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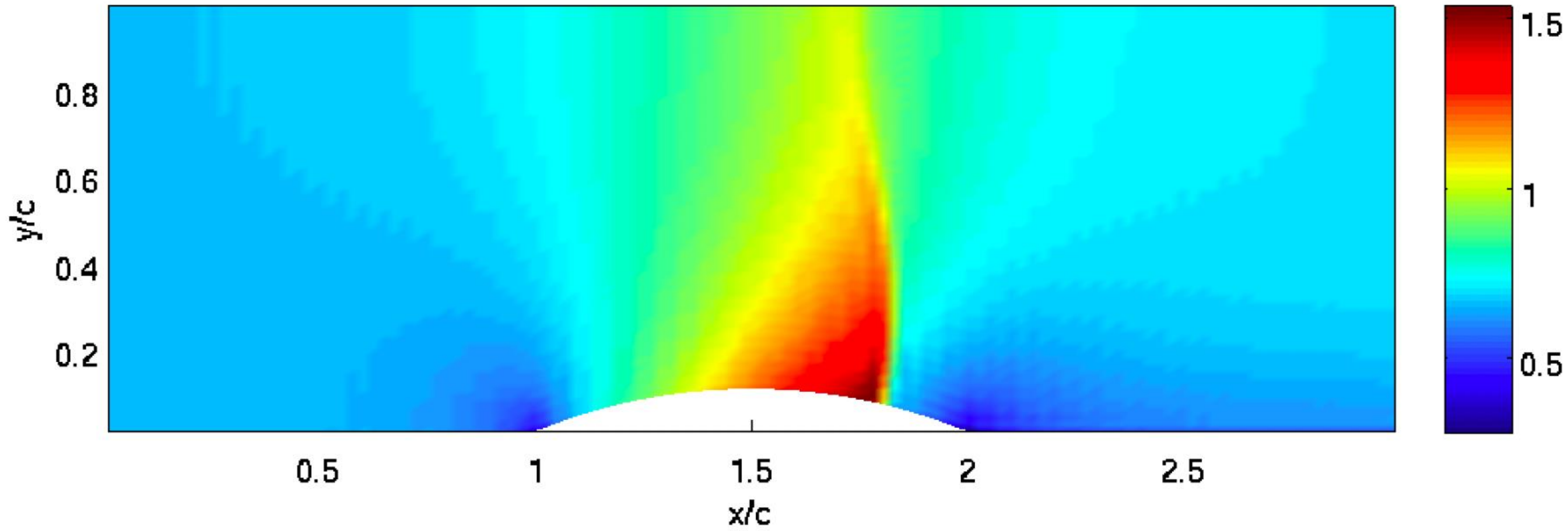
Numerische Methoden der Strömungsmechanik

Intro to compressible flows



Equations of compressible flow

- In a previous class, we already saw the equations for a Newtonian, perfect gas

$$
\boxed{
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \\
&\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho \boldsymbol{f_e} + \nabla \cdot \left(-p\bar{\bar{I}} + \bar{\bar{\tau}}\right) \\
&\frac{\partial \rho E}{\partial t} + \nabla \cdot \left(\rho H \boldsymbol{v} - k\nabla T - \bar{\bar{\tau}} \cdot \boldsymbol{v}\right) = W_f + q_H
\end{aligned}
}
$$

Newtonian

$$
\boxed{\tau_{ij} = \mu \left[\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} (\nabla \cdot \boldsymbol{v}) \delta_{ij} \right]}
$$

$$
h = e + \frac{p}{\rho}
$$

$$
H = h + \frac{v^2}{2} = e + \frac{p}{\rho} + \frac{v^2}{2} = E + \frac{p}{\rho}
$$

$$
W_f = \rho \boldsymbol{f_e} \cdot \boldsymbol{v}
$$

perfect gas

$$
\boxed{
\begin{aligned}
p &= \rho R T \\
e &= c_v T \\
h &= c_p T
\end{aligned}
}
$$

$$
\mu = \mu(T)\,, k = k(T)
$$

$$
R = c_p - c_v
$$

$$
\gamma = c_p / c_v
$$

Equations of compressible flow

- That we can rewrite

$$\boldsymbol{F} = (f, g, h)$$

$$\frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{F} = Q \longrightarrow \frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = Q$$

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{vmatrix} \qquad Q = \begin{vmatrix} 0 \\ \rho \left.\boldsymbol{f_e}\right|_x \\ \rho \left.\boldsymbol{f_e}\right|_y \\ \rho \left.\boldsymbol{f_e}\right|_z \\ W_f + q_H \end{vmatrix}$$

$$f = \begin{vmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho uv - \tau_{xy} \\ \rho uw - \tau_{xz} \\ \rho uH - \left(\overline{\overline{\tau}} \cdot \boldsymbol{v}\right)_x - k\partial_x T \end{vmatrix} g = \begin{vmatrix} \rho v \\ \rho uv - \tau_{yx} \\ \rho v^2 + p - \tau_{yy} \\ \rho vw - \tau_{yz} \\ \rho vH - \left(\overline{\overline{\tau}} \cdot \boldsymbol{v}\right)_y - k\partial_y T \end{vmatrix} h = \begin{vmatrix} \rho w \\ \rho uw - \tau_{zx} \\ \rho vw - \tau_{zy} \\ \rho w^2 + p - \tau_{zz} \\ \rho wH - \left(\overline{\overline{\tau}} \cdot \boldsymbol{v}\right)_z - k\partial_z T \end{vmatrix}$$

Equations of compressible flow

- The most general description of a single-phase fluid flow is given by the Navier-Stokes equations
- In 3D space, it is a system of 5 fully coupled time-dependent **partial differential equations**
- The equations are **non-linear**
- Dominant non-linearity is provided by the advection term in momentum equation → **turbulence** (we will study at the end of the course)
- In compressible flow, other non-linear effects like **shock waves in supersonic flows**
- Through a shock, pressure, velocity and temperature undergo a discontinuous jump
- In some cases, other approximations can be used →

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The Euler equations

- For flows at high Reynolds numbers outside viscous regions developing near solid surfaces → neglect viscous effects and heat conduction effects.

