Dissertation Proposal – Morse Theory for Wasserstein Spaces and Simplicial Metric Thickenings

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Contents

1 Introduction

The aim of this thesis is to braid together three mathematical threads: Morse theory from differential topology, simplicial metric thickenings from applied topology, and gradient flows in the theory of optimal transport. Pairwise these topics all intersect. A key element of Morse theory and of optimal transport is producing a curve, or family of curves, which flow along the gradient of a certain functional. Simplicial metric thickenings and optimal transport are related by their use of the Wasserstein metric, and the motivating question behind Morse theory and simplicial metric thickenings is to understand the homotopy type of certain topological spaces. Our contribution is to bring all three together. We do this by taking a Morse-theoretic approach to computing the homotopy type of certain simplicial metric thickenings (specifically, the Vietoris–Rips thickenings of the circle). Since these spaces are not manifolds but spaces of probability measures, we must adopt classical Morse theory to this new setting using the theory of gradient flows for Wasserstein spaces.

The next section consists of background on the three subjects mentioned here. Section 3 provides more detailed background on our particular problem. In Section 4 we discuss our current results and outline some future directions for this research. Finally, Appendices A and B briefly describe some of our related work that is not directly a part of the main argument of this thesis.

2 Background and Literature Review

2.1 Morse Theory

Marsten Morse began the study of the field that now bears his name in 1934 with [19]. The theory was extended through the 1950s and 1960s, eventually laying the foundations for Stephen Smale's proof of the generalized Poincaré conjecture in dimensions $n \geq 5$ in 1961. The full story of this remarkable theory is contained in Milnor's classic studies [16] and [17].

The premise of the theory is as follows: consider a smooth manifold M. We may suppose (for convenience, not necessity) by Whitney's theorem that it is embedded in some $\mathbb{R}^n$. Choose any vector $\vec{v} \in \mathbb{R}^n$. The function $L_{\vec{v}}\colon M \to \mathbb{R}$ given by $\vec{x} \mapsto \|\vec{v} - \vec{x}\|^2$ (where $\vec{x}$ denotes the Euclidean coordinates of a point in M) is a smooth function. Moreover, with probability 1, the choice of $\vec{v}$ gives a function $L_{\vec{v}}$ which has no degenerate critical points, meaning that at any point x where $DL_{\vec{v}} = 0$, the Hessian H of $L_{\vec{v}}$ is a full-rank matrix. In more generality, a *Morse function* is any smooth function $h\colon M \to \mathbb{R}$ without degenerate critical points. Morse functions are dense in the set of bounded, smooth functions. Now define for a given Morse function, h, the sublevel set of height a: $M^a := \{p \in M \mid f(p) \leq a\}$. For any increasing sequence of real numbers $a < b < c < \cdots$ there is an associated filtered sequence of topological spaces $M^a \subseteq M^b \subseteq M^c \subseteq \cdots$. It is quite natural to ask when M^a and M^b are homotopy equivalent, or even homeomorphic. (Indeed, this same question approached from a different angle led to the theory of persistent homology in the early 2000's.) The following lemmas characterize the main contribution of Morse theory [16]:

Lemma 2.1 (Morse Lemmas). *Let M be a smooth manifold, $h\colon M \to \mathbb{R}$ a Morse function, and D^k a k-dimensional disk. Then,*

(a) if $[a, b]$ does not contain any critical values of h, then $M^a \cong M^b$, and

(b) if a is a critical value of h such that H has k negative eigenvalues at the corresponding critical point p and $[a-\varepsilon, a+\varepsilon]$ contains no other critical values of h, then $M^{a+\varepsilon} \simeq M^{a-\varepsilon} \cup D^k$.

The second lemma can coherently speak of the critical point p because the non-degenerate condition implies that the critical point of h are isolated. The integer k is called the *index* of the critical point. The proof of (a), roughly speaking, involves taking the gradient vector field of h (this involves a choice of Riemannian structure on M) and flowing along it to construct a deformation retraction from M^b to M^a. The proof of (b) is more delicate, requiring six pages in [16] and the careful construction of a suitable "handle" to attach to $M^{a-\varepsilon}$.

Morse theory proves to be an incredibly powerful tool in differential topology for the computation of homotopy types. For example, suppose M is a compact n-dimensional manifold and h is a Morse function on M with exactly 2 critical points. Then M is necessarily homotopy equivalent to a sphere. The argument is simple. Call the critical points a and b with $a < b$. The first critical point is necessarily index-0, and so $M^a \simeq \emptyset \cup D^0 \simeq D^0$ by the second lemma. The second critical point is necessarily a global maximum, and so it has index equal to the dimension of M. Thus M is homotopy equivalent to a space with a cellular decomposition as a point and an n-dimensional disk, and the unique such space is the n-sphere. For details, see [16, Theorem 4.1].

Morse theory can be extended far beyond the statement of the two lemmas. Dr. Matthew Kahle, in talk at Colorado State University in October 2017 said that "every significant computation in topology uses either Morse theory or spectral sequences." Two extensions in particular are relevant to this thesis. First is Morse-Bott theory and the application of Morse theory to the loop-space of a manifold. This can be found in detail in [16], Part III. Let p and q be two points in a manifold M. Denote by $\Omega(M)$ the space of all smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. The space $\Omega(M)$ is not a manifold, but it is in some sense an infinite-dimensional analogue of one. The function $E\colon \Omega(M) \to \mathbb{R}$ given by length, $\gamma \mapsto \int_0^1 \|\dot{\gamma}(t)|^2 \mathrm{d}t$, functions much like $L_{\vec{v}}$ in the manifold setting. The calculus of variations shows how to compute the critical points and Hessian of E, and therefore demonstrate the Morse lemmas in the setting of $\Omega(M)$. This is of particular interest because $\pi_k(\Omega(M)) \cong \pi_{k+1}(M)$ and so it gives a method of studying the weak homotopy type of M. We hope to explore a different infinite-dimensional extension of classical Morse theory, and so draw some inspiration from this example.

