Tensor networks and quantum field theory

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Here we describe how tensor network states — a quantum information theory inspired tool originally arising in the study of quantum spin systems — can be used to understand the physics of quantum fields. In particular, we review the tensor network formalism and outline its application to the study of ground state physics and scattering amplitudes. We describe two approaches, one based on the Lie-Trotter expansion within the hamiltonian formalism, and a second based on the lagrangian formalism which represents the path integral as a tensor network. We find, exploiting the second approach, that all scattering amplitudes are determined by a special convex set. Our aim in this paper is to present recent developments in condensed matter theory with a quantum field theory audience in mind.

I. INTRODUCTION

Quantum field theory is now understood as an effective theory describing the large scale low energy physics [1, 2]. By exploiting the correspondence between a statistical physics system at criticality and a quantum field Wilson in one fell swoop resolved all the troubling "infinities" plaguing quantum field theory. And so it is that quantum field theory has become a tool par excellence in physics; to understand the large-scale effective physics of any system we need only identify the fixed point — often uniquely defined by the symmetries of the system — to which it belongs and, often enough, the resulting QFT describing that situation is amenable to perturbation theory allowed us to deploy the apparatus of Feynman diagrams.

Tensor network papers: [3][4–6][7] [8][9][10]

Perturbative QFT is useful and powerful but nonperturbative field theory still contains many mysteries: tantalising hints, via holography, of the power of QFT. Complex network of relationships via holographic duality.

The lattice is a good regulator allowing computers to be brought to bear on deep problems. Amazing insights [11] from confinement to the hadron spectrum [12].

The lattice

The path integral is the way to work with QFT [13][14][15][16][17][18][19][20][21][22][23]

Here we must cite Swingle, Vidal, Verstraete, Maldacena, Susskind, Takayanagi etc.....

Important points:

1. Path integral formalism to calculate n-point functions.
2. Path integral on the lattice = statistical mechanics.
3. Method 1: trotter
4. Method 2: Stat. mech. = lifshitz PEPS.
5. Scattering amplitudes as n-point functions.
6. Removing the cutoff: 2nd (higher) order phase transition; scaling limit.
7. Variational calculation of n-point functions
8. Path integrals+perturbation theory: good for weak coupling bad for strong coupling; TNS good for strong coupling bad for weak.

Since their emergence in 1987 in the condensed matter theory literature there has been an amazing amount of progress in the development of tools, both analytic and numeric, for the study of tensor network states. We cannot do proper justice to the literature here, and we only summarise a couple of the highlights.

In this paper we argue that tensor network methods are reaching a maturity so that they may be profitably employed as an alternative to the path integral in the study of correlated quantum fields. Here

In this sense tensor-network methods have become a natural alternative to the path integral appropriate for

The purpose of this letter is to interpret QFT in the language of tensor networks with a view to formalising possible approaches to studying holographic dualities and the calculation of To this end we describe two natural approaches to studying the statics and dynamics of QFT via TNS, namely, in the hamiltonian setting familiar to condensed matter theory and a second approach by directly representing the full path integral as a TNS. We then argue how to obtain continuum limits and briefly

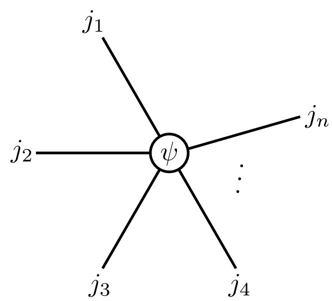
II. WHAT IS A TENSOR NETWORK?

Tensor networks provide an economic framework to parametrise tensors with many indices. Here we provide a brief overview of the tensor network formalism and sketch some calculational techniques. For some recent reviews of this material see, e.g., [4–6].

A. Tensor diagrams

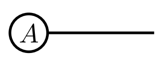
We begin our discussion by considering n-index tensors: these are, for us, nothing more than collections ψ of $d_1 d_2 \cdots d_k$ complex numbers $\psi_{j_1 j_2 \cdots j_n}$, where $j_k = 1, 2, \ldots, d_k$ and d_k is the dimension of the index. While we sometimes use the "upstairs" and "downstairs" notation for indices — basically in order to unclutter our expressions — this should not be interpreted as implying that the tensor transforms in any particular way under Lorentz transformations (whose action is not even defined here).

