Math 301 Assignment 6

These problems are due in class on Tuesday. If your homework takes up multiple pages, they must be stapled together. Your work must be legible, and any frills from notebook paper must be removed.

#1. For this problem use the encryption function $e(p) = (15p - 8) \bmod 26$ and the alphanumeric values

$$A = 1, B = 2, C = 3, ..., M = 13, N = 14, ..., X = 24, Y = 25, Z = 0$$

a. Encrypt the message ATTACK

b. Decrypt the message IYP

#2. Why is $e(p) = (2p) \bmod 26$ not a good choice for an encryption function? Specifically, what problem might occur?

<u>Prove all the remaining statements **by mathematical induction**</u>

#3. (Sum of the first n odd numbers) Show that for all $n \geq 1$

$$1 + 3 + 5 + ... + 2n - 1 = n^2$$

#4. (Sum of a geometric sequence) If $r \neq 1$, show that for any real number a and all integers $n \geq 0$

$$a + ar + ar^2 + ... + ar^n = \frac{a(r^{n+1} - 1)}{r - 1}$$

#5. Find a formula for

$$\frac{1}{1 * 2} + \frac{1}{2 * 3} + ... + \frac{1}{n(n+1)}$$

by evaluating the expression for small values of n. Then prove your formula is correct by induction.

#6. (Product of the first n odd numbers) Show that for all $n \geq 1$

$$1 * 3 * 5 * ... * (2n-1) = \frac{(2n)!}{2^n * n!}$$

(recall that $(k+1)! = (k+1) * k!$)

#7. (Powers of a square matrix) Let A be a square matrix. Then $A^2 = A * A$, $A^3 = A * A * A$, etc.

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Find a formula for A^n by evaluating the powers of A for small values of n. Then prove your formula is correct by induction.

For the next two problems, we define the Fibonacci sequence via the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

$$f_0 = 0, \quad f_1 = 1$$

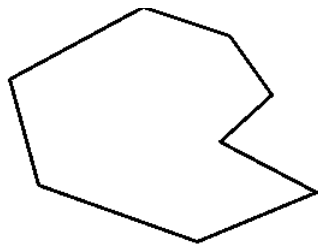
#8. (Fibonacci squares) Show that for $n \geq 0$,

$$f_0^2 + f_1^2 + ... + f_n^2 = f_n f_{n+1}$$

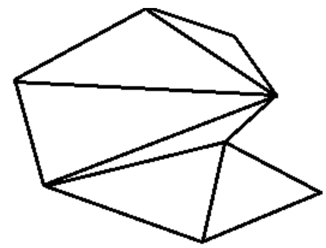
#9. (Fibonacci matrix) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Show that for $n \geq 1$,

$$A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

#10. A polygon is a geometric figure composed of line segments (that do not overlap). Triangulation is the process of dividing a polygon into triangles by adding diagonals (line segments that connect unconnected vertices).



an octagon (8-sided polygon)



a triangulated octagon

Use mathematical induction to show that for $n \geq 3$, an n-sided polygon can be triangulated into $n-2$ triangles. (This is not a strictly algebraic proof, but really a rather simple explanation. The main thing is to describe how we can use the triangulation of a k-gon to triangulate a $(k+1)$-gon.)

#11. BONUS: (10 points) Use mathematical induction to show that a convex n-sided polygon has $\frac{n(n-3)}{2}$ diagonals. (A convex polygon has no interior angles greater than 180°, so the octagon from the previous problem is concave rather than convex. You will need an explanation and a bit of algebra for this)

Practice Problems

Section 4.6: # 1, 3, 5 (Warning: the book uses $A = 0, B = 1, ...Z = 25$, which is obviously the wrong way to do it)

Section 5.1: # 5, 7, 9, 11, 13, 15