The Euler equations

$$\frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

$$U = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{vmatrix} \qquad f = \begin{vmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{vmatrix} \qquad g = \begin{vmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{vmatrix}$$

- Using the 1st and 4th equations above and using also $H = E + \frac{p}{\rho}$ we obtain

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial t}$$

- For steady flows the total enthalpy is constant along streamlines

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The Euler equations

- The entropy equation with the previous simplifications: inviscid flow, negligible heat conduction, perfect gas

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s = 0$$

- So that the entropy is constant along streamlines, but can vary from one streamline to another.
- Entropy variations will generate vorticity and inversely vorticity will create entropy gradients.

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Steady compressible flows

- For steady compressible flows, the total enthalpy and the entropy are constant along streamlines
- Looking at the thermodynamic relationships it is easy to see that all stagnation properties are constant along streamlines: the total temperature, the total pressure, etc.
- If the incoming flow is uniform, the stagnation properties are constant in the whole flow
- For reference all the useful expressions are repeated here

$$H = c_p T_0 = \text{const.} \qquad T_0 = T + \frac{|\boldsymbol{v}|^2}{2c_p} = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

$$\text{isentropic relations} \quad \frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{1/(\gamma-1)} = \left(\frac{p}{p_0}\right)^{1/\gamma}$$

$$p_0 = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

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Finite Volumes Euler eqns.

- Although we have already described how to employ the FVM in a general case, we are going to describe some practical details
- In the subsonic range, the solutions of the Euler equations for uniform inflow conditions should be identical to potential flow solutions
- The main difference is that Euler eqns. do not guarantee that the calculated flow remains isentropic
- Numerical dissipation generated by the numerical scheme will 'mimic' in some way the physical dissipation of viscous flows
- The computer cannot distinguish physical dissipation from numerical dissipation.

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Finite Volumes Euler eqns.

- So that instead of

$$T\left(\frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\nabla s\right) = 0$$

- the numerical solution will obey

$$T\left(\frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\nabla s\right) = \varepsilon_V$$

- where the right-hand side represents the dissipation of the numerical model
- As a consequence entropy will not remain constant
- We can use entropy as an indicator of the presence of numerical dissipation
- Where the calculated entropy increases, we can be sure that these regions are influenced by numerical dissipation

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Finite Volumes Euler eqns.

- This is a very important property
- It provides a direct measure of the quality of the numerical scheme on the selected grid
- Since the numerical dissipation is proportional to a power of the mesh size, we need to refine the mesh in regions where an excessive entropy generation would occur.
- This is a major requirement on accuracy. The grid must be refined in regions with high velocity-gradients, such as leading edges and trailing edges of airfoils, or regions with abrupt geometry changes, like sharp corners.
- The system of Euler eqns. is hyperbolic in space and time, and we will use a time-marching approach to obtain the steady state solution.

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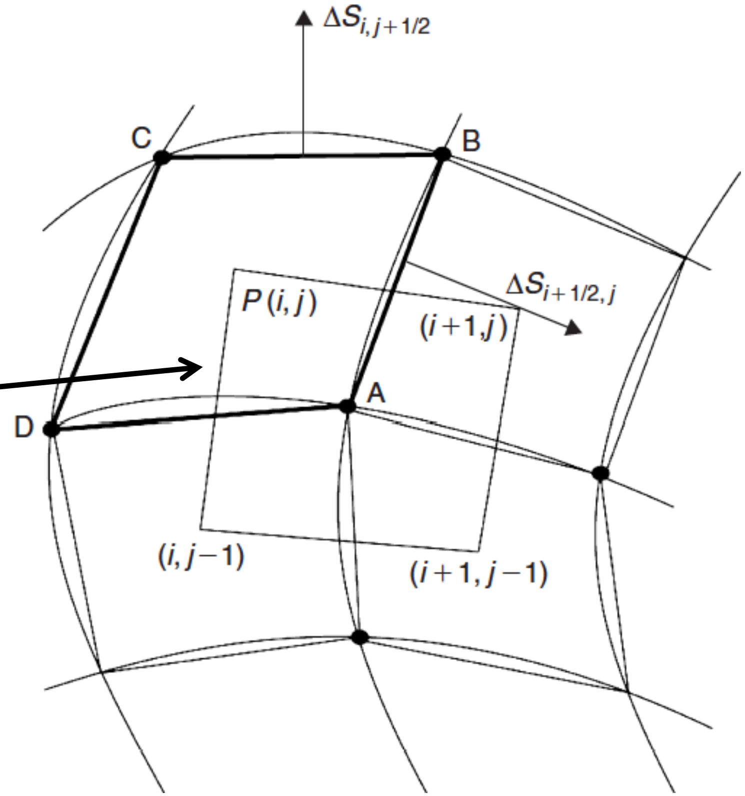
Mesh and notation



cell (i,j)

cell volume $\Omega_{i,j}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[U_{i,j}\Omega_{i,j}\right] = -\sum_{\text{faces}} \boldsymbol{F}^* \cdot \boldsymbol{\Delta S} \equiv -R_{i,j}$$

