Math 127
Suggested solutions to Homework Set 4

Problem 1. We check that $\mathbb{R}^n$ with coordinate-wise addition and multiplication satisfies all axioms of a commutative ring.

Addition is commutative: Consider two elements $\bar{x}, \bar{y} \in \mathbb{R}^n$. By definition, we have that the i-th component of $\bar{x}+\bar{y}$ is $x_i + y_i$. Similarly, the i-th component of $\bar{y}+\bar{x}$ is $y_i + x_i$. Since addition in $\mathbb{R}$ is commutative, it holds that $x_i + y_i = y_i + x_i$ for every index i. This shows that $\bar{x}+\bar{y}$ and $\bar{y}+\bar{x}$ have the same i-th component for every i, and hence $\bar{x}+\bar{y} = \bar{y}+\bar{x}$.

Addition is associative: Consider three elements $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^n$. By definition, we have that the i-th component of $(\bar{x}+\bar{y})+\bar{z}$ is $(x_i+y_i)+z_i$. Similarly, the i-th component of $\bar{x}+(\bar{y}+\bar{z})$ is $x_i+(y_i+z_i)$. Since addition in $\mathbb{R}$ is associative, it holds that $(x_i+y_i)+z_i = x_i+(y_i+z_i)$ for every index i. This shows that $(\bar{x}+\bar{y})+\bar{z}$ and $\bar{x}+(\bar{y}+\bar{z})$ have the same i-th component for every i, and hence $(\bar{x}+\bar{y})+\bar{z} = \bar{x}+(\bar{y}+\bar{z})$.

Additive Identity: We check that $\bar{0} = (0, 0, \ldots, 0)$ (the n-tuple all of whose components are equal to $0 \in \mathbb{R}$) is the neutral element of addition. Indeed, for every $\bar{x} \in \mathbb{R}^n$,

$$\bar{0}+\bar{x} = (0+x_1, 0+x_2, \ldots, 0+x_n) = (x_1, x_2, \ldots, x_n) = \bar{x},$$

and similarly

$$\bar{x}+\bar{0} = (x_1+0, x_2+0, \ldots, x_n+0) = (x_1, x_2, \ldots, x_n) = \bar{x}$$

(note that the latter also follows from combining the former with the commutativity of addition).

Additive Inverses: Consider an element $\bar{x} \in \mathbb{R}^n$, and set $\bar{w}$ to be the n-tuple with i-th component $w_i = -x_i$ (each x_i is a real number, so it has an additive inverse in $\mathbb{R}$). Then $\bar{w}$ is the additive inverse of $\bar{x}$:

$$\begin{aligned} \bar{x}+\bar{w} &= (x_1+w_1, x_2+w_2, \ldots, x_n+w_n) \\ &= \big(x_1+(-x_1), x_2+(-x_2), \ldots, x_n+(-x_n)\big) = \bar{0} \end{aligned}$$

(and similarly $\bar{w}+\bar{x} = \bar{0}$; note that this also follows from what we just checked and the commutativity of addition).

Multiplication is commutative: Consider two elements $\bar{x}, \bar{y} \in \mathbb{R}^n$. By definition, we have that the i-th component of $\bar{x} \cdot \bar{y}$ is $x_i y_i$. Similarly, the i-th component of $\bar{y} \cdot \bar{x}$ is $y_i x_i$. Since multiplication in $\mathbb{R}$ is commutative, it holds that $x_i y_i = y_i x_i$ for every index i. This shows that $\bar{x} \cdot \bar{y}$ and $\bar{y} \cdot \bar{x}$ have the same i-th component for every i, and hence $\bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x}$.

Multiplication is associative: Consider three elements $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^n$. By definition, we have that the i-th component of $(\bar{x} \cdot \bar{y}) \cdot \bar{z}$ is $(x_i y_i) z_i$. Similarly, the i-th component of $\bar{x} \cdot (\bar{y} \cdot \bar{z})$ is $x_i(y_i z_i)$. Since multiplication in $\mathbb{R}$ is associative, it holds that $(x_i y_i) z_i = x_i(y_i z_i)$ for every index i. This shows that $(\bar{x} \cdot \bar{y}) \cdot \bar{z}$ and $\bar{x} \cdot (\bar{y} \cdot \bar{z})$ have the same i-th component for every i, and hence $(\bar{x} \cdot \bar{y}) \cdot \bar{z} = \bar{x} \cdot (\bar{y} \cdot \bar{z})$.

