Fall 2021, Math 328, Homework 5

Due: End of day on 2021-10-25

1 10 points

Let $\varphi : G \to H$ be a surjective homomorphisms of groups. Suppose that G is cyclic. Prove that H is cyclic. Deduce that any group which is isomorphic to a cyclic group is cyclic.

2 10 points

Describe (with proof) the lattice of subgroups of D_8, S_3 and Q_8. In these lattices, identify all the cyclic subgroups.

3 10 points

Let n be a positive integer and let G be a cyclic group of order n with generator g.

1. Suppose that k is any integer. Prove that there is a unique homomorphism $\varphi_k : G \to G$ satisfying $\varphi_k(g) = g^k$.

2. Prove that φ_k is an automorphism of G if and only if $\gcd(k, n) = 1$.

3. Let $k, k' \in \mathbb{Z}$ be given. Prove that $\varphi_k \circ \varphi_{k'} = \varphi_{k \cdot k'}$. Prove that $\varphi_k = \varphi_{k'}$ if and only if $k \equiv k' \bmod n$.

4. Deduce that the map $\mathbb{Z}/n \to \mathrm{End}(G)$ defined by $(a \bmod n) \mapsto \varphi_a$ is well-defined, and that it restricts to a bijection

$$(\mathbb{Z}/n)^\times := \{a \bmod n \mid \gcd(a, n) = 1\} \cong \mathrm{Aut}(G).$$

5. Deduce that $(\mathbb{Z}/n)^\times$ is a group with respect to the operation

$$(a \bmod n,\ b \bmod n) \mapsto (a \cdot b \bmod n),$$

and that this group is isomorphic to $\mathrm{Aut}(G)$.

Remark: Here $\mathrm{End}(G)$ denotes the set of *endomorphisms of* G, i.e. the set of all homomorphisms φ from G to itself.

4 10 points

Let G be an infinite cyclic group. What is $\mathrm{Aut}(G)$?

5 10 points

Let G be a finite group, and let $x \in G$ be given. Prove that

$$N_G(\langle x \rangle) = \{g \in G \mid \exists a \in \mathbb{Z},\ g \cdot x \cdot g^{-1} = x^a\}.$$

6 10 points

Let $\varphi : G \to H$ be a surjective homomorphism of groups, and let $\psi : G \to N$ be homomorphism. In this exercise, we will describe necessary and sufficient conditions for the existence of a homomorphism $\delta : H \to N$ satisfying $\delta \circ \varphi = \psi$.

1. Suppose that δ exists as above. Prove that the kernel of φ is contained in the kernel of ψ.

2. Suppose that $\ker(\varphi) \subset \ker(\psi)$. Prove that there is a well-defined map

$$\delta : H \to N$$

   which satisfies $\delta(h) = \psi(\tilde{h})$ where $\tilde{h} \in G$ is some element satisfying $\varphi(\tilde{h}) = h$.

3. In the context of part 2, prove that δ is a homomorphism satisfying $\delta \circ \varphi = \psi$.

Put all the parts together to prove the following theorem.

Theorem: *Let $\varphi : G \to H$ be a surjective homomorphism of groups and $\psi : G \to N$ be any homomorphism of groups. Then there exists a homomorphism $\delta : H \to N$ satisfying $\delta \circ \varphi = \psi$ if and only if $\ker(\varphi) \subset \ker(\psi)$. If these conditions hold true, then δ is uniquely determined by φ and ψ.*