- This leads to an **inviscid model** known as the Euler equations

$$\frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{F} = Q \qquad \frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = Q$$

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{vmatrix} \qquad Q = \begin{vmatrix} 0 \\ \rho \, \boldsymbol{f_e}\big|_x \\ \rho \, \boldsymbol{f_e}\big|_y \\ \rho \, \boldsymbol{f_e}\big|_z \\ W_f + q_H \end{vmatrix}$$

$$f = \begin{vmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho u v - \tau_{xy} \\ \rho u w - \tau_{xz} \\ \rho u H - \left(\overline{\overline{\tau}} \cdot \boldsymbol{v}\right)_x - k\partial_x T \end{vmatrix} g = \begin{vmatrix} \rho v \\ \rho u v - \tau_{yx} \\ \rho v^2 + p - \tau_{yy} \\ \rho v w - \tau_{yz} \\ \rho v H - \left(\overline{\overline{\tau}} \cdot \boldsymbol{v}\right)_y - k\partial_y T \end{vmatrix} h = \begin{vmatrix} \rho w \\ \rho u w - \tau_{zx} \\ \rho v w - \tau_{zy} \\ \rho w^2 + p - \tau_{zz} \\ \rho w H - \left(\overline{\overline{\tau}} \cdot \boldsymbol{v}\right)_z - k\partial_z T \end{vmatrix}$$

The Euler equations

- The set of Euler eqns describe a non-viscous non-heat-conducting fluid

$$\frac{\partial U}{\partial t}+\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}=Q$$

$$f=\begin{vmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ \rho uH\end{vmatrix}\qquad g=\begin{vmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ \rho vH\end{vmatrix}\qquad h=\begin{vmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ \rho wH\end{vmatrix}$$

- The entropy equation in this approximation has a particularly simple form

$$\frac{\partial s}{\partial t}+\boldsymbol{v}\cdot\nabla s=0$$

- The entropy is constant along streamlines.
- In absence of discontinuities → the equations describe isentropic flows
- However, the Euler equations allow discontinuous solutions!
- These are: vortex sheets, contact discontinuities and **shock waves**

The Euler equations: discontinuous solutions

- The properties of the discontinuous solutions are analyzed from the integral form of the equations

- In the differential form there are gradients of the fluxes, undefined at discontinuities

$$\frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{F} = Q$$

- In the integral form, there are no gradients.

$$\frac{\partial}{\partial t} \int_{\Omega} U \, \mathrm{d}\Omega + \oint_{S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_{\Omega} Q \, \mathrm{d}\Omega$$

- Let's analyze the equations, ignoring the source terms, in 2D for simplicity

$$\frac{\partial}{\partial t} \int_{\Omega} U \, \mathrm{d}\Omega + \oint_{S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = 0$$

$$\boldsymbol{F} = (f, g)$$

$$f = \begin{vmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{vmatrix} \quad g = \begin{vmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{vmatrix} \quad U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{vmatrix}$$

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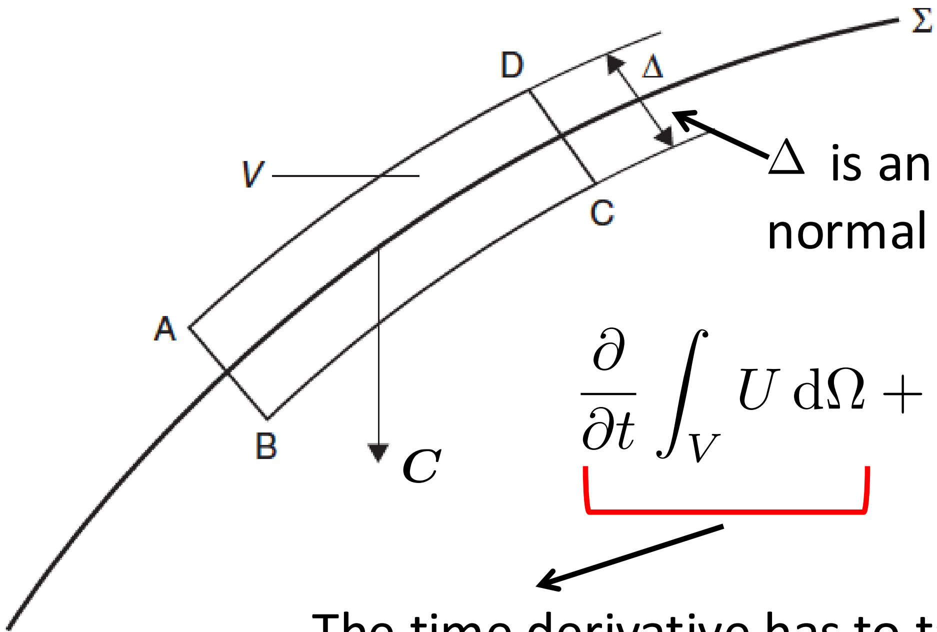
The Euler equations: discontinuous solutions

- Consider a control volume around a moving discontinuity surface Σ that moves with velocity $\boldsymbol{C}$ See Hirsch book p. 551-552 for mathematical details



Δ is an infinitesimal distance normal to Σ

$$\underbrace{\frac{\partial}{\partial t}\int_V U\,\mathrm{d}\Omega}\,+\oint_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}=0$$

The time derivative has to take into account the motion of the surface Σ and hence of the control volume

$$\frac{\partial}{\partial t}\int_V U\,\mathrm{d}\Omega=\int_V \frac{\partial U}{\partial t}\,\mathrm{d}\Omega+\oint_S U\boldsymbol{C}\cdot\mathrm{d}\boldsymbol{S}$$

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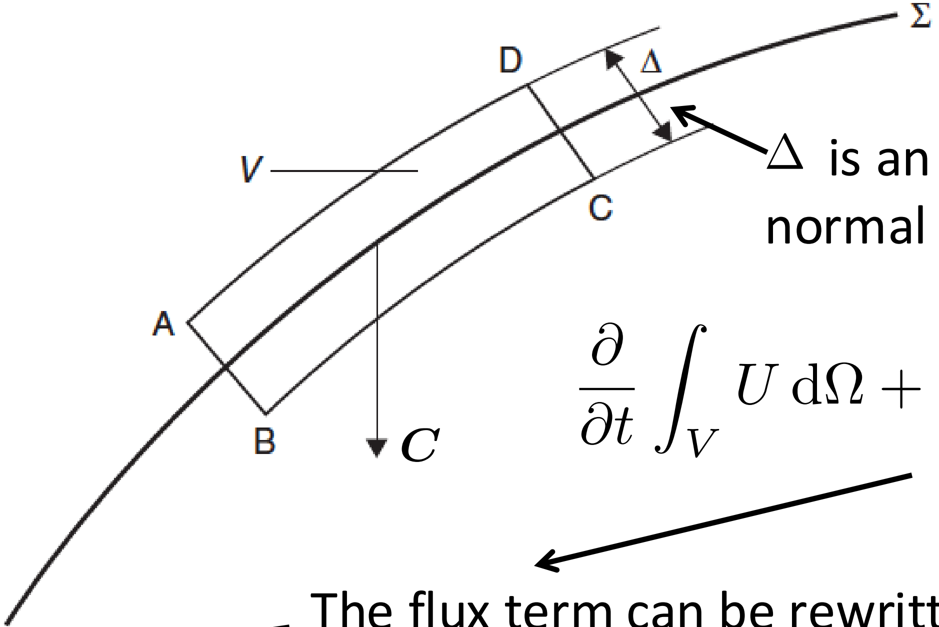
The Euler equations: discontinuous solutions