Multiplicative Identity: We check that $\bar{1} = (1, 1, \ldots, 1)$ (the n-tuple all of whose components are equal to $1 \in \mathbb{R}$) is the neutral element of multiplication. Indeed, for every $\bar{x} \in \mathbb{R}^n$,

$$\bar{1} \cdot \bar{x} = (1 \cdot x_1, 1 \cdot x_2, \ldots, 1 \cdot x_n) = (x_1, x_2, \ldots, x_n) = \bar{x}.$$

Similarly, we can see that $\bar{x} \cdot \bar{1} = \bar{x}$ (this also follows from what we just checked and the commutativity of multiplication).

Distributive law: Consider three elements $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^n$. By definition, we have that the i-th component of $(\bar{x} + \bar{y}) \cdot \bar{z}$ is $(x_i + y_i) z_i$. Similarly, the i-th component of $\bar{x} \cdot \bar{z} + \bar{y} \cdot \bar{z}$ is $x_i z_i + y_i z_i$. Since multiplication distributes over addition in $\mathbb{R}$, it holds that $(x_i + y_i) z_i = x_i z_i + y_i z_i$ for every index i. This shows that $(\bar{x} + \bar{y}) \cdot \bar{z}$ and $\bar{x} \cdot \bar{z} + \bar{y} \cdot \bar{z}$ have the same i-th component for every i, and hence $(\bar{x} + \cdot \bar{y}) \cdot \bar{z} = \bar{x} \cdot \bar{z} + \bar{y} \cdot \bar{z}$.

Similarly we can check that the left distributive law, stating that $\bar{z} \cdot (\bar{x} + \cdot \bar{y}) = \bar{z} \cdot \bar{x} + \bar{z} \cdot \bar{y}$, holds (alternatively, this follows from what we just confirmed combined with the commutativity of multiplication).

The above show that $\mathbb{R}^n$ with coordinate-wise addition and coordinate-wise multiplication is a commutative ring.

We finally check that this structure is not a field. Given that $n > 1$ by our assumptions, we can find **non-zero** n-tuples whose first component is 0: for instance, the element $(0, 1, \ldots, 1)$ (all of whose components are 1 except for the first one which is 0) is a non-zero element. Consider now any other element $\bar{x} \in \mathbb{R}^n$. Then

$$(0, 1, \ldots, 1) \cdot \bar{x} = (0 \cdot x_1, 1 \cdot x_2, \ldots, 1 \cdot x_n) = (0, x_2, \ldots, x_n) \neq \bar{1},$$

no matter what $\bar{x}$ is. This shows that there is no multiplicative inverse of $(0, 1, \ldots, 1)$ in $\mathbb{R}^n$, and hence this structure fails to be a field.

Problem 2. (a) We have to check that row equivalence on $\mathbb{F}^{m\times n}$ is reflexive, symmetric and transitive.

Reflexivity: we need to check that, for any $A \in \mathbb{F}^{m\times n}$, $A \sim A$. But for any $A \in \mathbb{F}^{m\times n}$, we have that $A = D_{1;1}A$, which gives the desired conclusion (*recall that we are using the notation $D_{i;\lambda}$ in this course to denote the diagonal matrix $\in \mathbb{F}^{m\times m}$ all of whose diagonal entries are equal to 1 except the i-th diagonal entry which is equal to some $\lambda \neq 0$; as we have said, this is an elementary matrix, and in fact, if $\lambda = 1$, it is equal to I_m*).

Symmetry: we need to check that, for any $A, B \in \mathbb{F}^{m\times n}$, if $A \sim B$, then $B \sim A$. Consider two matrices $A, B \in \mathbb{F}^{m\times n}$ satisfying $A \sim B$. Then, by definition there is some $k \geqslant 1$ and there are elementary matrices $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_k \in \mathbb{F}^{m\times m}$ so that

$$B = \mathcal{E}_k \cdots \mathcal{E}_2\mathcal{E}_1 A.$$

We now recall that elementary matrices are all invertible, and the inverses are also elementary matrices. Therefore, we can write

$$\begin{aligned} A &= \mathcal{E}_1^{-1}\mathcal{E}_2^{-1} \cdots \mathcal{E}_k^{-1}\big(\mathcal{E}_k \cdots \mathcal{E}_2\mathcal{E}_1 A\big) \\ &= \mathcal{E}_1^{-1}\mathcal{E}_2^{-1} \cdots \mathcal{E}_k^{-1} B, \end{aligned}$$

which shows that $B \sim A$, since, for every $1 \leqslant j \leqslant k$, $\mathcal{E}_j^{-1}$ is an elementary matrix.