A quite different take on Morse theory is *discrete* Morse theory. This applies the idea of Morse theory directly to simplicial or cellular complexes. The notion of a simplicial collapse was introduced by Whitehead in the 1930s [23]. A simplex $\sigma \in K$ is said to have an elementary free face if there is a proper face of σ which is contained in no other simplices of K. An elementary collapse is the removal from K of a free face and the interior of the unique simplex containing it. Then K is said to collapse to L, written $K \searrow L$, if there is a sequence of simplices $\sigma_1, \ldots, \sigma_n$ in K such that each can be removed in order by elementary collapse, and L is obtained by the sequence of removals. The collapsing operation is not a simplicial map but it does provide a homotopy equivalence between the geometric realizations of K and L. In the 1990s, Robin Forman developed a way of producing simplicial collapses using a Morse-theoretic idea; this has become known as discrete Morse theory [9]. This earns its name by using a type of discrete Morse function $f : K \to \mathbb{Q}$, for which the Morse lemmas hold: $f^{-1}((-\infty, b]) \searrow f^{-1}((-\infty, a])$ whenever $[a, b]$ does not contain a critical simplex, and at a critical simplex of dimension k the homotopy type changes by the gluing in of a k-cell. A description of this theory and its relation to the smooth setting can be found in [7].

Recently, a two-parameter version of discrete Morse theory, called Bestvina-Brady Morse theory, was used by Zaremsky to find the homotopy type of the Vietoris–Rips complexes of the sphere at small scale parameters [24]. Zaremsky's Morse function is the pair of natural parameters *diameter* and (negative) *dimension* for a simplex. This technique yields a nice proof that $\mathrm{VR}(\mathbb{S}^n; r) \simeq \mathbb{S}^n$ for $r \in (0, 1/4)$ (where $\mathbb{S}^n$ is given a metric with diameter $1/2$, see Section 2.2).

While discrete Morse theory is closer to studying the objects of interest in this thesis (simplicial complexes and generalizations thereof), it is actually the classical theory for manifolds which more closely resembles our approach. In particular, we work with spaces that admit some amount of differentiable structure, and for which the homotopy equivalences are given by working with flows of vector fields, rather than the more combinatorial arguments used in discrete Morse theory.

2.2 Homotopy Types of Simplicial Complexes

Persistent homology studies filtered topological spaces. As we remarked earlier, these can be built using sublevel sets, but in practice they are often filtered simplicial complexes. The Vietoris–Rips filtration of a metric space (X, d) is the sequence of simplicial complexes $\mathrm{VR}(X; r)$ with simplices defined by

$$[x_0, \ldots, x_n] \in \mathrm{VR}(X; r) \iff \max_{i,j} d(x_i, x_j) < r$$

for $r \in [0, +\infty]$.[1] The motivation for studying these complexes is that if X is data set, i.e. a finite subset of a manifold, M, then $\mathrm{VR}(X; r)$ thickens the discrete space X into something that might be topologically similar to M. A famous theorem of Hausmann shows that for small values of r,

[1] It is also possible to allow $\max_{i,j} d(x_i, x_j) \le r$. Many results hold for both conventions, but when necessary we will distinguish them by writing $\mathrm{VR}_<(X; r)$ and $\mathrm{VR}_\le(X; r)$.

$\mathrm{VR}(M;r) \simeq M$ when M is a Riemannian manifold [12], and quite a bit of work, most notably [14] and [20] have shown that $\mathrm{VR}(X;r)$ is homotopy equivalent to M when X is a sufficiently good sample from M and r is small.

The homotopy type of $\mathrm{VR}(X;r)$ when r is not sufficiently small has proven much more difficult to determine. One would like to know conditions under which $\mathrm{VR}(X;r) \simeq \mathrm{VR}(M;r)$ and at least some information about the homotopy type of $\mathrm{VR}(M;r)$ even when r is large. To date the most complete characterization of the latter type is in [1], where the homotopy type of $\mathrm{VR}(\mathbb{S}^1;r)$ was determined for all values of r.

Part of the difficulty is that when when X is not a discrete metric space the simplicial complex topology is not an altogether natural topology for $\mathrm{VR}(X;r)$. There is a canonical inclusion $X \hookrightarrow \mathrm{VR}(X;r)$ where points in X map to the vertices of $\mathrm{VR}(X;r)$. If X is not discrete, this inclusion is not a continuous map because the vertex set, in the simplicial complex topology, is discrete, and there are no non-constant continuous maps into a discrete space from a connected indiscrete one.

An alternative topology was proposed in [2] based on the following idea: points in the geometric realization of a Vietoris–Rips simplicial complex can be described with barycentric coordinates: let $x_0, \ldots, x_n$ be the vertices of some n-simplex σ (so each x_i is canonically identified with a point in X). A point in the interior of σ is given in barycentric coordinates by $(\lambda_0 x_0, \ldots, \lambda_n x_n)$, where $\sum \lambda_i = 1$ and $\lambda_i \geq 0$ is thought of as the weight on the i-th vertex. A point x in a metric space X can be canonically identified with the Dirac measure $\delta[x]$.[2] Then each point in the simplicial complex $\mathrm{VR}(X;r)$ can be identified with some finite probability distribution on X by

$$(\lambda_0 x_0, \ldots, \lambda_n x_n) \mapsto \sum_i \lambda_i \delta[x_i].$$

Now thinking of $\mathrm{VR}(X;r)$ as a space of probability measures, it can be equipped with the Wassertein metric (see Section 2.3). With this topology we refer to the space as $\mathcal{VR}(X;r)$, and it is a metric thickening of X (meaning that the inclusion map $x \mapsto \delta[x]$ is an isometry onto its image). Thus the space $\mathcal{VR}(X;r)$ retains the metric information about X which is lost in $\mathrm{VR}(X;r)$. It is these *simplicial metric thickenings* which we study here.

It is worth noting that when X is discrete, $\mathcal{VR}(X;r)$ is homeomorphic to $\mathrm{VR}(X;r)$ via the canonical map, and that when X is not discrete, these spaces need not be homeomorphic, or even homotopy equivalent ([2] gives an explicit example of this, as do we in Section 3). The most important result in [2] is that for r sufficiently small and M a Riemannian manifold, $\mathcal{VR}(M;r) \simeq M$. Our prior work [4] extends this also to manifolds embedded in Euclidean space, equipped with the Euclidean metric. These results mirror the classical result of Hausmann for the standard Vietoris–Rips complex [12]. The paper [2] concludes with a computation of the homotopy type of $\mathcal{VR}_{\leq}(\mathbb{S}^n;r_n)$ for all spheres, where r_n is the smallest scale parameter at which $\mathcal{VR}_{\leq}(\mathbb{S}^n;r_n)$ is *not* homotopy equivalent to $\mathbb{S}^n$. The homotopy type beyond this scale parameter remains conjectural.

[2]We use $\delta[x]$ rather than the more common δ_x for notational convenience later on.

