PERIODIC SOLUTIONS OF SECOND-ORDER EQUATIONS

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Abstract.

Contents

1. Introduction

1.1. Dirac Delta Function [1]. The dirac delta function δ is a generalized function over the real numbers where it is zero everywhere except at 0. We can construct it using a limit of the *delta*-sequence given below,

$$\delta_n(t) = \begin{cases} \frac{n}{2} & -\frac{1}{n} < t < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

From the above sequence we can see that $\delta_n(t)$ converges to a spike (as $n \to \infty$) at $t = 0$ with infinite height and zero width with the property

$$\int_{-\infty}^{\infty} \delta_n(t)dt = 0, \forall n \geq 1$$

Now assume a continuous function $f(t)$. We can write

$$\min_{t \in [\frac{-1}{n}, \frac{1}{n}]} f(t) \leq \int_{-\infty}^{\infty} f(t)\delta_n(t)dt \leq \max_{t \in [\frac{-1}{n}, \frac{1}{n}]} f(t)$$

Since,

$$\lim_{n\to\infty} \min_{t \in [\frac{-1}{n}, \frac{1}{n}]} f(t) = \lim_{n\to\infty} \max_{t \in [\frac{-1}{n}, \frac{1}{n}]} f(t) = f(0),$$

then by the squeeze theorem, we can write

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\delta_n(t)dt = f(0).$$

Definition 1. *The dirac delta function can be defined by the following equality:*

$$\int_{-a}^{a} f(t)\delta(t)dt = f(0)$$

for any $a > 0$ and any continuous function f. This gives us the following relation,

$$\int_{s-a}^{s+a} f(t)\delta(t-s)d\tau = f(s)$$

(This is also called the Sifting property.)

1.2. Spectral Theory.

Definition 2. *(pre-Hilbert) Let X be a complex vector space. A Hermitian inner product on X is a function $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{C}$, which is :*

1. *(positive non-degenerate)* $(\forall x \in X)\langle x, x\rangle \geq 0, \langle x, x\rangle = 0$ *iff* $x = 0$.
2. *(sesquilinear)* $(\forall \alpha, \beta \in \mathbb{C})(\forall x, y, z \in X), \langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$
3. *(conjugate-symmetric)* $(\forall x, y \in X)\langle y, x\rangle = \overline{\langle x, y\rangle}$

A space is complete if all cauchy sequences converge to a limit . A *Hilbert Space* is a complete inner product space. In fact every Hilbert space is a *Banach Space* but the reverse is not true.

Definition 3. *Banach space: Let V be a vector space. A norm is a mapping $\|\cdot\| : V \to [0, \infty)$ that satisfies:*

1. $\|x + y\| \leq \|x\| + \|y\|$.
2. $\|ax\| = a\|x\|$ *for all* $a \in \mathbb{R}$.
3. $\|x\| = 0$ *implies that* $x = 0$.

A complete normed space is called a Banach space.

We can now define an inner product in function space. An inner product in the vector space of continuous functions denoted by $\mathcal{C}^0([a,b],\mathbb{C})$ is defined as follows, given two arbitrary functions $u(x), v(x), x \in [a,b]$, introduce the inner product,

$$\langle u, v\rangle = \int_a^b uv^* dx, \quad *\text{: complex conjugation.}$$

Theorem 1. ***(Fredholm Alternative theorem)***
For a linear system $Ax = b$, A^* which is the adjoint satisfying the property $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all $x, y, \in \mathbb{C}^n$ exactly one is true:
i) Solution of $Ax = b$, if it exists, is unique if and only if $x = 0$ is the only solution of $Ax = 0$.
ii) The equation $Ax = b$ has a solution if and only if $\langle b, v\rangle = 0$ for all v satisfying $A^* v = 0$.