- Consider a control volume around a moving discontinuity surface Σ that moves with velocity C



Δ is an infinitesimal distance normal to Σ

$$\frac{\partial}{\partial t}\int_V U \,\mathrm{d}\Omega + \underbrace{\oint_S \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S}}_{} = 0$$

The flux term can be rewritten in the limit $\Delta \to 0$

$$\oint_S \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S} = \int_\Sigma (\boldsymbol{F}_{BC} - \boldsymbol{F}_{AD})\cdot \boldsymbol{n}\mathrm{d}\Sigma = \int_\Sigma [\boldsymbol{F}\cdot\boldsymbol{n}]\mathrm{d}\Sigma$$

The notation [A] indicates the jump in the variable A across the discontinuity

The Euler equations: discontinuous solutions

- In the limit $\Delta \to 0$, then $V \to 0$

$$\frac{\partial}{\partial t}\int_V U \,\mathrm{d}\Omega = \cancelto{0}{\int_V \frac{\partial U}{\partial t}\,\mathrm{d}\Omega} + \oint_S U\boldsymbol{C}\cdot \mathrm{d}\boldsymbol{S}$$

- This leads to

$$\int_\Sigma ([\boldsymbol{F}] + \boldsymbol{C}[U])\cdot \boldsymbol{n}\mathrm{d}\Sigma = 0$$

- And we obtain the local form of the conservation laws over a discontinuity, called the Rankine-Hugoniot relations

$$[\boldsymbol{F}]\cdot\boldsymbol{n} + \boldsymbol{C}[U]\cdot\boldsymbol{n} = 0$$

- If we change the reference frame, to a new one that moves with the discontinuity →

$$[\boldsymbol{F}]\cdot\boldsymbol{n} = 0$$

- Various forms of discontinuities are physically possible
  - **Shocks** (with non-zero mass flow across the discontinuity)
  - Contact discontinuities and vortex sheets (without mass flow)

The Euler equations: discontinuous solutions

- In the moving reference frame moving, the Rankine-Hugoniot relations for the Euler equations are

$$[\boldsymbol{F}]\cdot\boldsymbol{n} = 0 \qquad \boldsymbol{F} = (f, g) \qquad f = \begin{vmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{vmatrix} \qquad g = \begin{vmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{vmatrix}$$

$$\begin{aligned} [\rho v_n] &= 0 \\ [\rho v_n \boldsymbol{v}] + [p]\boldsymbol{n} &= 0 \\ \rho v_n [H] &= 0 \end{aligned} \qquad \text{with } v_n = \boldsymbol{v}\cdot\boldsymbol{n}$$

The total enthalpy always remains constant through the discontinuity

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The Euler equations: discontinuous solutions

- Contact discontinuities

- The condition of no mass flow through the discontinuity leads to:

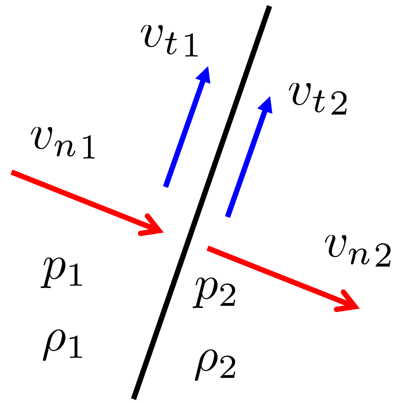
$$v_{n1} = v_{n2} = 0$$

- From the 2nd R-H relation, it follows that

$$p_1 = p_2 \quad \text{no pressure jump}$$

- From the 1st R-H relation, there is no restriction on density jump

$$[\rho] \neq 0$$



Rankine-Hugoniot

$$[\rho v_n] = 0$$
$$[\rho v_n \boldsymbol{v}] + [p]\boldsymbol{n} = 0$$
$$\rho v_n [H] = 0$$

- Projection of 2nd relation along tangential direction → no restriction on v_t

- A contact discontinuity is defined by the condition: $[v_t] = 0$

The Euler equations: discontinuous solutions

- Vortex sheets

- Also condition of no mass flow through the discontinuity:

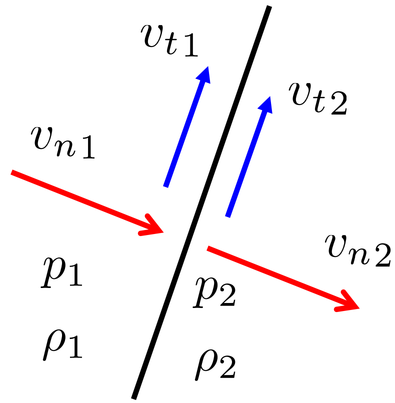
$$v_{n\,1} = v_{n\,2} = 0$$

- Therefore, it is also true that

$$p_1 = p_2 \quad \text{no pressure jump}$$

$$[\rho] \neq 0$$

- Vortex sheets are defined by the condition $[v_t] \neq 0$



Rankine-Hugoniot

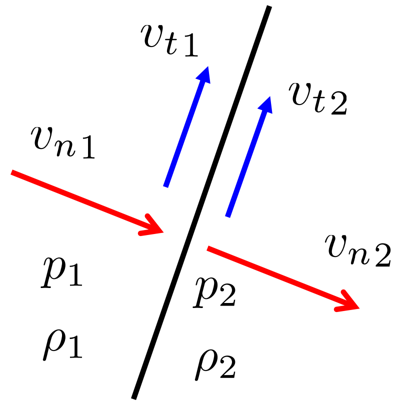
$$[\rho v_n] = 0$$

$$[\rho v_n \boldsymbol{v}] + [p]\boldsymbol{n} = 0$$

$$\rho v_n [H] = 0$$

The Euler equations: discontinuous solutions

- Shocks
- Shocks are solutions with non-zero mass flow through the discontinuity; they appear in supersonic flows.
- Pressure and normal velocity undergo discontinuous variations
- Tangential velocity remains continuous
- Conditions are: $$[v_n] \neq 0$$ $$[p] \neq 0$$ $$[\rho] \neq 0$$ $$[v_t] = 0$$
- Stagnation pressure is not constant across the shock
- This implies that across a shock there is an entropy jump



