MMF1941H: Stochastic Analysis - Assignment #2

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Problem 1. Calculate $\mathbb{P}(D_i)$ for $i = 1, \ldots, N$ and $\mathbb{E}(L)$.

Solution. Let $W \sim \mathcal{N}(0,1)$ be standard normally distributed and $U \sim unif[0,1]$ be uniformly distributed between 0 and 1. Since $\sqrt{\rho}Z + \sqrt{1-\rho}X_i \sim \mathcal{N}(0, \rho + (1-\rho))\,\mathcal{N}(0,1) \sim W$,

$$\begin{aligned}
\mathbb{P}(D_i) &= \mathbb{P}(\sqrt{\rho}Z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)) \\
&= \mathbb{P}(W \leq \Phi^{-1}(p)) \\
&= \mathbb{P}(\Phi(W) \leq p) \\
&= \mathbb{P}(U \leq p) \\
&= p.
\end{aligned}$$

Using this result,

$$\begin{aligned}
\mathbb{E}(L) &= \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N} 1_{D_i}\right) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(1_{D_i}) \\
&= \frac{1}{N}\sum_{i=1}^{N}\mathbb{P}(D_i) = \frac{1}{N}Np \\
&= p.
\end{aligned}$$

□

Problem 2. Calculate $p(Z) := \mathbb{E}(1_{D_i} \mid Z)$ for $i = 1, \ldots, N$.

Solution. For a given value of $Z = z$,

$$\begin{aligned}
p(Z = z) &= \mathbb{E}(1_{D_i} \mid Z = z) \\
&= \mathbb{E}\left(1_{\{\sqrt{\rho}Z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\}} \mid Z = z\right) = \mathbb{E}\left(1_{\{\sqrt{\rho}z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\}}\right) \\
&= \mathbb{P}\left(\sqrt{\rho}z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\right) = \mathbb{P}\left(X_i \leq \frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right) \\
&= \mathbb{P}\left(\Phi(X_i) \leq \Phi\left(\frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right) = \mathbb{P}\left(U \leq \Phi\left(\frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right) \\
&= \Phi\left(\frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right).
\end{aligned}$$

Thus,

$$p(Z) = \Phi\left(\frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right) = \Phi(V)$$

where we define a random variable $V = \frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}$ for conciseness. □

Problem 3. Using the explicit form of $p(Z)$ and the results from Question 2 in Revision Question Set 5, calculate $\mathbb{E}(p(Z))$.

Solution. The results from Question 2 in Revision Question Set 5 state that for $X \sim \mathcal{N}(0,1)$,

$$\mathbb{E}\left(\Phi(aX+b)\right) = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right).$$

Then,

$$\begin{aligned}
\mathbb{E}\left(p(Z)\right) &= \mathbb{E}\left(\Phi\left(\frac{-\sqrt{\rho}Z+\Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right) = \mathbb{E}\left(\Phi\left(-\frac{\sqrt{\rho}}{\sqrt{1-\rho}}Z+\frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right) \\
&= \Phi\left(\frac{\frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}}{\sqrt{1+\frac{\rho}{1-\rho}}}\right) = \Phi\left(\frac{\frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}}{\frac{1}{\sqrt{1-\rho}}}\right) = \Phi\left(\Phi^{-1}(p)\right) \\
&= p.
\end{aligned}$$

□

Problem 4. Calculate the partial derivative $\frac{\partial}{\partial\rho}p(Z)$ explicitly.

