Metric Thickenings of Euclidean Submanifolds

Henry Adams and Joshua Mirth, Colorado State University

SIAM Central States – September 30, 2017

Background

Definition

Let X be a metric space and $r > 0$ a scale parameter. The Vietoris–Rips complex, $\mathrm{VR}_{\leq}(X; r)$, of X, has vertex set X and a simplex for every finite subset $\sigma \subseteq X$ such that $\mathrm{diam}(\sigma) \leq r$.

Hausmann's Theorem

Theorem

Let M be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then $\mathrm{VR}(M; r) \simeq M$.

Hausmann's Theorem

Theorem

Let M be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then $\mathrm{VR}(M; r) \simeq M$.

- The bound on r depends upon the curvature of M.

Hausmann's Theorem

Theorem

Let M be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then $\mathrm{VR}(M; r) \simeq M$.

- The bound on r depends upon the curvature of M.
- $\mathrm{VR}(M; r)$ does not inherit the metric of M.

Hausmann's Theorem

Theorem

Let M be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then $\mathrm{VR}(M; r) \simeq M$.

- The bound on r depends upon the curvature of M.
- $\mathrm{VR}(M; r)$ does not inherit the metric of M. Thus:
  - Hausmann's proof only gives a map $T \colon \mathrm{VR}(M; r) \to M$, and proves the equivalence using algebraic techniques.

Theorem

Let M be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then $\mathrm{VR}(M;r) \simeq M$.

- The bound on r depends upon the curvature of M.
- $\mathrm{VR}(M;r)$ does not inherit the metric of M. Thus:
  - Hausmann's proof only gives a map $T\colon \mathrm{VR}(M;r) \to M$, and proves the equivalence using algebraic techniques.
  - T depends upon a total order of the points in M.

Hausmann's Theorem

Theorem

Let M be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then $\mathrm{VR}(M; r) \simeq M$.

- The bound on r depends upon the curvature of M.
- $\mathrm{VR}(M; r)$ does not inherit the metric of M. Thus:
  - Hausmann's proof only gives a map $T\colon \mathrm{VR}(M; r) \to M$, and proves the equivalence using algebraic techniques.
  - T depends upon a total order of the points in M.
  - In particular, the inclusion $\iota\colon M \hookrightarrow \mathrm{VR}(M; r)$ does not provide the inverse (in fact, ι is not even continuous.)

Metric Thickenings

Definition (Adamaszek, Adams, Frick)

For a metric space X and $r \geq 0$, the Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ is the set

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ x_i \in X, \text{ and } \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \right\}$$

equipped with the 1-Wasserstein metric.[1]

Metric Vietoris–Rips Thickenings

Definition (Adamaszek, Adams, Frick)

For a metric space X and $r \geq 0$, the Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ is the set

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ x_i \in X, \text{ and } \operatorname{diam}(\{x_0, \ldots, x_k\}) \leq r \right\}$$

equipped with the 1-Wasserstein metric.[1]

- As a set this is identical to the geometric realization of $\mathrm{VR}(X;r)$, but the topology is different.

Definition (Adamaszek, Adams, Frick)

For a metric space X and $r \geq 0$, the Vietoris–Rips thickening $\mathrm{VR}^m(X; r)$ is the set

$$\mathrm{VR}^m(X; r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ x_i \in X, \text{ and } \operatorname{diam}(\{x_0, \ldots, x_k\}) \leq r \right\}$$

equipped with the 1-Wasserstein metric.[1]

- As a set this is identical to the geometric realization of $\mathrm{VR}(X; r)$, but the topology is different.
- By identifying $x \in X$ with $\delta_x \in \mathcal{P}(X)$, we can view $\mathrm{VR}^m(X; r)$ as a subset of $\mathcal{P}(X)$, the set of all Radon probability measures on X.

Metric Vietoris–Rips Thickenings

Definition (Adamaszek, Adams, Frick)

For a metric space X and $r \geq 0$, the Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ is the set

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ x_i \in X, \text{ and } \operatorname{diam}(\{x_0, \ldots, x_k\}) \leq r \right\}$$

equipped with the 1-Wasserstein metric.[1]

- As a set this is identical to the geometric realization of $\mathrm{VR}(X;r)$, but the topology is different.
- By identifying $x \in X$ with $\delta_x \in \mathcal{P}(X)$, we can view $\mathrm{VR}^m(X;r)$ as a subset of $\mathcal{P}(X)$, the set of all Radon probability measures on X.
- This makes $\mathrm{VR}^m(X;r)$ a (metric) thickening of X.

