Vietoris–Rips Thickenings of Euclidean Submanifolds

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Motivation

Given a metric space, X, can we produce a simplicial complex which is homotopy equivalent to X? One solution to this question is the Vietoris–Rips complex, $\mathrm{VR}(X;r)$. Hausmann [3] proves that for X a Riemannian manifold and a sufficiently small scale parameter r there is a homotopy equivalence $\mathrm{VR}(X;r) \simeq X$. In fact, Latschev [4] shows that if $Y \subset X$ is a sufficiently dense sampling then $\mathrm{VR}(Y;r) \simeq X$. However, $\mathrm{VR}(X;r)$ loses the metric properties of X; if $\mathrm{VR}(X;r)$ is not locally finite, then it is not metrizable. We seek to address this with the *Vietoris–Rips thickening* of X.

Definitions

Let X be a metric space and r a scale parameter with $r \geq 0$. The **Vietoris–Rips complex** of X with scale parameter r, $\mathrm{VR}_{\leq}(X;r)$, has vertex set X and a simplex for every finite subset $\sigma \subset X$ such that $\mathrm{diam}(\sigma) \leq r$.

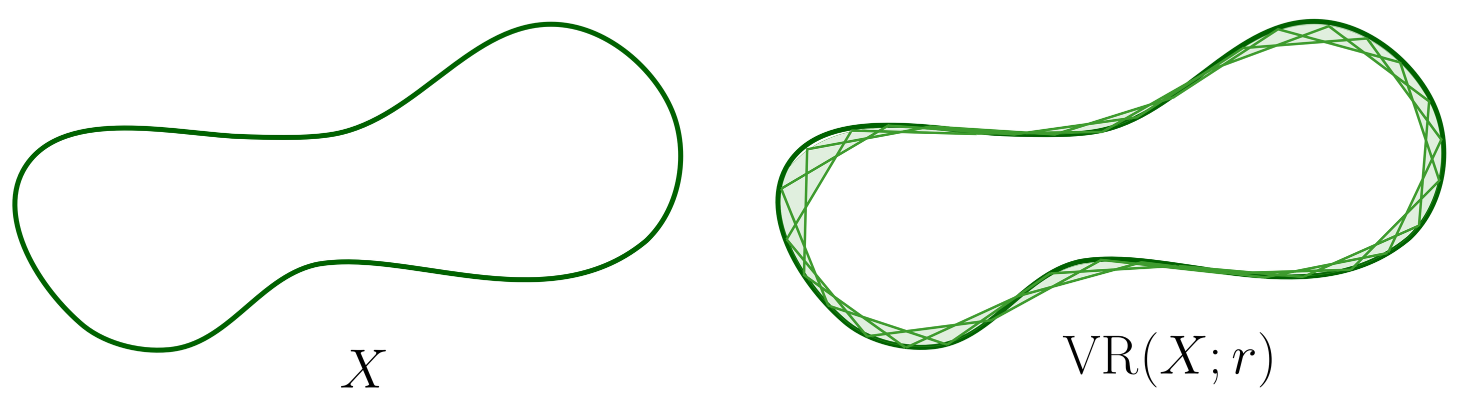

Figure 1: A manifold and (a subset of) its Vietoris–Rips complex.

The geometric realization $|\mathrm{VR}(X;r)|$ consists of all finite convex linear combinations of points in $\mathrm{VR}(X;r)$. The **Vietoris–Rips thickening**, $\mathrm{VR}^m(X;r)$, is $|\mathrm{VR}(X;r)|$ equipped with the 1-Wasserstein metric:

$$d_W\left(\sum_{i=0}^{k} \lambda_i x_i \,,\, \sum_{j=0}^{k'} \lambda'_j x'_j\right) = \inf_{p_{i,j}\geq 0} \left\{ \sum_{i,j} p_{i,j} d(x_i, x'_j) \;\middle|\; \sum_j p_{i,j} = \lambda_i, \sum_i p_{i,j} = \lambda'_j \right\}.$$

$\mathrm{VR}^m(X;r)$ is a metric thickening of X [1]. This means that in general $\mathrm{VR}^m(X;r)$ is not homeomorphic to $\mathrm{VR}(X;r)$.

Theorem (Adamaszek, Adams, Frick)

Let X be a complete Riemannian manifold and r sufficiently small. Then $\mathrm{VR}^m(X;r) \simeq X$.

Background

We are interested in the case where $X \subset \mathbb{R}^n$ is a set with positive reach [2]. In particular, for $k \geq 2$, all C^k Euclidean submanifolds have positive reach [5]. The **reach**, τ, of X is the distance to its medial axis. To define the medial axis, consider the set of points without the nearest point property:

$$Y = \{y \in \mathbb{R}^n \mid \exists\, x_1 \neq x_2 \in X \text{ with } d(y,x_1) = d(y,x_2) = d(y,X)\}\,.$$

The **medial axis** is then the closure of Y. The set of all points in $\mathbb{R}^n$ within τ of X is a tubular neighborhood, Tub_τ.

