Metric Thickenings of Euclidean Submanifolds

GSTGC 2018

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Colorado State University

Introduction

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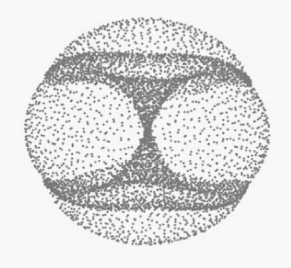
Energy landscape of cyclo-octane:



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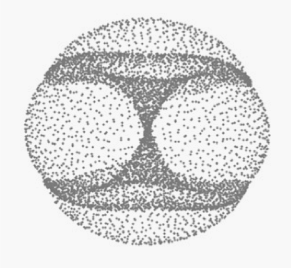

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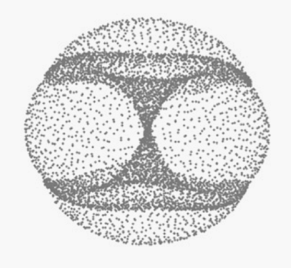

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Given a data set, can we describe the underlying space?

Reconstructing a Manifold

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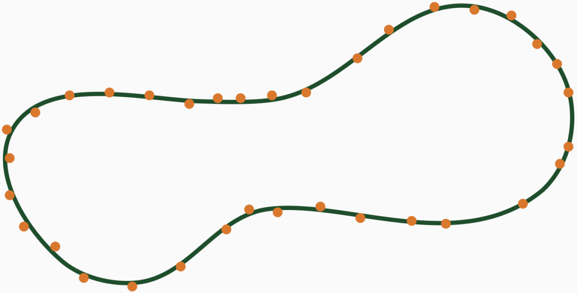

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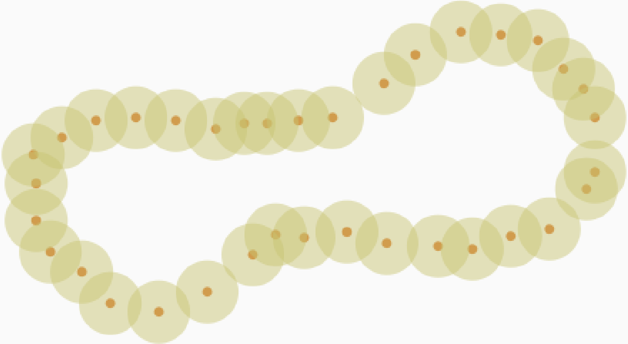

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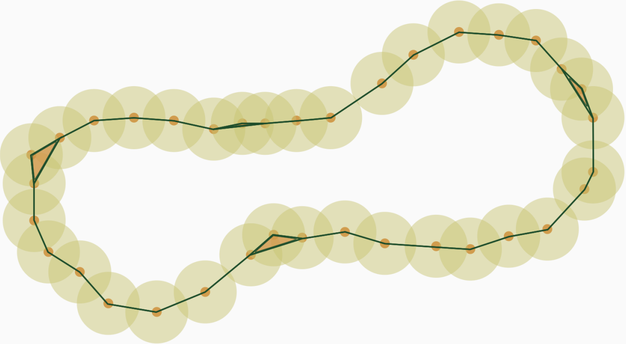

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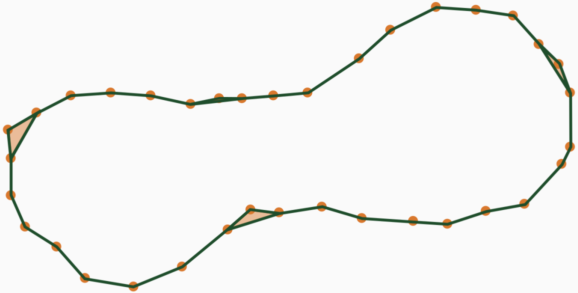

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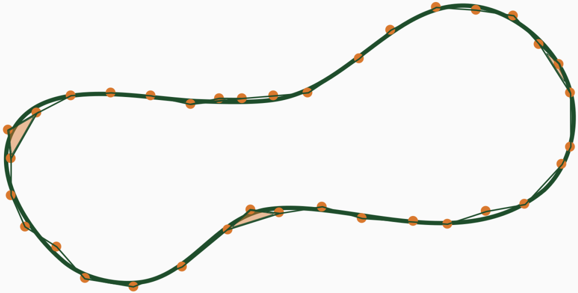

