LINEAR INDEPENDENCE AND STABILITY OF INTEGER SHIFTS OF FUNCTIONS IN $L_p(\mathbb{R})$

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Abstract. This proposal concerns the mathematical analysis of the stability of integer shifts of functions in the $L_p(\mathbb{R})$ space. It starts by introducing the necessary background information and then goes on to explain the research problem, research goals, discuss potential methods to solve it and some applications.

Applied Harmonic Analysis is the application of the mathematical principles and techniques of Harmonic Analysis to solve problems in various fields. Applied Harmonic Analysis is used to model and analyze complex systems, such as the vibrations of structures, the spread of waves in fluids and solids, and the behavior of financial markets. It is used in signal processing to decompose signals into their basic frequency components and in image processing for image compression and enhancement, object recognition and tracking in computer vision.

1. Background

1.1. **Fourier Transform.** The Fourier Transform is a powerful tool that helps us convert a function $f(x)$ defined in the (usually time) $x \in \mathbb{R}$ domain to another function $\widehat{f}(\xi)$ in the (frequency) ξ- domain which describes the frequency spectrum of the function f which is typically represented as a graph showing the amplitude or magnitude of each frequency component as a function of frequency. Mathematically, the Continuous Time Fourier Transform is $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx,\ \xi \in \mathbb{R}$ and the Discrete Fourier Transform is $\widehat{f}(\xi) = \sum_{k\in\mathbb{Z}} u(k)e^{-ik\xi},\ \xi \in \mathbb{R}$ where $u(j) \in \ell(\mathbb{Z})$ which is the space of all sequences. The Fourier Transform is a way to break down a signal, such as sound waves, into its individual frequency components, which are sine and cosine waves. This decomposition makes it easier to analyze and manipulate the signal and is used in many fields such as signal and image processing, noise reduction and feature extraction.

1.2. **The $L_p(\mathbb{R})$ space and sequence space $\ell_p(\mathbb{Z})$.** The L_p spaces, also known as Lebesgue spaces are used to measure the "size" of a function. Intuitively, the L_p spaces provide a way to measure how spread out a function is, with different values of p corresponding to different ways of measuring spread. For example, L_1 space is used to measure the absolute spread of a function, the L_2 space is used to measure the squared spread of a function i.e how much the function deviates from a specific value. Rigorously, the $L_p(\mathbb{R})$ space is defined as the set of all f defined on $\mathbb{R}$, such that $\|f\|_{L_p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p} < \infty$ and the sequence space $\ell_p(\mathbb{Z})$ is the set of complex sequences $\{(v_n)\}_{n\in\mathbb{Z}}$ such that $\|\{v_n\}\|_{\ell_p(\mathbb{Z})} := \left(\sum_{n\in\mathbb{Z}} |v_n|^p\right)^{1/p} < \infty,\ 0 < p < \infty$.

1.3. **Convolution.** Discrete convolution is a mathematical operation that is used to combine two sequences of values (signals) in a specific way. It basically allows one to shift and multiply one sequence by the other, and then sum the resulting values. For $v = \{v(k) = v_k\}_{k\in\mathbb{Z}} \in \ell(\mathbb{Z})$ and a compactly supported function ϕ (1) on $\mathbb{R}$, we define the discrete convolution [1] to be $(v * \phi)(\cdot) := \sum_{k\in\mathbb{Z}} v(k)\phi(\cdot - k)$, where $\phi(\cdot - k)$ is the integer shift by k. By a shift, we mean an integer translation.

1.4. **Compactly supported function.** The support of a function is the closure of the set of points where the function is non-zero. Compactness refers to the property of a set or a space being "compact" or "tightly bound". It means that any infinite sequence of elements within that set or space has a subsequence that converges to a specific point within the set or space. In other words, the set or space doesn't have any "large gaps" or "unbounded areas". Therefore, we say a function has compact support if the support of the function is compact. We denote the support of a function f by $\mathrm{supp}(f)$ which is defined as $\mathrm{supp}(f) = \mathrm{cl}_{\mathrm{X}}(\{x \in X : f(x) \neq 0\})$, where $f : X \to \mathbb{R}$ is a real-valued function whose domain is an arbitrary set X, and cl_X denotes the closure. The importance of compactly supported ϕ are:

(1) The summation $\sum_{k\in\mathbb{Z}} v(k)\phi(\cdot - k)$ is well-defined for every $x \in \mathbb{R}$ and the summation is actually finite.
(2) $\widehat{\phi}(\xi)$ is an entire function and can be defined for $\xi \in \mathbb{R}$. We get this result from **Schwartz's Paley - Weiner Theorem** [2]. An entire function is a function that has a derivative in every point of the complex plane and it's complex - differentiable in the entire complex plane.

1.5. **Linear Independence.** For $v = \{v_k\}_{k\in\mathbb{Z}} \in \ell(\mathbb{Z})$, we say $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is called $\ell(\mathbb{Z})$ - linearly independent [1] if $\sum_{k\in\mathbb{Z}} v_k\phi(\cdot - k) = 0$ implies $v_k = 0$ for all $k \in \mathbb{Z}$. Linear independence is important for integer shifts of compact support functions because it allows for stability and shift-invariance, meaning when the functions are linearly independent, the shifted versions are also linearly independent, which means that the shifted functions retain their unique characteristics and can be distinguished from one another. This property is handy in signal processing and image processing.

