Math 127
Suggested solutions to Homework Set 3

Problem 1. We can write

$$\begin{aligned}
(xy)\cdot(y^{-1}x^{-1}) &= x\left(y\,(y^{-1}x^{-1})\right) && \text{(associativity of multiplication)}\\
&= x\left((yy^{-1})\,x^{-1}\right) && \text{(again, associativity of multiplication)}\\
&= x\,(1_{\mathcal{S}}\cdot x^{-1}) && (\,y^{-1} \text{ is the multiplicative inverse of } y\,)\\
&= x\cdot x^{-1} && (\,1_{\mathcal{S}} \text{ is the multiplicative identity of } \mathcal{S}\,)\\
&= 1_{\mathcal{S}}\,. && (\,x^{-1} \text{ is the multiplicative inverse of } x\,)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(y^{-1}x^{-1})\cdot(xy) &= \left((y^{-1}x^{-1})\,x\right)y && \text{(associativity of multiplication)}\\
&= \left(y^{-1}\,(x^{-1}x)\right)y && \text{(again, associativity of multiplication)}\\
&= (y^{-1}\cdot 1_{\mathcal{S}})\,y && (\,x^{-1} \text{ is the multiplicative inverse of } x\,)\\
&= y^{-1}\cdot y && (\,1_{\mathcal{S}} \text{ is the multiplicative identity of } \mathcal{S}\,)\\
&= 1_{\mathcal{S}}\,. && (\,y^{-1} \text{ is the multiplicative inverse of } y\,)
\end{aligned}$$

Thus xy is invertible, and its multiplicative inverse is the element $y^{-1}x^{-1}$.

Problem 2. (i) In each field $\mathbb{F}$ there is a neutral element of addition, which we denote by 0, and a neutral element of multiplication, which we denote by 1, and these two are different elements. In fact, given that $\mathbb{Z}_2$ is one of the fields we have already seen, and that $\mathbb{Z}_2$ only has two elements, this implies that 0 and 1 are the only elements we can be sure to have in an arbitrary field $\mathbb{F}$.

Let $\mathbb{F}$ now be a field, and let $0_{\mathbb{F}}$ be the neutral element of addition in $\mathbb{F}$, $1_{\mathbb{F}}$ be the neutral element of multiplication in $\mathbb{F}$. Consider the matrices

$$A = \begin{pmatrix} 0_{\mathbb{F}} & 1_{\mathbb{F}} \\ 0_{\mathbb{F}} & 0_{\mathbb{F}} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1_{\mathbb{F}} & 1_{\mathbb{F}} \\ 0_{\mathbb{F}} & 0_{\mathbb{F}} \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 0_{\mathbb{F}} & 0_{\mathbb{F}} \\ 0_{\mathbb{F}} & 0_{\mathbb{F}} \end{pmatrix}, \qquad \text{while} \qquad BA = \begin{pmatrix} 0_{\mathbb{F}} & 1_{\mathbb{F}} \\ 0_{\mathbb{F}} & 0_{\mathbb{F}} \end{pmatrix}.$$

Since the $(1,2)$-entry of AB is different from the corresponding entry of BA, the two matrices are different. In other words, the matrices A and B do not commute.

(ii) Let us fix $n \geqslant 3$. We will rely on the matrices A, B we found for part (i) to come up with larger matrices $\widetilde{A}, \widetilde{B}$ in $\mathbb{F}^{n \times n}$ that do not commute either.

Set

$$\widetilde{A} = \left(\begin{array}{cc|cccc} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 & 0 & \cdots & 0 \\ \multicolumn{2}{c|}{} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right) = \left(\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right)$$

$$\text{and} \quad \widetilde{B} = \left(\begin{array}{cc|cccc} \multicolumn{2}{c|}{\multirow{2}{*}{B}} & 0 & 0 & \cdots & 0 \\ \multicolumn{2}{c|}{} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right) = \left(\begin{array}{cc|cccc} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right).$$

We can check that

$$\widetilde{A}\widetilde{B} = \left(\begin{array}{cc|cccc} \multicolumn{2}{c|}{\multirow{2}{*}{AB}} & 0 & 0 & \cdots\cdots & 0 \\ & & 0 & 0 & \cdots\cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \end{array}\right) \neq \left(\begin{array}{cc|cccc} \multicolumn{2}{c|}{\multirow{2}{*}{BA}} & 0 & 0 & \cdots\cdots & 0 \\ & & 0 & 0 & \cdots\cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \end{array}\right) = \widetilde{B}\widetilde{A}.$$

More simply, the $(1, 2)$-entry of $\widetilde{A}\widetilde{B}$ is the dot product of the 1st row of $\widetilde{A}$ and the 2nd column of $\widetilde{B}$, so it is equal to 0, while the $(1, 2)$-entry of $\widetilde{B}\widetilde{A}$ is the dot product of the 1st row of $\widetilde{B}$ and the 2nd column of $\widetilde{A}$, so it is equal to 1. This shows that the two products cannot be equal.

