MMF1941H: Stochastic Analysis - Assignment # 1

21 Oct 2022

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1 Instructions

1. Assignments are due on Monday Oct 31st, 7:30pm EST. Submissions are accepted late for 1 day with a 10% penalty, submissions after Nov 2nd will not be accepted.
2. Please have your final report typeset using LaTeX and submit your report individually. Provide code separate in a Python script file that you attach with your submission.
3. You may discuss these questions with your fellow students, however the write-up must be yours and yours alone, sharing of the write-up before the deadline is not allowed.

2 Problem: Bachelier Call Option Pricing

Let X be a standard normal random variable and let $\mu_X \in \mathbb{R}$ and variance $\sigma_X^2 > 0$ parameters. We are looking at the value of a call option in the *Bachelier Model* ie

$$v = \mathbb{E}\left((\mu_X + \sigma_X X - K)^+\right)$$

for a given strike K.

1. (10 pts) Show that for $K \in \mathbb{R}$

   $$\mathbb{E}((X-K)^+) = \varphi(K) - K(1-\Phi(K))$$

   holds where φ is the pdf of the standard normal distribution and Φ the respective cdf. (Hint: you can exploit that $\varphi'(x) = -x\varphi(x)$ holds).
2. (5 pts) Use the previous result to show that analytically

   $$v = \sigma_X \varphi(\frac{K-\mu_X}{\sigma_X}) - (K-\mu_X)\left(1-\Phi(\frac{K-\mu_X}{\sigma_X})\right)$$

   holds.

3. (10 pts) For a parameter $a \in \mathbb{R}$ we can define a measure $\mathbb{Q}_a$ via the definition

$$\mathbb{Q}_a(A) = \mathbb{E}\left(1_A \frac{\exp(aX)}{\mathbb{E}(\exp(aX))}\right).$$

For $x \in \mathbb{R}$ calculate the $\mathbb{Q}_a(X \leq x)$ and conclude that under $\mathbb{Q}_a$ X again follows a normal distribution and determine its parameters.

4. (5 pts) Write Python code to simulate the option value 1000 times under the measure $\mathbb{P}$ with a sample size of 5000 simulations each for $\mu_X = 2.5, \sigma_X^2 = 4$ and $K = 6$. Share the code and provide a histogram of the results. Also calculate the exact value analytically per the above formula.

5. (5 pts) Denoting v_j a single MC estimate (based on 5000 simulations) for $j = 1, \dots, M$ with $M = 1000$ we can define the sample variance as

$$s = \frac{1}{M-1} \sum_{j=1}^{M} \left(v_j - \frac{1}{M} \sum_{k=1}^{M} v_k \right)^2.$$

Calculate the sample variance for your previous experiment.

6. (10 pts) Note that for any a the option value can be re-written as

$$v = \mathbb{E}_{\mathbb{Q}_a}\left(\frac{d\mathbb{P}}{d\mathbb{Q}_a}(\mu_X + \sigma_X X - K)^+\right)$$

which mathematically will yield the same answer for any a. Define

$$g(a) = \mathbb{E}_{\mathbb{Q}_a}\left(\left(\frac{d\mathbb{P}}{d\mathbb{Q}_a}(\mu_X + \sigma_X X - K)^+\right)^2\right)$$

which can be evaluated through MC simulation given the fact the distribution of X under the measure $\mathbb{Q}_a$ is known. Write Python code plot the function $g(a)$ for the above selection of parameters and $a \in [0, 5]$. Note that equivalently,

$$g(a) = \mathbb{E}\left(\frac{d\mathbb{P}}{d\mathbb{Q}_a}\left((\mu_X + \sigma_X X - K)^+\right)^2\right)$$

holds which you could use alternatively for implementation purposes.

7. (5 pts) Determine the minimum of the function g numerically (and approximately) from the prior plot and repeat the experiment of simulating the option value but now calculated through

$$v = \mathbb{E}_{\mathbb{Q}_a^*}\left(\frac{d\mathbb{P}}{d\mathbb{Q}_a^*}(\mu_X + \sigma_X X - K)^+\right)$$

– where g attains its minimum at a^* – 1000 times with a sample size of 5000 simulations each and again determine the sample variance. Plot the histogram of this experiment again.