Transitivity: we need to check that, for any $A, B, C \in \mathbb{F}^{m\times n}$, if $A \sim B$ and $B \sim C$, then $A \sim C$. Consider three matrices $A, B, C \in \mathbb{F}^{m\times n}$ satisfying $A \sim B$ and $B \sim C$. Then, there are some integers $k_1, k_2 \geqslant 1$ and there are elementary matrices $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_{k_1}$, $\widetilde{\mathcal{E}}_1, \widetilde{\mathcal{E}}_2, \dots, \widetilde{\mathcal{E}}_{k_2} \in \mathbb{F}^{m\times m}$ so that

$$B = \mathcal{E}_{k_1} \cdots \mathcal{E}_2\mathcal{E}_1 A, \quad \text{and} \quad C = \widetilde{\mathcal{E}}_{k_2} \cdots \widetilde{\mathcal{E}}_2\widetilde{\mathcal{E}}_1 B.$$

Combining these two assumptions, we have that

$$C = \widetilde{\mathcal{E}}_{k_2} \cdots \widetilde{\mathcal{E}}_2\widetilde{\mathcal{E}}_1 \left(\mathcal{E}_{k_1} \cdots \mathcal{E}_2\mathcal{E}_1 A\right) = \left(\widetilde{\mathcal{E}}_{k_2} \cdots \widetilde{\mathcal{E}}_2\widetilde{\mathcal{E}}_1 \cdot \mathcal{E}_k \cdots \mathcal{E}_2\mathcal{E}_1\right) A,$$

which shows that $A \sim C$ as we wanted.

We conclude that row equivalence on $\mathbb{F}^{m\times n}$ is an equivalence relation.

(b) Fix an integer $m > 1$. We have to check that congruence modulo m is reflexive, symmetric and transitive.

Reflexivity: we need to check that, for any $k \in \mathbb{Z}$, $k \equiv k (\operatorname{mod} m)$. But for any k, $k - k = 0$, therefore m divides $k - k$.

Symmetry: we need to check that, for any $k, l \in \mathbb{Z}$, if $k \equiv l (\operatorname{mod} m)$, then $l \equiv k (\operatorname{mod} m)$. Consider two integers $k, l \in \mathbb{Z}$ satisfying $k \equiv l (\operatorname{mod} m)$. Then, by how congruence modulo m is defined, we have that m divides $l - k$. This means that there is some $s \in \mathbb{Z}$ such that $l - k = s \cdot m$. But then

$$k - l = -(l - k) = -s \cdot m = (-s) \cdot m,$$

which shows that $k - l$ is also a multiple of m. Therefore, $l \equiv k (\operatorname{mod} m)$, as we wanted.

Transitivity: we need to check that, for any $k, l, r \in \mathbb{Z}$, if $k \equiv l (\operatorname{mod} m)$ and $l \equiv r (\operatorname{mod} m)$, then $k \equiv r (\operatorname{mod} m)$. Consider three integers $k, l, r \in \mathbb{Z}$ satisfying $k \equiv l (\operatorname{mod} m)$ and $l \equiv r (\operatorname{mod} m)$. Then m divides both $l - k$ and $r - l$. In other words, there are $s_1, s_2 \in \mathbb{Z}$ such that $l - k = s_1 \cdot m$ and $r - l = s_2 \cdot m$. We then have that

$$r - k = r - l + l - k = (l - k) + (r - l) = s_1 \cdot m + s_2 \cdot m = (s_1 + s_2) \cdot m,$$

which shows that $r - k$ is also a multiple of m. Therefore, $k \equiv r (\operatorname{mod} m)$ as we wanted.

We conclude that congruence modulo m is an equivalence relation on $\mathbb{Z}$.

Problem 3. (a) We first observe that the first six rows of the system are non-zero rows for any combination of $\kappa, \lambda, \mu, u, v$ and w, so regardless of whether the last row is zero or non-zero, the condition that all non-zero rows should be above any non-zero row will be satisfied.