Problem 3. Consider matrices $A, B \in \mathbb{F}^{m\times n}$ and $C \in \mathbb{F}^{n\times l}$. Write

$$A = \big(a_{ij}\big)_{\substack{1\leqslant i\leqslant m,\\ 1\leqslant j\leqslant n}} \quad B = \big(b_{ij}\big)_{\substack{1\leqslant i\leqslant m,\\ 1\leqslant j\leqslant n}} \quad \text{and } C = \big(c_{rs}\big)_{\substack{1\leqslant r\leqslant n.\\ 1\leqslant s\leqslant l}}$$

Then $A + B = \big(a_{ij} + b_{ij}\big)_{\substack{1\leqslant i\leqslant m,\\ 1\leqslant j\leqslant n}}$ and therefore the product $(A+B)C$ is defined and is in $\mathbb{F}^{m\times l}$. Fix $1 \leqslant i \leqslant m$ and $1 \leqslant s \leqslant l$. The (i,s)-entry of $(A+B)C$ is equal to

$$\langle \mathrm{Row}_i(A+B),\, \mathrm{Col}_s(C)\rangle = \sum_{j=1}^{n}(a_{ij}+b_{ij})c_{js}. \tag{1}$$

Moving on, AC and BC are also defined and they are both in $\mathbb{F}^{m\times l}$. The (i,s)-entry of AC is equal to

$$\langle \mathrm{Row}_i(A),\, \mathrm{Col}_s(C)\rangle = \sum_{j=1}^{n} a_{ij}c_{js}\,,$$

while the (i,s)-entry of BC is equal to

$$\langle \mathrm{Row}_i(B),\, \mathrm{Col}_s(C)\rangle = \sum_{j=1}^{n} b_{ij}c_{js}\,.$$

Therefore the (i,s)-entry of $AC + BC$ is equal to

$$\left(\sum_{j=1}^{n} a_{ij}c_{js}\right) + \left(\sum_{j=1}^{n} b_{ij}c_{js}\right). \tag{2}$$

We now compare (1) and (2): by the distributive law in $\mathbb{F}$, we have that

$$\sum_{j=1}^{n}(a_{ij}+b_{ij})c_{js} = \sum_{j=1}^{n}\big(a_{ij}c_{js} + b_{ij}c_{js}\big)$$

which, by generalised commutativity and associativity in $\mathbb{F}$, is in turn

$$= \left(\sum_{j=1}^{n} a_{ij}c_{js}\right) + \left(\sum_{j=1}^{n} b_{ij}c_{js}\right).$$

In other words, the (i,s)-entry of $(A+B)C$ is equal to the (i,s)-entry of $AC + BC$. Since the indices i and s were arbitrary, we conclude that every entry of $(A+B)C$ is equal to the corresponding entry of $AC+BC$, and thus that $(A+B)C = AC+BC$.

Next we also consider a matrix $E \in \mathbb{F}^{k\times m}$ and write

$$E = \big(e_{tu}\big)_{\substack{1\leqslant t\leqslant k\\ 1\leqslant u\leqslant m}}.$$

$E(A+B)$ is defined and is in $\mathbb{F}^{k\times n}$. Fix $1 \leqslant t \leqslant k$ and $1 \leqslant j \leqslant n$. Then the (t,j)-entry of $E(A+B)$ is equal to

$$\langle \mathrm{Row}_t(E),\, \mathrm{Col}_j(A+B)\rangle = \sum_{u=1}^{m} e_{tu}(a_{uj}+b_{uj}). \tag{3}$$

Similarly EA and EB are defined and they are in $\mathbb{F}^{k\times n}$. The (t,j)-entry of EA is equal to

$$\langle \mathrm{Row}_t(E),\, \mathrm{Col}_j(A)\rangle = \sum_{u=1}^{m} e_{tu}a_{uj}$$

while the (t,j)-entry of EB is equal to

$$\langle \mathrm{Row}_t(E),\, \mathrm{Col}_j(B)\rangle = \sum_{u=1}^{m} e_{tu}b_{uj}\,.$$