Solution.

$$\begin{aligned}
\frac{\partial}{\partial\rho}p(Z) &= \frac{\partial}{\partial\rho}\Phi(V) = \phi(V)\frac{\partial}{\partial\rho}V \\
&= \phi(V)\cdot\frac{\frac{-Z}{2\sqrt{\rho}}\cdot\sqrt{1-\rho}-(-\sqrt{\rho}Z+\Phi^{-1}(p))\cdot\frac{-1}{2\sqrt{1-\rho}}}{1-\rho} \\
&= \phi(V)\cdot\frac{-(1-\rho)Z-\rho Z+\sqrt{\rho}\Phi^{-1}(p)}{2(1-\rho)\sqrt{\rho(1-\rho)}} \\
&= \phi(V)\cdot\frac{-Z+\sqrt{\rho}\Phi^{-1}(p)}{2(1-\rho)\sqrt{\rho(1-\rho)}} \\
&= \phi\left(\frac{-\sqrt{\rho}Z+\Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\cdot\frac{-Z+\sqrt{\rho}\Phi^{-1}(p)}{2(1-\rho)\sqrt{\rho(1-\rho)}}
\end{aligned}$$

where $\phi = \Phi'$ is the pdf of a standard normal distribution. □

Problem 5. For $k = 0,\dots,N$ calculate $\mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right)$.

Solution. Invoking the factorization lemma, for a given value of $Z = z$

$$\begin{aligned}
f(Z=z) &= \mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z=z\right) = \mathbb{E}\left(1_{\{NL=k\}} \mid Z=z\right) \\
&= \mathbb{E}\left(1_{\{\sum_{i=1}^N 1_{D_i}=k\}} \mid Z=z\right) \\
&= \mathbb{E}\left(1_{\{\sum_{i=1}^N 1_{\{\sqrt{\rho}Z+\sqrt{1-\rho}X_i\le\Phi^{-1}(p)\}}=k\}} \mid Z=z\right) \\
&= \mathbb{E}\left(1_{\{\sum_{i=1}^N 1_{\{\sqrt{\rho}z+\sqrt{1-\rho}X_i\le\Phi^{-1}(p)\}}=k\}}\right) \\
&= \mathbb{P}\left(\sum_{i=1}^N 1_{\{\sqrt{\rho}z+\sqrt{1-\rho}X_i\le\Phi^{-1}(p)\}}=k\right) = \mathbb{P}\left(\sum_{i=1}^N 1_{\{X_i\le\frac{-\sqrt{\rho}z+\Phi^{-1}(p)}{\sqrt{1-\rho}}\}}=k\right).
\end{aligned}$$

We define a random variable $I_i = 1_{\{X_i \leq \frac{-\sqrt{\rho}z+\Phi^{-1}(p)}{\sqrt{1-\rho}}\}}$. Then I_i are i.i.d. such that $\mathbb{P}(I_i = 1) = p(Z = z) = 1 - \mathbb{P}(I_i = 0)$. Hence $\sum_{i=1}^{N} I_i = Bin(N, p(Z = z))$ is a binomial distribution with

$$\mathbb{P}\left(\sum_{i=1}^{N} I_i = k\right) = \binom{N}{k} p(Z = z)^k \left(1 - p(Z = z)\right)^{N-k}$$

for $k = 0, \ldots, N$. Now returning back to the derivation of $f(Z = z)$,

$$\begin{aligned} f(Z = z) &= \mathbb{P}\left(\sum_{i=1}^{N} 1_{\{X_i \leq \frac{-\sqrt{\rho}z+\Phi^{-1}(p)}{\sqrt{1-\rho}}\}} = k\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{N} I_i = k\right) \\ &= \binom{N}{k} p(Z = z)^k \left(1 - p(Z = z)\right)^{N-k}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right) = f(Z) &= \binom{N}{k} p(Z)^k \left(1 - p(Z)\right)^{N-k} \\ &= \binom{N}{k} \Phi\left(\frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)^k \left(1 - \Phi\left(\frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right)^{N-k}. \end{aligned}$$

□

Problem 6. For $k = 0, \ldots, N$ calculate

$$\frac{\partial}{\partial\rho} \mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right). \tag{3}$$

Solution. We divide the problem into 3 cases:

Case 1: $k = 1, \ldots, N-1$

$$\begin{aligned} \frac{\partial}{\partial\rho} \mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right) = \binom{N}{k} \Bigg[&kp(Z)^{k-1} \left(\frac{\partial}{\partial\rho} p(Z)\right) (1 - p(Z))^{N-k} \\ &+ (N-k)(1 - p(Z))^{N-k-1} \left(-\frac{\partial}{\partial\rho} p(Z)\right) p(Z)^k \Bigg] \\ = \binom{N}{k} &\left[\frac{k}{p(Z)} p(Z)^k (1 - p(Z))^{N-k} - \frac{N-k}{1-p(Z)} p(Z)^k (1 - p(Z))^{N-k}\right] \frac{\partial}{\partial\rho} p(Z) \\ = \binom{N}{k} &p(Z)^k (1 - p(Z))^{N-k} \left[\frac{k}{p(Z)} - \frac{N-k}{1-p(Z)}\right] \frac{\partial}{\partial\rho} p(Z) \\ = \mathbb{E} &\left(1_{\{L=\frac{k}{N}\}} \mid Z\right) \left[\frac{k}{p(Z)} - \frac{N-k}{1-p(Z)}\right] \frac{\partial}{\partial\rho} p(Z). \end{aligned}$$

Case 2: $k = 0$

$$\begin{aligned} \frac{\partial}{\partial\rho} \mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right) &= \frac{\partial}{\partial\rho} (1 - p(Z))^N = N(1 - p(Z))^{N-1} \left(-\frac{\partial}{\partial\rho} p(Z)\right) \\ &= -\frac{N}{1-p(Z)} (1 - p(Z))^N \frac{\partial}{\partial\rho} p(Z) \end{aligned}$$

which is equivalent to the final expression from *Case 1* with $k = 0$.

Case 3: $k = N$

$$\frac{\partial}{\partial\rho}\mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right) = \frac{\partial}{\partial\rho}p(Z)^N = Np(Z)^{N-1}\frac{\partial}{\partial\rho}p(Z)$$
$$= \frac{N}{p(Z)}p(Z)^N\frac{\partial}{\partial\rho}p(Z)$$

which is equivalent to the final expression from *Case 1* with $k = N$.
Hence the general solution, for $k = 0, \ldots, N$, can be expressed as:

$$\frac{\partial}{\partial\rho}\mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right) = \mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right)\left[\frac{k}{p(Z)} - \frac{N-k}{1-p(Z)}\right]\frac{\partial}{\partial\rho}p(Z)$$

where $p(Z)$, $\frac{\partial}{\partial\rho}p(Z)$, and $\mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right)$ are solutions to *Problem 2*, *4*, and *5* respectively. □

Problem 7. Show first that for a random variable Y taking values in $\mathbb{N}_0$

$$\mathbb{E}(\min(Y,k)) = k + \sum_{j=0}^{k-1}(j-k)\mathbb{P}(Y=j) \tag{4}$$

holds for fixed $k \in \mathbb{N}$.

Solution. Since $\min(Y,k) = Y \cdot 1_{\{Y<k\}} + k \cdot 1_{\{Y \geq k\}}$,

$$\begin{aligned}
\mathbb{E}(\min(Y,k)) &= \mathbb{E}(Y \cdot 1_{\{Y<k\}}) + \mathbb{E}(k \cdot 1_{\{Y\geq k\}}) \\
&= \left(\sum_{j=0}^{k-1} j \cdot \mathbb{P}(Y=j)\right) + k \cdot \mathbb{E}[1_{\{Y\geq k\}}] = \left(\sum_{j=0}^{k-1} j \cdot \mathbb{P}(Y=j)\right) + k \cdot \left(1 - \mathbb{E}[1_{\{Y<k\}}]\right) \\
&= \left(\sum_{j=0}^{k-1} j \cdot \mathbb{P}(Y=j)\right) + k \cdot \left(1 - \sum_{j=0}^{k-1}\mathbb{P}(Y=j)\right) \\
&= k + \sum_{j=0}^{k-1}(j-k)\mathbb{P}(Y=j).
\end{aligned}$$

□

Problem 8. Fixing the parameters $p = 0.05$, $K = 0.08$, and $N = 100$, write Python code to calculate $\ell(K;\rho)$ by simulating 5000 outcomes of Z, leveraging formula (4). Provide the codoe and plot $\ell(K;\rho)$ for $\rho \in [0,1]$. For $\rho = 0$ the result can be calculated exactly, provide this exact result.