2.3 Gradient Flows in Wasserstein Space

The Wasserstein metric mentioned above has a long history [22, 21]. In 1781 Monge introduced the optimal transport problem [18]. Suppose that you are given a pile of dirt and an equivoluminal hole at some distance; in order to minimize effort (here thought of as mass times distance), which shovelful of dirt should be placed in which part of the hole? A rigorous formulation of this questions is to think of the pile and hole as probability distributions μ and ν. Then a transport plan is a function T such that the pushforward $T_{\#}\mu = \nu$. The associated effort is $\int d(x, T(x))\mathrm{d}\mu$, and the optimal transport map is that T which minimizes this. Unfortunately, this problem often fails to have a solution. (Suppose that $\mu = \delta[p]$ and ν is any measure supported on at least two points. Then it is impossible for any function to push μ onto ν.) Kantorovich in the 1940s showed that if the restriction to pushforward measures is removed, then there is always a solution [13]. In more detail, let $\Gamma(\mu, \nu)$ be the set of measures on $X \times X$ whose marginals are μ and ν, respectively. In this setting we might call it an optimal coupling, rather than a transport. At the very least, $\Gamma(\mu, \nu)$ contains the product measure $\mu \otimes \nu$; if a transport plan exists then $(\mu, T_{\#}\mu)$ is also in $\Gamma(\mu, \nu)$. Kantorovich asks for the optimum

$$W_p(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times X} d^p(x, y)\mathrm{d}\gamma(x, y). \tag{1}$$

His famous duality theorem shows that this is obtained [21].

The set of probability measures with finite p-th moment on a space X, denoted $\mathscr{P}_p(X)$, can be metrized with the Wasserstein metric W_p.[3] This is a true metric, satisfying non-degeneracy, symmetry, and the triangle inequality. The first two are elementary, and the last follows by disintegration of measure.

The field of optimal transport received new interest from the discovery that certain partial differential equations could be interpreted as gradient flow equations in Wasserstein space [6]. The key is in the continuity equation

$$\partial_t \mu + \nabla \cdot (v_t \mu_t) = 0. \tag{2}$$

Here μ_t is a continuous curve $\mathbb{R} \to \mathscr{P}_2(\mathbb{R}^n)$, and v_t is a vector field on $\mathbb{R}^n$. Given a sufficiently well-behaved function $F\colon \mathscr{P}_2(\mathbb{R}^n) \to \mathbb{R}$, it has a gradient with respect to a measure μ, in a sense made precise in [6] or Section 4. It is often possible to define a PDE by such a gradient, so that (2) is a weak form of the original PDE. If F satisfies certain convexity assumptions then there is a curve μ_t which satisfies (2), which curve is then a weak solution.

Perhaps surprisingly, the space $\mathscr{P}_2(\mathbb{R}^n)$ possesses quite a bit of additional differential structure. One excellent exposition of this is [11]. In particular, $\mathscr{P}_2(\mathbb{R}^n)$ has a well-defined notion of tangent

[3] Much of the literature assumes X is Polish, i.e. complete and separable, and all the spaces studied here will satisfy those assumptions. Removing that requires working only with Radon measures [8].

space at each point, differential forms, and de Rahm cohomology. Therefore there are potentially many tools available to study the geometry and topology of $\mathscr{P}_2(\mathbb{R}^n)$. We hope to contribute some steps toward extending this work to $\mathscr{P}_2(M)$ where M is a smooth Riemannian manifold.

3 Problem Statement

The main question we propose to study is:

Question 1. What is the homotopy type of $\mathscr{VR}(\mathbb{S}^1; r)$ for all $r \in [0, +\infty]$?

In particular, we offer the following conjecture:

Conjecture 3.1. *$\mathscr{VR}_<(\mathbb{S}^1; r) \simeq \mathrm{VR}_<(\mathbb{S}^1; r)$ for all r, and $\mathscr{VR}_\leq(\mathbb{S}^1; r) \simeq \mathscr{VR}_<(\mathbb{S}^1; r + \varepsilon)$ for all r and a sufficiently small $\varepsilon > 0$.*

The main theorem of [1] says that

$$\mathrm{VR}_<(\mathbb{S}^1; r) \simeq \mathbb{S}^{2k+1} \text{ when } \frac{k}{2k+1} < r \leq \frac{k+1}{2k+3} \tag{3}$$

and

$$\mathrm{VR}_\leq(\mathbb{S}^1; r) \simeq \begin{cases} \mathbb{S}^{2k+1} & \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ \bigvee^\infty \mathbb{S}^{2k} & r = \frac{k}{2k+1} \end{cases} \tag{4}$$

so proving Conjecture 3.1 would determine all but the homotopy types of $\mathscr{VR}_\leq(\mathbb{S}^1; r)$ for the scales $r = \frac{k}{2k+1}$. (Here $\mathbb{S}^1$ is assumed to have circumference 1.)

To understand the geometric plausibility of this conjecture, the following example may be illuminating: consider the space $X = [0, 1] \times \{0, 1\} \subset \mathbb{R}^2$. At scales $r < 1$, $\mathrm{VR}(X; r)$ is homotopy equivalent to two disjoint points, and so is $\mathscr{VR}(X; r)$, with either convention. At $r > 1$, all of the above are contractible, as there are simplices of arbitrarily high dimension connecting the two lines. However, at $r = 1$ there are three cases. Both $\mathrm{VR}_<(X; r)$ and $\mathscr{VR}_<(X; r)$ are still homotopy equivalent to a pair of points because there are no edges connecting the two lines. The metric complex $\mathscr{VR}_\leq(X; r)$ is contractible, as there are edges between the two lines filling in the square. The standard complex $\mathrm{VR}_\leq(X; r)$ is not contractible. Indeed, $H_1(\mathrm{VR}_\leq(X; r)) \cong \mathbb{Z}^\infty$ since for any $a, b \in [0, 1]$, there is a nontrivial cycle $[(a, 0), (a, 1)] + [(a, 1), (b, 1)] + [(b, 1), (b, 0)] + [(b, 0), (a, 0)]$. In general, $\mathrm{VR}_\leq(X; r)$ is prone to having wild homology at certain scale parameters, which all the other spaces avoid.

Further, the authors know of no examples where $\mathrm{VR}_<(X; r)$ and $\mathscr{VR}_<(X; r)$ are not homotopy equivalent. This leads one to strengthen the previous conjecture beyond $\mathbb{S}^1$ to all metric spaces:

Conjecture 3.2. *Let X be a (sufficiently nice) metric space. Then $\mathscr{VR}_<(X; r) \simeq \mathrm{VR}_<(X; r)$ for all r, and $\mathscr{VR}_\leq(X; r) \simeq \mathscr{VR}_<(X; r + \varepsilon)$ for all r and a sufficiently small $\varepsilon > 0$.*

We will demonstrate several settings in which this holds, though the question is open for general metric spaces X.

