

# MMF1941H: Stochastic Analysis - Assignment #2

Joshua (Ha Rim) Kim

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**Problem 1.** Calculate  $\mathbb{P}(D_i)$  for  $i = 1, \dots, N$  and  $\mathbb{E}(L)$ .

*Solution.* Let  $W \sim \mathcal{N}(0, 1)$  be standard normally distributed and  $U \sim \text{unif}[0, 1]$  be uniformly distributed between 0 and 1. Since  $\sqrt{\rho}Z + \sqrt{1-\rho}X_i \sim \mathcal{N}(0, \rho + (1-\rho)) \mathcal{N}(0, 1) \sim W$ ,

$$\begin{aligned}\mathbb{P}(D_i) &= \mathbb{P}(\sqrt{\rho}Z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)) \\ &= \mathbb{P}(W \leq \Phi^{-1}(p)) \\ &= \mathbb{P}(\Phi(W) \leq p) \\ &= \mathbb{P}(U \leq p) \\ &= p.\end{aligned}$$

Using this result,

$$\begin{aligned}\mathbb{E}(L) &= \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^N 1_{D_i}\right) = \frac{1}{N} \sum_{i=1}^N \mathbb{E}(1_{D_i}) \\ &= \frac{1}{N} \sum_{i=1}^N \mathbb{P}(D_i) = \frac{1}{N} Np \\ &= p.\end{aligned}$$

□

**Problem 2.** Calculate  $p(Z) := \mathbb{E}(1_{D_i} \mid Z)$  for  $i = 1, \dots, N$ .

*Solution.* For a given value of  $Z = z$ ,

$$\begin{aligned}p(Z = z) &= \mathbb{E}(1_{D_i} \mid Z = z) \\ &= \mathbb{E}\left(1_{\{\sqrt{\rho}Z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\}} \mid Z = z\right) = \mathbb{E}\left(1_{\{\sqrt{\rho}z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\}}\right) \\ &= \mathbb{P}\left(\sqrt{\rho}z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\right) = \mathbb{P}\left(X_i \leq \frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right) \\ &= \mathbb{P}\left(\Phi(X_i) \leq \Phi\left(\frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right) = \mathbb{P}\left(U \leq \Phi\left(\frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right) \\ &= \Phi\left(\frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right).\end{aligned}$$

Thus,

$$p(Z) = \Phi\left(\frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right) = \Phi(V)$$

where we define a random variable  $V = \frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}$  for conciseness.

□

**Problem 3.** Using the explicit form of  $p(Z)$  and the results from Question 2 in Revision Question Set 5, calculate  $\mathbb{E}(p(Z))$ .

*Solution.* The results from Question 2 in Revision Question Set 5 state that for  $X \sim \mathcal{N}(0, 1)$ ,

$$\mathbb{E}(\Phi(aX + b)) = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right).$$

Then,

$$\begin{aligned}\mathbb{E}(p(Z)) &= \mathbb{E}\left(\Phi\left(\frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right) = \mathbb{E}\left(\Phi\left(-\frac{\sqrt{\rho}}{\sqrt{1-\rho}}Z + \frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right) \\ &= \Phi\left(\frac{\frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}}{\sqrt{1+\frac{\rho}{1-\rho}}}\right) = \Phi\left(\frac{\frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}}{\frac{1}{\sqrt{1-\rho}}}\right) = \Phi(\Phi^{-1}(p)) \\ &= p.\end{aligned}$$

□

**Problem 4.** Calculate the partial derivative  $\frac{\partial}{\partial \rho} p(Z)$  explicitly.

*Solution.*

$$\begin{aligned}\frac{\partial}{\partial \rho} p(Z) &= \frac{\partial}{\partial \rho} \Phi(V) = \phi(V) \frac{\partial}{\partial \rho} V \\ &= \phi(V) \cdot \frac{\frac{-Z}{2\sqrt{\rho}} \cdot \sqrt{1-\rho} - (-\sqrt{\rho}Z + \Phi^{-1}(p)) \cdot \frac{-1}{2\sqrt{1-\rho}}}{1-\rho} \\ &= \phi(V) \cdot \frac{-(1-\rho)Z - \rho Z + \sqrt{\rho}\Phi^{-1}(p)}{2(1-\rho)\sqrt{\rho}(1-\rho)} \\ &= \phi(V) \cdot \frac{-Z + \sqrt{\rho}\Phi^{-1}(p)}{2(1-\rho)\sqrt{\rho}(1-\rho)} \\ &= \phi\left(\frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right) \cdot \frac{-Z + \sqrt{\rho}\Phi^{-1}(p)}{2(1-\rho)\sqrt{\rho}(1-\rho)}\end{aligned}$$

where  $\phi = \Phi'$  is the pdf of a standard normal distribution.