Therefore the (t,j)-entry of $EA+EB$ is equal to

$$\left(\sum_{u=1}^{m} e_{tu}a_{uj}\right) + \left(\sum_{u=1}^{m} e_{tu}b_{uj}\right). \tag{4}$$

We now compare (3) and (4): by the distributive law in $\mathbb{F}$, we have that

$$\sum_{u=1}^{m} e_{tu}(a_{uj}+b_{uj}) = \sum_{u=1}^{m}\big(e_{tu}a_{uj}+e_{tu}b_{uj}\big)$$

which, by generalised commutativity and associativity in $\mathbb{F}$, is in turn

$$= \left(\sum_{u=1}^{m} e_{tu}a_{uj}\right) + \left(\sum_{u=1}^{m} e_{tu}b_{uj}\right).$$

In other words, the (t,j)-entry of $E(A+B)$ is equal to the (t,j)-entry of $EA+EB$. Since the indices t and j were arbitrary, we conclude that every entry of $E(A+B)$ is equal to the corresponding entry of $EA+EB$, and thus that $E(A+B) = EA+EB$.

Problem 4. (i) Consider two upper triangular matrices $U, U' \in \mathbb{F}^{n\times n}$, and write

$$U = \big(u_{ij}\big)_{1\leqslant i,j\leqslant n}, \qquad U' = \big(u'_{ij}\big)_{1\leqslant i,j\leqslant n}.$$

By the definition of upper triangular matrix, we know that $u_{ij} = 0 = u'_{ij}$ if $i > j$.

We set $A = U + U'$, $B = UU'$, and we write

$$A = \big(a_{ij}\big)_{1\leqslant i,j\leqslant n}, \qquad B = \big(b_{ij}\big)_{1\leqslant i,j\leqslant n}.$$

To check that both A and B are upper triangular, we use the definition again: we need to check that $a_{ij} = 0$ if $i > j$, and similarly that $b_{ij} = 0$ if $i > j$.

But whenever $i > j$ we have that $a_{ij} = u_{ij} + u'_{ij} = 0 + 0 = 0$, thus we can conclude that A is upper triangular.

On the other hand,

$$\begin{aligned} b_{ij} = \sum_{r=1}^{n} u_{ir}u'_{rj} &= \sum_{\substack{1\leqslant r\leqslant n\\ i>r}} u_{ir}u'_{rj} + \sum_{\substack{1\leqslant r\leqslant n\\ i\leqslant r}} u_{ir}u'_{rj} \\ &= \sum_{\substack{1\leqslant r\leqslant n\\ i>r}} 0\cdot u'_{rj} + \sum_{\substack{1\leqslant r\leqslant n\\ i\leqslant r}} u_{ir}u'_{rj} = \sum_{\substack{1\leqslant r\leqslant n\\ i\leqslant r}} u_{ir}u'_{rj}. \end{aligned}$$

It remains to note that, $\underline{\textbf{if } \boldsymbol{i > j}}$, then $r \geqslant i$ implies that $r > j$, therefore the final sum above is also equal to 0:

$$b_{ij} = \sum_{\substack{1\leqslant r\leqslant n\\ i>r}} u_{ir}u'_{rj} + \sum_{\substack{1\leqslant r\leqslant n\\ i\leqslant r}} u_{ir}u'_{rj} = \sum_{\substack{1\leqslant r\leqslant n\\ i\leqslant r}} u_{ir}u'_{rj} = \sum_{\substack{1\leqslant r\leqslant n\\ i\leqslant r}} u_{ir}\cdot 0 = 0.$$

We conclude that B is upper triangular too.

(ii) Consider two lower triangular matrices $L, L' \in \mathbb{F}^{n\times n}$. Our proof here will be completely analogous to the above argument. We write

$$L = \big(l_{ij}\big)_{1\leqslant i,j\leqslant n}, \qquad L' = \big(l'_{ij}\big)_{1\leqslant i,j\leqslant n}.$$

By the definition of lower triangular matrix, we know that $l_{ij} = 0 = l'_{ij}$ if $i < j$.

We set $C = L + L'$, $E = LL'$, and we write

$$C = \big(c_{ij}\big)_{1\leqslant i,j\leqslant n}, \qquad E = \big(e_{ij}\big)_{1\leqslant i,j\leqslant n}.$$

To check that both C and E are lower triangular, we use the definition again: we need to check that $c_{ij} = 0$ if $i < j$, and similarly that $e_{ij} = 0$ if $i < j$.