Our proposed method for proving Conjectures 3.1 and 3.2 is to follow Morse-theoretic ideas. Let $r_k = \frac{k}{2k+1}$ for $k \in \mathbb{N}$ be called the critical scale parameters. We propose the following version of the first Morse lemma:

Conjecture 3.3. *For any $r_k < a < b < r_{k+1}$, there is a homotopy equivalence $\mathcal{VR}(\mathbb{S}^1; a) \simeq \mathcal{VR}(\mathbb{S}^1; b)$.*

It is not clear what the statement of the second Morse lemma should be in our setting. We expect that the homotopy type at r_k may have to be shown by more hands-on methods.

Adjacent to these are several questions that we do not expect to fully answer in this work, but which we might hope to develop in the future.

Question 2. Can the major lemmas of Morse theory be developed for $\mathcal{P}_2(M)$ for general Riemannian manifolds M?

The circle has a high-degree of symmetry which makes it easier than an arbitrary manifold. Whenever possible in Section 4 we state definitions for general manifolds in order to hone in upon the important differences between $\mathbb{S}^1$ and general M.

Finally, there are simplicial metric thickenings besides the Vietoris–Rips thickening. These spaces bear certain similarities to simplicial complexes. Our other questions ask about the relation between $\mathrm{VR}(X; r)$ and $\mathcal{VR}(X; r)$, but one can be more general and ask what properties of the category of simplicial complexes translate to properties of simplicial metric thickenings. Since this question is somewhat tangential, we address it in Appendix B.

4 Method and Results

We begin our investigation of the homotopy type of $\mathcal{VR}(\mathbb{S}^1; r)$ with a simpler, but helpful example.

Theorem 4.1. *Let Ω be a convex subset of $\mathbb{R}^n$. Then for any $r \in [0, +\infty]$, the simplicial metric thickening $\mathcal{VR}(\Omega; r)$ is contractible.*

Proof. First, recall that $\mathcal{VR}_{\le}(\Omega; 0) \cong \Omega$ by the canonical inclusion $\iota(x) = \delta[x]$, and that Ω is contractible. Thus it suffices to show that $\mathcal{VR}(\Omega; r) \simeq \mathcal{VR}_{\le}(\Omega; 0)$ for all $r > 0$. We will identify Ω with $\mathcal{VR}_{\le}(\Omega; 0)$ hereafter. Let $\mu = \sum_{i=0}^{k} \lambda_i \delta[x_i]$ denote any measure in $\mathcal{VR}(\Omega; r)$. The mean of μ is $\overline{\mu} = \sum_{i=0}^{k} \lambda_i x_i$ (which is in $\mathcal{VR}(\Omega; r)$ by convexity). Define a map $G\colon \mathcal{VR}(\Omega; r) \to \Omega$ by $\mu \mapsto \delta[\overline{\mu}]$. Since for any point mass $\overline{\delta[x]} = \delta[x]$, the composition $G \circ \iota\colon 0 \hookrightarrow \mathcal{VR}(\Omega; r)$ is the identity. Define a homotopy between $\iota \circ G$ and the identity by

$$H(\mu, t) := \sum_{i=0}^{k} \lambda_i \delta[(1-t)x_i + t\overline{\mu}] \tag{5}$$

(where $\mu = \sum_{i=0}^{k} \lambda_i \delta[x_i]$). Observe that $\operatorname{diam}(\operatorname{supp}(H(\mu, t))) \leq \operatorname{diam}(\operatorname{supp}(\mu))$. The continuity of H follows from the fact that $\mu \mapsto \overline{\mu}$ is continuous, and that $\overline{\mu} = \overline{H(\mu, t)}$ for any $t \in [0, 1]$. Thus $\mathscr{V}\mathscr{R}(\Omega; r) \simeq \Omega$. □

This proof method depends heavily upon the vector space structure of $\mathbb{R}^n$. (In fact, it extends *mutatis mutandis* to any Banach space over $\mathbb{R}$ or $\mathbb{C}$.) Therefore it cannot be directly adopted to other manifolds. However, since general manifolds can be locally approximated by vector spaces, it will be possible to adopt some of the technique. The difficulty arises from more global geometry which can prevent there being a well-behaved mean, at least when larger values of scale r are considered.

4.1 The Circle at Small Scales

The first question one must ask is how to define the mean of a measure on a general Riemannian manifold (M, g). Recall that on $\mathbb{R}^n$, the variance of a measure can be defined as $V(\mu) = \int_{\mathbb{R}^n} \|x - \overline{\mu}\| \mathrm{d}\mu(x)$. In fact, one can define the mean, $\overline{\mu}$, as the point in $\mathbb{R}^n$ which minimizes the quantity $\int_{\mathbb{R}^n} \|x - y\| \mathrm{d}\mu(x)$. This observation, due to Fréchet, characterizes the mean on a manifold.

Definition 4.2. *The* Fréchet *variance of $\mu \in \mathscr{P}(M)$ with respect to $p \in M$ is*

$$V_p(\mu) := \int_M d^2(x, p) \mathrm{d}\mu(x).$$

The absolute Fréchet variance is

$$V(\mu) := \inf_{p \in M} \int_M d^2(x, p) \mathrm{d}\mu(x).$$

The Fréchet mean of μ is

$$F(\mu) := \operatorname*{argmin}_{p \in M} \int_M d^2(x, p) \mathrm{d}\mu(x).$$

Note that the last is in general a set of points in M, and may be empty if M is not compact. If $M = \mathbb{R}^n$ then the argmin is unique and coincides with the usual mean.

It can be difficult to compute Fréchet means and variances, but the following lemma says that knowing how to compute them on one manifold can sometimes allows us to compute them on another.

Lemma 4.3 (Isometries preserve Fréchet means). *Suppose that $f : X \to Y$ is an isometry and μ is a measure on X with unique mean $F(\mu) = \overline{\mu}$. Then $f_\# \mu$ is a measure on Y with $F(f_\# \mu) = f(\overline{\mu})$.*

Proof. By the definitions of isometry and pushforward, $V_p(\mu) = V_{f(p)}(f_\#\mu)$ for all $p \in X$. Furthermore, as p varies over all points in X, of course $f(p)$ varies over all points in Y. Thus $F(f_\#\mu) = f(F(\mu)) = f(\overline{mu})$. $\square$

Corollary 4.4. *For any flat manifold, in particular $\mathbb{S}^1$, the mean of a measure μ with sufficiently small diameter is the pullback of the (Euclidean) mean of the pushforward of μ.*

This immediately gives us the homotopy type of $\mathcal{VR}(\mathbb{S}^1; r)$ when r is sufficiently small, simply by adopting the computations from the proof of Theorem 4.1.