We will now examine what the leading non-zero coefficient of every non-zero row is for the different combinations of the parameters.

To make sure we do not forget/omit to check any combinations, we consider two main cases which cover all possible combinations (not just the "good" ones that part (a) asks us to find).

Case 1: *combinations with* $\kappa \neq 0$. For any "good" combinations here, the first pivot of the corresponding system is κ. Therefore, the value of λ will not matter, because λ could not be a pivot coefficient anyway.

For the other rows we have:

- the leading non-zero coefficient of the second row is either μ if $\mu \neq 0$, or otherwise the coefficient 1 of x_3;
- the leading non-zero coefficient of the third row is the coefficient 2 of x_3;
- the leading non-zero coefficient of the fourth row is the coefficient 1 of x_4;
- the leading non-zero coefficient of the fifth row is the coefficient 1 of x_5;
- the leading non-zero coefficient of the sixth row is either v if $v \neq 0$, or otherwise the coefficient 2 of x_7;
- the seventh and last row is either zero, or its leading non-zero coefficient is $u^2 - 1$ if $u^2 - 1 \neq 0$, or otherwise it is w.

• We note that, if $\mu = 0$, then the leading non-zero coefficients of the second and of the third row would be in the same column. Thus, the corresponding system would not satisfy the definition of a staircase system, and hence such a combination would not be "good".

• Similarly, if $v \neq 0$, then the leading non-zero coefficients of the fifth and of the sixth row would be in the same column, and again the corresponding system would not be staircase.

The above show that any "good" combination in this case necessarily satisfies $\mu \neq 0$ and $v = 0$.

• The latter also implies that the leading non-zero coefficient of the sixth row is the coefficient 2 of x_7, and hence the coefficient $u^2 - 1$ of

x_7 in the next row should be 0 for the system to be staircase: given that $u \in \mathbb{Z}_5$, we see that we must have $u = 1$ or $u = 4$ for any "good" combinations.

• Finally, we note that if $w \neq 0$, then there is a leading non-zero coefficient in the last row, the coefficient w, which is to the right of all previous leading non-zero coefficients. On the other hand, if $w = 0$, then the last row is zero, which still allows for a staircase system.

We conclude that for all combinations for which $\kappa \neq 0$, if we also have

$$\mu \neq 0, \quad \text{and} \quad v = 0, \quad \text{and} \quad u \in \{1, 4\},$$

then the leading non-zero coefficient of every non-zero row is to the right of the leading non-zero coefficient of any previous row, and thus the corresponding system is staircase.

On the other hand, if at least one of the above conditions is not met, then the corresponding system is not staircase.

Case 2: *combinations with $\kappa = 0$.* We show that there are no "good" combinations in this case. Indeed, there are three subcases we could consider here:

Subcase 1: $\mu = 0$. As before, we can see that in this subcase the leading non-zero coefficients of the second and of the third row would be in the same column, thus any system here would not be staircase.

Subcase 2: $\mu \neq 0$ and $\lambda \neq 0$. Then the leading non-zero coefficients of the first and of the second row would both be in the second column, which again would violate the definition of a staircase system.

Subcase 3: $\mu \neq 0$ and $\lambda = 0$. Then the leading non-zero coefficient of the first row would be to the right of μ, the leading non-zero coefficient of the second row, and again we wouldn't get a staircase system.

We conclude that the only combinations giving an upper triangular system come from the first main case, and are the combinations satisfying the conditions

$$(1) \qquad \kappa \neq 0, \quad \text{and} \quad \mu \neq 0, \quad \text{and} \quad v = 0, \quad \text{and} \quad u \in \{1, 4\}.$$

(b) We consider two main subcases that the "good" combinations we found in part (a) belong to:

$w \neq 0$. For any combination here, the corresponding system has the following pivots: the coefficient κ of x_1 in the first row, the coefficient μ of x_2 in the second row, the coefficient 2 of x_3 in the third row, the coefficient 1 of x_4 in the fourth row, the coefficient 1 of x_5 in the fifth row, the coefficient 2 of x_7 in the sixth row, and the non-zero constant term w in the seventh row.

We thus see that any system we can consider in this case will have a pivot in the last column, and hence it will be inconsistent (in other words it will have no solutions).