Solution. Since $\mathbb{E}(\mathbb{E}(\min(L,K) \mid Z)) = \frac{1}{N}\mathbb{E}(\mathbb{E}(\min(NL,NK) \mid Z))$, we are interested in determining an expression for $\mathbb{E}(\min(NL,NK) \mid Z)$. Invoking the factorization lemma and using the result from *Problem 7*, for a given value of $Z = z$

$$\begin{aligned}
g(Z=z) = \mathbb{E}(\min(NL,NK) \mid Z=z) &= \mathbb{E}(\min(NL,NK) \mid Z=z) \\
&= \mathbb{E}\left(\min\left(\sum_{i=0}^{N} 1_{D_i}, NK\right) \mid Z=z\right) = \mathbb{E}\left(\min\left(\sum_{i=0}^{N} I_i, NK\right)\right) \\
&= NK + \sum_{j=0}^{NK-1}(j-NK)\mathbb{P}\left(\sum_{i=0}^{N} I_i = j\right) \\
&= NK + \sum_{j=0}^{NK-1}(j-NK)\binom{N}{j}p(Z=z)^j\left(1-p(Z=z)\right)^{N-j} \\
&= NK + \sum_{j=0}^{NK-1}(j-NK)\mathbb{E}\left(1_{\{L=\frac{j}{N}\}} \mid Z=z\right)
\end{aligned}$$

where $I_i = 1_{\{X_i \leq \frac{-\sqrt{\rho}z+\Phi^{-1}(p)}{\sqrt{1-\rho}}\}}$ as defined in *Problem 5*. Hence,

$$\mathbb{E}(\min(NL, NK) \mid Z) = g(Z) = NK + \sum_{j=0}^{NK-1} (j - NK)\mathbb{E}\left(1_{\{L=\frac{j}{N}\}} \mid Z\right)$$

where $\mathbb{E}\left(1_{\{L=\frac{j}{N}\}} \mid Z\right)$ is the solution to *Problem 5* with $k = j$.

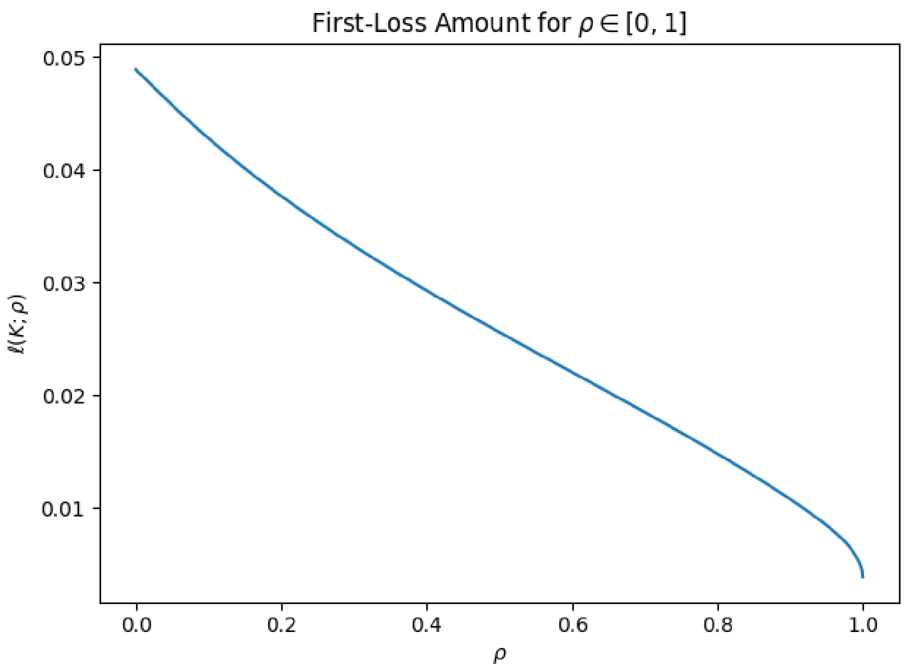

Figure 1: *first-loss amount $\ell(K;\rho)$ for $\rho \in [0,1]$, calculated by simulating 5000 outcomes of Z*