A key device we exploit in the following is to visualise an n-index tensor ψ as a vertex (really a circle) with n legs, one for each index j_k, $k = 1, 2, \ldots, n$:

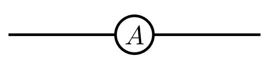


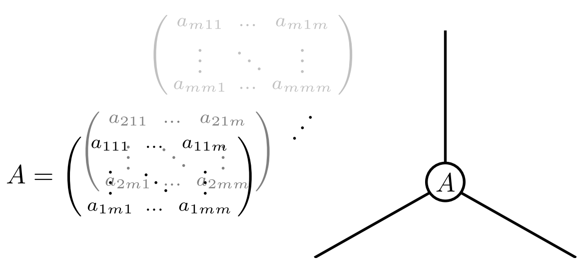
According to such a visualisation tensors such as vectors are represented as a circle with one leg, matrices as a circle with two legs, and so on:

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$



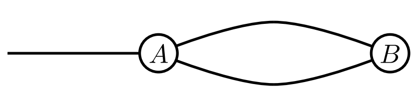
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}$$





The contraction of the indices of two tensors is then represented by joining the legs of the corresponding indices. For example:

$$\sum_{kl} a_{jkl} b_{kl}$$

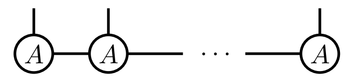


B. Tensor networks

A *tensor network* is then the tensor ψ associated with a diagram, actually a *graph* G, consisting of a collection of tensors, represented as *vertices*, whose indices are contracted according to the *contraction pattern* described by the edges of the graph. The result is a tensor with a number of indices given by the external or "dangling legs" of the diagram.

The canonical example of a tensor network is the *matrix product state* tensor:

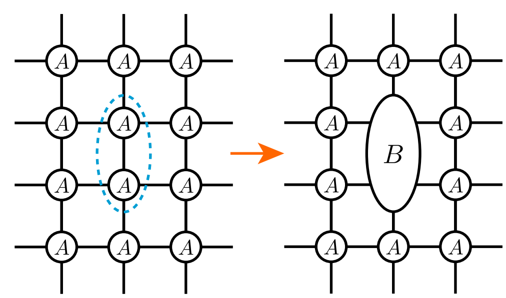
$$A^{j_1}_{\alpha_1} A^{j_2}_{\alpha_1\alpha_2} \cdots A^{j_n}_{\alpha_{n-1}} \equiv$$



Calculating the value of a tensor network for a given configuration $j_1 j_2 \cdots j_n$ of the external legs is called *contracting* a tensor network. The problem of contracting, or even approximating the contraction of, an arbitrary tensor network is, in general, *hard*: even *planar* tensor networks can be arbitrarily difficult [24]. Tensor networks are, in general, useful only when they have some additional structure, such as short-ranged correlations (more on this later), topological simplicity, i.e., containing only a few loops, or causal structure, e.g., the tensors are isometries.

III. OPERATIONS ON TENSOR NETWORKS

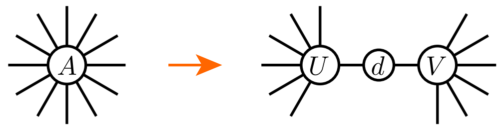
There are a couple of basic operations one can perform on a tensor network to obtain (sometimes simpler) new networks. The first operation is *partial contraction*: here a collection of vertices and their corresponding tensors are identified and the indices corresponding to all the edges connecting these vertices are contracted. The original tensors are then replaced by the resulting amalgamation:



The second key primitive is that of *tensor splitting*: here we first collect the external legs of a tensor into two groups which are then regarded as two multi-indices. For example, suppose we have the n-index tensor A inside our network: we could collect the first k indices together as the multi index $\alpha \equiv [j_1 j_2 \cdots j_k]$ — running, say, in lexicographic order through the $d_1 d_2 \cdots d_k$ assignments of the indices — and the remaining indices together as $\beta \equiv [j_{k+1} \cdots j_n]$. This bundling of indices gives us a 2-index tensor $A_{\alpha\beta}$, where α can be taken to run from 1 to $d_1 \cdots d_k$ and β from 1 to $d_{k+1} \cdots d_n$. We may now regard A as a *matrix*. The tensor splitting operation is then carried out by exploiting the *singular value decomposition* [25] to write A as a product of an isometry U, a diagonal matrix d with positive entries, and another isometry V:

$$A = UdV. \tag{1}$$

Graphically this is represented as:



Note that the original tensor is replaced with a contraction of three tensors and, in the process, the *degree* of the new vertices in the network is reduced. This procedure may be recursively applied to all the tensors in an arbitrary network to produce a new tensor network with maximal degree bounded above by 3. (Note: it is impossible, in general, to reduce the degree of all the vertices to 2 or below via the process.)