$w = 0$. For any combination here, the corresponding system has pivots in the first six rows, while the last row is zero. The pivots are: the coefficient κ of x_1 in the first row, the coefficient μ of x_2 in the second row, the coefficient 2 of x_3 in the third row, the coefficient 1 of x_4 in the fourth row, the coefficient 1 of x_5 in the fifth row, and the coefficient 2 of x_7 in the sixth row.

– We thus see that there is no pivot in the last column, so each system in this case is consistent.
– We also note that none of the pivots is a coefficient of x_6, therefore x_6 is a free variable.
– On the other hand, each of the remaining variables has a pivot coefficient.

The above show that each system in this case has 5 solutions (as many as the elements of $\mathbb{Z}_5$, or, in other words, as many as the choices we can make for the value of the unique free variable x_6).

Finally, we observe that there is no "good" combination that gives a system with a unique solution (since all "good" combinations lead to a linear system in which x_6 is a free variable).

Problem 4. (a) We consider the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & -2 \\ 2 & -3 & (k^2-8) & k-2 \end{array}\right)$$

and we try to find a REF of it.

We first note that $3^{-1} = 6$ in $\mathbb{Z}_{17}$ since $3 \cdot 6 = 18$ in $\mathbb{Z}$, and similarly $2^{-1} = 9$. Also $5^{-1} = 7$ since $5 \cdot 7 = 35 = 2 \cdot 17 + 1$ in $\mathbb{Z}$. Thus we have

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & -2 \\ 2 & -3 & (k^2-8) & k-2 \end{array}\right) \underset{\substack{6R_2 \to R_2' \\ 9R_3 \to R_3'}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 1 & -6 & 13 & -12 \\ 1 & -10 & 9(k^2-8) & 9k-1 \end{array}\right)$$

$$\underset{\substack{R_2 - R_1 \to R_2' \\ R_3 - R_1 \to R_3'}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -8 & 16 & -16 \\ 0 & -12 & 9(k^2-8)+3 & 9k-5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 9 & -1 & 1 \\ 0 & 5 & 9(k^2-8)+3 & 9k-5 \end{array}\right)$$

$$\underset{\substack{2R_2 \to R_2' \\ 7R_3 \to R_3'}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 12(k^2-8)+4 & 12k-1 \end{array}\right)$$

$$\underset{R_3 - R_2 \to R_3'}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 12(k^2-8)+6 & 12k-3 \end{array}\right).$$

The last matrix is in REF, so we can look at the number of its pivots to determine the size of the solution set of the corresponding linear system.

We distinguish three cases:

$12(k^2-8)+6 \neq 0$ Then any linear system contained in this case will have 3 pivots and no pivot in the last column, so it will have a unique solution.

$12(k^2-8)+6=0$ and $12k-3 \neq 0$ Then any linear system contained in this case will have a pivot in the last column (the column of the constant terms), and thus it will be inconsistent.

$12(k^2-8)+6=0$ and $12k-3=0$ Then any linear system contained in this case will have only two pivots (in the first and the second row), and no pivot in the last column; thus it will have more that one solutions.

It remains to see which values of k correspond to either of the cases above. We have that

$$12(k^2-8)+6=0 \;\Rightarrow\; 12(k^2-8) = -6 \;\Rightarrow\; k^2-8 = -60 = -9 = 8,$$

where we used that $12^{-1} = (-5)^{-1} = -(5^{-1}) = -7 = 10$, and also that $60 = 3 \cdot 17 + 9$. Furthermore, this implies that

$$k^2 = 16 \;\Rightarrow\; k = 4 \text{ or } k = -4 = 13.$$

This shows that the first case above corresponds to $k \notin \{4, 13\}$. In other words, for every $k \notin \{4, 13\}$ the corresponding system will have a unique solution.

On the other hand, $12k - 3 = 0 \;\Rightarrow\; 12k = 3 \;\Rightarrow\; k = 30 = -4 = 13$. We can now conclude that

- the second case above, in which we want $12(k^2 - 8) + 6 = 0$ and $12k - 3 \neq 0$, corresponds to $k = 4$,
- while the third case above, in which we want $12(k^2 - 8) + 6 = 0$ and $12k - 3 = 0$, corresponds to $k = 13$.

In other words, when $k = 4$ the corresponding system has no solutions, while if $k = 13$ the corresponding system has more than one solutions.