Wasserstein Metric

Let $x, x' \in \mathrm{VR}^m(X; r)$ with $x = \sum_{i=0}^{k} \lambda_i x_i$ and $x' = \sum_{i=0}^{k'} \lambda'_i x'_i$. Define a matching p between x and x' to be any collection of non-negative real numbers $\{p_{i,j}\}$ such that $\sum_{j=0}^{k'} p_{i,j} = \lambda_i$ and $\sum_{i=0}^{k} p_{i,j} = \lambda'_j$. Define the cost of the matching p to be $\mathrm{cost}(p) = \sum_{i,j} p_{i,j} d(x_i, x'_j)$.

Definition

The 1-Wasserstein metric on $\mathrm{VR}^m(X; r)$ is the distance d_W defined by

$$d_W(x, x') = \inf \left\{ \mathrm{cost}(p) \;\middle|\; p \text{ is a matching between } x \text{ and } x' \right\}.$$

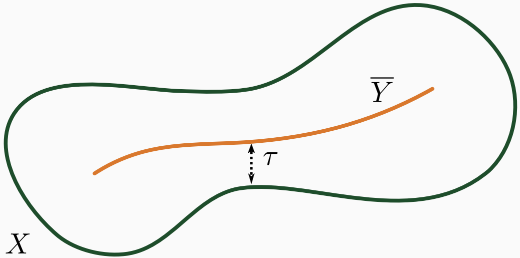
Euclidean Submanifolds

Positive Reach

The medial axis of $X \subseteq \mathbb{R}^n$ is the closure, $\overline{Y}$, of

$$Y = \{y \in \mathbb{R}^n \mid \exists\, x_1 \neq x_2 \in M \text{ with } d(y, x_1) = d(y, x_2) = d(y, X)\}\,.$$

The reach, τ, of X is the minimal distance $\tau = d(X, \overline{Y})$ between X and its medial axis.



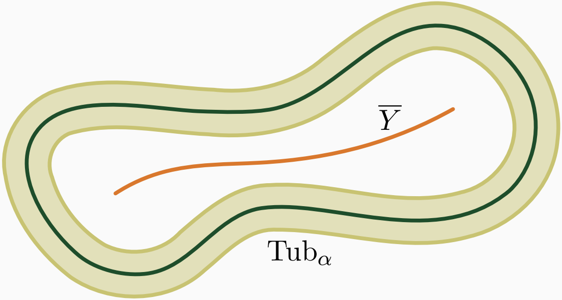
Smooth manifolds embedded in $\mathbb{R}^n$ have positive reach.

Nearest Point Projection

Define the α-offset of $X \subseteq \mathbb{R}^n$:

$$\mathrm{Tub}_\alpha = \{x \in \mathbb{R}^n \mid d(x, X) < \alpha\} = \bigcup_{x \in X} \mathrm{B}(x, \alpha).$$

If X has reach τ, then $\pi \colon \mathrm{Tub}_\tau \to X$ where x maps to its nearest point in X is well-defined and continuous [3].



Results

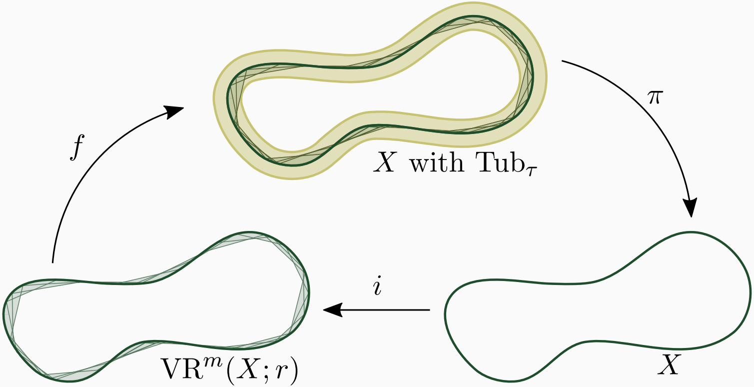
Main Theorem

Theorem (Adams and M.)

Let $X \subseteq \mathbb{R}^n$ and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Vietoris–Rips thickening $\mathrm{VR}^m(X; r)$ is homotopy equivalent to X.

Theorem (Adams and M.)

Let $X \subseteq \mathbb{R}^n$ and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Vietoris–Rips thickening $\mathrm{VR}^m(X; r)$ is homotopy equivalent to X.