Theorem 4.5. *Consider the circle $\mathbb{S}^1$ with the geodesic metric of circumference 2π. Then $\mathcal{VR}(\mathbb{S}^1; r) \simeq \mathbb{S}^1$ for any $r \in [0, \frac{2\pi}{3})$.*

Proof. The circle is a flat manifold with the property that any metric ball of radius $\rho < \pi/2$ is isometric to the interval $I = (0, 2\rho)$. For $r \in [0, \frac{2\pi}{3})$, any measure $\mu \in \mathcal{VR}(\mathbb{S}^1; r)$ has its support contained in such a ball. Denote this ball B_μ and call the coordinate charts giving the isometry $\phi_\mu \colon B_\mu \to (0, \pi)$. (For concreteness, one could consider the circle as a subset of $\mathbb{C}$ and choose a branch of the complex logarithm.) By Lemma 4.3, the Fréchet mean of μ is $\overline{\mu} = \phi^{-1}(\overline{\phi_\#\mu})$ and so $\overline{\mu} \in \mathbb{S}^1$ exists and is unique. As above, define a map $G \colon \mathcal{VR}(\mathbb{S}^1; r) \to \mathbb{S}^1$ by $G(\mu) = \overline{\mu}$. There is a canonical inclusion $\iota \colon \mathbb{S}^1 \to \mathcal{VR}(\mathbb{S}^1; r)$, and $G \circ \iota = \mathrm{id}$ exactly. Define a homotopy between $\iota \circ G$ and the identity by

$$H(\mu, t) := \sum_{i=0}^{k} \lambda_i \delta[\phi^{-1}((1-t)\phi(x_i) + t(\overline{\phi_\#\mu}))] \tag{6}$$

where $\mu = \sum_{i=0}^{k} \lambda_i \delta[x_i]$. Observe that $\mathrm{diam}(\mathrm{supp}(H(\mu, t))) \leq \mathrm{diam}(\mathrm{supp}(\mu))$. The continuity of H follows from the fact that G is continuous, and that $G(\mu) = G(H(\mu, t))$ for any $t \in [0, 1]$. Thus $\mathcal{VR}(\mathbb{S}^1; r) \simeq \mathbb{S}^1$. $\square$

While Theorems 4.1 and 4.5 are not new results, the proofs given here are, and they have original consequences.

Corollary 4.6. *Let $\mathcal{VR}^{(k)}(X; r)$ denote the k-skeleton of $\mathcal{VR}(X; r)$—the set of measures in $\mathcal{VR}(X; r)$ supported on at most $k+1$ points in X. Then for any $k \in \mathbb{N}$,*

(a) If Ω is a convex subset of a (real or complex) Banach space $\mathcal{B}$, then $\mathcal{VR}^{(k)}(\Omega; r)$ is contractible, and

(b) If $r \in [0, \frac{2\pi}{3})$, then $\mathcal{VR}^{(k)}(\mathbb{S}^1; r) \simeq \mathbb{S}^1$.

Proof. Repeat the proofs of Theorem 4.1 and Theorem 4.5, respectively, with all maps restricted to $\mathcal{VR}^{(k)}(\Omega; r)$ and $\mathcal{VR}^{(k)}(\mathbb{S}^1; r)$. In particular, the cardinality of the supporting set does not increase, $\#\mathrm{supp}(H(\mu, t)) \leq \#\mathrm{supp}(\mu)$, so H is well-defined on this restriction. $\square$

This is not the case for the linear homotopies used in [2, 4], which depend upon adding a point to the support. This can potentially be used to get bounds not only on the diameter but also the cardinality of Carathéodory sets of generalized Borsuk–Ulam theorems in [3].

The proof method also extends to larger cardinalities!

Corollary 4.7. *Let $\mathcal{VR}^{(\infty)}(X;r)$ denote the set of all measures μ in $\mathcal{P}(X)$ such that* $\mathrm{diam}(\mathrm{supp}(\mu)) < r$ *(in particular, without restricting the cardinality of the support to be finite). Then,*

(a) If Ω is a convex subset of a (real or complex) Banach space $\mathcal{B}$, then $\mathcal{VR}^{(\infty)}(\Omega;r)$ is contractible, and

(b) If $r \in [0, \frac{2\pi}{3})$, then $\mathcal{VR}^{(\infty)}(\mathbb{S}^1;r) \simeq \mathbb{S}^1$.

Proof. Starting with (a), note that the mean $\overline{\mu}$ still exists and is unique for any $\mu \in \mathcal{VR}^{(\infty)}(\Omega;r)$, so $G(\mu) = \delta[\overline{\mu}]$ is still well-defined and continuous. Define $L_p(x,t)\colon \Omega \times [0,1] \to \Omega$ by $(x,t) \mapsto (1-t)x + tp$ for $x \in \Omega$, $t \in [0,1]$, and $p \in \Omega$. The only change is that now the homotopy between $\iota \circ G$ and the identity needs to be

$$H(\mu,t) := L_{\overline{\mu}}(x,t)_{\#}\mu.$$

(Note that restricted to $\mathcal{VR}(\Omega;r)$ this is the same as the homotopy defined in Equation (5)). For (b), make a similar adaptation on $\mathbb{S}^1$, with the homotopy being pushforward along a geodesically straight-line path toward the mean. □

4.2 Homotopies as Gradient Flows

By fixing a choice of measures, Equations (5) and (6) define curves in the Vietoris–Rips thickening. We expect that these curves are the flow along the gradient of the Fréchet variance functional. Before making precise what a gradient flow is in such a space, we give several computations which make this conjecture plausible.

Starting with Equation (5), define a curve $\alpha_\mu(t)\colon [0,1] \to \mathcal{VR}(\Omega;r)$ by $\alpha_\mu(t) = H(\mu,t)$. As in Section 4.1, let F be the Fréchet mean and let V denote the absolute Fréchet variance.

Proposition 4.8. *Along any of the curves $\alpha_\mu(t)$ in $\mathcal{VR}(\Omega;r)$,*

(a) The Fréchet mean is constant, and

(b) the Fréchet variance is decreasing.