Rankine-Hugoniot

$$[\rho v_n] = 0$$
$$[\rho v_n \boldsymbol{v}] + [p]\boldsymbol{n} = 0$$
$$\rho v_n [H] = 0$$

The Euler equations: discontinuous solutions

- Shocks

- Both compression shocks (positive entropy jump) and expansion shocks (negative entropy jump) are mathematical solutions

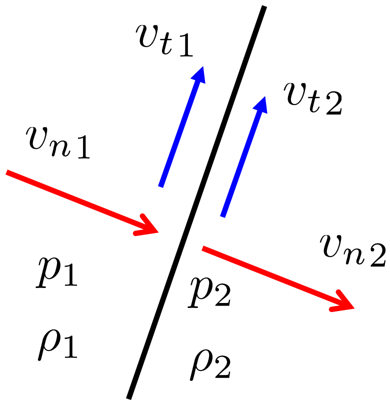
- However, **only compression shocks are physical**. This is connected to the 2^{nd} law of thermodynamics

- There is no mathematical mechanism to distinguish discontinuities with positive or negative entropy jump → sometimes an additional condition is required

- This is called an **entropy condition**.

- Any solution of the Euler equation has to satisfy the inequality:

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s \geq 0$$



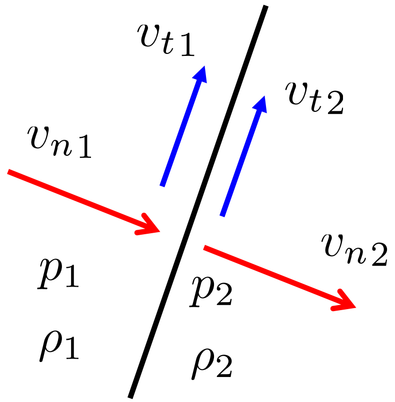
Rankine-Hugoniot

$$[\rho v_n] = 0$$
$$[\rho v_n \boldsymbol{v}] + [p]\boldsymbol{n} = 0$$
$$\rho v_n [H] = 0$$

The Euler equations: discontinuous solutions

- Summary discontinuous solutions

Contact discontinuity	Vortex sheets	Shocks
$v_{n1} = v_{n2} = 0$	$v_{n1} = v_{n2} = 0$	$[v_n] \neq 0$
$[p] = 0$	$[p] = 0$	$[p] \neq 0$
$[\rho] \neq 0$	$[\rho] \neq 0$	$[\rho] \neq 0$
$[v_t] = 0$	$[v_t] \neq 0$	$[v_t] = 0$



Rankine-Hugoniot

$$\begin{aligned} [\rho v_n] &= 0 \\ [\rho v_n \boldsymbol{v}] + [p]\boldsymbol{n} &= 0 \\ \rho v_n [H] &= 0 \end{aligned}$$

These expressions are valid in a reference frame that is moving with the discontinuity!

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The 1D Euler equations: characteristics

- The Euler equations are hyperbolic
- They are dominated by advection, since diffusion has been neglected
- We may try to apply the knowledge we acquired using the advection equation and the **Burgers equation**.
- The 1D Euler equations can be written as $\dfrac{\partial U}{\partial t} + \dfrac{\partial f}{\partial x} = 0$

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho E \end{vmatrix} \qquad f = \begin{vmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{vmatrix}$$

- The third equation can be substituted by the entropy equation $\dfrac{\partial s}{\partial t} + u\dfrac{\partial s}{\partial x} = 0$
- Recall also the thermodynamic definition of the speed of sound:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \quad \rightarrow \quad c^2 = \gamma p/\rho = \gamma RT \quad \text{[perfect gas]}$$

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The 1D Euler equations: characteristics

- Recall that for the advection eq. $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$
- Information propagates along the characteristic lines: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = a$
- This means that *u* is constant along the characteristics lines
- For the **Burgers** equation $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$ characteristics lines are $\dfrac{\mathrm{d}x}{\mathrm{d}t} = u$
- We discussed that the non-linearity may lead to discontinuities
- Now, it is possible to re-write the Euler equations in characteristic form, where

$$
\begin{aligned}
s \quad & \text{conserved along characteristic lines} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u \\
R^+ = u + \frac{2c}{\gamma - 1} \quad & \text{conserved along characteristic lines} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u + c \\
R^- = u - \frac{2c}{\gamma - 1} \quad & \text{conserved along characteristic lines} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u - c
\end{aligned}
$$

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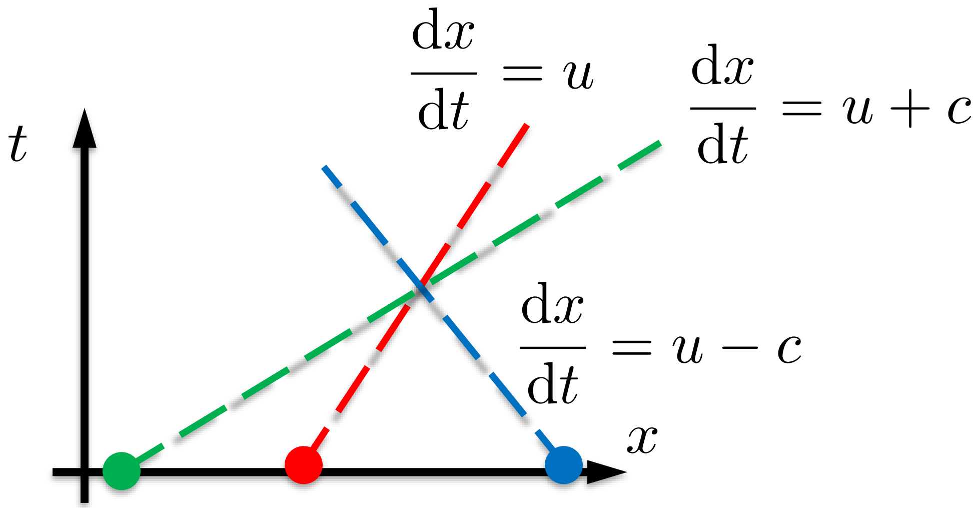
The 1D Euler equations: characteristics

- For a subsonic flow, *u<c*, information travels to the right and to the left simultaneously



- If we want to use an upwind method to solve the equations, it is not so easy!

