynomials (its partial sums), and the coefficients of those polynomials are related to the derivatives of f at the centre.

This will require some preparation.

6.1.1 General Rolle's Theorem

Recall that Rolle's Theorem states that for f differentiable, if $f(a) = f(b)$ for some $a < b$, there must be $c \in (a, b)$ for which $f'(c) = 0$.

Is there an analogue for higher derivatives? Indeed, there is.

Theoem (General Rolle's Theorem)
Let $I = [a, b]$ be an interval, and f a continuous function on I, n times continuously differentiable on $(a, b]$, where $n \geq 0$ is an integer, and $f(a) = f(b)$. Suppose $f^{(n+1)}$ exists in at least (a, b) and $f^{(k)}(b) = 0$ for $k = 1,2, \dots, n$. Then there is $c \in (a, b)$ such that $f^{(n+1)}(c)$=0.
A similar result holds for intervals $[a, b)$ with the roles of a and b interchanged. EOT.

Proof. We proceed by induction on n. If $n = 0$ this is Rolle's Theorem. Now suppose the theorem holds for a particular n, and let a, b such that $f(a) = f(b)$ and $f^{(k)}(b) = 0$ for all $1 \leq k \leq n + 1$. We must show that there is $c \in (a, b)$ such that $f^{(n+2)}(c) = 0$.

By induction, there exists $c' \in (a, b)$ such hat $f^{(n+1)}(c') = 0$. Then $c' < b$ and applying Rolle's Theorem to $f^{(n+1)}$ on the interval $[c', b]$, there must be $c \in (c', b) \subseteq (a, b)$ such that $f^{(n+1)'}(c) = f^{(n+2)}(c) = 0$. QED.

Corollary (of Proof)
Let $I = [a, b]$ be an interval and f a continuous function on I, $n + 1$ times differentiable on $[a, b)$, where $n \geq 0$ is an integer, and $f(a) = f(b)$. Suppose $f^{(k)}(a) = 0$ for $k = 1,2, \dots, n$. Then there is $c \in (a, b)$ such that $f^{(n+1)}(c) = 0$. EOC.

Proof. Exercise. QED.

Original Rolle was used to prove the Mean Value Theorem. General Rolle is then used to prove its natural generalization, known as Taylor's Theorem.

6.1.2 Taylor polynomials

Definition

Let f be a function defined on an interval I. Let $c \in I$. If f is n-times differentiable at c, then the polynomial $P_{f,n,c}(x) = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(x-c)^n$ is called the **degree n Taylor polynomial** (or nth Taylor polynomial) of f at c. It is always a polynomial of degree **at most** n.

The polynomials $P_{f,n.c}$ are often also referred to as **Taylor expansions** of f at c. EOD.

Exercise

Show that if f itself is a polynomial function, then $P_{f,c,n} = f$ for all $c \in \mathbb{R}$ as long as $n \geq \deg f$. EOE.

Example

1. Let $f(x) = \sin x$ defined on $\mathbb{R}$. Then $P_{f,0,n} = x - \frac{1}{6}x^3 + \frac{1}{105}x^5 + \cdots = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(2k-1)!}x^{2k-1}$.
2. Let $f(x) = \log(1+x)$ defined on $(-1,1)$. Then $f'(0) = 1$, $f''(x) = \frac{1}{1+x}' = -\frac{1}{(1+x)^2} = -(1+x)^{-2}$, so $f''(0) = -1$. Continuing, $f^{(n)}(0) = (-1)^{n+1}\,(n-1)!$, and $P_{f,n,0} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \pm \cdots + \frac{(-1)^{n+1}}{n}x^n$.

EOE.

Exercise

Show that $P_{f,n,c}^{(k)}(c) = f^{(k)}(c)$ for all $1 \leq k \leq n$. EOE.

6.1.3 Taylor's Theorem

One of the remarkable facts of analysis is that one often can say a lot of meaningful things about the difference $\left|f(x) - P_{f,n,c}(x)\right|$. If that difference is small, then the Taylor polynomial can serve as a good approximation for f.

Theorem (Taylor's Theorem; TT)

Suppose f is n times continuously differentiable on an interval $[a,b]$, and suppose $f^{(n+1)}$ exists on at least (a,b). For every $u \in [a,b)$ there is d strictly between u and b such that

$$f(u) - P_{f,n,c}(u) = \frac{(u-c)^{n+1}}{(n+1)!}f^{(n+1)}(d)$$

A similar theorem holds for intervals of the form $[a,b)$ with the roles of a and b interchanged.
EOT.

For $n = 0$, this is essentially the Mean Value Theorem, where we say a function f is "0-times continuously differentiable" if it is continuous.

Proof. The idea (not mine) is to apply General Rolle. For this we must find a function h such that $h(u) = h(b)$ and $h^{(k)}(b) = 0$ for all $1 \leq k \leq n$, and such that $h^{(n+1)}(d) = 0$ for d strictly between u and b if and only if $f(u) - P_{f,n,c}(u) = \frac{(x-c)^{n+1}}{(n+1)!}f^{(n+1)}(d)$.

Note that $g(x) = f(x) - P_{f,n,b}(x)$ satisfies that $g^{(k)}(b) = 0$ for $0 \leq k \leq n$. The only thing that is missing is that $g(u) \neq g(b) = 0$ in general.

To fix this define $h(x) = f(x) - P_{f,n,c}(x) - \frac{(x-b)^{n+1}}{(u-b)^{n+1}}\left(f(u) - P_{f,n,b}(u)\right)$.