But whenever $i < j$ we have that $c_{ij} = l_{ij} + l'_{ij} = 0 + 0 = 0$, thus we can conclude that C is lower triangular.

On the other hand,

$$\begin{aligned} e_{ij} = \sum_{r=1}^{n} l_{ir} l'_{rj} &= \sum_{\substack{1 \leqslant r \leqslant n \\ i<r}} l_{ir} l'_{rj} + \sum_{\substack{1 \leqslant r \leqslant n \\ i \geqslant r}} l_{ir} l'_{rj} \\ &= \sum_{\substack{1 \leqslant r \leqslant n \\ i<r}} 0 \cdot l'_{rj} + \sum_{\substack{1 \leqslant r \leqslant n \\ i \geqslant r}} l_{ir} l'_{rj} = \sum_{\substack{1 \leqslant r \leqslant n \\ i \geqslant r}} l_{ir} l'_{rj}. \end{aligned}$$

It remains to note that, **if $\boldsymbol{i < j}$**, then $r \leqslant i$ implies that $r < j$, therefore the final sum above is also equal to 0:

$$e_{ij} = \sum_{\substack{1 \leqslant r \leqslant n \\ i<r}} l_{ir} l'_{rj} + \sum_{\substack{1 \leqslant r \leqslant n \\ i \geqslant r}} l_{ir} l'_{rj} = \sum_{\substack{1 \leqslant r \leqslant n \\ i \geqslant r}} l_{ir} l'_{rj} = \sum_{\substack{1 \leqslant r \leqslant n \\ i \geqslant r}} l_{ir} \cdot 0 = 0.$$

We conclude that E is lower triangular too.

Problem 5. One example is the following system with coefficients from $\mathbb{R}$:

$$\left\{\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 8 \\ x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & = & 4 \\ & & x_2 & + & x_3 & + & x_4 & = & 4 \end{array}\right\}.$$

This system has 3 equations (with the desired properties) and 4 unknowns, so it is underdetermined. To see that it is inconsistent, note that

$$\left\{\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 8 \\ x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & = & 4 \\ & & x_2 & + & x_3 & + & x_4 & = & 4 \end{array}\right\} \xLeftrightarrow{E_2 - E_1 \to E_2'}$$

$$\left\{\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 8 \\ & & x_2 & + & x_3 & + & x_4 & = & -4 \\ & & x_2 & + & x_3 & + & x_4 & = & 4 \end{array}\right\} \xLeftrightarrow{E_3 - E_2 \to E_3'}$$

$$\left\{\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 8 \\ & & x_2 & + & x_3 & + & x_4 & = & -4 \\ & & & & & & 0x_4 & = & 8 \end{array}\right\}.$$

Thus the last equivalent system contains an inconsistent equation, the equation $0x_4 = 8$, and so it is inconsistent too.

Problem 6. 1. A_1 has 8 columns. Given that it is the augmented matrix of a linear system, its last column is the column of the constant terms. Therefore, the corresponding system is in 7 unknowns, and thus the solutions to it, if any exist, will be vectors in $\mathbb{R}^7$.

A_1 is in row echelon form, or equivalently the corresponding system is staircase. Indeed, all the rows of the matrix are non-zero, and the first non-zero entry of the first row is 2 in the first column, the first non-zero entry of the second row is 2 in the second column, the first non-zero entry of the third row is 35 in the third column, the first non-zero entry of the fourth row is 4 in the fourth column, and finally the first non-zero entry of the fifth row is 8 in the sixth column; given that each such entry is to the right of the previous first non-zero entries, the matrix is indeed in REF and these entries are its pivots.

We finally observe that there is no pivot in the fifth, seventh and eighth columns. The latter shows that the corresponding system is consistent, while the former show that there are two free variables. Therefore the system has infinitely many solutions (given that $\mathbb{R}$ is infinite).

2. A_2 has 5 columns. Therefore, the corresponding system is in 4 unknowns, and thus the solutions to it, if any exist, will be vectors in $\mathbb{Z}_{13}^4$.

A_2 is not in row echelon form; equivalently the corresponding system is not staircase. Indeed, the first non-zero entry of the first row and the first non-zero entry of the second row are both in the second column, which does not agree with the definition of a matrix in REF.

3. A_3 has 5 columns. Therefore, the corresponding system is in 4 unknowns, and thus the solutions to it, if any exist, will be vectors in $\mathbb{Z}_7^4$.

