Linear Independence and Stability of Integer Shifts of Functions

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Intoduction

Applied Harmonic Analysis:

- Study the spread of waves in fluids and solids.
- Study of the **Fourier series** and **Fourier transforms**.
- Used in Signal and Image Processing.

Fourier Transform of a function f is a tool that helps us convert a function $f(x)$ defined in the (usually time) $x \in \mathbb{R}$ domain to another function $\hat{f}(\xi)$ in the (frequency) ξ-domain which describes the frequency spectrum of the function f. It is defined as:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx,\ \xi \in \mathbb{R}$$

Background Cont...

Compactly Supported Functions

Theorem (É-Borel '95) : A set is compact iff it is closed and bounded.

The support of a function is the set of points where the function is non-zero. It is defined as,

$$\mathrm{supp}(f) = (\{x \in X : f(x) \neq 0\})$$

where $f : X \to \mathbb{R}$ is a real-valued function whose domain is an arbitrary set X.

A function whose support is compact is called a compactly supported function.

A sequence u is just a mapping $u : \mathbb{Z} \to \mathbb{C}$. The space of all such sequences is denoted as $\ell(\mathbb{Z})$. For $v = v(k)_{k\in\mathbb{Z}} \in \ell(\mathbb{Z})$ and a compactly supported function ϕ on $\mathbb{R}$, we define the Semi - Discrete Convolution as,

$$(v * \phi)(\cdot) := \sum_{k\in\mathbb{Z}} v(k)\phi(\cdot - k)$$

where $\phi(\cdot - k)$ is the integer shift by k. By integer shift, we just mean a translate.

For $v = \{v_k\}_{k\in\mathbb{Z}} \in \ell(\mathbb{Z})$, we say $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is called $\ell(\mathbb{Z})$ - linearly independent, if $\sum_{k\in\mathbb{Z}} v_k\phi(\cdot - k) = 0$ implies $v_k = 0$ for all $k \in \mathbb{Z}$.

The L_p spaces, also known as Lebesgue spaces are used to measure the "size" of a function. Intuitively, the L_p spaces provide a way to measure how spread out a function is, with different values of p corresponding to different ways of measuring spread.

The $L_p(\mathbb{R})$ space is defined as the set of all f defined on $\mathbb{R}$, such that $\|f\|_{L_p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p} < \infty$.

The sequence space $\ell_p(\mathbb{Z})$ is the set of complex sequences $\{(v_n)\}_{n\in\mathbb{Z}}$ such that $\|\{v_n\}\|_{\ell_p(\mathbb{Z})} := \left(\sum_{n\in\mathbb{Z}} |v_n|^p\right)^{1/p} < \infty, \ 0 < p < \infty$.

Research Goals

For $1 \leq p \leq \infty$, we say that the integer shift of $\phi \in L_p(\mathbb{R})$ is stable in $L_p(\mathbb{R})$ if there exist positive constants C_1 and C_2 such that for all $\{v_k\}_{k\in\mathbb{Z}} \in \ell_p$ we have,

$$C_1\|\{v_k\}\|_{\ell_p(\mathbb{Z})} \leq \left\|\sum_{k\in\mathbb{Z}} v_k\phi(\cdot - k)\right\|_{L_p(\mathbb{R})} \leq C_2\|\{v_k\}\|_{\ell_p(\mathbb{Z})}$$

The proof of the stability of integer shifts of functions has already been proven using **Linear Independence.**

Research Goals Cont...

Our research goals include:

- The stability condition for a finite number of functions with compact support has already been established . Our goal for this project is to verify already known results and potentially discover new ones, but this time focusing on a single function with compact support. We plan to study the proofs for the case of finite functions and apply the same steps and reasoning to our specific function.

Research Goals Cont...

- Find simple examples of functions ϕ such that ϕ has stability but are not linearly independent.
- If $v * \phi = 0$ for any non-trivial v, then try to study the structure of v, that is consider $V = \{v \in \ell(\mathbb{Z}) : v * \phi = 0\}$.

Applications

Stability is an important concept in the study of integer shifts of functions because it ensures that small perturbations in the input function do not result in large changes in the shifted output function.

Applications Cont...

A few applications include:

- Theorem (H-Nyquist '28): A continuous-time signal can be perfectly reconstructed from its samples if the sampling frequency is greater than twice the highest frequency present in the signal.
- In fields like signal processing and control systems, where even little input changes can have a big effect on the final product.
- Essential for obtaining accurate and reliable solutions to partial differential equations.

References

- Han, Bin. Framelets and wavelets. *Algorithms, Analysis, and Applications, Applied and Numerical Harmonic Analysis. Birkhäuser xxxiii* Cham (2017).
- Vaidyanathan, P. P. "Generalizations of the sampling theorem: Seven decades after Nyquist." *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* 48.9 (2001): 1094-1109.

Thank you