□

**Problem 5.** For  $k = 0, \dots, N$  calculate  $\mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z\right)$ .

*Solution.* Invoking the factorization lemma, for a given value of  $Z = z$

$$\begin{aligned}f(Z = z) &= \mathbb{E}\left(1_{\{L=\frac{k}{N}\}} \mid Z = z\right) = \mathbb{E}\left(1_{\{NL=k\}} \mid Z = z\right) \\ &= \mathbb{E}\left(1_{\{\sum_{i=1}^N 1_{D_i}=k\}} \mid Z = z\right) \\ &= \mathbb{E}\left(1_{\{\sum_{i=1}^N 1_{\{\sqrt{\rho}Z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\}}=k\}} \mid Z = z\right) \\ &= \mathbb{E}\left(1_{\{\sum_{i=1}^N 1_{\{\sqrt{\rho}z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\}}=k\}}\right) \\ &= \mathbb{P}\left(\sum_{i=1}^N 1_{\{\sqrt{\rho}z + \sqrt{1-\rho}X_i \leq \Phi^{-1}(p)\}} = k\right) = \mathbb{P}\left(\sum_{i=1}^N 1_{\{X_i \leq \frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\}} = k\right).\end{aligned}$$

We define a random variable  $I_i = 1_{\{X_i \leq \frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\}}$ . Then  $I_i$  are i.i.d. such that  $\mathbb{P}(I_i = 1) = p(Z = z) = 1 - \mathbb{P}(I_i = 0)$ . Hence  $\sum_{i=1}^N I_i = \text{Bin}(N, p(Z = z))$  is a binomial distribution with

$$\mathbb{P}\left(\sum_{i=1}^N I_i = k\right) = \binom{N}{k} p(Z = z)^k (1 - p(Z = z))^{N-k}$$

for  $k = 0, \dots, N$ . Now returning back to the derivation of  $f(Z = z)$ ,

$$\begin{aligned} f(Z = z) &= \mathbb{P}\left(\sum_{i=1}^N 1_{\{X_i \leq \frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\}} = k\right) \\ &= \mathbb{P}\left(\sum_{i=1}^N I_i = k\right) \\ &= \binom{N}{k} p(Z = z)^k (1 - p(Z = z))^{N-k}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}\left(1_{\{L = \frac{k}{N}\}} \mid Z\right) &= f(Z) = \binom{N}{k} p(Z)^k (1 - p(Z))^{N-k} \\ &= \binom{N}{k} \Phi\left(\frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)^k \left(1 - \Phi\left(\frac{-\sqrt{\rho}Z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right)\right)^{N-k}. \end{aligned}$$

□

**Problem 6.** For  $k = 0, \dots, N$  calculate

$$\frac{\partial}{\partial \rho} \mathbb{E}\left(1_{\{L = \frac{k}{N}\}} \mid Z\right). \quad (3)$$

*Solution.* We divide the problem into 3 cases:

Case 1:  $k = 1, \dots, N - 1$

$$\begin{aligned} \frac{\partial}{\partial \rho} \mathbb{E}\left(1_{\{L = \frac{k}{N}\}} \mid Z\right) &= \binom{N}{k} \left[ k p(Z)^{k-1} \left( \frac{\partial}{\partial \rho} p(Z) \right) (1 - p(Z))^{N-k} \right. \\ &\quad \left. + (N - k) (1 - p(Z))^{N-k-1} \left( -\frac{\partial}{\partial \rho} p(Z) \right) p(Z)^k \right] \\ &= \binom{N}{k} \left[ \frac{k}{p(Z)} p(Z)^k (1 - p(Z))^{N-k} - \frac{N - k}{1 - p(Z)} p(Z)^k (1 - p(Z))^{N-k} \right] \frac{\partial}{\partial \rho} p(Z) \\ &= \binom{N}{k} p(Z)^k (1 - p(Z))^{N-k} \left[ \frac{k}{p(Z)} - \frac{N - k}{1 - p(Z)} \right] \frac{\partial}{\partial \rho} p(Z) \\ &= \mathbb{E}\left(1_{\{L = \frac{k}{N}\}} \mid Z\right) \left[ \frac{k}{p(Z)} - \frac{N - k}{1 - p(Z)} \right] \frac{\partial}{\partial \rho} p(Z). \end{aligned}$$

Case 2:  $k = 0$

$$\begin{aligned} \frac{\partial}{\partial \rho} \mathbb{E}\left(1_{\{L = \frac{k}{N}\}} \mid Z\right) &= \frac{\partial}{\partial \rho} (1 - p(Z))^N = N(1 - p(Z))^{N-1} \left( -\frac{\partial}{\partial \rho} p(Z) \right) \\ &= -\frac{N}{1 - p(Z)} (1 - p(Z))^N \frac{\partial}{\partial \rho} p(Z) \end{aligned}$$

which is equivalent to the final expression from *Case 1* with  $k = 0$ .

Case 3:  $k = N$

$$\begin{aligned}\frac{\partial}{\partial \rho} \mathbb{E} \left( 1_{\{L=\frac{k}{N}\}} \mid Z \right) &= \frac{\partial}{\partial \rho} p(Z)^N = N p(Z)^{N-1} \frac{\partial}{\partial \rho} p(Z) \\ &= \frac{N}{p(Z)} p(Z)^N \frac{\partial}{\partial \rho} p(Z)\end{aligned}$$

which is equivalent to the final expression from *Case 1* with  $k = N$ .

Hence the general solution, for  $k = 0, \dots, N$ , can be expressed as:

$$\frac{\partial}{\partial \rho} \mathbb{E} \left( 1_{\{L=\frac{k}{N}\}} \mid Z \right) = \mathbb{E} \left( 1_{\{L=\frac{k}{N}\}} \mid Z \right) \left[ \frac{k}{p(Z)} - \frac{N-k}{1-p(Z)} \right] \frac{\partial}{\partial \rho} p(Z)$$

where  $p(Z)$ ,  $\frac{\partial}{\partial \rho} p(Z)$ , and  $\mathbb{E} \left( 1_{\{L=\frac{k}{N}\}} \mid Z \right)$  are solutions to *Problem 2*, *4*, and *5* respectively.  $\square$

**Problem 7.** Show first that for a random variable  $Y$  taking values in  $\mathbb{N}_0$

$$\mathbb{E}(\min(Y, k)) = k + \sum_{j=0}^{k-1} (j - k) \mathbb{P}(Y = j) \quad (4)$$

holds for fixed  $k \in \mathbb{N}$ .

*Solution.* Since  $\min(Y, k) = Y \cdot 1_{\{Y < k\}} + k \cdot 1_{\{Y \geq k\}}$ ,

$$\begin{aligned}\mathbb{E}(\min(Y, k)) &= \mathbb{E}(Y \cdot 1_{\{Y < k\}}) + \mathbb{E}(k \cdot 1_{\{Y \geq k\}}) \\ &= \left( \sum_{j=0}^{k-1} j \cdot \mathbb{P}(Y = j) \right) + k \cdot \mathbb{E}[1_{\{Y \geq k\}}] = \left( \sum_{j=0}^{k-1} j \cdot \mathbb{P}(Y = j) \right) + k \cdot (1 - \mathbb{E}[1_{\{Y < k\}}]) \\ &= \left( \sum_{j=0}^{k-1} j \cdot \mathbb{P}(Y = j) \right) + k \cdot \left( 1 - \sum_{j=0}^{k-1} \mathbb{P}(Y = j) \right) \\ &= k + \sum_{j=0}^{k-1} (j - k) \mathbb{P}(Y = j).\end{aligned}$$

$\square$

**Problem 8.** Fixing the parameters  $p = 0.05$ ,  $K = 0.08$ , and  $N = 100$ , write Python code to calculate  $\ell(K; \rho)$  by simulating 5000 outcomes of  $Z$ , leveraging formula (4). Provide the code and plot  $\ell(K; \rho)$  for  $\rho \in [0, 1]$ . For  $\rho = 0$  the result can be calculated exactly, provide this exact result.