Proof. Let $x_1, \ldots, x_n$ be the points in Ω on which μ is supported. Recall that the mean, $\overline{\mu}$, minimizes the (weighted) sum of squared distances. Thus,

$$\begin{aligned}
F(\alpha_\mu(t)) &= \sum_{i=1}^{n} \lambda_i((1-t)x_i + t\overline{\mu}) \\
&= \sum_{i=1}^{n} \lambda_i x_i - t\sum_{i=1}^{n} \lambda_i x_i + t\sum_{i=1}^{n} \lambda_i \overline{\mu} \\
&= \overline{\mu}
\end{aligned}$$

because $\sum_{i=0}^{n} \lambda_i x_i = \overline{\mu}$ and $\sum_{i=0}^{n} \lambda_i = 1$. The same principle shows that V is decreasing along α_μ:

$$\begin{aligned}
V(\alpha_\mu(t)) &= \frac{1}{2}\sum_{i=1}^{n} \lambda_i \|\overline{\alpha_\mu(t)} - ((1-t)x_i - t\overline{\mu})\|^2 \\
&= \frac{1}{2}\sum_{i=1}^{n} \lambda_i \|\overline{\mu} - ((1-t)x_i - t\overline{\mu})\|^2 \\
&= \frac{(1-t)^2}{2}\sum_{i=1}^{n} \lambda_i \|\overline{\mu} - x_i\|^2,
\end{aligned}$$

which is, of course, decreasing in t. □

Proposition 4.9. *Along any of the curves $\alpha_\mu(t)$ in $\mathscr{V}\mathscr{R}(\mathbb{S}^1; r)$ with $r \in [0, \frac{2\pi}{3})$,*

(a) The Fréchet mean is constant, and

(b) the Fréchet variance is decreasing.

Proof. The proof is the same, except for an invocation of Lemma 4.3 to justify doing the computations in $\mathbb{R}$ and then lifting them to $\mathbb{S}^1$. □

Propositions 4.8 and 4.9 have lead us to consider a different approach to finding a homotopy equivalence. Is there a way to find a collection of paths (which together form a homotopy) along which the variance decreases? In many settings the way to find such a path is to compute the gradient of some functional. That is exactly how the first lemma of classical Morse theory is proven. As section 2.3 mentions, there is a gradient in Wasserstein spaces. Thus we need to outline how that operates within the Vietoris–Rips thickening.

We will start by only giving definitions for $\mathscr{V}\mathscr{R}(\mathbb{R}^n; r)$, for the sake of simplicity. The definitions that follow are adopted from the standard ones found in [11, 6]. They consider always the full space $\mathscr{P}(\mathbb{R}^n)$, but we will emphasize the restricted definitions only for $\mathscr{F}(\mathbb{R}^n)$ (that is, the space of finitely-supported measures). Note that much of the literature restricts attention to the subspace of absolutely-continuous measures instead.

For any measure $\mu \in \mathscr{F}(\mathbb{R}^n)$, the *tangent space* of $\mathscr{F}(\mathbb{R}^n)$ at μ is the set of vector fields on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n}\langle X, X\rangle \mathrm{d}\mu < +\infty$. Quotienting by the relation $X \sim Y$ if $\int_{\mathbb{R}^n}\langle X-Y, X-Y\rangle \mathrm{d}\mu = 0$ makes this a Hilbert space. Explicitly, a tangent vector to $\mathscr{F}(\mathbb{R}^n)$ at μ is a collection of tangent vectors (to $\mathbb{R}^n$), $\vec{v}_1, \dots, \vec{v}_k$ such that $\vec{v}_i \in T_{x_i}\mathbb{R}^n$ for each $x_i \in \operatorname{supp}(\mu)$. The *tangent bundle* of $\mathscr{F}(\mathbb{R}^n)$ is the disjoint union $T\mathscr{F}(\mathbb{R}^n) = \sqcup_{\mu\in\mathscr{F}(\mathbb{R}^n)} T_\mu\mathscr{F}(\mathbb{R}^n)$. We add here that the tangent bundle is canonically in bijection with $\mathscr{F}(T\mathbb{R}^n)$, and therefore is itself a Wasserstein space. A *vector field* on $\mathscr{F}(\mathbb{R}^n)$ is a continuous section of the projection map $\pi\colon T\mathscr{F}(\mathbb{R}^n) \to \mathscr{F}(\mathbb{R}^n)$. (In the classical setting we should want a *smooth* section, but that is not necessarily well-defined here.)

The *subdifferential* at μ of $\partial_\mu F$ of a functional $F\colon \mathscr{F}(\mathbb{R}^n) \to \mathbb{R}$ is the set of vector fields ξ (on $\mathscr{F}(\mathbb{R}^n)$) defined by

$$\lim_{\nu\to\mu} \frac{F(\nu) - F(\mu) - \sup_{\gamma\in\Gamma_0(\mu,\nu)} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \langle \xi(x), y - x\rangle \mathrm{d}\gamma(x,y)}{W_2(\nu,\mu)} \geq 0 \tag{7}$$

and the *superdifferential*, $\partial^\mu F$ is the set of vector fields ξ such that

$$\lim_{\nu\to\mu} \frac{F(\nu) - F(\mu) - \sup_{\gamma\in\Gamma_0(\mu,\nu)} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \langle \xi(x), y - x\rangle \mathrm{d}\gamma(x,y)}{W_2(\nu,\mu)} \leq 0 \tag{8}$$

The functional F is said to be differentiable at μ if there is a (not necessarily unique) vector field in $\partial_\mu F \cap \partial^\mu F$, and that vector field is the gradient of F at μ, denoted $\nabla_\mu F$.

Vector fields in the manifold setting define curves, by the existence and uniqueness theorems for ordinary differential equations, and the velocity of any curve defines a vector field along that curve. There is a similar relation for a certain class of curves in Wasserstein space.

Definition 4.10. *Let $\mu(t)\colon (0,1) \to \mathscr{F}(X)$ be a curve. It is* 2-absolutely continuous *if for all $0 < s < t < 1$ there exists a $\beta(t) \in L^2(0,1)$ such that*

$$W_2(\mu(s), \mu(t)) \leq \int_s^t \beta(x)\mathrm{d}x.$$

The metric derivative *of a 2-absolutely continuous curve is*

$$|\mu'|(t) \coloneqq \lim_{s\to t} \frac{W_2(\mu(t),\mu(s))}{|t-s|}.$$

This can be shown to exist Lebesgue-almost everywhere for any 2-absolutely continuous curve.

The metric derivative is more analogous to the speed of a curve than the velocity, but the following theorem from [6] shows that every 2-absolutely continuous curve has an associated velocity vector field.