2. Project Objectives

The concept of stability plays an important role in approximation theory and wavelet analysis. For $1 \le p \le \infty$, we say that the integer shift of $\phi \in L_p(\mathbb{R})$ is stable in $L_p(\mathbb{R})$ if there exist positive constants C_1 and C_2 such that for all $\{v_k\}_{k\in\mathbb{Z}} \in \ell_p$ we have,

$$C_1\|\{v_k\}\|_{\ell_p(\mathbb{Z})} \le \left\| \sum_{k\in\mathbb{Z}} v_k\phi(\cdot - k)\right\|_{L_p(\mathbb{R})} \le C_2\|\{v_k\}\|_{\ell_p(\mathbb{Z})}$$

The proof for the stability of integer shifts of functions in $L_p(\mathbb{R})$ [1] is done via characterization which states that if ϕ is a compactly supported function defined on $\mathbb{R}$ and let $\widehat{\phi}$ be it's Fourier transform (2), then, the integer shift of ϕ is stable if and only if $\widehat{\phi}$ does not possess in $\mathbb{C}$ any $2\pi k$-periodic zeros, i.e., the set $S = \left\{\xi \in \mathbb{C} : \hat{\phi}(\xi + 2\pi k) = 0,\ \forall k \in \mathbb{Z}\right\}$ is empty.

Our research will focus on the following questions:

(1) The stability condition for a finite number of functions with compact support has already been established [1]. Our goal for this project is to verify already known results and potentially discover new ones, but this time focusing on a single function with compact support. We plan to study the proofs for the case of finite functions and apply the same steps and reasoning to our specific function with compact support.
(2) Given a sequence $\{u_j\}_{j\in\mathbb{Z}} \in \ell(\mathbb{Z})$. Consider a compactly supported distribution $\phi = \sum_{j\in\mathbb{Z}} u_j\delta_j(\cdot) = \sum_{j\in\mathbb{Z}} u_j\delta(\cdot - j)$ where δ is the the Kronecker/Dirac sequence on $\mathbb{Z}$ such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Then, $(v * \phi)(\cdot) = \sum_{k\in\mathbb{Z}} v_k\phi(\cdot - k) = \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} v_ku_j\delta_{j-k}(\cdot)$. Let $j' = j + k$. Then we have $\sum_{k\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}} v_ku_{j'-k}\delta_{j'}(\cdot) = \sum_{j'\in\mathbb{Z}}(v * u)(j')\delta_{j'}(\cdot)$. We prove that if $v * u = 0$ then this is equivalent to saying $(v * u)(j) = 0$ for all $j \in \mathbb{Z}$ which by characterization implies $\widehat{\phi}(\xi) = \sum_{j\in\mathbb{Z}} u(j)e^{-ij\xi} = 0$ for all $\xi \in \mathbb{C} \Leftrightarrow \sum_{j\in\mathbb{Z}} u(j)z^j = 0$ for all $z \in \mathbb{C} \setminus \{0\}$ has only one term.
(3) Find simple examples of functions ϕ such that ϕ has stability but are not linearly independent.
(4) If $v * \phi = 0$ for any non-trivial v, then try to study the structure of v, that is consider $V = \{v \in \ell(\mathbb{Z}) : v * \phi = 0\}$. If $V \ne \emptyset$, then prove that there exists an exponential sequence $v(k) = e^{ck}, c \in \mathbb{C}$. Why? Consider the set S (3) as described above, if S is not empty, then for $\xi \in S$, construct an exponential sequence $v \in \ell(\mathbb{Z})$ such that $v * u = 0$.
(5) If time permits, try to prove some of the results and the stability conditions in other spaces such as Sobolev Spaces $H^\tau(\mathbb{R})$ with $\tau \in \mathbb{R}$.

3. Applications

Stability is an important concept in the study of integer shifts of functions in L_p spaces because it ensures that small perturbations in the input function do not result in large changes in the shifted output function. This is crucial in fields like signal processing and control systems, where even little input changes can have a big effect on the final product. A few applications are:

(1) The **Nyquist-Shannon sampling theorem** [3], states that a continuous-time signal can be perfectly reconstructed from its samples if the sampling frequency is greater than twice the highest frequency present in the signal. One of the important assumptions in this theorem is that the signal is band-limited, meaning that it has no frequency components above a certain cut off frequency. This is important because it allows us to reason about the signal's behavior in terms of its frequency content. The stability of integer shifts of functions allows us to reason about the signal's behavior under time translations. It ensures that the sampling process preserves the temporal structure of the signal, allowing us to reconstruct the original signal from the samples. Without the stability of integer shifts of functions, the reconstructed signal might be distorted or incorrect.

(2) Linearly independent integer shifts of a function are important for interpolation because they form a set of basis functions that can be used to represent any function that is band-limited to the same cut off frequency. In interpolation, the goal is to reconstruct a continuous-time signal from a set of discrete samples. To achieve this, we use interpolation functions, which are functions that pass through the sampled points and are band-limited to the same cut off frequency as the original signal. By using linearly independent integer shifts of a function as the interpolation functions, we can represent any band-limited signal as a linear combination of these functions. This is important because it guarantees that the reconstructed signal will be an exact replica of the original signal, as long as the samples were taken at the correct rate and are free from noise.

(3) Linearly independent and stability of integer shifts are important concepts in the numerical solution of partial differential equations. These concepts are closely related to the concept of numerical stability, which is essential for obtaining accurate and reliable solutions to partial differential equations. For example, if a solution is stable under integer shifts, then it can be discretized effectively on a grid with a finite number of points.

References

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