Turnstile Puzzles are PSPACE-Complete Even When Restricted to Thick L-shape or l-shape Turnstiles

CSCI 361 Theory of Computation (Prof. Aaron Williams)

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***Abstract*—This report examines the computational complexity of turnstile puzzles, with a particular focus on cases where the puzzle is limited to certain types of turnstiles. We report gadget constructions that show turnstile puzzles limited to thick L-shape turnstiles or thick l-shape turnstiles are still PSPACE-Complete.**

I. Introduction

Turnstile puzzles are single-player puzzle games played on a grid, where the goal of the player is to move their token from the start position to the end position. The movement of the token is obstructed by either immovable walls or movable turnstiles, which the player may push to rotate them in 90° increments.

We can further classify different types of turnstiles based on their *thickness* and *shape*. First, there are two types of turnstiles, *thick* or *thin*, depending on whether the turnstile occupies grid cells or only grid lines. Also, there are five shapes of turnstiles with varying arm lengths. A full summary of different turnstile types is presented in [1] and duplicated here as Figure 1.

In this project, we study the computational complexity of turnstile puzzles with restrictions on the types of turnstiles that can be included in the puzzle. Specifically, we show that turnstile puzzles are PSPACE-Complete even when restricted to either thick L-shaped turnstiles or thick l-shaped turnstiles. Our findings extend the results of Greenblatt el al. that proved the PSPACE-Completeness of cases limited to thick or thin T-shaped turnstiles [1]. Together, the results hint at the hardness of solving turnstile puzzles in general.

II. Previous Results

In proving the PSPACE-Completness of turnstile puzzles with only thick or thin T-shape turnstiles, previous literature presented a reduction proof based on the Crossing 2-Toggle Path Problem (C2TPP) [1]. C2TPP originates from a more general gadget model that serves as a great framework for formulating hardness proofs for *single agent motion planning problems*, problems where a player tries to move a robot from some start location to a designated end location [2]. Turnstile puzzles are indeed one of such motion planning problems, and results from Demaine et al. that motion planning with any type of 2-toggle gadget is PSPACE-Complete can be well applied to our analysis of turnstile puzzles.

In this report, however, we base our approach on an even more generalized result. Demaine et al. report that 1-player motion planning with any interacting-k-tunnel reversible deterministic gadget is PSPACE-Complete [3]. While the restrictions on reversibility and determinism persist, this finding extends the number of states allowed in the gadgets we will be constructing with thick L-shape and thick l-shape turnstiles.

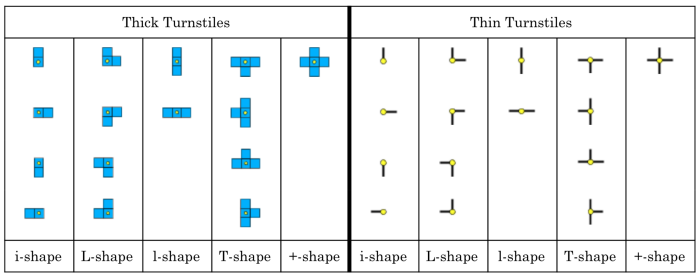

Fig. 1. Rotations of the five different shapes of turnstiles, both thick and thin.

III. Gadgets for PSPACE-Completeness Proof

Now, we show that an interacting-k-tunnel reversible deterministic gadget can be constructed using turnstile puzzles that only involve thick L-shape turnstile or thick l-shape turnstiles. Figure 2 describes a gadget that uses *both* thick L-shape and thick l-shape turnstiles with three states 1, 2, and 3.

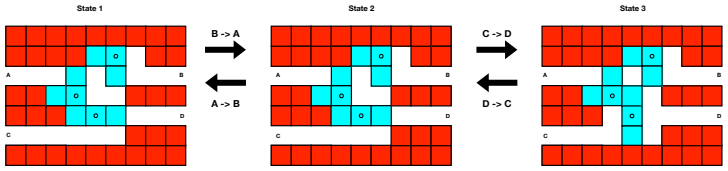

Fig. 2. Gadget constructed with both thick L-shape and thick l-shape turnstiles. Between states 1 and 2: player may only enter and exit A and B. Between states 2 and 3: player may only enter and exit C and D.

As in [1], we can verify the following points:

- If the gadget is in state 1, then a B-to-A traversal changes the gadget to state 2.
- If the gadget is in state 2, then an A-to-B traversal changes the gadget to state 1.
- If the gadget is in state 2, then a C-to-D traversal changes the gadget to state 3.
- If the gadget is in state 3, then a D-to-C traversal changes the gadget to state 2.