(b) We can find the solution sets of those linear systems that we found are consistent by working with the matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 12(k^2-8)+6 & 12k-3 \end{array}\right)$$

which we saw is row equivalent to the original augmented matrix (and hence its corresponding system will have the same solution set).

Recall that in case (i) we found that we need to have $12(k^2 - 8) + 6 \neq 0$, and hence we will be able to divide by $12(k^2 - 8) + 6$. Thus, for any value of k contained in this case, we will be able to write

$$(12(k^2 - 8) + 6)x_3 = 12k - 3 \;\Rightarrow\; x_3 = (12(k^2 - 8) + 6)^{-1} \cdot (12k - 3),$$

and moreover

$$x_2 - 2(12(k^2 - 8) + 6)^{-1} \cdot (12k - 3) = 2 \;\Rightarrow\; x_2 = 2(12(k^2 - 8) + 6)^{-1} \cdot (12k - 3) + 2,$$

$$x_1 + 2\big(2(12(k^2 - 8) + 6)^{-1} \cdot (12k - 3) + 2\big) - 3(12(k^2 - 8) + 6)^{-1} \cdot (12k - 3) = 4 \;\Rightarrow$$
$$x_1 + 4(12(k^2 - 8) + 6)^{-1} \cdot (12k - 3) + 4 - 3(12(k^2 - 8) + 6)^{-1} \cdot (12k - 3) = 4 \;\Rightarrow$$
$$x_1 = -(12(k^2 - 8) + 6)^{-1} \cdot (12k - 3) = -x_3.$$

In other words, the unique solution to any linear system in this case will be

$$(x_1, x_2, x_3) = (-\lambda_k,\ 2\lambda_k+2,\ \lambda_k), \qquad \text{where } \lambda_k = (12(k^2-8)+6)^{-1}\cdot(12k-3).$$

Next, we find the solution set to the system we get if $k = 13$. Then, as we have seen, the variable x_3 will be a free variable, and for the same reason the last row of the matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 12(k^2-8)+6 & 12k-3 \end{array}\right)$$

will be a zero row. Thus, we can parametrise the solution set here by setting $x_3 = \mu$, and then observing that the non-zero equations of the system give us

$$x_2 - 2x_3 = x_2 - 2\mu = 2 \ \Rightarrow\ x_2 = 2\mu + 2 \qquad \text{and}$$

$$x_1 + 2x_2 - 3x_3 = 4 \ \Rightarrow\ x_1 = -2x_2 + 3x_3 + 4 = (-4\mu - 4) + 3\mu + 4 = -\mu.$$

In other words, the solution set to the only system belonging to case (iii) is the following set:

$$\big\{(-\mu,\ 2\mu+2,\ \mu) : \mu \in \mathbb{Z}_{17}\big\}.$$

Problem 5. (a) Suppose $D, D' \in \mathbb{F}^{n\times n}$ are two diagonal matrices. Write

$$D = (d_{ij})_{1\leqslant i,j\leqslant n}\,, \qquad D = \left(d'_{ij}\right)_{1\leqslant i,j\leqslant n}$$

and recall that $d_{ij} = 0 = d'_{ij}$ if $i \neq j$.

We first observe that DD' and $D'D$ are both diagonal matrices. We could check this directly, or we could refer to HW3, Problem 4: we recall that a matrix is diagonal if and only if it is both upper triangular and lower triangular.

But then we know that D and D' are upper triangular, so as we showed in HW3, Problem 4 the matrices DD' and $D'D$ are upper triangular. Similarly, we can conclude that the latter matrices are lower triangular, therefore in the end DD' and $D'D$ are both diagonal.

It now suffices to check that the corresponding entries of DD' and $D'D$ on the diagonal coincide: consider $1 \leqslant i \leqslant n$ and note that the (i,i)-entry of DD' is the dot product of the i-th row of D and the i-th column of D', that is, the (i,i)-entry of DD' is equal to

$$\sum_{r=1}^{n} d_{ir}d'_{ri} = d_{ii}d'_{ii} + \sum_{r\neq i} d_{ir}d'_{ri} = d_{ii}d'_{ii}.$$

Similarly the (i,i)-entry of $D'D$ is equal to

$$\sum_{r=1}^{n} d'_{ir}d_{ri} = d'_{ii}d_{ii} + \sum_{r\neq i} d'_{ir}d_{ri} = d'_{ii}d_{ii}.$$

By commutativity of multiplication in $\mathbb{F}$, we have that $d_{ii}d'_{ii} = d'_{ii}d_{ii}$.

