MATH 217 (Fall 2021)

Honors Advanced Calculus, I

Solutions #1

1. Let $+$ and $\cdot$ be defined on $\{\spadesuit, \dagger, \bigcirc, \mathrm{A}\}$ through:

$+$	$\spadesuit$	$\dagger$	$\bigcirc$	A
$\spadesuit$	$\spadesuit$	$\dagger$	$\bigcirc$	A
$\dagger$	$\dagger$	$\bigcirc$	A	$\spadesuit$
$\bigcirc$	$\bigcirc$	A	$\spadesuit$	$\dagger$
A	A	$\spadesuit$	$\dagger$	$\bigcirc$

$\cdot$	$\spadesuit$	$\dagger$	$\bigcirc$	A
$\spadesuit$	$\spadesuit$	$\spadesuit$	$\spadesuit$	$\spadesuit$
$\dagger$	$\spadesuit$	$\dagger$	$\bigcirc$	A
$\bigcirc$	$\spadesuit$	$\bigcirc$	$\spadesuit$	$\bigcirc$
A	$\spadesuit$	A	$\bigcirc$	$\dagger$

.

Do these turn $\{\spadesuit, \dagger, \bigcirc, \mathrm{A}\}$ into a field?

Solution: The neutral element of $\{\spadesuit, \dagger, \bigcirc, \mathrm{A}\}$ with respect to $+$, i.e., the zero, is $\spadesuit$. According to the second table, $\bigcirc \cdot \bigcirc = \spadesuit$ holds, which is impossible in a field.

2. Show that

$$\mathbb{Q}[i] := \{p + i\,q : p, q \in \mathbb{Q}\} \subset \mathbb{C}$$

with $+$ and $\cdot$ inherited from $\mathbb{C}$, is a field. Is there a way to turn $\mathbb{Q}[i]$ into an ordered field?

(*Hint*: Many of the field axioms are true for $\mathbb{Q}[i]$ simply because they are true for $\mathbb{C}$; in this case, just point it out and don't verify the axiom in detail.)

Solution: Let $p, q, r, s \in \mathbb{Q}$. Then

$$(p + i\,q) + (r + i\,s) = (p + r) + i\,(q + s) \in \mathbb{Q}[i]$$

and

$$(p + i\,q)(r + i\,s) = \underbrace{(pr - qs)}_{\in\mathbb{Q}} + i\,\underbrace{(qr + ps)}_{\in\mathbb{Q}} \in \mathbb{Q}[i]$$

hold, so that (F 1) is satisfied.

Since (F 2), (F 3), and (F 4) hold for $\mathbb{C}$, they also hold for $\mathbb{Q}[i]$.

Since $0 = 0 + i\,0, 1 = 1 + i\,0 \in \mathbb{Q}[i]$, (F 5) is satisfied as well.

Let $p, q \in \mathbb{Q}$, and let $x = p + i\,q$. Then $-x = -p + i\,(-q) \in \mathbb{Q}[i]$ as well. Suppose that $x \neq 0$, so that $p^2 + q^2 \neq 0$. Set

$$y := \frac{p}{p^2 + q^2} - i\,\frac{q}{p^2 + q^2} \in \mathbb{Q}[i].$$

It is immediate that $xy = 1$. Hence, (F 6) is also satisfied.

Assume that there is $P \subset \mathbb{Q}[i]$ as in the definition of an ordered field. Then either $i \in P$ or $-i \in P$ holds, so that in either case $-1 = i^2 = (-i)^2 \in P$, which contradicts the fact that $1 \in P$.

3. Let $\varnothing \neq S \subset \mathbb{R}$ be bounded below, and let $-S := \{-x : x \in S\}$. Show that:

   (a) $-S$ is bounded above;

   (b) S has an infimum, namely $\inf S = -\sup(-S)$.

   Solution:

   (a) Let L be a lower bound for S, i.e., $L \leq x$ for all $x \in S$. It follows that $-x \leq -L$ for each $x \in S$ and thus $x \leq -L$ for each $x \in -S$. Hence, $-L$ is an upper bound for $-S$.

   (b) Let $C := \sup(-S)$, so that $x \leq C$ for all $x \in -S$. It follows that $-x \geq -C$ for all $x \in -S$, i.e., $x \geq -C$ for all $x \in S$. Hence, $-C$ is a lower bound for S. Let C' be another other lower bound for S. In the solution to (a), we have seen that $-C'$ is an upper bound for $-S$, and thus $-C' \geq C$ by the definition of a supremum. It follows that $C' \leq -C$. Hence, $-C = \inf S$ holds.

4. Find $\sup S$ and $\inf S$ in $\mathbb{R}$ for

$$S := \left\{ (-1)^n \left(1 - \frac{1}{n}\right) : n \in \mathbb{N} \right\}.$$

   Justify, i.e., *prove*, your findings.