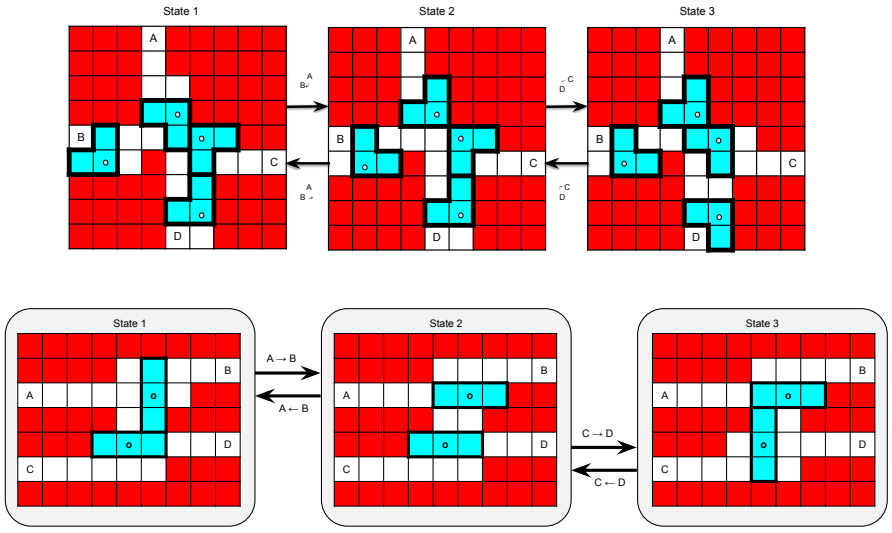

Fig. 3. Interacting tunnel gadgets with 3 states. (a) Construction with only thick L-shape turnstiles. (b) Construction with only thick l-shape turnstiles.

These points ensure that the traversals in A/B tunnel or C/D tunnel cannot be performed at once from the same state. Furthermore, no other traversals other than the ones listed are possible in any of the states, and therefore, our construction is an interacting-k-tunnel gadget.

Similarly, we can also construct a gadget with only thick L-shape or thick l-shape turnstiles as shown in Figure 3 (a) and (b). The same, if not similar, points we discussed above apply to our new gadgets as well. Notice that no other traversals other than the ones listed are possible in any of the states.

Therefore, combining the two gadget constructions, we state the following theorem.

Theorem: The turnstile puzzle is PSPACE-Complete even when restricted to thick L-shaped turnstiles or thick l-shape turnstiles.

IV. Open Problems and Outlook

While this report has shown PSPACE-Completeness for thick L-shape and thick l-shape turnstiles, the question of how to construct similar gadgets with other types of turnstiles remain as open problems. Specifically, constructions with:

- thin L-shape turnstiles
- thin l-shape turnstiles
- thick and thin i-shape turnstiles

would be the immediate next steps towards a more comprehensive understanding of the hardness of turnstile puzzles.

Working with thin turnstiles also seem to pose additional challenges with respect to defining their precise model of physics–that is, when and how are thin turnstiles allowed to overlap with one another. Ideally, one would be able to construct a 2-toggle or interacting-tunnel gadget that does not involve any overlaps, yet it remains unclear if such construction is possible with thin L-shape, l-shape or i-shape turnstiles.

Figure 4 illustrates a preliminary attempt on constructing a reverasible, deterministic 3-state interacting-tunnel gadget with both thin L-shape and thin l-shape turnstiles. The implied turnstile physics is that the grid walls of the thin turnstiles themselves occupy space, such that no overlap between turnstiles are allowed.

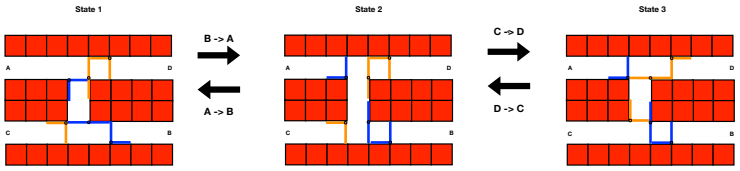

Fig. 4. Gadget constructed with both thin L-shape and thin l-shape turnstiles. Between states 1 and 2: player may only enter and exit A and B. Between states 2 and 3: player may only enter and exit C and D.

For additional open problems in this field of research, we direct the readers to [1].

References

[1] A. Greenblatt, O. Hernandez, R. A. Hearn, Y. Hou, H. Ito, A. Williams, and A. Winslow, "Around and Around: Turnstile Puzzles are PSPACE-Complete," *Draft*, 2019.

[2] E. D. Demaine, I. Grosof, J. Lynch, and M. Rudoy, "Computational complexity of motion planning of a robot through simple gadgets," *FUN 2018*, June 2018.

[3] E. D. Demaine, D. H. Hendrickson, and J. Lynch, "A General Theory of Motion Planning Complexity: Characterizing Which Gadgets Make Games Hard," *arXiv:1812.03592*, Dec 2018.