A_3 is in row echelon form, or equivalently the corresponding system is staircase. Indeed, all the rows of the matrix are non-zero, and the first non-zero entry of the first row is 4 in the first column, the first non-zero entry of the second row is -3 in the third column, the first non-zero entry of the third row is 4 in the fourth column, and the first non-zero entry of the fourth row is 3 in the last column; given that each such entry is to the right of the previous first non-zero entries, the matrix is indeed in REF and these entries are its pivots.

We finally observe that, since the last pivot is in the last column, which corresponds to the column of constant terms of the system, we will have that the corresponding system is inconsistent.

4. A_4 has 6 columns. Therefore, the corresponding system is in 5 unknowns, and thus the solutions to it, if any exist, will be vectors in $\mathbb{Z}_7^5$.

A_4 is in row echelon form, or equivalently the corresponding system is upper triangular/staircase. Indeed, the only zero row of the matrix is the last one, while the first non-zero entry of the first row is 3 in the first column, the first non-zero entry of the second row is 2 in the second column, and the first non-zero entry of the third row is 4 in the fourth column; given that each such entry is to the right of the previous first non-zero entries, the matrix is indeed in REF and these entries are its pivots.

We now observe that there is no pivot in the third, fifth and sixth columns. The latter shows that the corresponding system is consistent, while the former show that there are two free variables. Therefore the system has more than one solutions, and in fact has $7^2 = 49$ solutions (given that $|\mathbb{Z}_7| = 7$).

5. A_5 has 6 columns. Therefore, the corresponding system is in 5 unknowns, and thus the solutions to it, if any exist, will be vectors in $\mathbb{Q}^5$.

A_5 is in row echelon form, or equivalently the corresponding system is upper triangular/staircase. Indeed, all the rows of the matrix are non-zero, and the first non-zero entry of the first row is 2 in the first column, the first non-zero entry of the second row is 13 in the second column, the first non-zero entry of the third row is -99 in the third column, the first non-zero entry of the fourth row is 5 in the fourth column, and finally the first non-zero entry of the fifth row is 7 in the fifth column; given that each such entry is to the right of the previous first non-zero entries, the matrix is indeed in REF and these entries are its pivots.

We finally observe that there is no pivot in the last column, but there is a pivot in each of the remaining columns. These show that the corresponding system is consistent and has no free variables, therefore it has a unique solution.

Problem 7. (i) We have

$$A^2 = \begin{pmatrix} -8 & -7 & 2 \\ 59 & 1 & 25 \\ -14 & -16 & 7 \end{pmatrix}.$$

Next, we observe that it suffices to check that A is invertible in order to determine whether A^2 is invertible. Indeed we have the following

Claim. A^2 is invertible if and only if A is invertible. *(This claim can be proven in several different ways, although it is not necessary in this problem to give a proof; below one is given for completeness and it is based only on the definition of invertibility.)*

Proof of Claim. If A is invertible, then A^{-1} exists and we can check that $(A^2)^{-1} = A^{-1}A^{-1} = \left(A^{-1}\right)^2$ (this is a special case of Problem 1 in this homework set).

Conversely, if A^2 is invertible, then there should exist $B \in \mathbb{R}^{3\times 3}$ such that $A^2B = I_3 = BA^2$. But this would imply that $A(AB) = I_3 = (BA)A$, so A would have both a right inverse C_1 ($C_1 = AB$ here) and a left inverse C_2 ($C_2 = BA$ here). But in such a case, the two inverses are equal given that

$$C_1 = I_3C_1 = (C_2A)C_1 = C_2(AC_1) = C_2I_3 = C_2.$$

Therefore A would be invertible too. □

We now check whether A is invertible. By one of the main theorems about matrix arithmetic (Theorem 2 of Lecture 28), it suffices to check whether a Row Echelon Form of A has 3 pivots. But

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 11 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix} \underset{\sim}{\overset{R_2-11R_1}{\scriptstyle R_3-2R_1}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 15 & -9 \\ 0 & 0 & 1 \end{pmatrix},$$

and the last matrix is in REF and has 3 pivots. Therefore, A is invertible, and so is A^2.

(ii) We have

$$A^2 = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Analogously to part (i), we observe that, in order to check whether A^2 is invertible, it would suffice to check if A is invertible. However here it may be more efficient to work directly with A^2 (given that A^2 has more zero entries than A). We note that

$$A^2 = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{}{\overset{\substack{R_2-3R_1 \\ R_3-R_1}}{\sim}} \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{R_2 \leftrightarrow R_4}{\sim} \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, A^2 is not invertible since at least one Row Echelon Form of it does not have 4 pivots (in fact, as we will see, this shows that all Row Echelon Forms of A^2 will have only 2 pivots).