MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Assignment #8

1. Let I be a compact interval, and let $f = (f_1, \ldots, f_M) : I \to \mathbb{R}^M$. Show that f is Riemann integrable if and only if $f_j : I \to \mathbb{R}$ is Riemann integrable for each $j = 1, \ldots, M$ and that, in this case,
$$\int_I f = \left(\int_I f_1, \ldots, \int_I f_M \right)$$
holds.

2. Let $I \subset \mathbb{R}^N$ be a compact interval, and let $f : I \to \mathbb{R}^M$ be Riemann integrable. Show that f is bounded.

3. Let $\varnothing \neq D \subset \mathbb{R}^N$ be bounded, and let $f, g \colon D \to \mathbb{R}$ be Riemann-integrable. Show that $fg \colon D \to \mathbb{R}$ is Riemann-integrable.

   Do we necessarily have
$$\int_D fg = \left(\int_D f \right) \left(\int_D g \right)?$$
   (*Hint*: First, treat the case where $f = g$ and then the general case by observing that $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2.)$

4. Let $\varnothing \neq D \subset \mathbb{R}^N$ have content zero, and let $f \colon D \to \mathbb{R}^M$ be bounded. Show that f is Riemann-integrable on D such that
$$\int_D f = 0.$$

5. Let $\varnothing \neq U \subset \mathbb{R}^N$ be open with content, and let $f : U \to [0, \infty)$ be bounded and continuous such that $\int_U f = 0$. Show that $f \equiv 0$ on U.

6*. The function
$$f \colon [0,1] \times [0,1] \to \mathbb{R}, \quad (x, y) \mapsto xy$$
is continuous and thus Riemann integrable. Evaluate $\int_{[0,1]\times[0,1]} f$ using only the definition of the Riemann integral, i.e., in particular, without using Fubini's Theorem.

Due Thursday, November 18, 2020, at 5:00 p.m.; no late assignments.