- There are methods that split the flux into right-travelling and left-travelling information → flux-vector splitting methods

$$f = f^{+} + f^{-}$$

- Approximate f^{+} with backward derivatives and f^{-} with forward derivatives

The 1D Euler equations: numerical solution

- There are other alternatives, like the MacCormack scheme

$$U_j^* = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_j^n\right)$$
$$U_j^{n+1} = \frac{1}{2}\left(U_j^n + U_j^*\right) - \frac{\Delta t}{2\Delta x}\left(F_j^* - F_{j-1}^*\right)$$

- or the two-stage Lax-Wendroff

$$U_{j+1/2}^* = \frac{1}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_j^n\right)$$
$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^* - F_{j-1/2}^*\right)$$

- Both methods are stable under the CFL condition, the advection speed needs to be determined using the maximum propagation speed $|u| + c$

$$\frac{(|u| + c)\Delta t}{\Delta x} \leq 1$$

- However, if discontinuities develop, both schemes will develop oscillations → additional diffusion needs to be added near the discontinuity as discussed for the advection equation.

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The 1D Euler equations: numerical solution

- Artificial dissipation methods

- We use central derivatives for the flux term and we add a diffusive term to the equations

- For example, if we use 2nd order central differences the truncation error of the scheme is $O(\Delta x^2)$

- Then we add a term, with a lower error so that this does not affect the accuracy of the scheme like for example:

$$\gamma \Delta x^3 \left(\frac{\partial^4 U}{\partial x^4} \right)_i$$

- Again, if there are discontinuities this is not enough and we need to add more diffusion near the discontinuity

- A popular scheme was designed by Jameson-Schmidt-Turkel (JST)

The 1D Euler equations: numerical solution

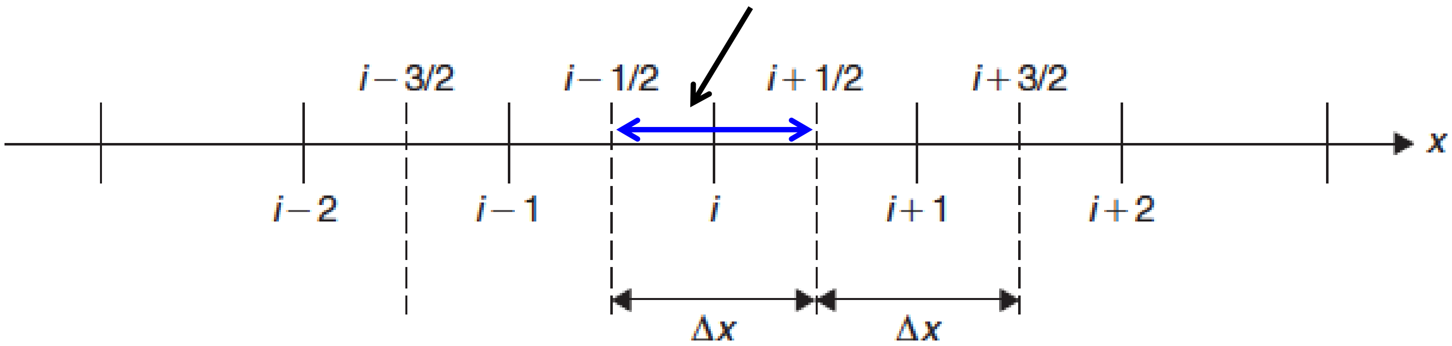
- The JST scheme [More details in Computational Aerodynamics by Jameson, 2022]

- It blends a high and a low diffusion term → near the discontinuity the method is first order and far from it the method remains second order accurate

- A switch is needed to "detect" the discontinuity

- For example, if we solve the 1D equations using a finite volume method

$$\text{The } i\text{-th cell is denoted } \mathcal{C}_i = [x_{i-i/2}, x_{i+1/2}]$$



$$\Delta x \frac{\mathrm{d}U_i}{\mathrm{d}t} + F_{j+1/2} - F_{j-1/2} = 0$$

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The 1D Euler equations: numerical solution

- The JST scheme $$\Delta x \frac{\mathrm{d}U_i}{\mathrm{d}t} + F_{j+1/2} - F_{j-1/2} = 0$$

- The numerical flux is defined as

$$F_{j+1/2} = \underbrace{\frac{1}{2}(F_j + F_{j+1})}_{\text{2nd order central approximation}} - \underbrace{D_{j+1/2}}_{\text{Artificial dissipation term}}$$

- The latter term has the form

$$\Delta U_{j+1/2} = U_{j+1} - U_j$$

$$D_{j+1/2} = \epsilon^{(2)}_{j+1/2}\Delta U_{j+1/2} - \epsilon^{(4)}_{j+1/2}(\Delta U_{j+3/2} - 2\Delta U_{j+1/2} + \Delta U_{j-1/2})$$

needed to prevent oscillations near shocks

has to be switched off far from shocks to preserve accuracy

needed to obtain consistent convergence to steady state

has to be switched off near shocks

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The 1D Euler equations: numerical solution

- The JST scheme

$$\Delta x \frac{\mathrm{d}U_i}{\mathrm{d}t} + F_{j+1/2} - F_{j-1/2} = 0 \qquad F_{j+1/2} = \frac{1}{2}(F_j + F_{j+1}) - D_{j+1/2}$$

$$D_{j+1/2} = \epsilon^{(2)}_{j+1/2}\Delta U_{j+1/2} - \epsilon^{(4)}_{j+1/2}(\Delta U_{j+3/2} - 2\Delta U_{j+1/2} + \Delta U_{j-1/2})$$

- In order to detect the singularity, a pressure sensor is employed:

$$s_i = \left|\frac{p_{i+1} - 2p_i + p_{i-1}}{p_{i+1} + 2p_i + p_{i-1}}\right| \quad \rightarrow s_i \leq 1 \quad \text{and} \quad s_i \sim O(\Delta x^2)\ [\text{in a smooth region}]$$

- The sensor at the interface is calculated → $s_{i+1/2} = \max(s_{i+2}, s_{i+1}, s_i, s_{i-1})$

