Theorem. *$Fib(n)$ is the closest integer to $\phi^n/\sqrt{5}$, where $\phi = (1+\sqrt{5}/2)$.*

Proof. Let $\psi = (1-\sqrt{5})/2$. Then two Lemmas follow:

Lemma 1. $\phi^2 = (1+\sqrt{5})^2/4 = (6+2\sqrt{5})/4 = (3+\sqrt{5})/2 = 1+(1+\sqrt{5})/2 = \phi+1$

Lemma 2. $\psi^2 = (1-\sqrt{5})^2/4 = (6-2\sqrt{5})/4 = (3-\sqrt{5})/2 = 1+(1-\sqrt{5})/2 = \psi+1$

Now we'll prove a third lemma by induction:

Lemma 3. *$Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$ for all nonnegative integers n.*

Proof. First, $(\phi^0 - \psi^0)/\sqrt{5} = (1-1)/\sqrt{5} = 0 = Fib(0)$.
Second, $(\phi^1 - \psi^1)/\sqrt{5} = ((1+\sqrt{5}-1+\sqrt{5})/2)/\sqrt{5} = 1 = Fib(1)$.
Now assume $Fib(a) = (\phi^a - \psi^a)/\sqrt{5}$ and $Fib(a+1) = (\phi^{a+1} - \psi^{a+1})/\sqrt{5}$ for some nonnegative integer a.
Then using Lemmas 1-2:

$$\begin{aligned}
Fib(a+2) &= Fib(a) + Fib(a+1) \\
&= (\phi^a - \psi^a + \phi^{a+1} - \psi^{a+1})/\sqrt{5} \\
&= (\phi^a(\phi+1) - \psi^a(\psi+1))/\sqrt{5} \\
&= (\phi^a(\phi^2) - \psi^a(\psi^2))/\sqrt{5} \\
&= (\phi^{a+2} - \psi^{a+2})/\sqrt{5}
\end{aligned}$$

Thus by induction, $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$ for all nonnegative integers n. □

Rearranging Lemma 3, $\phi^n/\sqrt{5} = Fib(n) + \psi^n/\sqrt{5}$. So we need to show $0.5 < \psi^n/\sqrt{5} < 0.5$ for all nonnegative integers n to finish the proof.
Now $\psi = (1-\sqrt{5})/2 \approx -0.618$.
Since $-1 < \psi < 1$, then $-1 < \psi^n \leq 1$, and $-\frac{1}{\sqrt{5}} < \frac{\psi^n}{\sqrt{5}} \leq \frac{1}{\sqrt{5}}$
That is, $-0.447 < \frac{\psi^n}{\sqrt{5}} \leq 0.447$, so $-0.5 < \frac{\psi^n}{\sqrt{5}} < 0.5$. With Lemma 3, this shows that there is less than a 0.5 difference between $Fib(n)$ and $\phi^n/\sqrt{5}$, which means $Fib(n)$ is the closest integer to $\phi^n/\sqrt{5}$. □