Proof: i) Assume that $Ax = 0$ for $x \neq 0$ and $Ax_0 = b$. Then $A(x_0 + \alpha x) = b$ for all α. Therefore, the solution is not unique. Conversely, if there are two different solutions, x_1 and x_2, satisfying $Ax_1 = b$ and $Ax_2 = b$, then one has a nonzero solution $x = x_1 - x_2$ such that $Ax = A(x_1 - x_2) = 0$.
ii) Let $A^* v = 0$ and $Ax_0 = b$. Then we have

$$\langle b, v\rangle = \langle Ax_0, v\rangle = \langle x_0, A^* v\rangle = 0$$

For the second part we assume that $\langle b, v\rangle = 0$ for all v such that $A^* v = 0$. Write b as the sum of a part that is in the range of A and a part that in the space orthogonal to the range of $A, b = b_r + b_o$. Then, $0 = \langle b_o, Ax\rangle = \langle A^* b, x\rangle$ for all x. Since $\langle b, v\rangle = 0$ for all v in the nullspace of A^*, then $\langle b, b_o\rangle = 0$ Therefore, $\langle b, v\rangle = 0$ implies that

$$0 = \langle b, b_o\rangle = \langle b_r + b_o, b_o\rangle = \langle b_o, b_o\rangle$$

This means that $b_o = 0$, giving $b = b_r$ is in the range of A. So, $Ax = b$ has a solution. □

The same can be said about linear operators, *let L be a bounded linear operator on a Hilbert space with adjoint $L^\dagger$. Then exactly one of the following is true:*
i) The inhomogeneous problem

$$Lu = f$$

has a unique solution u.
ii) The homogeneous adjoint problem

$$L^\dagger u = 0$$

has a non-trivial solution.

Fredholm Alternative theorem is established by taking the inner product of (1.1) with the adjoint null space function $v(x)$ and we obtain,

$$\begin{aligned} &\langle v, Lu\rangle = \langle v, f\rangle \\ &\langle L^\dagger v, u\rangle = \langle v, f\rangle \end{aligned}$$

1.3. **Green's Functions.** The focus of this paper is to construct the appropriate Green's function for the following BVP (A):

$$\begin{cases} x'' + \lambda^2 x = \delta(t-s) \\ x(0) = x(\omega) \\ x'(0) = x'(\omega) \end{cases}$$

We first solve $Ly = f$, a differential equation with homogenous boundary conditions. The Sifting property mentioned earlier can be written as,

$$\langle f(t), \delta(t-s)\rangle = \int_0^l f(t)\delta(t-s)\,dt = f(s), \quad t \in [0,l]$$

Now we can consider the following problem,

$$\begin{aligned} &Lu = f \\ &L^\dagger G(t,s) = \delta(t-s) \end{aligned}$$

where $x \in [0,l]$ and $L^\dagger$ is the adjoint operator with its boundary conditions. $G(t,s)$ is the Greens function. Taking the inner product of $Lu = f$ with respect to $G(t,s)$ we get,

$$\begin{aligned} &\langle Lu, G\rangle = \langle u, L^\dagger G\rangle = \langle f, G\rangle \\ &\langle u, \delta(t-s)\rangle = \langle f, G\rangle \\ &u(t) = \langle f, G\rangle. \end{aligned}$$

This gives us

$$u(t) = \int_0^l f(t)G(t,s)dt$$

and thus the inverse operator

$$L^{-1}[f] = \int_0^l f(t)G(t,s)ds$$

can be computed.

2. Solution of $x'' + \lambda^2 x = \delta(t-s)$

From the above section we can define the Greens function $G(t,s)$ of $\mathcal{L}$ to be the unique soltion to the problem $\mathcal{L}G = \delta(t-s)$, where $\mathcal{L}$ is the general linear second order differential operator. We now construct the green function for A: From the definition of the dirac delta function, at all points $t \neq s$, $x'' + \lambda^2 x = 0$. Solving this we get homogenous solution is $k\cos(\lambda t) + c\sin(\lambda t)$. Therefore our Greens function is,

$$G(t,s) = \begin{cases} c_1\cos(\lambda t) + c_2\sin(\lambda t) & 0 \le t < s \le \omega \\ c_3\cos(\lambda t) + c_4\sin(\lambda t) & 0 \le s < t \le \omega \end{cases} \tag{1}$$