Theorem 4.11. *Let μ_t be any 2-absolutely continuous curve in $\mathscr{P}(\mathbb{R}^n)$. Then there exists a Borel map $v_t \colon (a,b) \times \mathbb{R}^n \to \mathbb{R}^n$ such that v_t is in $L^2(\mu_t)$ for Lebesgue almost every $t \in (a,b)$, and for which $\frac{\partial \mu}{\partial t} = -\mathrm{div}_\mu(v_t)$, in a weak sense. Explicitly, for all compactly-supported $\phi \in C^\infty((a,b) \times \mathbb{R}^n)$,*

$$\int_a^b \int_{\mathbb{R}^n} \left(\frac{\partial \phi}{\partial t} + d\phi(v_t) \right) \mathrm{d}\mu_t \mathrm{d}t = 0.$$

Moreover, one can choose v_t such that $\|v_t\|_\mu = |\mu'|(t)$ for Lebesgue almost-all t.

We would like to show that the gradient of the variance, V, is the velocity vector field for the curves $\alpha_\mu(t)$ defined earlier. It should be possible to prove the first Morse lemma for (certain subsets of) $\mathscr{F}(M)$ using this technique. The details become more subtle with general manifolds.

4.3 Larger Scale Parameters

The proof of Theorem 4.5 for the homotopy type of $\mathscr{V}\mathscr{R}(\mathbb{S}^1; r)$ fails when a measure μ with $\mathrm{diam}(\mathrm{supp}(\mu)) = \frac{2\pi}{3}$ is included. Indeed, it fails as soon as any measure with support *not* contained in a metric ball of diameter at most π is included, as no isometry with a subset of $\mathbb{R}$ can be used to compute the mean. Computation is not the only difficulty, however. Let τ be a measure supported on three equally spaced points on $\mathbb{S}^1$, write $\tau = \frac{1}{3}\delta[x_0] = \frac{1}{3}\delta[x_1] + \frac{1}{3}\delta[x_2]$. The diameter of the support of τ is $\frac{2\pi}{3}$. A direct computation shows that $F(\mu) = \{x_0, x_1, x_2\}$. Even though the mean can be computed, a homotopy cannot be defined as in the previous examples because that mean is not a unique point!

The homotopy type of $\mathscr{V}\mathscr{R}_{\leq}(\mathbb{S}^1; \frac{2\pi}{3}) \simeq \mathbb{S}^3$ was determined in [2]. The method used there does not extend to larger scales. Our hope is to use the method of Section 4.2 to prove a version of the first Morse lemma; however, that would not completely determine the homotopy types of $\mathscr{V}\mathscr{R}(\mathbb{S}^1; r)$ since we expect all of the scales $r_k = \frac{2\pi k}{2k+1}$ to be critical. It may be possible to use hands-on methods to compute the homotopy types of $\mathscr{V}\mathscr{R}(\mathbb{S}^1; r_k)$. In the spirit of Morse theory, they could also be determined by a version of the second Morse lemma. That lemma depends upon knowing the index of critical points, which is second-order information. (In the classical setting one needs the Hessian matrix of the Morse function.) At this point in time a second-order theory for Wasserstein spaces has not been fully developed. The papers [5] and [15] begin work on a second-order theory, including a definition of curvature and a construction of parallel transport. We are not aware of any construction of a Hesssian operator in $\mathscr{P}(M)$. Developing such a theory would be an extensive, but valuable, future project.

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A Nearest Points Are Not Unique

A measure $\mu \in \mathscr{V}\mathscr{R}(X;r)$ possesses a unique Fréchet mean if and only if it has a unique nearest point (in the Wasserstein sense) in the underlying metric space. Indeed,

$$F(\mu) = \operatorname*{argmin}_{p \in X} \int_X d^2(x,p) \mathrm{d}\mu$$

is unique if and only if

$$\operatorname*{argmin}_{p \in X} W_2(\mu, \delta[p]) = \operatorname*{argmin}_{p \in X} \left(\int_X d^2(x,p) \mathrm{d}\mu \right)^{1/2}$$

is unique. Necessarily the minimizing point is the same.

In Section 4 we saw that the measure $\tau \in \mathcal{VR}(\mathbb{S}^1; \frac{2\pi}{3})$ consisting of three equally spaced and weighted points does not have a unique Fréchet mean. We can make the same argument directly. Let $G\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the $\mathbb{Z}_2$-action (reflection) on $\mathbb{S}^1$ which fixes $\pm x_0$. Then the only points which could possibly be the unique minimum of W_2 to τ are x_0 and its antipode, $-x_0$ (the fixed points of G). An easy computation shows that

$$W_2(\tau, \delta[x_0]) = \frac{4\pi}{27} = W_2(\tau, \delta[x_1]) = W_2(\tau, \delta[x_2])$$

so the minimum cannot be unique.

One might hope that this only fails for measures with rotational symmetry, such as the τ mentioned there. These occur exactly with the scale parameters at which $\mathrm{VR}(\mathbb{S}^1; r)$ changes homotopy type. Call these "critical" scale parameters $c_1, c_2, \ldots$. It then seems plausible that one could show $\mathcal{VR}(\mathbb{S}^1; c_i) \simeq \mathrm{VR}(\mathbb{S}^1; c_i)$ by a direct argument, and also that $\mathcal{VR}(\mathbb{S}^1; r) \simeq \mathcal{VR}(\mathbb{S}^1; c_i)$ for the largest $c_i < r$ by mapping to a nearest point in $\mathcal{VR}(\mathbb{S}^1; c_i)$, and thereby understand $\mathcal{VR}(\mathbb{S}^1; r)$ at all scales.