The third primitive is *truncation*. Focus on one edge e in a tensor network: this represents a contraction of two specific indices of two tensors. That is, we perform a sum over all the values of the index α indicated by the edge e and shared by the two tensors at the ends of the edge:

$$\sum_{\alpha=1}^{d} \cdots A_{\cdots\alpha} B_{\alpha\cdots} \cdots . \tag{2}$$

The truncation procedure is an approximation technique where we simply sum over fewer of the possible values of α:

$$\sum_{\alpha=1}^{d} \cdots A_{\cdots\alpha} B_{\alpha\cdots} \cdots \mapsto \sum_{\alpha=1}^{d'} \cdots A_{\cdots\alpha} B_{\alpha\cdots} \cdots , \tag{3}$$

with $d' < d$. Truncation is particularly effective in combination with tensor splitting: when the diagonal matrix d in Eq. (1) has many small entries we can simply replace the sum over β with a sum over a smaller range, including only the indices corresponding to large entries.

A final approximation operation frequently useful in studying quantum states generated by local dynamics is the *Lie-Trotter* expansion:

$$e^{A+B} \approx (e^{A/m} e^{B/m})^m, \tag{4}$$

where n is a positive integer and A and B are operators on some hilbert space $\mathcal{H}$. This approximation improves as n is increased and becomes an identity in the limit $m \to \infty$. This result allows us to approximate the propagator $U(t) = e^{-itH}$ — an n-index tensor — for a quantum spin system with a *local* hamiltonian H by a tensor network involving small-index tensors. To see how this works suppose that H is a hamiltonian for a chain of n quantum spins (similar results hold in higher dimensions):

$$H = \sum_{j=1}^{n-1} h_j, \tag{5}$$

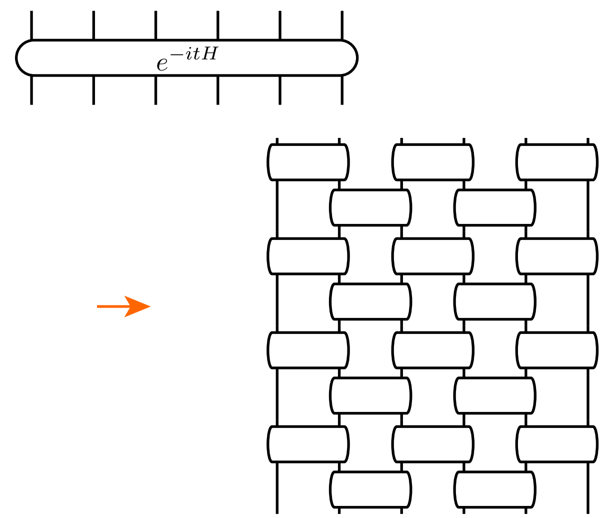
where h_j is a local interaction term which acts nontrivially only on spins j and $j+1$. Next collect together the even-numbered (respectively, odd numbered) interactions:

$$A = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} h_{2k} \quad \text{and} \quad B = \sum_{k=1}^{\lceil (n-1)/2 \rceil} h_{2k-1}. \tag{6}$$

Now A and B only contain terms which mutually commute with each other, so that $e^A = \prod_k e^{h_{2k}}$ and $e^B = \prod_k e^{h_{2k+1}}$. Applying the Lie-Trotter expansion allows us to replace e^{zH} with the product

$$e^{zH} \approx \left(\prod_k e^{\frac{z}{m} h_{2k}} \prod_{k'} e^{\frac{z}{m} h_{2k'}} \right)^m, \tag{7}$$

which is a tensor network of mn degree-4 tensors. Graphically this is represented by:



We've thus reduced the contraction of a single degree $2n$ tensor to contracting a network of smaller tensors. In general the difficulty of these two different tasks is roughly equivalent, however, there are cases where the Lie-Trotter network can be exploited to develop useful approximation methods