*Solution.* Since  $\mathbb{E}(\mathbb{E}(\min(L, K) \mid Z)) = \frac{1}{N} \mathbb{E}(\mathbb{E}(\min(NL, NK) \mid Z))$ , we are interested in determining an expression for  $\mathbb{E}(\min(NL, NK) \mid Z)$ . Invoking the factorization lemma and using the result from *Problem 7*, for a given value of  $Z = z$

$$\begin{aligned}g(Z = z) &= \mathbb{E}(\min(NL, NK) \mid Z = z) = \mathbb{E}(\min(NL, NK) \mid Z = z) \\ &= \mathbb{E} \left( \min \left( \sum_{i=0}^N 1_{D_i}, NK \right) \mid Z = z \right) = \mathbb{E} \left( \min \left( \sum_{i=0}^N I_i, NK \right) \right) \\ &= NK + \sum_{j=0}^{NK-1} (j - NK) \mathbb{P} \left( \sum_{i=0}^N I_i = j \right) \\ &= NK + \sum_{j=0}^{NK-1} (j - NK) \binom{N}{j} p(Z = z)^j (1 - p(Z = z))^{N-j} \\ &= NK + \sum_{j=0}^{NK-1} (j - NK) \mathbb{E} \left( 1_{\{L=\frac{j}{N}\}} \mid Z = z \right)\end{aligned}$$

where  $I_i = 1_{\{X_i \leq \frac{-\sqrt{\rho}z + \Phi^{-1}(p)}{\sqrt{1-\rho}}\}}$  as defined in *Problem 5*. Hence,

$$\mathbb{E}(\min(NL, NK) \mid Z) = g(Z) = NK + \sum_{j=0}^{NK-1} (j - NK) \mathbb{E} \left( 1_{\{L = \frac{j}{N}\}} \mid Z \right)$$

where  $\mathbb{E} \left( 1_{\{L = \frac{j}{N}\}} \mid Z \right)$  is the solution to *Problem 5* with  $k = j$ .

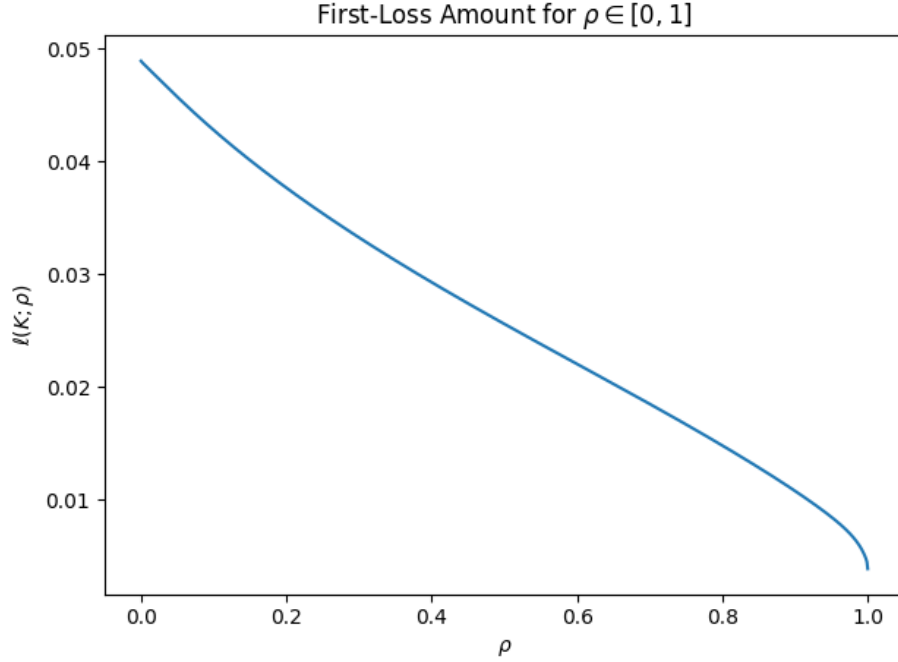


Figure 1: first-loss amount  $\ell(K; \rho)$  for  $\rho \in [0, 1]$ , calculated by simulating 5000 outcomes of  $Z$