To solve for $G(t,s)$ we have to solve for c_1, c_2, c_3, c_4, we can do this by solving the following system of equations,

The periodic boundary conditions,

$$x(0) = x(\omega),\ k\cos(0) + c\sin(0) = k\cos(\lambda\omega) + c\sin(\lambda\omega) \tag{2}$$

$$x'(0) = x'(\omega),\ -k\lambda\sin(0) + c\lambda\cos(0) = -k\lambda\sin(\lambda\omega) + c\lambda\cos(\lambda\omega) \tag{3}$$

At $t = s$ (as $G(t,s)$ is continuous) ,

$$\lim_{t\to s^+} G(t,s) - \lim_{t\to s^-} G(t,s) = 0$$

$$c_3\cos(\lambda s) + c_4\sin(\lambda s) - c_1\cos(\lambda s) - c_2\sin(\lambda s) = 0 \tag{4}$$

and the "derivative" jump of $G(t,s)$,

$$\lim_{t\to s^+} G'(t,s) - \lim_{t\to s^-} G'(t,s) = 1$$

$$-c_3\lambda\sin(\lambda s) + c_4\lambda\cos(\lambda s) - (-c_1\lambda\sin(\lambda s) + c_2\lambda\cos(\lambda s)) = 1 \tag{5}$$

Remark 1. *Why* $\lim_{t\to s^+} G'(t,s) - \lim_{t\to s^-} G'(t,s) = 1$*?*
Consider $\mathcal{L}G = \delta(t-s)$*, this is zero when* $t < s$ *and* $t > s$ *and* δ *is infinity when* $t = s$*. Which tells us the first derivative must be discontinuous and when we take the second derivative it diverges. Now integrate the given ode from* $s-\epsilon$ *to* $s+\epsilon$ *and let* $\epsilon \to 0$*. We get,*

$$\int_{s-\epsilon}^{s+\epsilon} \frac{\partial G}{\partial t^2}dt + \int_{s-\epsilon}^{s+\epsilon} \lambda^2 G = \int_{s-\epsilon}^{s+\epsilon} \delta(t-s)dt = 1$$

The second term on the l.h.s vanishes as $\epsilon \to 0$ *as the integrands are finite and so we get,*

$$\left.\frac{\partial G}{\partial t}\right|_{t=s^+} - \left.\frac{\partial G}{\partial t}\right|_{t=s^-} = 1$$

Solving (2), (3), (4) and (5) we get our Greens Function,

$$G(t,s) = \begin{cases} \dfrac{\sin(\lambda(t-s+\omega)) + \sin(\lambda(s-t))}{2\lambda(1-\cos(\lambda\omega))} & 0 \le t < s \le \omega \\ \dfrac{\sin(\lambda(s-t+\omega)) + \sin(\lambda(t-s))}{2\lambda(1-\cos(\lambda\omega))} & 0 \le s < t \le \omega \end{cases} \tag{6}$$

$$G(t,s) = \begin{cases} G_1(t,s) & 0 \le t < s \le \omega \\ G_2(t,s) & 0 \le s < t \le \omega \end{cases} \tag{7}$$

We can verify this as,

$$\lim_{t\to s^+} G(t,s) - \lim_{t\to s^-} G(t,s) = 0$$

$$\lim_{t\to s^+} G'(t,s) - \lim_{t\to s^-} G'(t,s) = 1$$

$$G(t,s) = G(s,t)$$

3. Solution of $x'' = f(t, x(t), x'(t))$

Consider the following B.V.P (B),

$$\begin{cases} x'' = f(t, x(t), x'(t)) \\ x(0) = x(\omega) \\ x'(0) = x'(\omega) \end{cases}$$