IV. TENSOR NETWORK STATES

Tensor network states (TNS) are quantum states of a many body quantum system associated with a tensor network. The systems we typically think about are quantum spin systems comprised of n distinguishable quantum spins with local hilbert spaces $\mathcal{H}_j$, $j = 1, 2, \ldots, n$; the total hilbert space is then $\mathcal{H} \cong \bigotimes_{j=1}^{n} \mathcal{H}_j$. It is common to assume that $\mathcal{H}_j$ is finite dimensional, with dimension d_j, although there is no fundamental reason preventing everything from straightforwardly applying to infinite dimensional systems. An arbitrary state $|\psi\rangle$ of $\mathcal{H}$ can be written as

$$|\psi\rangle = \sum_{j_1=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} \psi_{j_1 j_2 \cdots j_n} |j_1\rangle \cdots |j_n\rangle, \tag{8}$$

where $|j_k\rangle$ is an arbitrary orthonormal basis for $\mathcal{H}_j$. The state $|\psi\rangle$ is encoded in the $d_1 d_2 \cdots d_n$ components of the n-index tensor $\psi_{j_1 j_2 \cdots j_n}$. When the dangling legs of a tensor ψ are associated with an orthonormal basis of a quantum spin system we obtain a *tensor network state.*

A. Basic properties of tensor network states

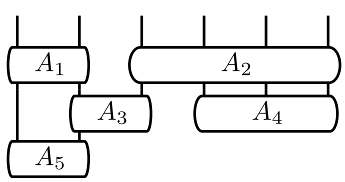
The tensor network formalism exposes some very powerful structural information about quantum states.

The first property is that a tensor network state is always manifestly a *state*. This is an extremely useful property not shared by other representations (e.g., a truncated perturbation series around some mixed state does not usually determine a positive state).

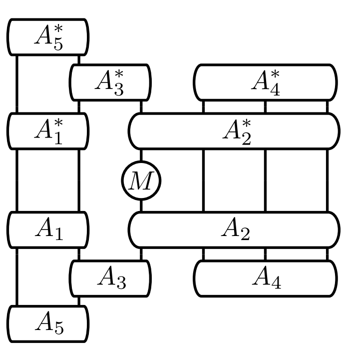
The second property is that expectation values of (local) operators may be computed by contracting a related tensor network: let $|\psi\rangle$ be a TNS and M some (local) operator on the lth spin. Then the expectation value $\langle M\rangle$ is determined by

$$\sum_{j_1=1}^{d_1} \cdots \sum_{j_l,k_l=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} \psi^*_{j_1 j_2 \cdots j_n} \psi_{j_1 \cdots j_{l-1} k_l j_{l+1} \cdots j_n} M_{j_l k_l}, \tag{9}$$

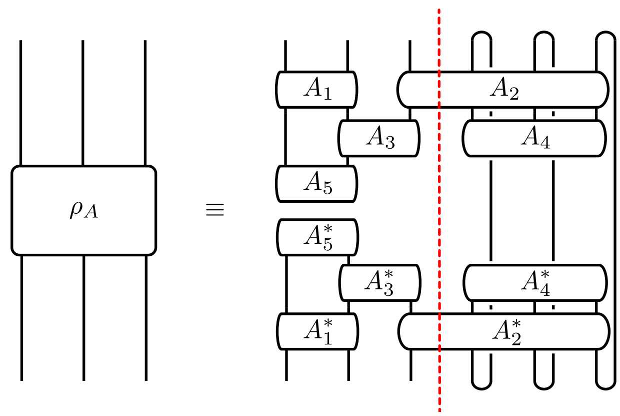
which is the contraction of three tensor networks: one for ψ^*, one for M, and one for ψ. For example, suppose we have a TNS whose coefficients are determined by the network

$\psi \equiv$ 

Then the expectation value of a hermitian operator M on the 3rd spin is given by the contraction

$\langle\psi|M|\psi\rangle \equiv$ 

Closely related to the calculation of expectation values is the construction of the *reduced density operator*. Let $|\psi\rangle$ be a tensor network state. Form the corresponding density operator $\rho \equiv |\psi\rangle\langle\psi|$: this is a tensor network with twice the number of external legs and every tensor doubled. Take the partial trace over all spins outside of region A; this operation is found by connecting the indices corresponding to the legs in the complement in A as follows:



The third remarkable property is that we obtain an upper bound on the entropy of entanglement associated a region $\mathcal{R}$ in the network. This result may be argued as follows.