$\longleftarrow$ residual $R_{i,j}$as balance of the fluxes over all faces of cell (i,j)

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Summary of FVM Euler eqns.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[U_{i,j}\Omega_{i,j}\right] = -\sum_{\text{faces}} \boldsymbol{F}^* \cdot \boldsymbol{\Delta S} \equiv -R_{i,j}$$

- The solution $U_{i,j}$ is the cell-averaged value
- In post-processing we will assign the cell-averaged value to the center of the cell
- This introduces an error, typically a 2nd order error
- The numerical flux $\boldsymbol{F}^*$ represents the discretization of the physical fluxes, as defined by the selected numerical scheme

- In the following slides, some recommendations on the decisions to take are given

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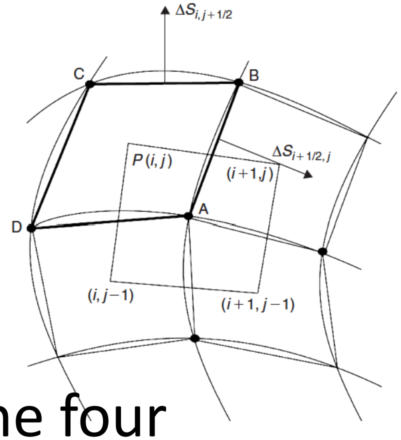
Space discretization



- Use a cell-centered finite volume discretization
- Define the grid lines
- Then compute the cell centers as the average of the four corners $\boldsymbol{x}_P = \frac{1}{4}(\boldsymbol{x}_A + \boldsymbol{x}_B + \boldsymbol{x}_C + \boldsymbol{x}_D)$

- Define the cell areas and the face normals based on the formulas we saw on the chapter on FVM
- Use a central discretization for the numerical flux and add an artificial viscosity term to stabilize the computation
- The numerical flux for the east face $(i+1/2, j)$ should look like:

$$(\boldsymbol{F}^* \cdot \boldsymbol{\Delta S})_{i+1/2,j} = \left[\frac{1}{2}(\boldsymbol{F}_{i,j} + \boldsymbol{F}_{i+1,j})\right] \cdot \boldsymbol{\Delta S} - D_{i+1/2,j}$$

artificial dissipation

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Numerical fluxes

- The numerical flux for all faces are then:

$$(\boldsymbol{F}^* \cdot \boldsymbol{\Delta S})_{i+1/2,j} = \left[\frac{1}{2}(\boldsymbol{F}_{i,j} + \boldsymbol{F}_{i+1,j})\right] \cdot \boldsymbol{\Delta S} - D_{i+1/2,j}$$

east

$$(\boldsymbol{F}^* \cdot \boldsymbol{\Delta S})_{i-1/2,j} = \left[\frac{1}{2}(\boldsymbol{F}_{i-1,j} + \boldsymbol{F}_{i,j})\right] \cdot \boldsymbol{\Delta S} + D_{i-1/2,j}$$

west

$$(\boldsymbol{F}^* \cdot \boldsymbol{\Delta S})_{i,j+1/2} = \left[\frac{1}{2}(\boldsymbol{F}_{i,j} + \boldsymbol{F}_{i,j+1})\right] \cdot \boldsymbol{\Delta S} - D_{i,j+1/2}$$

north

$$(\boldsymbol{F}^* \cdot \boldsymbol{\Delta S})_{i,j-1/2} = \left[\frac{1}{2}(\boldsymbol{F}_{i,j-1} + \boldsymbol{F}_{i,j})\right] \cdot \boldsymbol{\Delta S} + D_{i,j-1/2}$$

south

- In the following we give the expressions for the artificial dissipation of the east face only. The other faces are analogous

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Artificial dissipation

- In subsonic flow, it is often enough to add a 3rd derivative to the flux
- In supersonic flow, a blend of 1st and 3rd derivatives is recommended
- The 4th derivative is introduced as the difference of the fluxes with 3rd derivatives

$$D_{i+1/2,j} = -\gamma_{i+1/2,j}(U_{i+2,j} - 3U_{i+1,j} + 3U_{i,j} - U_{i-1,j})$$

- This term needs values from 2 additional rows of cells
- The coefficient is given by

$$\gamma_{i+1/2,j} = \frac{1}{2}\kappa^{(4)}\left[|\boldsymbol{v}\cdot\boldsymbol{\Delta S}| + c|\boldsymbol{\Delta S}|\right]_{i+1/2,j}$$

speed of sound

non-dimensional coefficient of dissipation
for example $\kappa^{(4)} = \frac{1}{256}$

Artificial dissipation

- In subsonic flow, it is often enough to add a 3rd derivative to the flux

$$D_{i+1/2,j} = -\gamma_{i+1/2,j}(U_{i+2,j} - 3U_{i+1,j} + 3U_{i,j} - U_{i-1,j})$$