Notice the above ODE can be written as,

$$x'' + \lambda^2 x(t) = \lambda^2 x(t) + f(t, x(t), x'(t)) \tag{8}$$

Note: The solution to the differential equation $x'' +^2 x = \delta(t-s)$ is the Greens function we obtained above. I claim the solution to the above differential equation (20) is,

$$x(t) = \int_0^\omega G(t,s)\left\{f\left(s, x(s), x'(s)\right) + \lambda^2 x(s)\right\} ds \tag{9}$$

Why so? *Existence discussed later*
Proof: Need to show $x(t)$ satisfies the BVP (B),

$$\begin{aligned}
x(t) &= \int_0^\omega G(t,s)\left\{f\left(s, x(s), x'(s)\right) + \lambda^2 x(s)\right\} ds \\
x'(t) &= \frac{d}{dt}\int_0^\omega G(t,s)h(s)ds, \quad h(s) = \left\{f\left(s, x(s), x'(s)\right) + \lambda^2 x(s)\right\} \\
&= \frac{d}{dt}\int_0^{t^-} G_2(t,s)h(s)ds + \frac{d}{dt}\int_{t^+}^\omega G_1(t,s)h(s)ds \\
&= G_2(t,t^-)h(t^-) + \int_0^{t^-} \frac{\partial G_2(t,s)}{\partial t}h(s)ds - G_1(t,t^+)h(t^+) + \frac{d}{dt}\int_{t^+}^\omega \frac{\partial G_1(t,s)}{\partial t}h(s)ds \\
&= \int_0^{t^-} \frac{\partial G_2(t,s)}{\partial t}h(s)ds + \int_{t^+}^\omega \frac{\partial G_1(t,s)}{\partial t}h(s)ds
\end{aligned}$$

$$\begin{aligned}
x''(t) &= \frac{\partial G_2(t,t^-)}{\partial t}h(t^-) + \int_0^{t^-} \frac{\partial^2 G_2(t,s)}{\partial t^2}h(s)ds - \frac{\partial G_1(t,t^+)}{\partial t}h(t^+) + \int_{t^+}^\omega \frac{\partial^2 G_2(t,s)}{\partial t^2}h(s)ds \\
&= h(t) + \int_0^{t^-} \frac{\partial^2 G_2(t,s)}{\partial t^2}h(s)ds + \int_{t^+}^\omega \frac{\partial^2 G_1(t,s)}{\partial t^2}h(s)ds \\
&= h(t) + \lambda^2 x(t) - \lambda^2 x(t) + \int_0^{t^-} \frac{\partial^2 G_2(t,s)}{\partial t^2}h(s)ds + \int_{t^+}^\omega \frac{\partial^2 G_1(t,s)}{\partial t^2}h(s)ds \\
&= h(t) + \lambda^2 x(t) + \int_0^{t^-} \frac{\partial^2 G_2(t,s)}{\partial t^2}h(s)ds + \int_{t^+}^\omega \frac{\partial^2 G_1(t,s)}{\partial t^2}h(s)ds - \lambda^2 x(t) \\
&= h(t) + \lambda^2\left(\int_0^{t^-} G_2(t,s)h(s)ds + \int_{t^+}^\omega G_1(t,s)h(s)ds)\right) + \\
&\int_0^{t^-} \frac{\partial^2 G_2(t,s)}{\partial t^2}h(s)ds + \int_{t^+}^\omega \frac{\partial^2 G_1(t,s)}{\partial t^2}h(s)ds - \lambda^2 x(t) \\
&= h(t) + \int_0^{t^-}\left(\frac{\partial^2 G_2(t,s)}{\partial t^2} + \lambda^2 G_2(t,s)\right)h(s)ds + \\
&\int_{t^+}^\omega \left(\frac{\partial^2 G_1(t,s)}{\partial t^2} + G_1(t,s)\right)h(s)ds - \lambda^2 x(t) \\
&= h(t) + \int_0^{t^-}\underbrace{\left(\frac{\partial^2 G_2(t,s)}{\partial t^2} + \lambda^2 G_2(t,s)\right)}_{0}h(s)ds + \\
&\int_{t^+}^\omega \underbrace{\left(\frac{\partial^2 G_1(t,s)}{\partial t^2} + G_1(t,s)\right)}_{0}h(s)ds - \lambda^2 x(t) \\
x'' &= h(t) - \lambda^2 x(t)
\end{aligned}$$