- We then set

$$\epsilon^{(2)}_{i+1/2} = \min\left(\frac{1}{2}, k^{(2)} s_{i+1/2}\right) r_{i+1/2} \qquad r_{i+1/2} = (|u| + c)_{i+1/2}$$

$$\epsilon^{(4)}_{i+1/2} = \max\left(0, k^{(4)} - 2s_{i+1/2}\right) r_{i+1/2} \qquad k^{(2)} = 1, k^{(4)} = 1/32$$

- Both diffusive terms have a magnitude proportional to Δx^3 in smooth regions

- $k^{(2)} = 1$ is a compromise between accuracy and robustness
- $k^{(4)}$ can be optimized in conjunction with the time integration scheme

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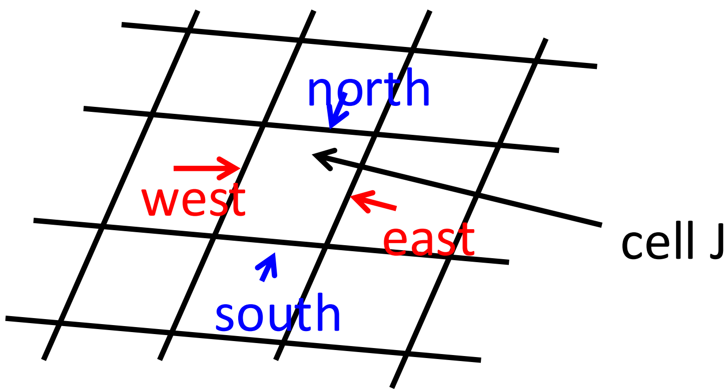
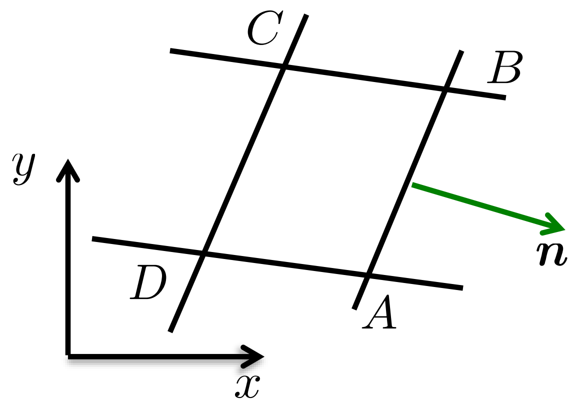
The 2D Euler equations: practical application

- Finally, some hints on the practical application using a finite volume method
- Euler equations can be recast in the form $\dfrac{\partial}{\partial t}\displaystyle\int_{\Omega_J} U \,\mathrm{d}\Omega + \oint_{S_J} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S} = 0$

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{vmatrix} \qquad \boldsymbol{F} = (f, g) \qquad f = \begin{vmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{vmatrix} \quad g = \begin{vmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{vmatrix}$$





$$\boldsymbol{F} = (f, g)$$

$$\mathrm{d}\boldsymbol{S} = \mathrm{d}S\boldsymbol{n} = (\mathrm{d}y, -\mathrm{d}x)$$

$$\oint_{S_J} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S} = \oint_{ABCD} (f\mathrm{d}y - g\mathrm{d}x)$$

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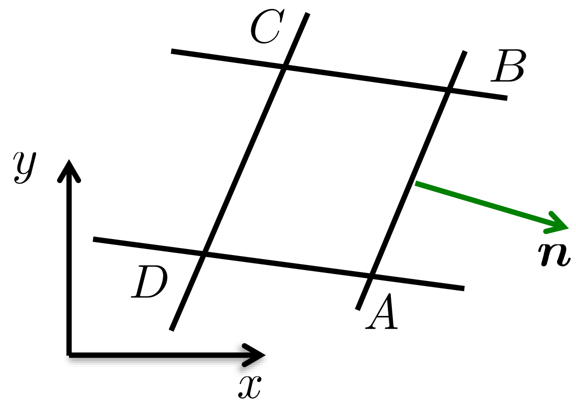
The 2D Euler equations: practical application

- Using a finite volume approach

$$\frac{\partial}{\partial t}\int_{\Omega_J} U\,\mathrm{d}\Omega + \oint_{S_J} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} = 0 \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(U_J\Omega_J) + \sum_{faces} \boldsymbol{F}\cdot\Delta\boldsymbol{S} = 0$$

- Since $\oint_{S_J} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} = \oint_{ABCD} (f\,\mathrm{d}y - g\,\mathrm{d}x)$

$$\begin{aligned}\sum_{ABCD} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} =& f_{AB}(y_B - y_A) - g_{AB}(x_B - x_A) + \\ &+ f_{BC}(y_C - y_B) - g_{BC}(x_C - x_B) + \\ &+ f_{CD}(y_D - y_C) - g_{CD}(x_D - x_C) + \\ &+ f_{DA}(y_A - y_D) - g_{DA}(x_A - x_D)\end{aligned}$$



The 2D Euler equations: practical application

- Using a finite volume approach

$$\frac{\partial}{\partial t}\int_{\Omega_J} U \,\mathrm{d}\Omega + \oint_{S_J} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S} = 0 \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(U_J\Omega_J) + \sum_{faces} \boldsymbol{F}\cdot\Delta\boldsymbol{S} = 0$$

- Since $\displaystyle \oint_{S_J} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S} = \oint_{ABCD} (f\mathrm{d}y - g\mathrm{d}x)$

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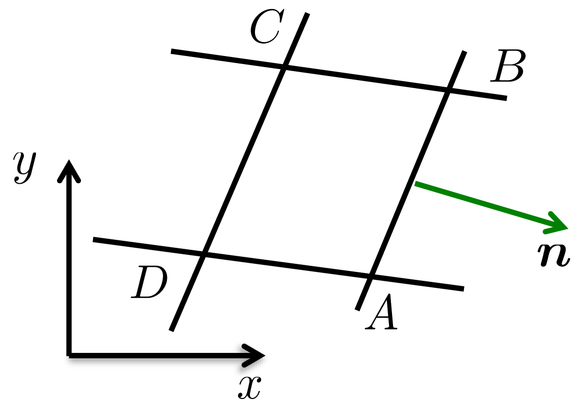

- Compute the fluxes in each face, using a 2D version of the JST scheme for example → details can be found in Hirsch book

The 2D Euler equations: practical application

- After spatial discretization, time integration can be performed with a Runge-Kutta scheme, for example:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U_J\Omega_J) + \sum_{faces} \boldsymbol{F}\cdot\Delta\boldsymbol{S} = 0 \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[U_{i,j}\Omega_{i,j}\right] = -\sum_{\text{faces}} \boldsymbol{F}^*\cdot\boldsymbol{\Delta S} \equiv -R_{i,j}$$