V. TENSOR NETWORKS FROM HAMILTONIANS

In this section we'll discuss a now-standard way to obtaining a tensor network state representation of the ground state and low-lying excitations of a complex quantum system. As we'll discuss, in the quantum field setting there are additional complications arising when removing (lattice) regulator. The discussion here is framed in the *hamiltonian* setting: our input is the regulated *quantum* hamiltonian for the system. We use as regulator the spatial lattice $L_a \equiv a\mathbb{Z}^D \subset \mathbb{R}^D$, with lattice spacing a. The discussion throughout this paper is illustrated in terms of the running example of ϕ^4 theory; additional fields and fermions do not present any new major conceptual difficulties (beyond fermion doubling). To further simplify our discussion in this section we only consider this model in $(1+1)$ dimensions (the extension to higher dimensions is only sketched).

The hamiltonian of ϕ^4 theory on a lattice with lattice spacing a is expressible as

$$H_a = a \sum_{j\in L_a} \frac{\widehat{p}_j}{2a^2} + \frac{(\widehat{x}_j - \widehat{x}_{j+1})^2}{2a^2} + \frac{\mu_0^2}{2}\widehat{x}_j^2 + \frac{\lambda}{4!}\widehat{x}_j^4, \tag{10}$$

where $[\widehat{x}_j, \widehat{p}_k] = i\delta_{jk}$. To a tensor-network practitioner the most natural way to approach the approximation of the ground state is to first make a guess $|\Phi_0\rangle$ for the ground state as a simple tensor network state and then to improve this guess. One tried and true approach to doing this is to use *mean-field theory* to obtain $|\Phi_0\rangle$ as a trivial matrix product state and then to simulate its *imaginary time evolution*

$$|\Phi(\beta)\rangle \equiv \frac{e^{-\beta H_a}|\Phi_0\rangle}{\|e^{-\beta H_a}|\Phi_0\rangle\|}. \tag{11}$$

If H has a spectral gap and if the initial condition $|\Phi(0)\rangle \equiv |\Phi_0\rangle$ has some overlap with the true ground state $|\Omega\rangle$, i.e., $\langle\Phi(0)|\Omega\rangle \neq 0$, then we obtain an exponentially improving approximation $|\Omega(\beta)\rangle$ to $|\Omega\rangle$. As we saw in the previous section, the Lie-Trotter expansion can be utilised to approximate $e^{-\beta H_a}$ with a tensor network. When this tensor network is contracted against the tensor network for a product state we obtain a matrix product state $|\Psi\rangle$ with bond dimension equal to const.$^{\#\text{ of layers}}$:

VI. TENSOR NETWORKS FROM PATH INTEGRALS

Here we must cite lifschitz theories; ardonne; Horava

The basic input to any calculation in QFT is the lagrangian $\mathcal{L}$. To keep a concrete example in mind think of ϕ^4 theory in $D+1$ dimensions:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{12}$$

however, everything we discuss here is valid in generality (in particular, everything we say here will apply equally to relativistic and nonrelativistic settings). Using the lagrangian we can calculate any n-point correlation function using the path integral according to the standard formula

$$\langle T[\phi(x_1)\cdots\phi(x_n)]\rangle \equiv \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n)e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}}{\int \mathcal{D}\phi\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}}. \tag{13}$$

Before such a path integral can be evaluated — or even approximated — it is generally necessary to *regulate* the theory by applying a cutoff Λ. Now, in the context of tensor networks and quantum information, the most natural regulator is the lattice, with lattice spacing $a = \frac{1}{\Lambda}$, and we don't hesitate to use it throughout. In this way we discretise our continuous field(s) $\phi(x)$ onto the integer lattice $a\mathbb{Z}^{D+1}$; we restrict our coordinates to elements of the form $x_\mu = an_\mu$, $n_\mu \in \mathbb{Z}^{D+1}$. Write the field evaluated at lattice points as $\phi_{n_\mu} = \phi(x_\mu)$; thus we discretise derivatives as $\partial_\mu\phi(x_\nu) \mapsto (\phi_{n_\nu+e_\mu} - \phi_{n_\nu})/a$, where e_μ is the unit lattice vector in the direction μ. Thus our potentially ill-defined path integral becomes the better-behaved iterated integral

$$\int \mathcal{D}\phi \mapsto \int [d\phi] \equiv \int \left(\prod_{n\in\mathbb{Z}^{D+1}} d\phi_n\right). \tag{14}$$