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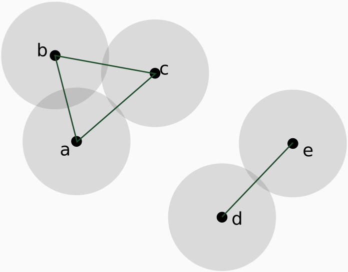
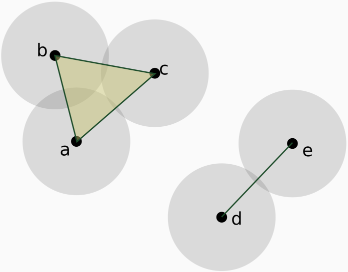
  - ▷ What information is contained at different scale parameters?
- Manifold Reconstruction:
  - ▷ Does any scale parameter give a simplicial complex with the “correct” homology (or homotopy type)?
  - ▷ Does the simplicial complex have predictable structure at “bad” scale parameters?

Technical Aside #1

(Čech complex)



(Vietoris–Rips complex)



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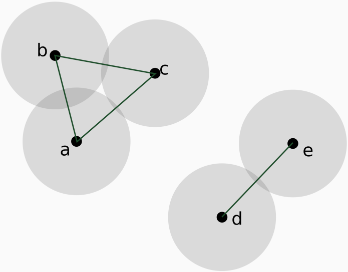

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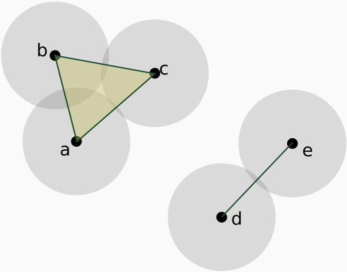

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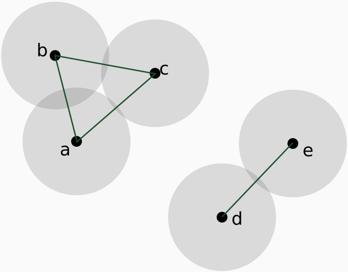

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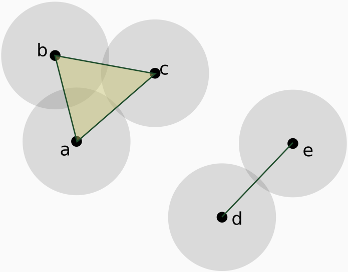

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- Vietoris–Rips complex, $\mathrm{VR}(X; r)$, contains an n-simplex for every set of $n+1$ points with diameter $< r$.
- Write $K(X; r)$ when the distinction is unimportant.

Model Theorems

Theorem (Hausmann '95)

Let M be a compact Riemannian manifold and $r > 0$ be sufficiently small (depending on curvature of M). Then $\mathrm{VR}(M; r) \simeq M$.

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Theorem (Niyogi, Smale, Weinberger '05)

Let Y be a sufficiently dense sampling (possibly with noise) of a Euclidean submanifold M, and $r > 0$ sufficiently small. Then $\check{\mathrm{C}}(Y; r) \simeq M$.

Metric Thickenings

Let X be a metric space, $r \geq 0$, and $K(X;r)$ either a Vietoris–Rips of Čech complex.

Definition

The Metric thickening $K^m(X;r)$ is the set

$$\left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1,\ [x_0, \ldots, x_k] \text{ a simplex in } K(X;r) \right\},$$

equipped with the 1-Wasserstein metric.