- In a simplified treatment, it is possible to use a simpler formulation (which will be less accurate)

$$D_{i+1/2,j} = \gamma_{i+1/2,j}(U_{i+1,j} - U_{i,j})$$

- Note the difference in the sign

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Artificial dissipation

- In supersonic flow with shocks the following model is recommended

$$D_{i+1/2,j} = \eta_{i+1/2,j}(U_{i+1,j} - U_{i,j}) \\ - \gamma_{i+1/2,j}(U_{i+2,j} - 3U_{i+1,j} + 3U_{i,j} - U_{i-1,j})$$

- with

$$\eta_{i+1/2,j} = \frac{1}{2}\kappa^{(2)} \left[|\boldsymbol{v} \cdot \boldsymbol{\Delta S}| + c|\boldsymbol{\Delta S}|\right]_{i+1/2,j} \max(\nu_{i-1}, \nu_i, \nu_{i+1}, \nu_{i+2})$$

$$\gamma_{i+1/2,j} = \max\left(0, \frac{1}{2}\kappa^{(4)} \left[|\boldsymbol{v} \cdot \boldsymbol{\Delta S}| + c|\boldsymbol{\Delta S}|\right]_{i+1/2,j} - \eta_{i+1/2,j}\right)$$

- where the variables ν_i are sensors that activate the 2nd order dissipation in regions of strong gradients. They are based on pressure variations

$$\nu_i = \left|\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{p_{i+1,j} + 2p_{i,j} + p_{i-1,j}}\right|$$

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Artificial dissipation

- In the shock region the sensors are of order 1.
- In that case the artificial viscosity term is then 1st order
- If the pressure variations are linear, the sensor vanishes, since in smooth regions the numerator is a 2nd order discretization of the 2nd derivative

$$\nu_i = \left| \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{p_{i+1,j} + 2p_{i,j} + p_{i-1,j}} \right|$$

- In that case the dissipation term becomes identical as the one used in subsonic applications
- The coefficient $k^{(2)}$ is typically of order 1.

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Artificial dissipation

- In supersonic flow with shocks the following model is recommended

$$D_{i+1/2,j} = \eta_{i+1/2,j}(U_{i+1,j} - U_{i,j}) \\ - \gamma_{i+1/2,j}(U_{i+2,j} - 3U_{i+1,j} + 3U_{i,j} - U_{i-1,j})$$

- Again, we can use a simplified treatment that will be less accurate but simpler to include in your code:

$$D_{i+1/2,j} = \eta_{i+1/2,j}(U_{i+1,j} - U_{i,j}) + \gamma_{i+1/2,j}(U_{i+1,j} - U_{i,j})$$

- With $\eta_{i+1/2,j} = \frac{1}{2}\kappa^{(2)} \left[|\boldsymbol{v} \cdot \boldsymbol{\Delta S}| + c|\boldsymbol{\Delta S}|\right]_{i+1/2,j} \max(\nu_i, \nu_{i+1})$

$$\gamma_{i+1/2,j} = \max\left(0, \frac{1}{2}\kappa^{(4)} \left[|\boldsymbol{v} \cdot \boldsymbol{\Delta S}| + c|\boldsymbol{\Delta S}|\right]_{i+1/2,j} - \eta_{i+1/2,j}\right)$$

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Time integration

- Low-storage Runge-Kutta methods of 4th order are recommended
- It is also possible to use the McCormack scheme with smaller time step
- The recommended R-K method is

$$Y_1 = U_{i,j}^n$$

$$Y_2 = U_{i,j}^n - \frac{\Delta t}{\Omega_{i,j}} \alpha_2 R_{i,j}(Y_1)$$

$$Y_3 = U_{i,j}^n - \frac{\Delta t}{\Omega_{i,j}} \alpha_3 R_{i,j}(Y_2)$$

$$Y_4 = U_{i,j}^n - \frac{\Delta t}{\Omega_{i,j}} \alpha_4 R_{i,j}(Y_3)$$

$$U_{i,j}^{n+1} = U_{i,j}^n - \frac{\Delta t}{\Omega_{i,j}} R_{i,j}(Y_4)$$

Equation to be solved is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[U_{i,j}\Omega_{i,j}\right] = -\sum_{\text{faces}} \boldsymbol{F}^* \cdot \boldsymbol{\Delta S} \equiv -R_{i,j}$$

Two possible sets of coefficients are:

$$\alpha_2 = \frac{1}{4},\ \alpha_3 = \frac{1}{3},\ \alpha_4 = \frac{1}{2}$$

$$\alpha_2 = \frac{1}{8},\ \alpha_3 = 0.306,\ \alpha_4 = 0.587$$

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Time integration

- Recall to choose the CFL number under the stability condition
- For R-K 4, CFL<2.8
- For McCormack CFL<1
- The time step should be based on the fact that the physical domain of dependence should be contained in the numerical domain of dependence.
- We need to define the maximum propagation velocity, for example in the x-direction $u_{max} = \max(|u+c|, |u-c|)$
- And a possible formula is $$\Delta t = \frac{\sigma}{\dfrac{u_{max}}{\Delta x} + \dfrac{v_{max}}{\Delta y}}$$ CFL
- More complex formulas are needed for highly distorted cell volumes.

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