STA 2503 / MMF 1928 Project – 3: Stochastic Interest Rates and Swaptions

In this project you will investigate several aspects of the stochastic interest rates. Let $r = (r_t)_{t \geq 0}$ denote the short rate of interest and suppose that $r_t = \phi_t + x_t + y_t$ where x and y satisfy the SDEs

$$\begin{aligned} dx_t &= -\alpha\, x_t\, dt + \sigma\, dW_t^1, \qquad \text{and} \\ dy_t &= -\beta\, y_t\, dt + \eta\, dW_t^2, \end{aligned}$$

$W^{1,2} = (W_t^{1,2})_{t \geq 0}$ are independent risk-neutral Brownian motions, and ϕ_t is a deterministic function of time. This is called a two-factor interest rate model.

For base parameters use the following:

$$x_0 = -0.5\%, \quad \alpha = 3, \quad \sigma = 1\%, \quad y_0 = 0.5\%, \quad \beta = 1, \quad \eta = 0.5\%,$$

and

$$\phi_t = a + b\left(\frac{1 - e^{-\lambda t}}{\lambda t} - e^{-\lambda t}\right), \qquad a = 2\%, \quad b = 5\%, \quad \lambda = 0.75$$

1. This is an affine model, therefore writing T-maturity bond price process as $P(T) = (P_t(T))_{t \in [0,T]}$, there exist deterministic functions $A_t(T)$, $B_t(T)$ and $C_t(T)$ such that

   $$P_t(T) = \exp\{A_t(T) - B_t(T)\, r_t - C_t(T)\, \theta_t\}\ .$$

   Determine the functions A, B and C by either solving for the distribution of $\int_0^T r_u du$ or using the PDE approach and draw the term structure of interest rates with the base parameters.

2. Use an Euler-scheme to discretize the SDE (with discretization of 1 step per day, 252 trading days in a year) and generate 1,000 simulations of risk-neutral interest rate paths out to 10 years and plot the quantiles of the paths. Investigate x, y, and r on separate plots.

3. Use 10,000 paths to obtain Monte Carlo estimate of bond yields (with confidence bands). Compute the yields for maturities ranging from one month to 10 years in steps of one month, and compare with the analytical formula you derived in Q1 using the parameters above.

4. Investigate what role the parameters α, β, σ, and η play in determining the term structure (using the analytical formula).

5. Suppose we have an IRS with tenure structure $\tau = \{3, 3.25, \ldots, 6\}$ where 3 is the first reset date (no payment) and the first payment is at 3.25 and every 0.25 after that. Determine the Black implied volatility of a swaption with strike equal to today's swap-rate. The black implied volatility is the volatility in an LSM model that makes the LSM price equal the price you obtained. How does this change as a function of strike?