Rearranging we get,

$$x'' + \lambda^2 x(t) = \lambda^2 x(t) + f(t, x(t), x'(t)) \implies x'' = f(t, x(t), x'(t))$$

Now we have to show that the boundary conditions are satisfied, i.e

$$x(0) = x(\omega)$$
$$x'(0) = x'(\omega)$$

Consider $x(0) = x(\omega)$,

$$\begin{aligned} x(t) &= \int_0^{t^-} G_2(t,s)h(s)ds + \int_{t^+}^{\omega} G_1(t,s)h(s)ds \\ x(0) &= \int_0^{\omega} G_1(0,s)h(s)ds \\ x(\omega) &= \int_0^{\omega} G_2(\omega,s)h(s)ds \end{aligned} \tag{10}$$

Well $x(0) = x(\omega)$ as we know $G(t,s)$ satisfies the BVP (A). Another way to verify is by just evaluating $G_1(0,s)$ and $G_2(\omega,s)$.
Consider $x'(0) = x'(\omega)$,

$$\begin{aligned} x'(t) &= \int_0^{t^-} \frac{\partial G_2(t,s)}{\partial t}h(s)ds + \int_{t^+}^{\omega} \frac{\partial G_1(t,s)}{\partial t}h(s)ds \\ x'(0) &= \int_0^{\omega} \frac{\partial G_1(0,s)}{\partial t}h(s)ds \\ x'(\omega) &= \int_0^{\omega} \frac{\partial G_2(\omega,s)}{\partial t}h(s)ds \end{aligned} \tag{11}$$

By a similar argument we get $x'(0) = x'(\omega)$.
Therefore we showed the integral satisfies the ODE. □

4. Degree Theory

Let A be a mapping: $A : \bar{U} \subset \mathbb{R}^N \to \mathbb{R}^N$, where U is an open bounded set. We defined the degree of the mapping at p,

$$\deg(A,U,p) = \sum_{x_i \in A^{-1}(p)} \operatorname{sign}\left(\det J_A\left(x_i\right)\right)$$

as long as $A^{-1}(p)$ are regular points. Let $h : \mathbb{R}^N \to \mathbb{R}$ is a smooth function such that

$$\int_{\mathbb{R}^N} h(x)dx = 1,$$

and $h(x) = 0$ outside of a ball $B_\varepsilon(0)$ for some small $\varepsilon > 0$. Here $x = (x_1,\dots,x_N)$ and $dx = dx_1 \cdots dx_N$.

$$\deg(A,U,0) = \int_{\mathbb{R}^N} h(A(x)) \det J_A(x)dx$$

for $\boldsymbol{x} = (x_1,\dots,x_n)$. We show the integral is independent of h.
Proof. By induction. For $n = 1$, Let $\eta(x)$ be another function with support in $(-\varepsilon,\varepsilon)$ and

$$\int_{\mathbb{R}} \eta(x)dx = 1$$

Therefore, $\omega = h(x) - \eta(x)$ has the property

$$\int_{\mathbb{R}} \omega(x)dx = 0$$

We show

$$\int_{\mathbb{R}} \{h(A(x)) - \eta(A(x))\}A'(x)dx = 0$$

For simplicity, let us denote $h(A(x)) - \eta(A(x))\}A'(x)dx = f(x)dx$. We show, there is g with support in $(-\varepsilon,\varepsilon)$ such that

$$f(x)dx = d(g)(x)$$

It is enough to take g as

$$g(x) = \int_{-\infty}^{x} f(x)dx$$

Therefore, as g has a support in $(-\varepsilon,\varepsilon), g$ vanishes outside any sufficiently large interval in $\mathbb{R}$ by the Fundamental Theorem of Calculus,

$$\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} d(g)(x) = g(\infty) - g(-\infty) = 0$$

and therefore

$$\int_{\mathbb{R}} h(A(x))A'(x)dx = \int_{\mathbb{R}} \eta(A(x))A'(x)dx.$$

For $n = 2$, let $h(x_1, x_2)$ and $\eta(x_1, x_2)$ are real valued functions with support in $B_\varepsilon(0)$ such that

$$\int_{\mathbb{R}^2} h(x_1, x_2)\, dx_1 dx_2 = \int_{\mathbb{R}^2} \eta(x_1, x_2)\, dx_1 dx_2 = 1$$

