MMF1941H: Stochastic Analysis - Assignment # 2

21 Nov 2022

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1 Instructions

1. Assignments are due on Wednesday Nov 30th, 7:30pm EST. Submissions are accepted late for 1 day with a 10% penalty, submissions after Dec 2nd will not be accepted.
2. Please have your final report typeset using LaTeX and submit your report individually. Provide code separate in a Python script file that you attach with your submission.
3. You may discuss these questions with your fellow students, however the write-up must be yours and yours alone, sharing of the write-up before the deadline is not allowed.

2 Problem: Portfolio Loss Modelling and Correlation Risk

For $N \in \mathbb{N}$ let $X_1, \ldots, X_N$ and Z be independent, identically, standard normally distributed random variables. Define a portfolio loss fraction L as

$$L := \frac{1}{N} \sum_{i=1}^{N} 1_{D_i} \tag{1}$$

where the set D_i is defined as

$$D_i = \{\sqrt{\rho} Z + \sqrt{1-\rho} X_i \leq \Phi^{-1}(p)\} \tag{2}$$

for $i = 1, \ldots, N$ where $\rho \in [0,1]$ is a correlation parameter, $p \in (0,1)$ denotes a default probability and Φ denotes the cumulative distribution function of the standard normal distribution.

1. (2 pts) Calculate $\mathbb{P}(D_i)$ for $i = 1, \ldots, N$ and $\mathbb{E}(L)$.
2. (5 pts) Calculate $p(Z) := \mathbb{E}\left(1_{D_i} \mid Z\right)$ for $i = 1, \ldots, N$.
3. (5 pts) Using the explicit form of $p(Z)$ and the results from Question 2 in Revision Question Set 5, calculate $\mathbb{E}(p(Z))$.
4. (5 pts) Calculate the partial derivative $\frac{\partial}{\partial \rho} p(Z)$ explictly.

5. (5 pts) For $k = 0, \ldots, N$ calculate $\mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right)$. (Hint: Conditionally on Z the events D_i are independent, meaning that the distribution of $L \cdot N$ is known analytically as a Binomial distribution with suitable parameters).

6. (5 pts) For $k = 0, \ldots, N$ calculate
$$\frac{\partial}{\partial \rho}\mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right). \tag{3}$$

7. (5 pts) We are now interested in calculating the so-called *first-loss* amount, defined as
$$\ell(K;\rho) = \mathbb{E}\left(\min(L, K)\right)$$
for some percentage $K \in [0,1]$ where we consider it a function of ρ for all other parameters fixed. Show first that for a random variable Y taking values in $\mathbb{N}_0$
$$\mathbb{E}\left(\min(Y, k)\right) = k + \sum_{j=0}^{k-1}(j-k)\mathbb{P}\left(Y = j\right) \tag{4}$$
holds for fixed $k \in \mathbb{N}$.

8. (8 pts) Through the tower property of conditional expectation it is clear that
$$\ell(K;\rho) = \mathbb{E}\left(\mathbb{E}\left(\min(L, K) \mid Z\right)\right)$$
holds. Fixing the parameters $p = 0.05$, $K = 0.08$ and $N = 100$, write Python code to calculate $\ell(K;\rho)$ by simulating 5000 outcomes of Z, leveraging formula (4). Provide the code and plot $\ell(K;\rho)$ for $\rho \in [0,1]$. For $\rho = 0$ the result can be calculated exactly, provide this exact result.

9. (5 pts) Extend the previous simulation result to also include the *correlation risk* $\frac{\partial}{\partial \rho}\ell(K;\rho)$ in the simulation, utilising
$$\frac{\partial}{\partial \rho}\ell(K;\rho) = \mathbb{E}\left(\frac{\partial}{\partial \rho}\mathbb{E}\left(\min(L, K) \mid Z\right)\right)$$
and using the explicit form established in (3). Plot the correlation risk for $\rho \in [0,1]$.

10. (5 pts) Finally, we are aiming to validate the prior result for the correlation risk against re-calculating the value under a small perturbation of ρ, ie $\frac{\partial}{\partial \rho}\ell(K;\rho) \approx \frac{1}{\varepsilon}\left(\ell(K;\rho+\varepsilon) - \ell(K;\rho)\right)$ for small values of ε. Extend your Monte Carlo simulation to include this "bump-and-reval" correlation risk and include this result in the plot as a comparison against the previously calculated risk.