MMF1941H: Stochastic Analysis - Assignment #1

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Problem 1. Show that for $K \in \mathbb{R}$

$$\mathbb{E}((X-K)^+) = \varphi(K) - K(1-\Phi(K))$$

holds where φ is the pdf of the standard normal distribution and Φ the respective cdf.

Solution. Let $f_K(X) = (X-K)^+$. Given that $X \sim \mathcal{N}(0,1)$ is a standard normal random variable

$$\begin{aligned}
\mathbb{E}(f_K(X)) = \mathbb{E}((X-K)^+) &= \int_{-\infty}^{\infty} (x-K)^+ \varphi(x)dx = \int_K^{\infty} (x-K)\varphi(x)dx \\
&= \int_K^{\infty} x\varphi(x)dx - K\int_K^{\infty} \varphi(x)dx \\
&= \int_K^{\infty} -\varphi'(x)dx - K(1-\Phi(K)) \\
&= -\varphi(x)|_{x=K}^{\infty} - K(1-\Phi(K)) \\
&= \varphi(K) - K(1-\Phi(K)).
\end{aligned}$$

□

Problem 2. Use the previous result to show that analytically

$$v = \sigma_X \varphi\left(\frac{K-\mu_X}{\sigma_X}\right) - (K-\mu_X)\left(1-\Phi\left(\frac{K-\mu_X}{\sigma_X}\right)\right)$$

holds.

Solution.

$$\begin{aligned}
(\mu_X + \sigma_X X - K)^+ &= (\sigma_X X - (K-\mu_X))^+ \\
&= \left(\sigma_X\left(X - \frac{K-\mu_X}{\sigma_X}\right)\right)^+ \\
&= \sigma_X\left(X - \frac{K-\mu_X}{\sigma_X}\right)^+ \\
&= \sigma_X f_{K'}(X)
\end{aligned}$$

where $K' = \frac{K-\mu_X}{\sigma_X}$. Given that $v = \mathbb{E}\left((\mu_X + \sigma_X X - K)^+\right)$

$$\begin{aligned}
v &= \mathbb{E}((\sigma_X f_{K'}(X))) = \sigma_X \mathbb{E}(f_{K'}(X)) && \text{(by linearity of expectation)} \\
&= \sigma_X(\varphi(K') - K'(1-\Phi(K'))) \\
&= \sigma_X\left(\varphi\left(\frac{K-\mu_X}{\sigma_X}\right) - \frac{K-\mu_X}{\sigma_X}\left(1-\Phi\left(\frac{K-\mu_X}{\sigma_X}\right)\right)\right) \\
&= \sigma_X\varphi\left(\frac{K-\mu_X}{\sigma_X}\right) - (K-\mu_X)\left(1-\Phi\left(\frac{K-\mu_X}{\sigma_X}\right)\right)
\end{aligned}$$

□

Problem 3. For a parameter $a \in \mathbb{R}$ we can define a measure $\mathbb{Q}_a$ via the definition

$$\mathbb{Q}_a(A) = \mathbb{E}\left(1_A \frac{\exp(aX)}{\mathbb{E}(\exp(aX))}\right).$$

For $x \in \mathbb{R}$ calculate the $\mathbb{Q}_a(X \leq x)$ and conclude that under $\mathbb{Q}_a$, X again follows a normal distribution and determine its parameters.

Solution. Using the knowledge of moment generating functions (mgf), $\mathbb{E}(\exp(aX)) = m_X(a) = \exp\left(\frac{1}{2}a^2\right)$ where m_X is the mgf of X. Then

$$\begin{aligned}
\mathbb{Q}_a(X \leq x) &= \mathbb{E}\left(1_{X\leq x} \frac{\exp(aX)}{\exp\left(\frac{1}{2}a^2\right)}\right) \\
&= \mathbb{E}\left(1_{X\leq x} \exp\left(aX - \frac{1}{2}a^2\right)\right) \\
&= \int_{-\infty}^{x} \exp\left(as - \frac{1}{2}s^2\right)\varphi(s)ds \\
&= \int_{-\infty}^{x} \exp\left(as - \frac{1}{2}s^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}s^2\right)ds \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(s^2 - 2as + a^2)\right)ds \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(s-a)^2\right)ds \\
&= \Phi(x-a)
\end{aligned}$$

which is the cdf of $\mathcal{N}(a,1)$. Therefore $X \overset{\mathbb{Q}_a}{\sim} \mathcal{N}(a,1)$. $\square$

Problem 4. Write Python code to simulate the option value 1000 times under the measure $\mathbb{P}$ with a sample size of 5000 simulations each for $\mu_X = 2.5$, $\sigma_X^2 = 4$, and $K = 6$. Share the code and provide a histogram of the results. Also calculate the exact value analytically per the above formula.