For the special case when  $\rho = 0$ ,

$$\mathbb{E} \left( 1_{\{L = \frac{j}{N}\}} \mid Z \right) = \binom{N}{j} \Phi \left( \Phi^{-1}(p) \right)^j \left( 1 - \Phi \left( \Phi^{-1}(p) \right) \right)^{N-j} = \binom{N}{j} p^j (1-p)^{N-j}$$

and the first-loss amount when  $\rho = 0$  is:

$$\begin{aligned} \ell(K; \rho = 0) &= \mathbb{E}(\mathbb{E}(\min(L, K) \mid Z)) = \frac{1}{N} \mathbb{E}(\mathbb{E}(\min(NL, NK) \mid Z)) \\ &= \frac{1}{N} \mathbb{E} \left( NK + \sum_{j=0}^{NK-1} (j - NK) \mathbb{E} \left( 1_{\{L = \frac{j}{N}\}} \mid Z \right) \right) \\ &= \frac{1}{N} \mathbb{E} \left( NK + \sum_{j=0}^{NK-1} (j - NK) \binom{N}{j} p^j (1-p)^{N-j} \right) \\ &= K + \frac{1}{N} \sum_{j=0}^{NK-1} (j - NK) \binom{N}{j} p^j (1-p)^{N-j}. \end{aligned}$$

Using Python, the exact value calculated analytically, up to 6 decimal places, is  $\ell(K; \rho = 0) = 0.048909$ .  $\square$

**Problem 9.** Extend the previous simulation result to also include the *correlation risk*  $\frac{\partial}{\partial \rho} \ell(K; \rho)$  in the simulation, utilising

$$\frac{\partial}{\partial \rho} \ell(K; \rho) = \mathbb{E} \left( \frac{\partial}{\partial \rho} \mathbb{E}(\min(L, K) \mid Z) \right)$$

and using the explicit form established in (3). Plot the correlation risk for  $\rho \in [0, 1]$ .

*Solution.*

$$\begin{aligned} \frac{\partial}{\partial \rho} \ell(K; \rho) &= \mathbb{E} \left( \frac{\partial}{\partial \rho} \mathbb{E}(\min(L, K) \mid Z) \right) = \frac{1}{N} \mathbb{E} \left( \frac{\partial}{\partial \rho} \mathbb{E}(\min(NL, NK) \mid Z) \right) \\ &= \frac{1}{N} \mathbb{E} \left[ \frac{\partial}{\partial \rho} \left( NK + \sum_{j=0}^{NK-1} (j - NK) \mathbb{E} \left( 1_{\{L = \frac{j}{N}\}} \mid Z \right) \right) \right] \\ &= \frac{1}{N} \mathbb{E} \left[ \sum_{j=0}^{NK-1} (j - NK) \frac{\partial}{\partial \rho} \mathbb{E} \left( 1_{\{L = \frac{j}{N}\}} \mid Z \right) \right] \end{aligned}$$

where  $\frac{\partial}{\partial \rho} \mathbb{E} \left( 1_{\{L = \frac{j}{N}\}} \mid Z \right)$  is the solution to Problem 6 with  $k = j$ .