Main Theorem

Let $X \subset \mathbb{R}^n$ with reach $\tau > 0$. Then for all $r < \tau$ we have $\mathrm{VR}^m(X;r) \simeq X$.

Proof Outline

Let $f\colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ be the linear projection map, $\pi\colon \mathbb{R}^n \to X \subset \mathbb{R}^n$ the nearest-point map, and i the inclusion of $X \hookrightarrow \mathrm{VR}^m(X;r)$. Then πf and i are homotopy inverses:

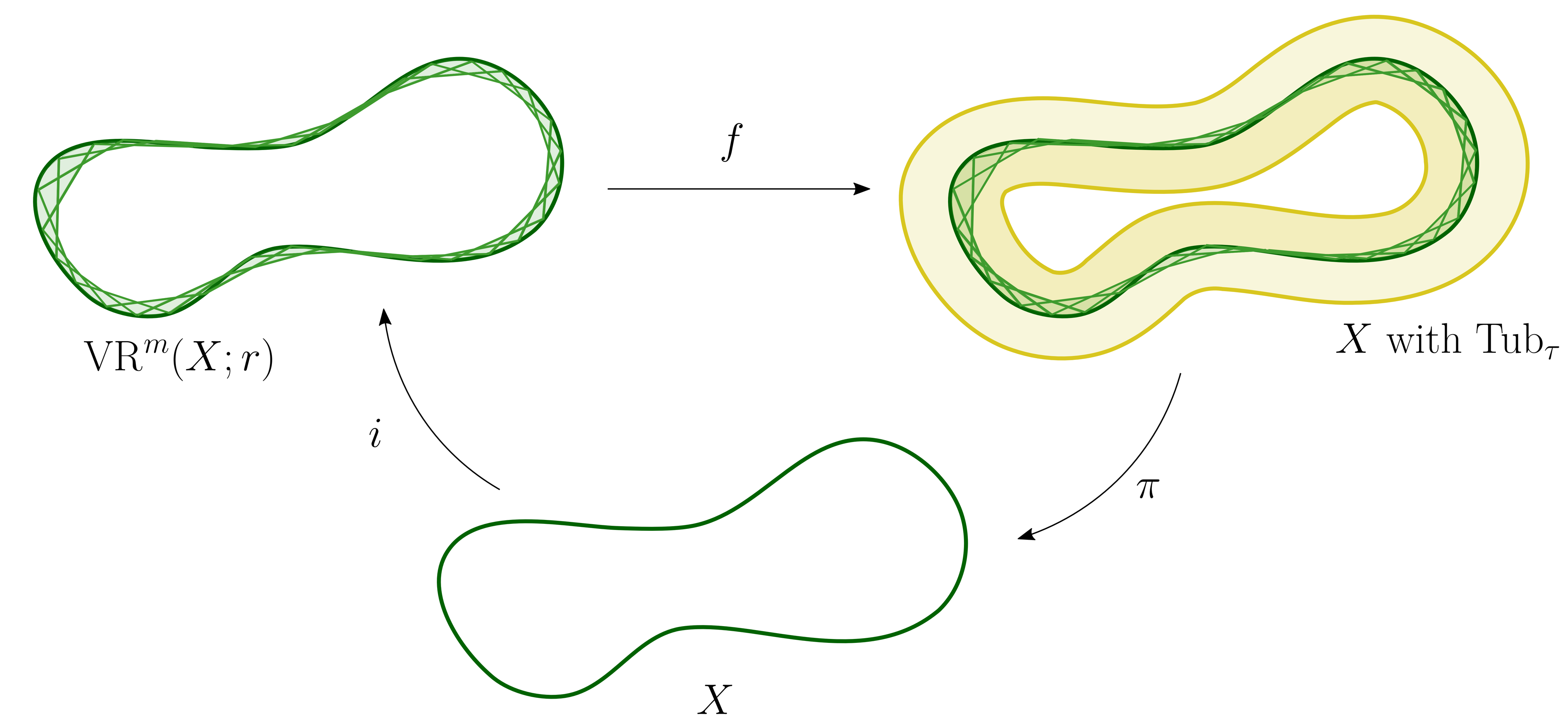

Figure 2: The homotopy equivalence between $\mathrm{VR}^m(X;r)$ and X.

Note that π is well-defined and continuous because $f(\mathrm{VR}^m(X;r))$ lies within Tub_τ. We have $\pi f \circ i = \mathrm{id}_X$ and $i \circ \pi f \simeq \mathrm{id}_{\mathrm{VR}^m(X;r)}$ via a linear homotopy.

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