MATH 217 (Fall 2021)
Honors Advanced Calculus, I

Midterm Model Solutions

1. Let $S \subset \mathbb{R}^N$, and let $x \in \mathbb{R}^N$. Show that x is a cluster point of S if and only if there is a sequence in $S \setminus \{x\}$ converging to x.

   Solution: Suppose that x is a cluster point of S. Then, for each $n \in \mathbb{N}$, there is $x_n \in B_{\frac{1}{n}}(x) \cap (S \setminus \{x\})$. It is clear that $x_n \to x$.

   Conversely, suppose that there is a sequence $(x_n)_{n=1}^\infty$ in $S \setminus \{x\}$ such that $x_n \to x$. Let $\epsilon > 0$. Then there is $n_\epsilon \in \mathbb{N}$ such that $\|x_{n_\epsilon} - x\| < \epsilon$, so that $x_{n_\epsilon} \in B_\epsilon(x) \cap (S \setminus \{x\})$. Therefore, x is a cluster point for S.

2. Let $\varnothing \neq U \subset \mathbb{R}^N$ be open, and let $f, g : U \to \mathbb{R}$ be twice partially differentiable. Show that

$$\Delta(fg) = f\Delta g + 2(\nabla f) \cdot (\nabla g) + (\Delta f)g.$$

   Solution: Let $j \in \{1, \ldots, N\}$, and note that

$$\begin{aligned}
\frac{\partial^2}{\partial x_j^2} fg &= \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j} fg\right) \\
&= \frac{\partial}{\partial x_j}\left(f\,\frac{\partial g}{\partial x_j} + \frac{\partial f}{\partial x_j}\,g\right), \qquad \text{by the Product Rule,} \\
&= \frac{\partial}{\partial x_j}\left(f\,\frac{\partial g}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}\,g\right) \\
&= f\,\frac{\partial^2 g}{\partial x_j^2} + \frac{\partial g}{\partial x_j}\,\frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_j}\,\frac{\partial g}{\partial x_j} + \frac{\partial^2 f}{\partial x_j^2}\,g, \qquad \text{again by the Product Rule,} \\
&= f\,\frac{\partial^2 g}{\partial x_j^2} + 2\,\frac{\partial g}{\partial x_j}\,\frac{\partial f}{\partial x_j} + \frac{\partial^2 f}{\partial x_j^2}\,g.
\end{aligned}$$

   Taking the sum over j from 1 to N yields the claim.

3. Recall that $C \subset \mathbb{R}^N$ is called *path connected* if, for any $x, y \in C$, there is a continuous $\gamma \colon [0,1] \to C$ with $\gamma(0) = x$ and $\gamma(1) = y$.

   Show that:

   (a) if $C_1 \subset \mathbb{R}^{N_1}$ and $C_2 \subset \mathbb{R}^{N_2}$ are path connected, then so is $C_1 \times C_2 \subset \mathbb{R}^{N_1+N_2}$;

   (b) if $C \subset \mathbb{R}^N$ is path connected and $f \colon C \to \mathbb{R}^M$ is continuous, then $f(C)$ is path connected;

   (c) if $C_1, C_2 \subset \mathbb{R}^N$ is path connected, then so is $C_1 + C_2$.

   Solution:

(a) Let $(x_1, x_2), (y_1, y_2) \in C_1 \times C_2$. As C_1 and C_2 are path connected, there are, for $j = 1, 2$, continuous $\gamma_j : [0, 1] \to C_j$ with $\gamma_j(0) = x_j$ and $\gamma_j(1) = y_j$. Consequently,

$$\gamma\colon [0,1] \to C_1 \times C_2, \quad t \mapsto (\gamma_1(t), \gamma_2(t))$$

is continuous with $\gamma(0) = (x_1, x_2)$ and $\gamma(1) = (y_1, y_2)$.

(b) Let $x, y \in f(C)$. Choose, $u, v \in C$ such that $f(u) = x$ and $f(v) = y$. As C is path connected, there is a continuous $\sigma : [0, 1] \to C$ with $\sigma(0) = u$ and $\sigma(1) = y$. Consequently, $\gamma := f \circ \sigma$ is continuous with $\gamma(0) = x$ and $\gamma(1) = y$.

(c) By (a), $C_1 \times C_2 \subset \mathbb{R}^{2N}$ is path connected. As

$$f\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, \quad (x, y) \mapsto x + y$$

is continuous, $C_1 + C_2 = f(C_1 \times C_2)$ is also path connected.