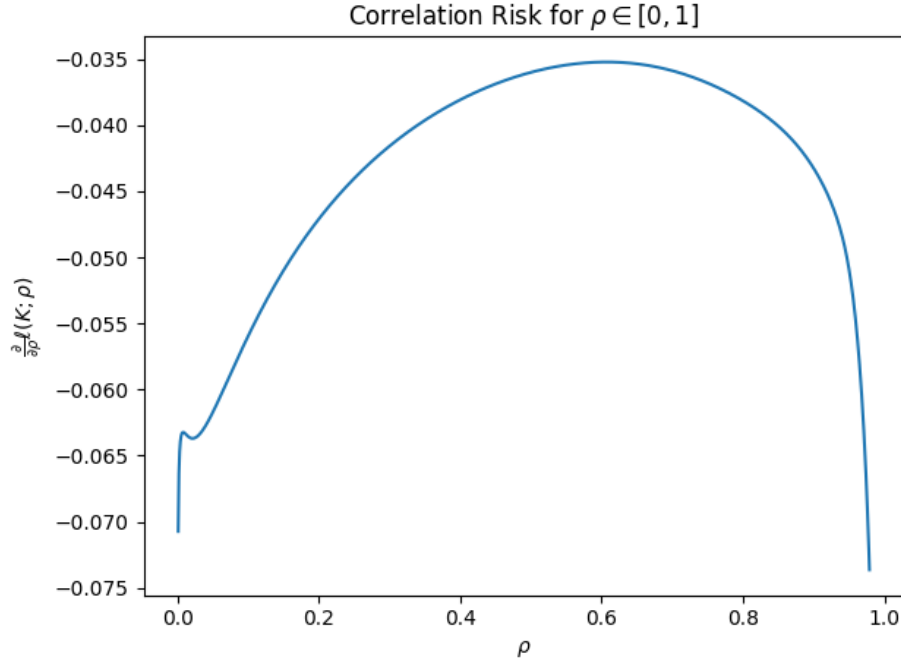


Figure 2: correlation risk value  $\frac{\partial}{\partial \rho} \ell(K; \rho)$  for  $\rho \in [0, 1]$ , calculated by simulating 5000 outcomes of  $Z$

□

**Problem 10.** Finally, we are aiming to validate the prior result for the correlation risk against re-calculating the value under a small perturbation of  $\rho$ , i.e.  $\frac{\partial}{\partial \rho} \ell(K; \rho) \approx \frac{1}{\epsilon} (\ell(K; \rho + \epsilon) - \ell(K; \rho))$  for small values of  $\epsilon$ . Extend your Monte Carlo simulation to include this “bump-and-reval” correlation risk and include this result in the plot as a comparison against the previously calculated risk.

*Solution.* Using  $\epsilon = 10^{-6}$  as the perturbation, the following is the “bump-and-reval” simulation result.

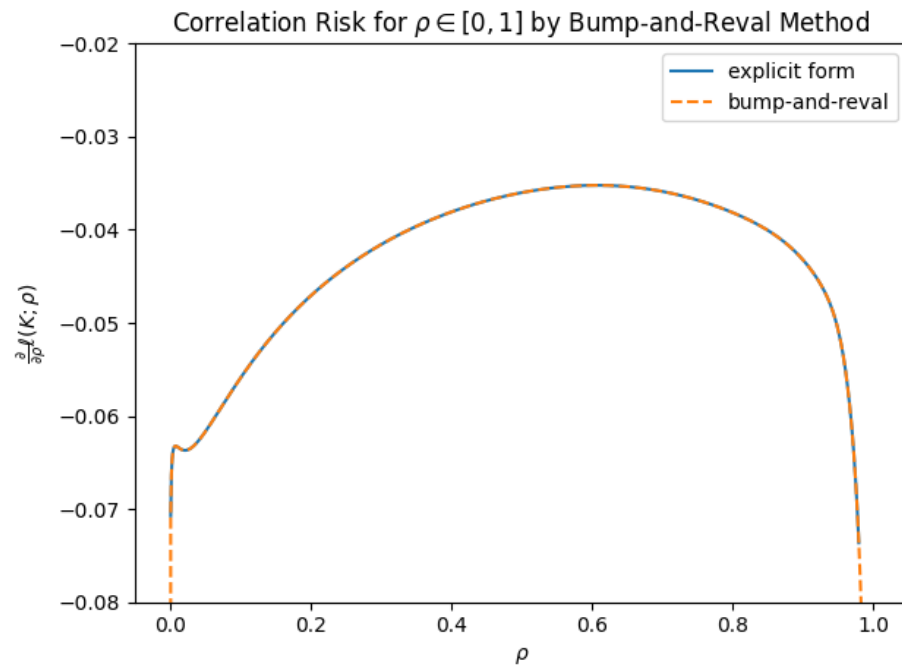


Figure 3: correlation risk value  $\frac{\partial}{\partial \rho} \ell(K; \rho)$  for  $\rho \in [0, 1]$ , calculated using “bump-and-reval” method

□