Fall 2021, Math 328, Homework 8

Due: End of day on 2021-12-09

1 10 points

Let n and m be two nonnegative integers. Prove that the groups $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are isomorphic if and only if $m = n$.

2 10 points

Let G be a group and suppose that $[G : Z(G)] = n$, with n finite. Prove that every conjugacy class of G has at most n elements.

3 10 points

Let $\sigma_1 \in S_5$ denote the 5-cycle (12345) and let $\sigma_2 \in S_5$ denote the 4-cycle (1234). For each $i \in \{-3, -2, -1, 1, 2, 3\}$ and $j \in \{1, 2\}$, either find an explicit $\tau \in S_5$ such that $\tau \cdot \sigma_j \cdot \tau^{-1} = \sigma_j^i$, or prove that no such τ exists.

4 10 points

Let G be a finite group, and let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of G. Suppose that for all $1 \leq i \leq j \leq k$, one has $g_i \cdot g_j = g_j \cdot g_i$. Prove that G is abelian.

5 10 points

Let G be a group and H a subgroup of G. We say that H is a *characteristic subgroup* of G provided that for all $\varphi \in \mathrm{Aut}(G)$, one has $\varphi(H) = H$.

1. Prove that any characteristic subgroup of G must be normal in G.
2. Given an explicit example of a normal subgroup of a group which is not characteristic.

3. Suppose that H is a subgroup of G such that $\#H = n$, and that H is the unique subgroup of G whose order is n. Prove that H is characteristic, hence normal.

4. Suppose that H is a subgroup of G such that $[G : H] = n$, and that H is the unique subgroup of G whose index is n. Prove that H is characteristic, hence normal.

6 10 points

1. Prove that any group of order 56 has a normal subgroup of order 7 or 8.

2. Suppose p, q, r are distinct primes. Prove that any group of order $p \cdot q \cdot r$ has a normal subgroup of order p, q or r.

7 10 points

Suppose that G is a group of order 1575 which has a normal subgroup of order 9.

1. Prove that G has a normal subgroup of order 25.

2. Prove that G has a normal subgroup of order 7.

3. Prove that G is abelian.

Hint: If N is a normal subgroups of order 9, consider the Sylow subgroups of G/N.