For $\omega = h - \eta$,

$$\int_{\mathbb{R}^2} \omega(x)dx = 0$$

For $f(x_1, x_2) = \{h(A(x_1, x_2)) - \eta(A(x_1, x_2))\} \det J_A(x_1, x_2)$, we show there is an expression

$$g := g_1(x_1, x_2)\, dx_1 + g_2(x_1, x_2)\, dx_2,$$

with support in $B_\varepsilon(0)$ such that

$$dg(x_1, x_2) = f(x_1, x_2)\, dx_1 dx_2$$

If we are able to show that, then

$$\int_{\mathbb{R}^2} f(x_1, x_2)\, dx_1 dx_2 = \int_{\mathbb{R}^2} dg(x_1, x_2) = 0.$$

as g, ω have compact support. Define g,

$$g(x_1, x_2) = \int_{-\infty}^{x_1} \left\{ f(t, x_2) - \left(\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 \right) \tau(x_1) \right\} dt$$

where $\tau(x_1)$ is a function with the property

$$\int_{-\infty}^{\infty} \tau(x_1)\, dx_1 = 1$$

It is simply seen that $g(\infty, y) = g(x, \infty) = 0$. We have

$$\frac{\partial g_1}{\partial x_1} + \left(\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 \right) \tau(x_1) = f(x_1, x_2)$$

Let us denote $g_3(x_2)$ by

$$g_3(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1.$$

This is a function of a single variable x_2 and thus, there is $g_4(x_2)$ such that

$$g_3(x_2) = dg_4(x_2),$$

and therefore

$$f(x_1, x_2) = \frac{\partial g_1}{\partial x_1} + \frac{\partial (g_4(x_2)\tau(x_1))}{\partial x_2}$$

Now assume for integral is independent of h for $n = N$. This means in $\mathbb{R}^N$ there exists an $\omega = f(x)dx$ such that $\int_{\mathbb{R}^N} \omega(x)dx = 0$ with support in some $B_\varepsilon(0)$ and there exists a g with support in $B_\varepsilon(0)$ such that $\omega = d(g)$.

We show the property is true for $n = N + 1$. Let $x_1 = t,\ (t, x) = (t, x_2 \ldots x_{N+1})$. Consider,

$$g(t, x) = \int_{-\infty}^{t} (f(s, x) - \tau(s)r(x))ds$$

where $\tau(s)$ has the property,

$$\int_{-\infty}^{\infty} \tau(t)dt = 1$$

and set

$$r(x) = \int_{-\infty}^{\infty} f(t, x)dt$$

This is a function of N dimensions. By induction hypothesis, $\int r(x)dx = 0$ and there exist $g_1, \ldots, g_N$ such that

$$r(x) = \sum_{j=1}^{N} \frac{\partial g_j}{\partial x_j}$$

and each g_j are supported in $B_\varepsilon(0)$. Now

$$g(t,x) = \int_{-\infty}^{t} \left(f(s,x) - \tau(s) \sum_{j=1}^{N} \frac{\partial g_j}{\partial x_j} \right) ds$$

This integral vanishes in t as g has support in $B_\varepsilon(0)$, Thus

$$\frac{\partial g(t,x)}{\partial x_t} = f(t,x) - \tau(t) r(x)$$

which gives us

$$f\,(t,x) = \frac{\partial g(t,x)}{\partial x_t} + \sum_{j=1}^{N} \frac{\partial (g_j(x)\tau(t))}{\partial x_j}$$

$\square$

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