For the special case when $\rho = 0$,

$$\mathbb{E}\left(1_{\{L=\frac{j}{N}\}} \mid Z\right) = \binom{N}{j}\Phi\left(\Phi^{-1}(p)\right)^j\left(1 - \Phi\left(\Phi^{-1}(p)\right)\right)^{N-j} = \binom{N}{j}p^j(1-p)^{N-j}$$

and the first-loss amount when $\rho = 0$ is:

$$\begin{aligned}
\ell(K;\rho = 0) = \mathbb{E}(\mathbb{E}(\min(L, K) \mid Z)) &= \frac{1}{N}\mathbb{E}(\mathbb{E}(\min(NL, NK) \mid Z)) \\
&= \frac{1}{N}\mathbb{E}\left(NK + \sum_{j=0}^{NK-1}(j - NK)\mathbb{E}\left(1_{\{L=\frac{j}{N}\}} \mid Z\right)\right) \\
&= \frac{1}{N}\mathbb{E}\left(NK + \sum_{j=0}^{NK-1}(j - NK)\binom{N}{j}p^j(1-p)^{N-j}\right) \\
&= K + \frac{1}{N}\sum_{j=0}^{NK-1}(j - NK)\binom{N}{j}p^j(1-p)^{N-j}.
\end{aligned}$$

Using Python, the exact value calculated analytically, up to 6 decimal places, is $\ell(K;\rho = 0) = 0.048909$. $\square$

Problem 9. Extend the previous simulation result to also include the *correlation risk* $\frac{\partial}{\partial\rho}\ell(K;\rho)$ in the simulation, utilising

$$\frac{\partial}{\partial\rho}\ell(K;\rho) = \mathbb{E}\left(\frac{\partial}{\partial\rho}\mathbb{E}(\min(L,K) \mid Z)\right)$$

and using the explicit form established in (3). Plot the correlation risk for $\rho \in [0,1]$.

Solution.

$$\begin{aligned}
\frac{\partial}{\partial\rho}\ell(K;\rho) &= \mathbb{E}\left(\frac{\partial}{\partial\rho}\mathbb{E}(\min(L,K) \mid Z)\right) = \frac{1}{N}\mathbb{E}\left(\frac{\partial}{\partial\rho}\mathbb{E}(\min(NL,NK) \mid Z)\right) \\
&= \frac{1}{N}\mathbb{E}\left[\frac{\partial}{\partial\rho}\left(NK + \sum_{j=0}^{NK-1}(j-NK)\mathbb{E}\left(1_{\{L=\frac{j}{N}\}} \mid Z\right)\right)\right] \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{j=0}^{NK-1}(j-NK)\frac{\partial}{\partial\rho}\mathbb{E}\left(1_{\{L=\frac{j}{N}\}} \mid Z\right)\right]
\end{aligned}$$

where $\frac{\partial}{\partial\rho}\mathbb{E}\left(1_{\{L=\frac{j}{N}\}} \mid Z\right)$ is the solution to *Problem 6* with $k = j$.