Lemmas

Lemma

For $X \subseteq \mathbb{R}^n$ *and* $r > 0$, *the linear projection map* $f\colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ *has its image contained in* $\overline{\mathrm{Tub}_r}$.

Lemma

For $X \subseteq \mathbb{R}^n$ and $r > 0$, the linear projection map $f \colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ has its image contained in $\overline{\mathrm{Tub}_r}$.

Proof.

Let $x = \sum_{i=0}^{k} \lambda_i x_i \in \mathrm{VR}^m(X;r)$; we have

$$\mathrm{diam}(\mathrm{conv}\{x_0, \ldots, x_k\}) = \mathrm{diam}([x_0, \ldots, x_k]) \leq r.$$

Since $f(x) \in \mathrm{conv}\{x_0, \ldots, x_k\}$, it follows that $d(f(x), X) \leq d(f(x), x_0) \leq r$, and so $f(x) \in \overline{\mathrm{Tub}_r}$. $\square$

Lemmas

Lemma

For $X \subseteq \mathbb{R}^n$ and $r > 0$, the linear projection map $f\colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ has its image contained in $\overline{\mathrm{Tub}_r}$.

Proof.

Let $x = \sum_{i=0}^{k} \lambda_i x_i \in \mathrm{VR}^m(X;r)$; we have

$$\mathrm{diam}(\mathrm{conv}\{x_0, \ldots, x_k\}) = \mathrm{diam}([x_0, \ldots, x_k]) \leq r.$$

Since $f(x) \in \mathrm{conv}\{x_0, \ldots, x_k\}$, it follows that $d(f(x), X) \leq d(f(x), x_0) \leq r$, and so $f(x) \in \overline{\mathrm{Tub}_r}$. $\square$

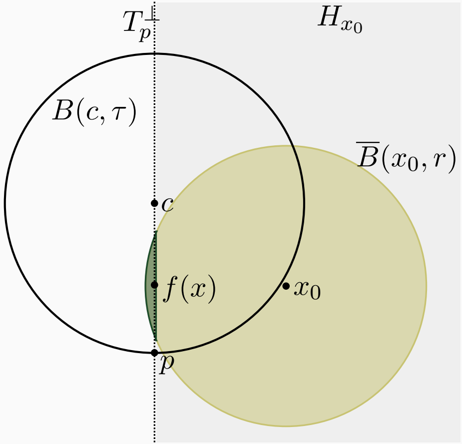
Lemma

Let $x_0, \ldots, x_k \in \mathbb{R}^n$, let $y \in \mathrm{conv}\{x_0, \ldots, x_k\}$, and let C be a convex set with $y \notin C$. Then there is at least one x_i with $x_i \notin C$.

Lemma

Let $X \subseteq \mathbb{R}^n$ have positive reach τ, let $[x_0, \ldots x_k]$ be a simplex in $\mathrm{VR}(X; r)$ with $r < \tau$, let $x = \sum \lambda_i x_i \in \mathrm{VR}^m(X; r)$, and let $p = \pi(f(x))$. Then the simplex $[x_0, \ldots, x_k, p]$ is in $\mathrm{VR}(X; r)$.

Proof.



Main Result

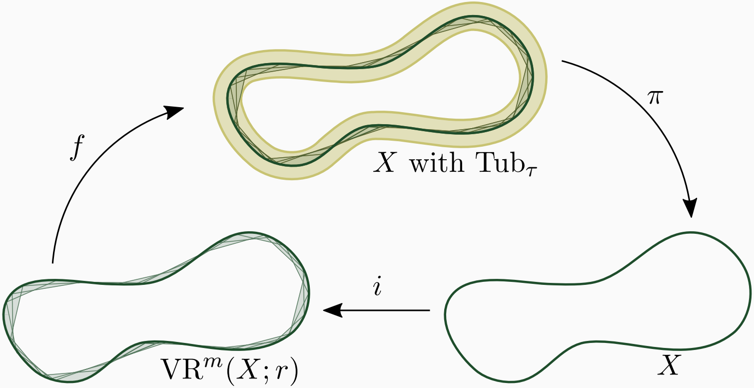
We are now prepared to prove our main result.

Theorem

Let X be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Vietoris–Rips thickening $\mathrm{VR}^m(X; r)$ is homotopy equivalent to X.

Proof.