   Solution: For odd $n \in \mathbb{N}$, $(-1)^n \left(1 - \frac{1}{n}\right)$ is negative, and for even n, we have

$$(-1)^n \left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n} \leq 1.$$

   Hence, S is bounded above by 1. Assume that $\sup S < 1$, and let $\epsilon := 1 - \sup S$. In class, we saw that there is $n \in \mathbb{N}$ with $0 < \frac{1}{n} < \epsilon$, so that

$$\underbrace{1 - \frac{1}{2n}}_{\in S} > 1 - \frac{1}{n} > 1 - \epsilon = \sup S,$$

   which is impossible.

   Similarly, one sees that $\inf S = -1$.

5. Let $S, T \subset \mathbb{R}$ be non-empty and bounded above. Show that

$$S + T := \{x + y : x \in S,\ y \in T\}$$

is also bounded above with

$$\sup(S+T) = \sup S + \sup T.$$

Solution: Let $x \in S$ and $y \in T$. Then $x \leq \sup S$ and $y \leq \sup T$. It follows that

$$x + y \leq \sup S + \sup T,$$

so that $\sup S + \sup T$ is an upper bound for $S + T$. Consequently,

$$\sup(S+T) \leq \sup S + \sup T$$

holds.

Assume that $\sup(S+T) < \sup S + \sup T$. Let $\epsilon := \frac{1}{2}(\sup S + \sup T - \sup(S+T))$. Choose $x \in S$ and $y \in T$ such that

$$x > \sup S - \epsilon \qquad \text{and} \qquad y > \sup T - \epsilon.$$

It follows that

$$x + y > \sup S + \sup T - 2\epsilon = \sup(S+T),$$

which is a contradiction.

6*. An ordered field $\mathbb{O}$ is said to have the *nested interval property* if $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$ for each decreasing sequence $I_1 \supset I_2 \supset I_3 \supset \cdots$ of closed intervals in $\mathbb{O}$.

Show that an Archimedean ordered field with the nested interval property is complete.

Solution: Let $\varnothing \neq S \subset \mathbb{O}$ be bounded above. Choose $a_1 \in S$ and let $b_1 > a_1$ be an upper bound for S. Let $I_1 := [a_1, b_1]$, and let $c_1 := \frac{1}{2}(b_1 - a_1)$. There are two possibilities:

Case 1: c_1 is an upper bound for S. In this case, let $a_2 := a_1$, $b_2 := c_1$, and $I_2 := [a_2, b_2]$.

Case 2: c_1 is not an upper bound for S. In this case, there is $a_2 \in S$ with $a_2 > c_1$. Let $b_2 := b_1$, and define $I_2 := [a_2, b_2]$.

Let $c_2 := \frac{1}{2}(b_2 - a_2)$. Depending on whether c_2 is an upper bound for S or not, we find a_3 and b_3 as we found a_2 and b_2 and define $I_3 := [a_3, b_3]$.

Continuing in this fashion, we obtain a decreasing sequence $I_1 \supset I_2 \supset I_3 \supset \cdots$ of closed intervals in $\mathbb{O}$ with the following properties for all $n \in \mathbb{N}$:

- $I_n = [a_n, b_n]$, where $a_n \in S$ and $b_n \in \mathbb{O}$ is an upper bound for S;
- $(b_{n+1} - a_{n+1}) \leq \frac{1}{2}(b_n - a_n)$.

This second fact yields that

$$(b_{n+1} - a_{n+1}) \leq \frac{1}{2^n}(b_1 - a_1) \leq \frac{1}{n}(b_1 - a_1)$$

for all $n \in \mathbb{N}$ by induction on n.

Since $\mathbb{O}$ has the nested interval property, there is $x \in \bigcap_{n=1}^{\infty} I_n$. We claim that x is the supremum of S in $\mathbb{O}$.

Assume that x is not an upper bound for S, i.e., there is $y \in S$ such that $y > x$. Use the fact that $\mathbb{O}$ is Archimedean to find $n \in \mathbb{N}$ such that

$$(b_{n+1} - a_{n+1}) \leq \frac{1}{n}(b_1 - a_2) < y - x.$$

Since $x \geq a_{n+1}$, we obtain

$$y - x > b_{n+1} - a_{n+1} \geq b_{n+1} - x,$$

and adding x on both sides yields $y > b_{n+1}$, which contradicts b_{n+1} being an upper bound for S.

Hence, x is an upper bound for S.

Assume that there is an upper bound $y \in \mathbb{O}$ with $y < x$. Again use the fact that $\mathbb{O}$ is Archimedean to find $n \in \mathbb{N}$ such that

$$(b_{n+1} - a_{n+1}) \leq \frac{1}{n}(b_1 - a_2) < x - y.$$

Since $b_{n+1} \geq x$, we obtain

$$x - y > b_{n+1} - a_{n+1} \geq x - a_{n+1},$$

and subtracting x and multiplying with -1 on both sides yields that $a_{n+1} > y$ which contradicts y being an upper bound for S.

Hence, x is the least upper bound for S, i.e., $x = \sup S$.