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TDA231 - Homework 1

Theoretical Problems

Problem 1.1

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Problem 1.1

$N(\mu, \sigma^2 I)$

$$L = p(x) = \frac{1}{(2\pi)^{\frac{n}{2}} |\sigma^2 I|^{\frac{1}{2}}} \cdot \exp\left(-\frac{1}{2}(x-\mu)^T (\sigma^2 I)^{-1} (x-\mu)\right)$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \cdot \sigma^n} \cdot \exp\left(-\frac{1}{2}(x-\mu)^T \left(\frac{1}{\sigma^2} I\right)(x-\mu)\right)$$

$$\ln L = \ln(1) - \left(\ln(2\pi)^{\frac{n}{2}} + \ln(\sigma)^n\right) + \left(-\frac{1}{2}(x-\mu)^T\left(\frac{1}{\sigma^2} I\right)(x-\mu)\right)$$

$$= -\frac{n}{2}\ln(2\pi) - n\ln(\sigma) - \frac{1}{2\sigma^2}(x-\mu)^T I (x-\mu)$$

$$\frac{d \ln L}{d\sigma} = 0 - \frac{n}{\sigma} + \frac{1}{\sigma^3}(x-\mu)^T I (x-\mu) = 0$$

$$\Leftrightarrow \frac{1}{\sigma^2}(x-\mu)^T I (x-\mu) = n$$

$$\Leftrightarrow \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_i)^2 = \sigma^2$$

$$\Rightarrow \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_i)^2}$$

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Problem 1.2

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1.2 a) $\mathcal{D} = \{x_1, \dots, x_n\}$

$$p(\sigma^2 = s \mid \mathcal{D}) \propto \prod_{i=1}^{n} p(X = x_i \mid \sigma^2)\, P(\sigma^2 = s \mid \alpha, \beta)$$

$$= \frac{\beta^{\alpha} s^{-\alpha-1}}{2\pi s\, \Gamma(\alpha)} \exp\left(\frac{\sum_{i=1}^{n} (x_i - \mu_i)^T (x_i - \mu_i) - 2\beta}{s} \right)$$

b)

$$BF = \frac{P(\mathcal{D} \mid M_A)}{P(\mathcal{D} \mid M_B)} = \frac{\int P(\mathcal{D} \mid \sigma^2 = s)\, P(\sigma^2 = s \mid \alpha_A, \beta_A)\, ds}{\int P(\mathcal{D} \mid \sigma^2 = s)\, P(\sigma^2 = s \mid \alpha_B, \beta_B)\, ds}$$

c) $P(M_A) = P(M_B) = \frac{1}{2}$

$$\hat{s}_{MAP} := \frac{d}{d\sigma}\left(\operatorname{argmax} \log\left(P(\mathcal{D} \mid \sigma^2, \mu = \bar{x})\, P(\sigma^2 = s \mid \alpha, \beta) \right) \right) = 0$$

$$\Leftrightarrow \quad \sigma^2 = \frac{1}{2\alpha} \cdot \left(\sum_{i=1}^{n} \left[(x_i - \bar{x})^T (x_i - \bar{x}) \right] + 2\beta \right)$$

$$P(M \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid M)\, P(M)}{P(\mathcal{D})} \Leftrightarrow P(\mathcal{D} \mid M) = \frac{P(M \mid \mathcal{D})\, P(\mathcal{D})}{P(M)}$$

$$BF = \frac{P(\mathcal{D} \mid M_A)}{P(\mathcal{D} \mid M_B)} = \frac{P(M_A \mid \mathcal{D})\, P(M_B)}{P(M_B \mid \mathcal{D})\, P(M_A)} = \frac{P(M_A \mid \mathcal{D})}{P(M_B \mid \mathcal{D})}$$

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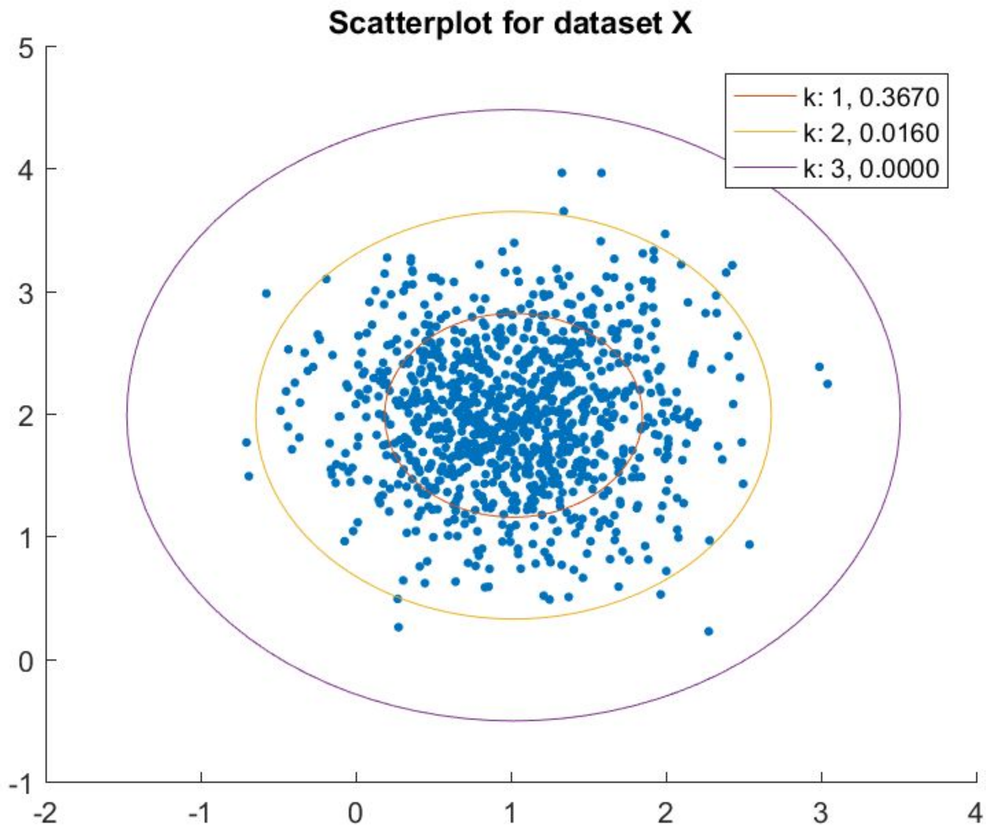
Matlab problems