Since i was arbitrary, we conclude that the corresponding diagonal entries of DD' and $D'D$ coincide, and hence that the matrices are equal.

(b) This is false. Consider for example the matrices

$$D = \left(\begin{array}{cc|cccc} 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 1 & 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \end{array}\right) \quad \text{and} \quad E = \left(\begin{array}{cc|cccc} 1 & 1 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \end{array}\right).$$

Then D is a diagonal matrix, but it does not commute with E. Indeed,

$$DE = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \end{pmatrix} \neq \left(\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \end{array}\right) = ED.$$

Note that the above example works no matter which field $\mathbb{F}$ we consider, and no matter what $n \geqslant 1$.

That said, we can also give more specific counterexamples. For instance, consider the following matrices from $\mathbb{R}^{3\times 3}$:

$$\widetilde{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \widetilde{E} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\widetilde{D} \cdot \widetilde{E} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \widetilde{E} \cdot \widetilde{D}.$$

Problem 6. (a) Based on the given assumption, we can find $\bar{b} \in \mathbb{F}^n$ such that the system $A\bar{x} = \bar{b}$ has a unique solution.

Then, if we consider a REF $\big(A' \mid \bar{b}'\big)$ of the augmented matrix $\big(A \mid \bar{b}\big)$ of the system, we know that $\big(A' \mid \bar{b}'\big)$ will have n pivots, and none of those pivots will be in its last column (or in other words all n pivots of $\big(A' \mid \bar{b}'\big)$ will be contained in the submatrix A').

We now observe that A' is a REF of A. Indeed, $A \sim A'$, given that if $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_k$ are elementary matrices such that

$$\big(A' \mid \bar{b}'\big) = \mathcal{E}_k \cdots \mathcal{E}_2\mathcal{E}_1\big(A \mid \bar{b}\big) = \big(\mathcal{E}_k \cdots \mathcal{E}_2\mathcal{E}_1 A \mid \mathcal{E}_k \cdots \mathcal{E}_2\mathcal{E}_1\bar{b}\big),$$

then it follows that

$$A' = \mathcal{E}_k \cdots \mathcal{E}_2\mathcal{E}_1 A.$$

We also have that the first non-zero entry of each row of $\big(A' \mid \bar{b}'\big)$ is found in A'. Thus, A' satisfies the condition that the first non-zero entry of each of its rows is found to the right of previous first non-zero entries.

We can conclude that there is a REF A' of A which has n pivots, and hence A is invertible.

(b) Assume towards a contradiction that all linear systems with coefficient matrix B are consistent, but there exists at least one vector $\bar{c}_0 \in \mathbb{F}^n$ for which the system $B\bar{y} = \bar{c}_0$ has more than one solutions.

This will imply that, if $\big(B' \mid \bar{c}'\big)$ is a REF of the augmented matrix $\big(B \mid \bar{c}_0\big)$ of the system, then $\big(B' \mid \bar{c}'\big)$ has fewer than n pivots. Moreover, none of these pivots will be in the last column, since the system must be consistent.

As in part (a), we can observe that B' is a REF of B, and thus we can find elementary matrices $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_s$ such that

$$B = \mathcal{Q}_s \cdots \mathcal{Q}_2\mathcal{Q}_1 B'.$$

Note now that, since B' has fewer than n pivots, it also has fewer than n non-zero rows, and hence the system $B'\bar{y} = \bar{e}_n$ is certainly inconsistent (since its last linear equation is of the form $0y_1 + 0y_2 + \cdots + 0y_n = 1$).

But then the system with augmented matrix

$$\big(B \mid \mathcal{Q}_s \cdots \mathcal{Q}_2\mathcal{Q}_1\bar{e}_n\big) = \mathcal{Q}_s \cdots \mathcal{Q}_2\mathcal{Q}_1\big(B' \mid \bar{e}_n\big)$$

will be equivalent to the previous system (given that their augmented matrices are row equivalent), and hence it will be inconsistent too.

This contradicts the assumption that every linear system with coefficient matrix B is consistent. We thus see that the assumption that at least one system of the form $B\bar{y} = \bar{c}_0$ has more than one solutions was incorrect. We conclude that each such system will have a unique solution (note that we already knew that each such system will have at least one solution).