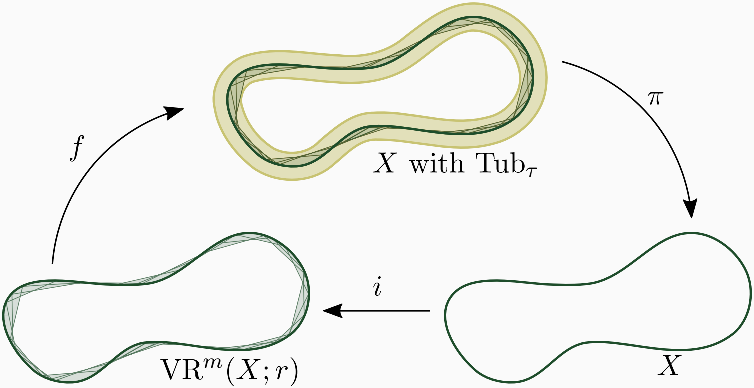
By [1, Lemma 5.2], map $f\colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ is 1-Lipschitz and hence continuous. It follows from Lemma 6 that the image of f is a subset of Tub_τ. Let $i\colon X \to \mathrm{VR}^m(X;r)$ be the inclusion map. Note that $\pi \circ f \circ i = \mathrm{id}_X$.



$\square$

Proof.

Consider $H\colon \mathrm{VR}^m(X;r) \times I \to \mathrm{VR}^m(X;r)$ defined by $H(x,t) = t \cdot \mathrm{id}_{\mathrm{VR}^m(X;r)} + (1-t) i \circ \pi \circ f$. H is well-defined by Lemma 8, and continuous by [1, Lemma 3.8].It follows that H is a homotopy equivalence from $i \circ \pi \circ f$ to $id_{\mathrm{VR}^m(X;r)}$.



□

Conclusions

- Analogue of Hausmann in Euclidean space.

Conclusions

- Analogue of Hausmann in Euclidean space.
- For a Riemannian version see [1]. Or:

Corollary

If N is a smooth, compact, Riemannian manifold, there exists a $\tau > 0$ such that $\mathrm{VR}^m(N; r) \simeq N$ for all $0 < r < \tau$.

Proof.

This follows from the Nash Embedding theorem [7]. $\square$

Conclusions

- Analogue of Hausmann in Euclidean space.
- For a Riemannian version see [1]. Or:

Corollary

If N is a smooth, compact, Riemannian manifold, there exists a $\tau > 0$ such that $\mathrm{VR}^m(N; r) \simeq N$ for all $0 < r < \tau$.

Proof.

This follows from the Nash Embedding theorem [7]. $\square$

- The same techniques hold for metric Čech thickenings.

Conclusions

- Analogue of Hausmann in Euclidean space.
- For a Riemannian version see [1]. Or:

Corollary

If N is a smooth, compact, Riemannian manifold, there exists a $\tau > 0$ such that $\mathrm{VR}^m(N; r) \simeq N$ for all $0 < r < \tau$.

Proof.

This follows from the Nash Embedding theorem [7]. $\square$

- The same techniques hold for metric Čech thickenings.
- Worth considering version for dense-samplings [6][2].

References

[1] M. Adamaszek, H. Adams, and F. Frick, *Metric reconstruction via optimal transport*. **Preprint,** `arxiv/1706.04876`**.**

[2] F. Chazal and S. Oudot, *Towards persistence-based reconstruction in Euclidean spaces*, in Proceedings of the 24th Annual Symposium on Computational Geometry, ACM, 2008, pp. 232–241.

[3] H. Federer, *Curvature measures*, Transactions of the American Mathematical Society, 93 (1959), pp. 418–491.

[4] J.-C. Hausmann, *On the Vietoris–Rips complexes and a cohomology theory for metric spaces*, in Prospects In Topology, F. Quinn, ed., vol. 138 of Annals of Mathematics Studies, Princeton University Press, 1995, pp. 175–188.

[5] H. Karcher, *Riemannian center of mass and mollifier smoothing*, Communications on pure and applied mathematics, 30 (1977), pp. 509–541.

[6] J. Latschev, *Vietoris–Rips complexes of metric spaces near a closed Riemannian manifold*, Archiv der Mathematik, 77 (2001), pp. 522–528.

[7] J. Nash, *The imbedding problem for Riemannian manifolds*, Annals of Mathematics, 63 (1956), pp. 20–63.

[8] P. Niyogi, S. Smale, and S. Weinberger, *Finding the homology of submanifolds with high confidence from random samples*, Discrete Computational Geometry, 39 (2008), pp. 419–441.