A further important device that we also initially exploit is to make a Wick rotation of time $t \mapsto -i\beta$: this turns our path integral expression for the correlation functions into a statistical mechanical problem:

$$\langle\phi(x_1^E)\cdots\phi(x_n^E)\rangle \equiv \frac{1}{\mathcal{Z}}\int [d\phi]\, \phi_{m_1}\cdots\phi_{m_n}e^{-S}, \tag{15}$$

where $\mathcal{Z} = \int [d\phi]\, e^{-S}$ and, e.g., $S = a^{D+1}\sum_{\langle m,n\rangle\in\mathbb{Z}^{D+1}}(\phi_m - \phi_n)^2/(2a^2) + \sum_n m^2\phi_n^2/2 + \lambda\phi_n^4/4!$.

To practitioners in tensor networks an expression such as Eq. (15) is rather suggestive because we can introduce a special pure quantum state $|\Phi\rangle$ which encodes all the information we can extract from a partition function. To explain this construction suppose we have a classical system which can be in one of N different configurations, $x = 1, 2, \ldots, N$. (One example to keep in mind is the classical Ising model on an $L \times L$ lattice, here x runs over all the configurations of the spins, labelled from 1 to 2^{L^2}.) Write the hamiltonian as $H = \sum_{x=1}^N E_j(x)|x\rangle\langle x|$, where we exploit quantum notation for classical systems: they are, for us, simply systems diagonal in a given "position" basis. The thermal state for the system is simply $\rho = e^{-\beta H}/\mathcal{Z}$. The trick we exploit is to take the square root of the probabilities $p(x) = e^{-\beta E(x)}/\mathcal{Z}$ [17, 18] and build the *pure* quantum state $|\Phi\rangle \equiv \sum_{x=1}^N e^{-\frac{1}{2}E(x)}|x\rangle/\sqrt{\mathcal{Z}}$, where $|x\rangle$ is an orthonormal basis corresponding to the N possible configurations of the classical system. The state $|\Phi\rangle$ encodes the expectation value of any observable: suppose $O : \{1, 2, \ldots, N\} \to \mathbb{C}$ is an observable for the classical system (e.g., a multi-point correlation function such as $\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_N}$). We can build a quantum observable which gives us the same expectation values:

$$\widehat{O} \equiv \sum_{x=1}^N O(x)|x\rangle\langle x|. \tag{16}$$

We hence obtain $\langle\Phi|\widehat{O}|\Phi\rangle = \langle O\rangle$.

This construction is easily extended to the case of imaginary time path integrals: so long as the integral does not have a sign problem the above construction goes through unmodified: when we apply the procedure described here to a Wick-rotated path integral we obtain the following state of a lattice of a $(D+1)$-dimensional spatial lattice of harmonic oscillators

$$|\Phi\rangle \equiv \frac{1}{\mathcal{Z}}\int [d\phi]\, e^{-\frac{1}{2}S[\phi]}|\boldsymbol{\phi}\rangle, \tag{17}$$

where

$$|\boldsymbol{\phi}\rangle \equiv \bigotimes_{x_\mu\in a\mathbb{Z}^{D+1}} |\phi(x_\mu)\rangle, \tag{18}$$

and $|\phi(x)\rangle$ is an improper *position eigenstate* of the harmonic oscillator for site x_μ localised at the harmonic oscillator position $\phi(x_\mu)$.

One might wonder whether this construction can be extended to deal with the case of arbitrary path integrals.

The naive answer is no: if we just take the square roots of the integrand e^{iS} in a path integral then the resulting state $|\Phi\rangle$, constructed as above, has the unfortunate property that $\langle\Phi|\widehat{O}|\Phi\rangle$ is independent of $|\Phi\rangle$! However if we are willing to introduce an ancillary quantum spin we can make an analogous construction. Suppose we have a path integral expression of the form

$$\langle O\rangle = \frac{\sum_{x=1}^{N} O(x)e^{iS(x)}}{\sum_{x=1}^{N} e^{iS(x)}}, \tag{19}$$

where S is the action and O an observable. We can again construct a pure quantum state encoding all the information about arbitrary observables O as follows:

$$|\Phi\rangle = \frac{1}{\sqrt{2\mathcal{Z}_+}}\sum_{x=1}^{N} e^{\frac{i}{2}S(x)}|x\rangle|0\rangle + \frac{1}{\sqrt{2\mathcal{Z}_-}}\sum_{x=1}^{N} e^{-\frac{i}{2}S(x)}|x\rangle|1\rangle, \tag{20}$$

where

$$\mathcal{Z}_\pm = \sum_{x=1}^{N} e^{\pm iS(x)}. \tag{21}$$

It is now possible to generalise the construction above to construct quantum operators $\widehat{O}$ whose expectation values give those of their corresponding classical counterparts O:

$$\widehat{O} \equiv 2\sum_{x=1}^{N} O(x)|x\rangle\langle x| \otimes |1\rangle\langle 0|. \tag{22}$$

The auxiliary spin here is thought of as indicating the arrow of time.