- A recommended R-K method is a 4th order low-storage method:

$$\begin{aligned}
Y_1 &= U^n_{i,j} \\
Y_2 &= U^n_{i,j} - \frac{\Delta t}{\Omega_{i,j}}\alpha_2\, R_{i,j}(Y_1) \\
Y_3 &= U^n_{i,j} - \frac{\Delta t}{\Omega_{i,j}}\alpha_3\, R_{i,j}(Y_2) \\
Y_4 &= U^n_{i,j} - \frac{\Delta t}{\Omega_{i,j}}\alpha_4\, R_{i,j}(Y_3) \\
U^{n+1}_{i,j} &= U^n_{i,j} - \frac{\Delta t}{\Omega_{i,j}}\, R_{i,j}(Y_4)
\end{aligned}$$

Two possible sets of coefficients are:

$$\alpha_2 = \frac{1}{4},\ \alpha_3 = \frac{1}{3},\ \alpha_4 = \frac{1}{2}$$

$$\alpha_2 = \frac{1}{8},\ \alpha_3 = 0.306,\ \alpha_4 = 0.587$$

- A **key issue** is how to impose boundary conditions.

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The 2D Euler equations: boundary conditions

- The time-dependent Euler equations are hyperbolic.
- Therefore, they are dominated by wave propagation and we have seen that information may travel in any direction –>
- How many conditions of physical origin are to be imposed at a given boundary?
- What physical quantities are to be imposed at a boundary?
- How are the remaining variables to be defined at the boundaries?
- We discussed that in 1D there are 3 characteristic lines associated to the three quantities *u*, *u+c* and *u-c*. These are the 3 eigenvalues of the problem.
- In 2D, similarly, 4 eigenvalues are obtained, associated with the corresponding characteristic surfaces. The eigenvalues are

$$
\begin{aligned}
\lambda_1 &= v_n \\
\lambda_2 &= v_n \\
\lambda_3 &= v_n + c \\
\lambda_4 &= v_n - c
\end{aligned}
$$

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The 2D Euler equations: boundary conditions

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\lambda_3 &= v_n + c \\
\lambda_4 &= v_n - c
\end{aligned}
$$

- The transport properties at a face are given by the normal component of the fluxes

- The behaviour of the Euler system of equations will be determined by the propagation of waves with speeds corresponding to the 4 eigenvalues

- This defines quasi-1D propagation→ look at propagation of information at a face

- The first 2 eigenvalues are equal to the normal component of the velocity → correspond to entropy and vorticity waves.
- The 2 remaining eigenvalues correspond to acoustic waves.

The 2D Euler equations: boundary conditions

- When information is propagated from outside toward the inside of the computational domain → this information has to be defined from outside. This is **a physical b.c.**

- When the eigenvalue is positive, the information propagates toward the inside → physical b.c.

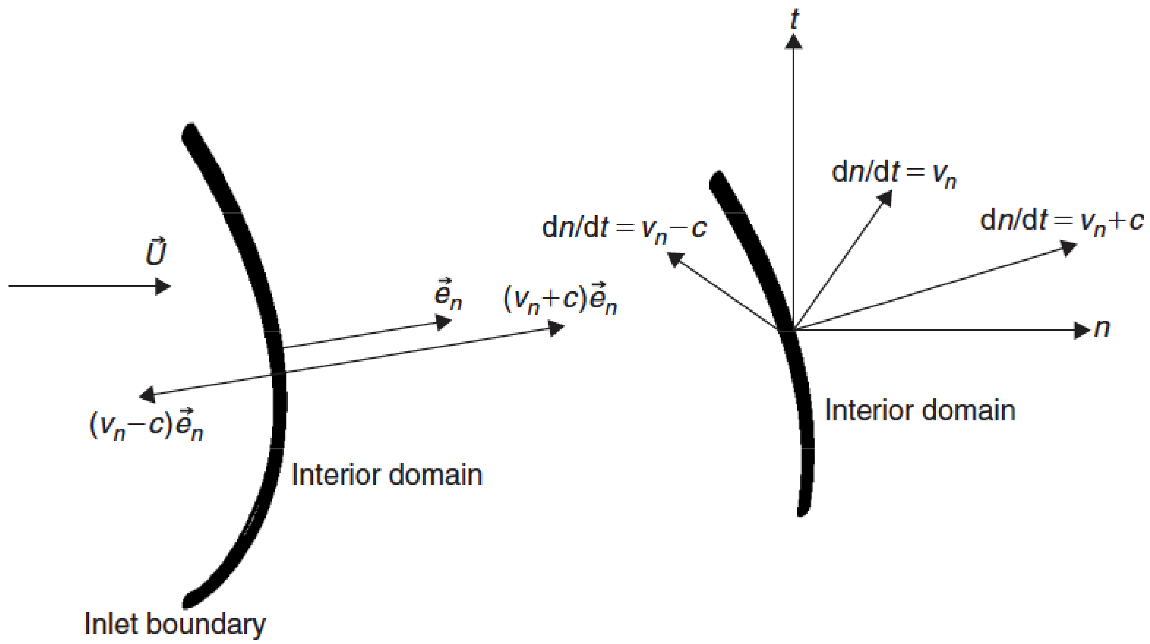
- When the eigenvalue is negative, the information at the boundary is determined from the interior values → **numerical b.c.**

- The number of physical conditions to be imposed at the boundary is defined by the number of characteristics entering the domain.

- Let's clarify with some examples →

The 2D Euler equations: boundary conditions

- Subsonic flow, **inlet** boundary



- 3 eigenvalues are positive, 1 eigenvalue is negative

$$\lambda_1 = v_n$$
$$\lambda_2 = v_n$$
$$\lambda_3 = v_n + c$$
$$\lambda_4 = v_n - c < 0 \quad \text{in subsonic flow}$$

3 quantities will have to be fixed by the physical b.c.'s at the inlet and 1 will have to be fixed by a numerical b.c.