Now $h(u) = 0 = h(b)$. Also $h^{(k)}(b) = 0$ for $1 \le k \le n$. So, there is d between u, c such that $h^{(n+1)}(d) = 0$. (See also the last exercise in the previous section.)

But $h^{(n+1)}(d) = f^{(n+1)}(d) - 0 - \frac{(n+1)!}{(u-b)^{n+1}}\left(f(u) - P_{f,n,b}(u)\right)$. Solving this equation for $f^{(n+1)}(d)$ gives the result. QED.

Definition
The "error term" $\frac{(u-b)^{n+1}}{(n+1)!} f^{(n+1)}(d)$ is often called the **Lagrange remainder**. EOD.

Because of the "high" degree in $(u-b)$ the Lagrange remainder may seem large. But note that if u is close to c, then $(u-b)^{n+1}$ is small, made even smaller by the division by $(n+1)!$. Thus, the behaviour of $f^{(n+1)}(x)$ is crucial to analyzing the error term. EOD.

Warning
The d in the Lagrange remainder depends on n and x. Different x means different d in general. EOW.

In particular, if we know that $\left|f^{(n+1)}(x)\right| \le \alpha C^{n+1}$ for some $\alpha, C > 0$ and all n (assuming that f is smooth), this yields an error term that always converges to 0.

Example

1. Let $f(x) = \sin x$. Then for $k = 0,1, \dots$ $f^{(2k)}(x) = (-1)^k \sin x$, and $f^{(2k+1)}(x) = (-1)^k \cos x$. It follows that the Lagrange remainder is always bounded by
$$\frac{|(u-b)^{n+1}|}{(n+1)!}$$
In particular, for fixed $u, b \in \mathbb{R}$, we have $\lim_{n\to\infty} P_{f,n,b}(u) = f(u)$.
2. Let $f(x) = \log(1+x)$ on $(-1,1)$. Recall $f'(x) = \frac{1}{1+x}$, and for $n \ge 1$
$$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(1+x)^{-n}$$
Then for $u \in (-1,1)$, we have that the Lagrange remainder (with $b = 0$) is
$$R_n(x) = \frac{u^{n+1}}{(n+1)!}(-1)^{n+1} n!\,(1+d)^{-n-1}$$
for some d strictly between u and 0. Note that for the Lagrange remainder we get $\frac{|u|^{n+1}}{(n+1)|1+d|^{n+1}} \le \frac{1}{n+1}$ as long as $x \in [0,1)$, and so $d \ge 0$. Thus for such x, we have $\lim_{n\to\infty} R_n(x) = 0$.
It follows that for $x \in [0,1)$ we have
$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$
What about $x \in (-1,0)$? We know that the right hand side still converges (it is a power series with radius of convergence 1).
Consider $g(x) = \frac{1}{1+x}$. Then on $(-1,1)$ g is a power series, namely $g(x) = \sum_{n=0}^{\infty}(-1)^n x^n$.
Indeed, $g(-x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Let $G(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$. Then $D(G) = g$ so G and g have the same radius of convergence. In particular, $G' = g = f'$, so $G = f + C$ for some $C \in \mathbb{R}$. Then $C = G(0) - f(0) = 0$. It follows that $\log(1+x) = G(x)$ on $(-1,1)$.

EOE.

Remark

Let $f(x) = \log(1+x)$. We know that $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ on $(-1,1)$. We also know that the power series diverges for $x = -1$ (harmonic series), and still converges for $x = 1$ (Leibniz Criterion). Could it be that $f(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$? The Lagrange Remainder gives the answer: We know that $R_n(1) = f(1) - P_{f,n,0}(1) = \frac{(-1)^{n+1} 1^{n+1}}{(n+1)(1+d)^{n+1}}$ for some $d \in (0,1)$. Observe here it is crucial that Taylor's Theorem applies for the *closed* interval $[0,1]$. It follows that $R_n(x) \to 0$ for $n \to \infty$. We conclude that

$$f(1) = \log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

EOR.

6.1.4 Taylor series

One of the most important examples of so called power series are Taylor series associated to smooth functions. Recall that a function f defined on an interval I is called **smooth** if $f^{(n)}$ exists on all of I for all $n \in \mathbb{N}$.

Definition

Let I be an interval, $c \in I^\circ$, and f a function defined on I, such that $f^{(n)}(c)$ exists for all $n \in \mathbb{N}$. Then the **Taylor series** of f at c is the formal power series

$$T_{f,c}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

EOD.

Subtlety alert

The existence of $f^{(n)}$ at c for all n implies that for each n there is a $\delta_n > 0$ such that $f, f', f'', \dots, f^{(n)}$ are defined on $(c - \delta_n, c + \delta_n)$. But nothing prevents δ_n from being a zero sequence a priori. So $f^{(n)}$ to exist for all n does not directly imply that there is an open interval containing c where $f^{(n)}$ is defined for all $n \in \mathbb{N}$. EOS.

The convergence radius of T_f may be 0. Even if T_f converges at a point x_0, it may converge to a value other than $f(x_0)$. We identify the Taylor series with the function it induces on the interval centered at c (of lenth twice the convergence radius).

Example

1. Since $\exp'(\mathrm{x}) = \exp(x)$, $T_{\exp,0}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \exp(x)$.
2. More generally, if f is any power series centered at c with positive convergence radius, then f coincides with $T_{f,c}$.
3. For any real number $a \in \mathbb{R}$ and any $n \in \mathbb{N}_0$ we define the **generalized binomial coefficient** $\binom{a}{n}$ as