Problem 2.1

$$\mu = \frac{\sum x}{n}$$

$$\sigma = \frac{1}{n}\sqrt{\sum_{i=1}^{n} (x_i - \mu_i) \cdot (x_i - \mu_i)^T}$$



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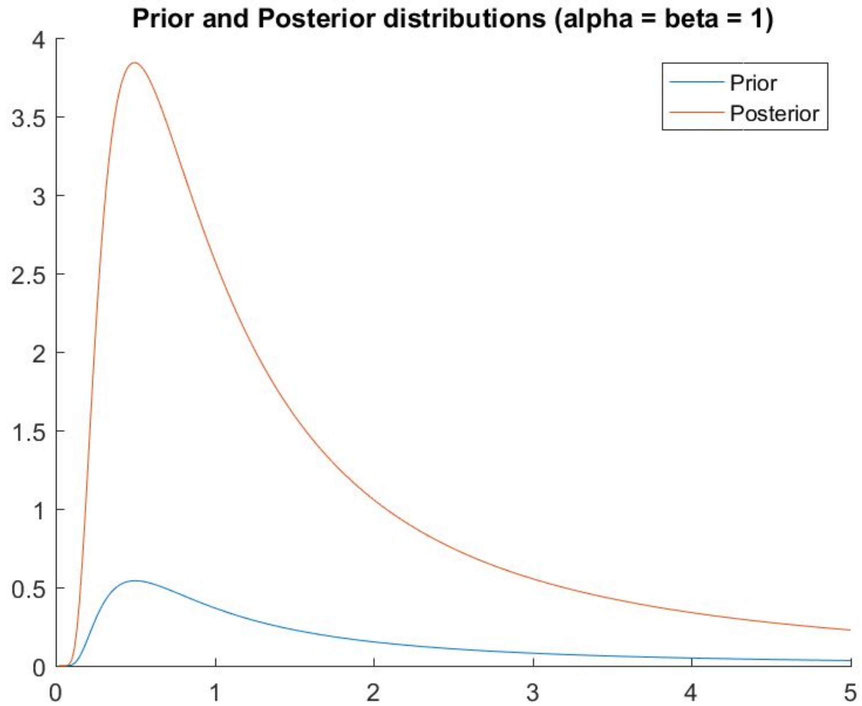
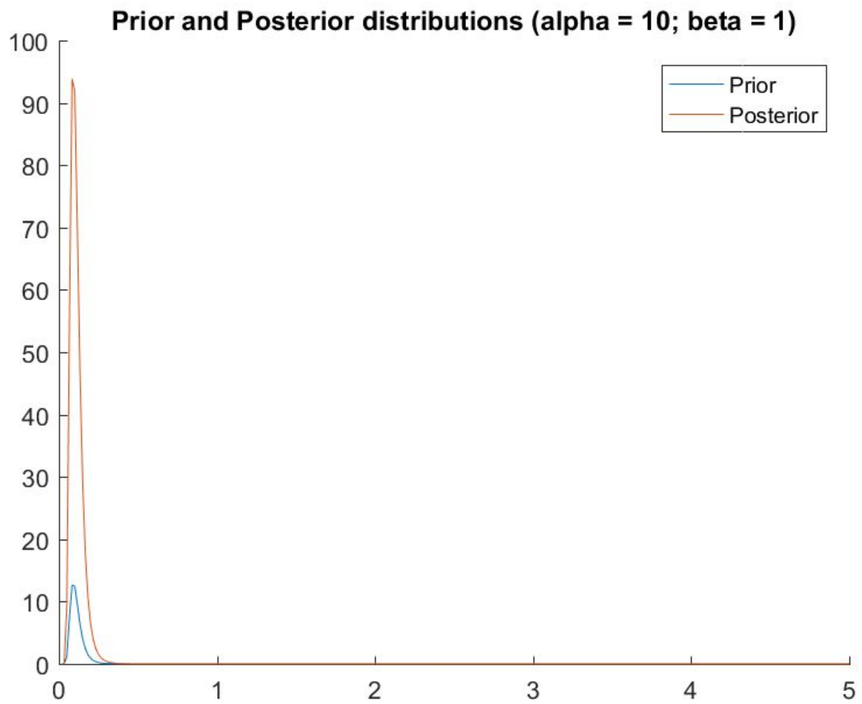
Problem 2.2

a)





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b) Map estimate for $\boldsymbol{\sigma}^2$ = s

The analytical model has been derived in Task 1.2 c).

Model A (alpha=beta=1)

$s_{MAP} =$ 346.3813

Model B (alpha=10, beta=1)

$s_{MAP} =$ 34.63813

c) Bayes Factor

$$BF = \frac{P(M_A|D)}{P(M_B|D)} = 2.61717 * 10^{17} > 1$$

Since the bayes factor BF is greater than 1 Model A is the better model.