The pure state Eq. (20) is perfect for approximation via tensor network states. Indeed, we will go further: the principle contribution of this project is to promote tensor networks of the form Eq. (20) to the status of a *definition* for a class of path integrals. The justification of this idea will take us on a long journey.

VII. PARENT HAMILTONIANS AND CONVEX SETS

Remarkably, we can also realise $|\Phi\rangle$ as the *ground state* of a natural hamiltonian [13, 14, 26, 27]. The key to this construction is to introduce a reversible markov chain M which has $p_j \equiv e^{-E_j}/\mathcal{Z}$ as its stationary distribution [28], i.e., $\sum_{j=1}^{N} p_j M_{jk} = p_k$, $\forall k$. Reversibility is the condition that $p_j M_{jk} = p_k M_{kj}$, $\forall j,k$. This motivates us to introduce the matrix K with matrix elements:

$$K_{jk} \equiv \sqrt{p_j}M_{jk}\frac{1}{\sqrt{p_k}}. \tag{23}$$

Because of the reversibility condition we find that K is real and symmetric, i.e., hermitian. Further, we have that operator $\widehat{K} \equiv \sum_{j,k=1}^{N} K_{jk}|k\rangle\langle j|$ has $|\Phi\rangle$ as an eigenvector corresponding to its largest eigenvalue. We obtain our desired hamiltonian by writing $\widehat{H} \equiv \mathbb{I} - \widehat{K}$, this operator has $|\Phi\rangle$ as a ground state with eigenvalue 0.

Given a probability distribution of exponential form, i.e., $p_j \equiv e^{-E_j}/\mathcal{Z}$, there is a special reversible Markov chain having p_j as its stationary distribution, namely, the Metropolis algorithm [29]. To define this Markov chain we introduce the quantity

$$c_{jk} \equiv \min\{1, e^{-E_j+E_k}\}. \tag{24}$$

Let F be an operator which flips classical configurations, a *flip operator*. In our cases it is convenient to let F be a permutation of $j = 1, 2, \ldots, N$. We then use the flip operator to construct

$$\widehat{M} = \sum_{j\neq k} c_{jk}\langle j|F|k\rangle \tag{25}$$

VIII. REFLECTION POSITIVITY AND ANALYTIC CONTINUATION

IX. CONCLUSIONS

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Appendix A: A generalised Metropolis algorithm for the Lifschitz state

In this appendix we write out some of the calculations involved in constructing the parent hamiltonian for the Lifschitz state.

The Lifschitz state for a classical hamiltonian of the form $H = \sum_{x=1}^{N} E(x)|x\rangle\langle x|$ is given by

$$|\Phi\rangle \equiv \sum_{x=1}^{N} e^{-\frac{1}{2}E(x)}|x\rangle/\sqrt{\mathcal{Z}}. \tag{A1}$$

Recall that, associated to any flip operator F we have the operator

$$\widehat{M} = \sum_{x\neq y} c_{xy}\langle x|F|y\rangle|x\rangle\langle y| - \sum, \tag{A2}$$

where

$$c_{jk} \equiv \min\{1, e^{-E(x)+E(y)}\}. \tag{A3}$$

When we act on $|\Phi\rangle$ with $\widehat{M}$ we obtain

$$\begin{aligned}\widehat{M}|\Phi\rangle &= \frac{1}{\sqrt{Z}}\sum_{xy} e^{-\frac{1}{2}E(y)}c_{xy}\langle x|F|y\rangle|x\rangle \\ &= \frac{1}{\sqrt{Z}}\sum_{xy} \min\{e^{-\frac{1}{2}E(y)+\frac{1}{2}E(x)}, e^{-\frac{1}{2}E(x)+\frac{1}{2}E(y)}\}\langle x|F|y\rangle e^{-\frac{1}{2}E(x)}|x\rangle\end{aligned} \tag{A4}$$