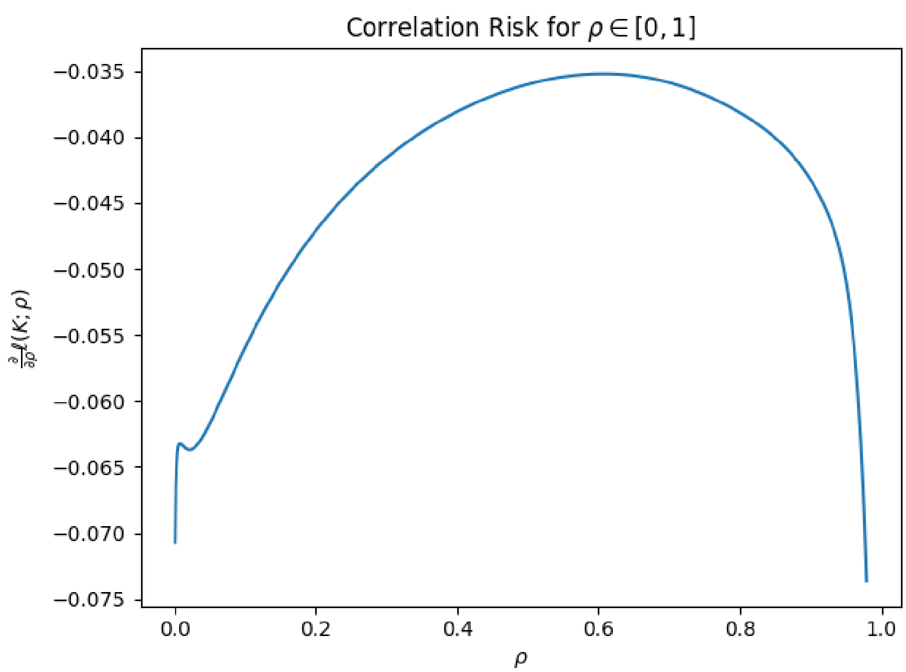

Figure 2: *correlation risk value $\frac{\partial}{\partial\rho}\ell(K;\rho)$ for $\rho \in [0,1]$, calculated by simulating 5000 outcomes of Z*

□

Problem 10. Finally, we are aiming to validate the prior result for the correlation risk against re-calculating the value under a small perturbation of ρ, i.e. $\frac{\partial}{\partial\rho}\ell(K;\rho) \approx \frac{1}{\varepsilon}(\ell(K;\rho+\varepsilon) - \ell(K;\rho))$ for small values of ε. Extend your Monte Carlo simulation to include this "bump-and-reval" correlation risk and include this result in the plot as a comparison against the previously calculated risk.

Solution. Using $\epsilon = 10^{-6}$ as the perturbation, the following is the "bump-and-reval" simulation result.

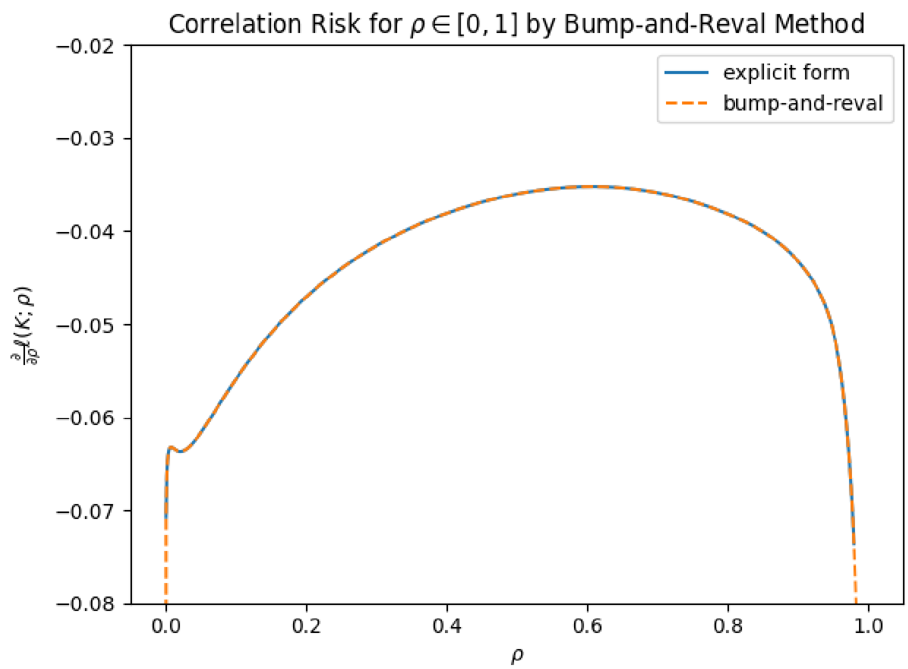

Figure 3: *correlation risk value* $\frac{\partial}{\partial \rho}\ell(K;\rho)$ *for* $\rho \in [0,1]$, *calculated using the "bump-and-reval" method*

$\square$