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Definition

The 1-Wasserstein metric on $\mathrm{VR}^m(X;r)$ is the distance defined by

$$d_W(x, x') = \inf\left\{\mathrm{cost}(p) \;\middle|\; p \text{ is a matching between } x \text{ and } x'\right\}.$$

Established Facts

Theorem (Adamaszek, Adams, Frick '17)

- $\mathrm{VR}^m(X;r) \cong \mathrm{VR}(X;r)$ *if and only if* $\mathrm{VR}(X;r)$ *is locally finite.*
- $\mathrm{VR}^m(X;r)$ *is an* r*-thickening of* X*: The metric of* X *extends to that of* $\mathrm{VR}^m(X;r)$ *and* $d(x.\mathrm{VR}^m(X;r)) < r$ *for all* $x \in X$*.*
- *Hausmann's theorem holds: if* X *is a Riemannian manifold, then for* r *sufficiently small* $\mathrm{VR}^m(X;r) \simeq X$*.*

Results

Main Theorem

Theorem (Adams, M. '17)

Let $X \subseteq \mathbb{R}^n$ *and suppose the reach,* τ*, of* X *is positive. Then for all* $r < \tau$*, the metric Vietoris–Rips thickening* $\mathrm{VR}^m(X; r)$ *is homotopy equivalent to* X*.*

For all $r < 2\tau$*, the metric Čech thickening* $\check{\mathrm{C}}^{\mathrm{m}}(X; r)$ *is homotopy equivalent to* X*.*

Technical Aside #2

The medial axis of $X \subseteq \mathbb{R}^n$ is the closure, $\overline{Y}$, of

$$Y = \{y \in \mathbb{R}^n \mid \exists\, x_1 \neq x_2 \in M \text{ with } d(y, x_1) = d(y, x_2) = d(y, X)\}\,.$$

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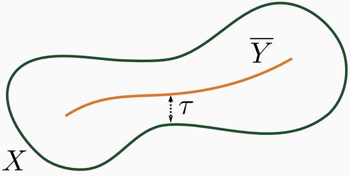

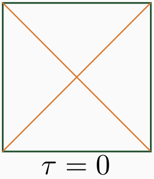

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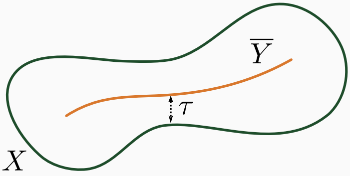
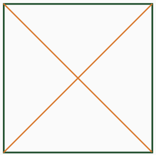


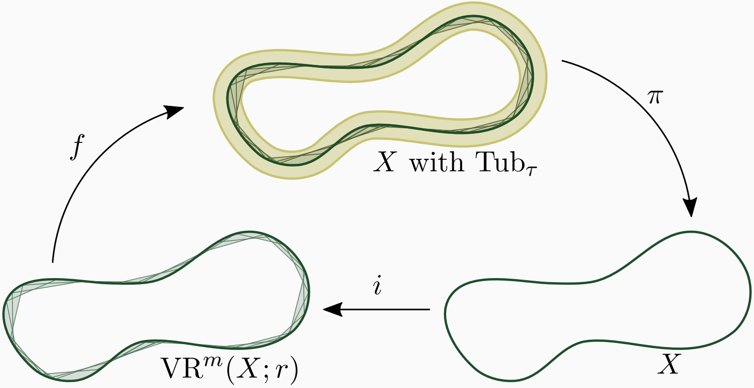


$\tau = 0$

Smooth manifolds (embedded in $\mathbb{R}^n$) have positive reach. Sets with corners have zero reach.

Proof.

$\pi \circ f$ and i are homotopy inverses:



$\square$

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- Similar results for (infinite) dense samplings.
- Show stability with regard to persistence.

References

[1] M. Adamaszek, H. Adams, and F. Frick, *Metric reconstruction via optimal transport*, arXiv preprint arXiv:1706.04876, (2017).

[2] H. Adams and J. Mirth, *Metric thickenings of Euclidean submanifolds*, arXiv:1709.02492 [math], (2017).

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[5] S. Martin, A. Thompson, E. A. Coutsias, and J.-P. Watson, *Topology of cyclo-octane energy landscape*, The Journal of Chemical Physics, 132 (2010), p. 234115.

[6] P. Niyogi, S. Smale, and S. Weinberger, *Finding the Homology of Submanifolds with High Confidence from Random Samples*, Discrete & Computational Geometry, 39 (2008), pp. 419–441.

Slides available at `http://www.math.colostate.edu/~mirth/talks.html`