Unfortunately, there is not in general a unique nearest point in $\mathcal{VR}(\mathbb{S}^1; c_i)$. Here is a direct example: let y_0, y_1, and y_2 be points on $\mathbb{S}^1$ such that $d(y_0, y_1) = d(y_0, y_2) = \frac{3\pi}{4}$ and $d(y_1, y_2) = \frac{\pi}{2}$. Consider $\sigma = \frac{1}{100} y_0 + \frac{199}{200} y_1 + \frac{199}{200} y_2$. (Of course, these coefficients are not particularly special, the point is simply that much less mass is on y_0, and that the others are weighted equally. Choosing specific masses makes the computation clearer.) As above, let G be the $\mathbb{Z}_2$-action on $\mathbb{S}^1$, except now fixing y_0. We want to find a unique measure $\mu \in \mathcal{VR}(\mathbb{S}^1; \frac{2\pi}{3})$ which minimizes $W_2(\sigma, \mu)$. Clearly any such measure must be a fixed-point of G. Moreover, it must be supported on at most three points, due to the following lemma:

Lemma A.1. *Let $\mu = \sum_{i=0}^k \lambda_i x_i \in \mathcal{VR}(X; \infty)$ and suppose there exists a $\nu \in \mathcal{VR}(X; r)$ for some $r < +\infty$ such that $W^2(\mu, \mathcal{VR}(X; r)) = W^2(\mu, \nu)$. Then there exists an $\nu' \in \mathcal{VR}(X; r)$ achieving the distance and with $\#\mathrm{supp}(\nu') \leq \#\mathrm{supp}(\mu)$.*

Proof. Let $\nu = \sum_{j=0}^n \xi_j y_j$ be a point achieving the minimal distance. Let Θ denote the matching between $\mathrm{supp}(\mu)$ and $\mathrm{supp}(\nu)$ giving the minimal distance. Construct a new measure ν' be choosing, for each x_i, the element y_i' of $\Theta(x_i)$ such that $d(x_i, y_i')$ is minimal over all $y_i \in \Theta(x_i)$, and with mass $\xi_i' = \sum_{y_i \in \Theta(x_i)} \xi_i$. (It may be possible that two y_i''s are actually the same point; this causes no issues. Likewise the minimal distance may be tied between multiple points, in which case any one can be chosen.) By construction, $d(\mu, \nu') \leq d(\mu, \nu)$ and $\#\mathrm{supp}(\nu') \leq \#\mathrm{supp}(\mu)$. ◻

Therefore the only symmetric measures which could achieve the minimum distance are the "triangle" supported on three points with at x_0, measures supported on two points equidistant from x_0, and the delta mass at $-x_0$. We now exhibit an asymmetric measure which is better than any of these.

Let $\rho = \frac{1}{100}\delta[p] + \frac{199}{200}\delta[x_1] + \frac{199}{200}\delta[x_2]$ where p is the point determined by $d(x_0, p) = \frac{7\pi}{12}$ and $d(x_1, p) = \frac{\pi}{6}$. Numerical computation of $W_2(\sigma, \rho)$ and $W_2(\sigma, \mu)$ for μ in each of the three classes of measures given above shows that $W_2(\sigma, \rho) < W_2(\sigma, \mu)$ for all possible μ.

B Simplicial Metric Thickenings

Another direction in which many of the questions in this thesis can be taken is to study an abstract notion of simplicial metric thickening, divorced from the particularities of the Vietoris–Rips complex. We do this be introducing an abstraction definition of simplicial metric thickenings.

B.1 Background: The Kantorovich Monad and Categorical Probability

Here we drew considerable inspiration from categorical probability theory. In particular, [10] introduces the Kantorovich monad on the category of (complete) metric spaces. A monad on a category C consists of an endofunctor T and natural transformations $e\colon \mathrm{id} \to T$ and $m\colon TT \to T$ satisfying certain coherence conditions. The canonical example is that of powerset (P, e, m) on Set. Here P is the powerset functor, e_S sends S to the subset of PS consisting of all singletons, and m_{PPS} takes a set of subsets of subsets and "unions" out". For example, m_{PPS} would take the element $\{\{x\}, \{x, y\}, \{y, z\}\}$ in PPS (with $S = \{w, x, y, z\}$) to $\{x\} \cup \{x, y\} \cup \{y, z\} = \{x, y, z\}$.

The Kantorovich monad $(\mathcal{P}(), \delta, E)$ is an analogous structure on Met, the category of metric spaces and short (1-Lipschitz) functions. The functor $\mathcal{P}()$ sends a metric space to its space of Radon probability measures with finite p-th moment, as defined in Section 2.3. Functions $f : X \to Y$ naturally extend to $\mathcal{P}(f)\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ by pushforward: $\mathcal{P}(f)(\mu) = f_\#\mu$. The unit δ is the map $\delta_X\colon X \to \mathcal{P}(X)$ given by $x \mapsto \delta_x$, the canonical inclusion of X into $\mathcal{P}(X)$. The multiplication is an averaging, with $E_{\mathcal{P}(\mathcal{P}(X))}\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$ defined by sending μ to the measure acting on any measurable subset $A \subseteq \mathcal{P}(X)$ by

$$E\mu(A) = \int_{\mathcal{P}(X)} \rho(A) \mathrm{d}\mu(\rho).$$

The main theorem of [10] is a categorical interpretation of the density of $\mathcal{F}(X)$ in $\mathcal{P}(X)$.

B.2 Simplicial Metric Thickenings

A simplicial complex can be defined as follows: let S be a set and let $F(S)$ denote all the finite subsets of S. Then a simplicial complex with vertex set S is a collection $K \subseteq F(S)$ which contains all singletons and is closed under $\subseteq$. Now, let X be a metric space and $\mathcal{F}(X)$ be the set of all finitely-supported measures on X. Then a simplicial metric thickening with vertex space X is a subspace $K \subseteq \mathcal{F}(X)$ which contains all δ measures and is closed under $\ll$ (here $\ll$ means absolute continuity). Since K is a subspace of $\mathscr{F}(X)$ it is a metric space with the Wasserstein metric. By defining morphisms to be 1-Lipschitz maps which respect the closure structure, we form a category of simplicial metric thickenings.

The Vietoris–Rips metric thickening is a natural example of this structure, as are spaces of measures of bounded variance, spaces defined using the analogue of the Čech simplicial complex, and others.

A simplicial complex can be constructed from any simplicial metric thickening in a functorial way by defining simplices as the sets on which measures are supported. Conversely, any simplicial complex can be turned into a simplicial metric thickening by putting the discrete metric on its vertex set. Moreover, if the vertex set of a simplicial complex is a metric space, the simplicial complex can always be viewed as a metric simplicial complex.

B.3 Results and Conjectures

In a separate work we are studying the properties of the category of simplicial metric thickenings. We are particularly interested in the behaviour of limits and colimits. Some preliminary results are as follows:

Proposition B.1. *Let M be the functor taking a simplicial complex with metric vertex set to the corresponding metric simplicial complex. This functor factors over products up to homotopy: for any simplicial metric thickenings K and L,*

$$M(K) \times M(L) \simeq M(K \times L).$$

This implies that the Vietoris–Rips thickening of a product is (up to homotopy) the product of the Vietoris–Rips thickenings. The same holds for wedge sums of simplicial metric thickenings.

We conjecture that a Dowker-type theorem (the two simplicial complexes defined by a relation are homotopy equivalent) holds in this category.