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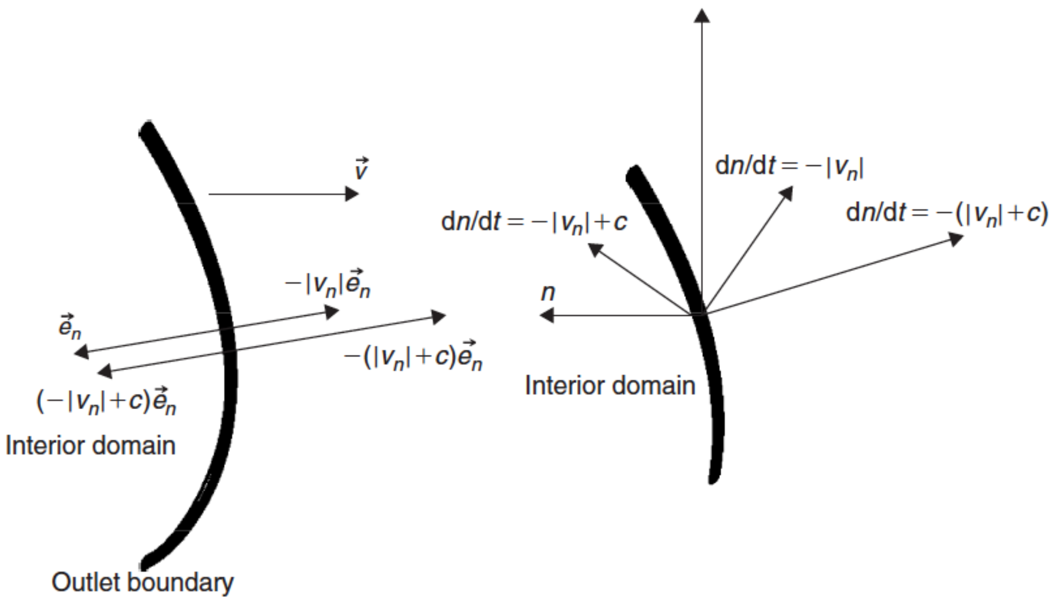
The 2D Euler equations: boundary conditions

- Subsonic flow, **outlet** boundary



- 3 eigenvalues are positive, 1 eigenvalue is negative

$$\lambda_1 = v_n < 0$$
$$\lambda_2 = v_n < 0$$
$$\lambda_3 = v_n + c > 0 \quad \text{subsonic flow}$$
$$\lambda_4 = v_n - c < 0$$

1 quantitiy will have to be fixed by the physical b.c. at the inlet and 3 will have to be fixed by numerical b.c.'s

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The 2D Euler equations: boundary conditions

- This is a very important result
- We cannot fix all four quantities at a subsonic inlet. Only 3 of them

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{vmatrix}$$

- At the subsonic outlet we can only impose 1 physical condition
- The next question is what quantities should be fixed as physical boundary conditions?

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The 2D Euler equations: boundary conditions

- This is a very important result
- We cannot fix all four quantities at a subsonic inlet. Only 3 of them

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{vmatrix}$$

- At the subsonic outlet we can only impose 1 physical condition
- The next question is what quantities should be fixed as physical boundary conditions?
- This question is more tricky than it seems, and has no unique answer
- It forms a complex subject outside of the scope of this brief introduction
- However, some recommendations follow →

The 2D Euler equations: boundary conditions

Inlet boundary

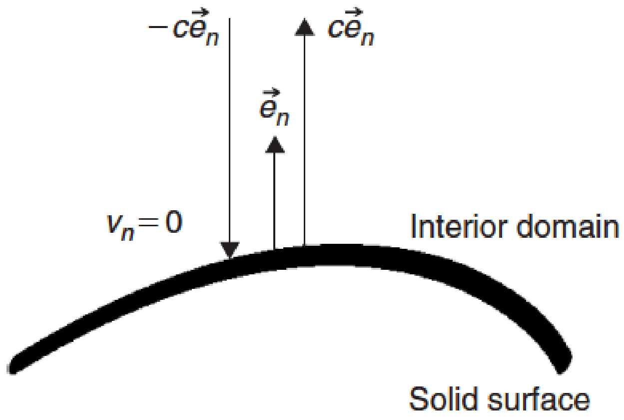
- For an external flow problem, you can fix the inlet velocity and inlet temperature
- For an internal flow problem, it is common practice to specify 2 thermodynamic variables, such as upstream stagnation pressure and temperature, and an inlet Mach number or velocity magnitude. In this case the inlet flow angle is a result of the calculation. Alternatively it is possible to fix the incident flow angle and the inlet Mach is a result.

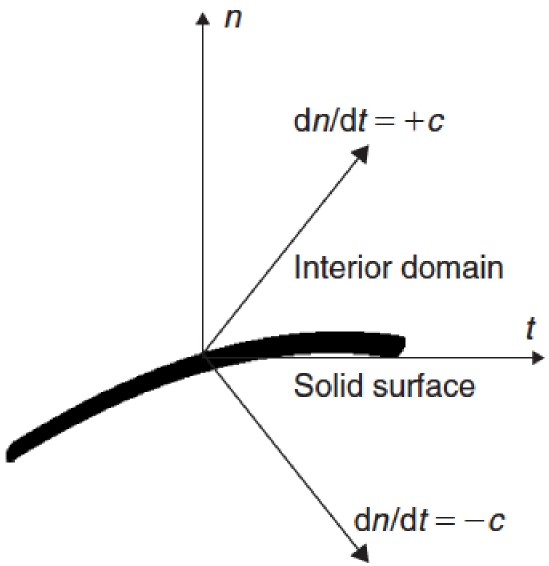
Outlet boundary

- The most appropriate b.c., particularly for internal flows, consists in fixing the downstream static pressure.
- For external flows, sometimes instead of this, the free-stream velocity is imposed.

The 2D Euler equations: boundary conditions

- **Wall** boundary





- The normal velocity is 0 → only one eigenvalue is positive. The only condition to be imposed is exactly that $v_n = 0$

$$\lambda_1 = v_n = 0$$
$$\lambda_2 = v_n = 0$$
$$\lambda_3 = v_n + c > 0$$
$$\lambda_4 = v_n - c < 0$$

The other variables are to be extrapolated from the interior and are therefore numerical b.c.'s

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The 2D Euler equations: boundary conditions

$$\lambda_1 = v_n$$
$$\lambda_2 = v_n$$
$$\lambda_3 = v_n + c$$
$$\lambda_4 = v_n - c$$

- Supersonic flow
- At the **inlet** boundary all 4 eigenvalues are positive, since $v_n > c$
- At the oulet all eigenvalues are negative
- Then at the inlet the 4 variables are imposed

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{vmatrix}$$

- At the outlet, no physical boundary conditions are imposed

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Numerische Methoden der Strömungsmechanik

Intro to compressible flows