Solution. The exact option value calculated analytically, up to 6 decimal places, is $v = 0.032348$.

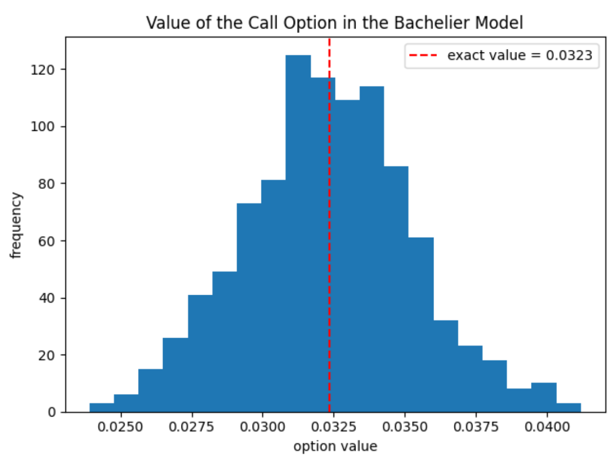

Figure 1: *histogram of 1000 simulated option value under* $\mathbb{P}$

$\square$

Problem 5. Calculate the sample variance for your previous experiment.

Solution. The sample variance is $s = 8.56136 \times 10^{-6}$. □

Problem 6. Note that for any a the option value can be re-written as

$$v = \mathbb{E}_{\mathbb{Q}_a}\left(\frac{d\mathbb{P}}{d\mathbb{Q}_a}(\mu_X + \sigma_X X - K)^+\right)$$

which mathematically will yield the same answer for any a. Define

$$g(a) = \mathbb{E}_{\mathbb{Q}_a}\left(\left(\frac{d\mathbb{P}}{d\mathbb{Q}_a}(\mu_X + \sigma_X X - K)^+\right)^2\right)$$

which can be evaluated through MC simulation given the fact the distribution of X under the measure $\mathbb{Q}_a$ is known. Write Python code to plot the function $g(a)$ for the above selection of parameters and $a \in [0, 5]$.

Solution. The minimum of g is attained at $a = 2.475$.

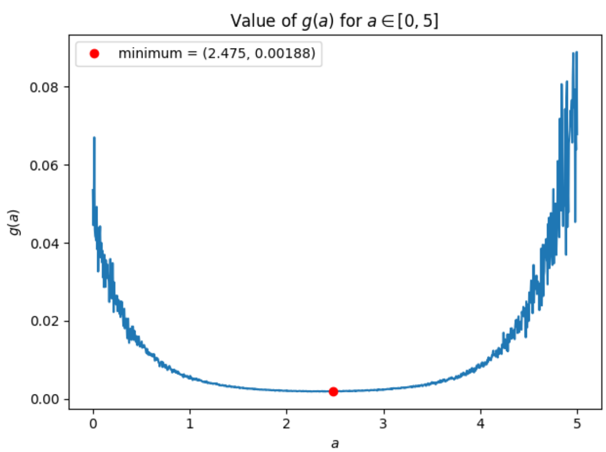

Figure 2: *values of $g(a)$ for $a \in [0, 5]$*

□

Problem 7. Determine the minimum of the function g numerically (and approximately) from the prior plot and repeat the experiment of simulating the option value but now calculated through

$$v = \mathbb{E}_{\mathbb{Q}_a^*}\left(\frac{d\mathbb{P}}{d\mathbb{Q}_a^*}(\mu_X + \sigma_X X - K)^+\right)$$

- where g attains its minimum at a^* - 1000 times with a sample size of 5000 simulations each and again determine the sample variance. Plot the histogram of this experiment again.

Solution. Using $a^* = 2.475$, the sample variance is $s = 1.90081 \times 10^{-7}$.

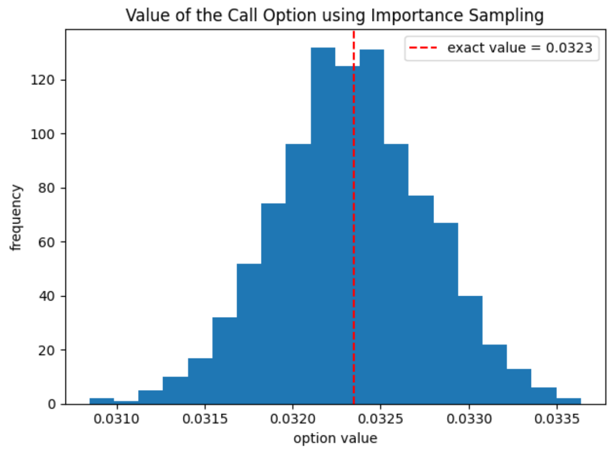

Figure 3: *histogram of 1000 simulated option value using importance sampling under* $\mathbb{Q}_a^*$

$\square$