2018·04·25

Yoneda Lemma — Category Theory Reading Group(oid)

Conventions: always assume locally small, write $\mathcal{C}(a,b) = \mathrm{Hom}_{\mathcal{C}}(a,b)$.

Defn: A functor $F : \mathcal{C} \to \mathrm{Set}$ is <u>representable</u> if there exists a $c \in \mathcal{C}$ s.t.
$\mathcal{C}(c,-) \cong F$ (or $\mathcal{C}(-,c)$ if F contravariant).

E.g. (a) Forgetful $U : \mathrm{Top} \to \mathrm{Set}$ is represented by space $\{*\}$. $\mathrm{Top}(*, X) \cong X$ as set.

(b) $U : \mathrm{Grp} \to \mathrm{Set}$ is represented by $\mathbb{Z}$.

Yoneda generalizes a fact we know...

E.g. Let G be a group, BG be G viewed as a one-object category.
$F : BG \to \mathrm{Set}$ sends $*$ to some set X, and g to $F_g : X \to X$, i.e. a group action. (F defines a G-set.)

$$\begin{tikzcd} * \arrow[loop left, "g"] \arrow[loop below, "h"] \arrow[r, "F"] & X \arrow[loop right, "F(gh)"] \end{tikzcd}$$

The represented functor $BG(*,*)$ is G, w/ g_* being multiplication by G.

– What is a natural transformation $\eta : BG(*,-)$ to F? It is a G-equivariant map $\eta : G \to X$.

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[d, "g_*"'] & X \arrow[d, "F_g"] \\ G \arrow[r, "\eta"'] & X \end{tikzcd}$$

– Any η is defined by $\eta(1)$, b/c $\eta(g) = g \cdot \eta(1)$
$= F_g(\eta(1))$.

– $\eta(1)$ can be any $x \in X$. So $\mathrm{Nat}(BG(*,-), F) \cong X = F(*)$ //

<u>Lemma</u>: (Yoneda) For any $F : \mathcal{C} \to \mathrm{Set}$ and any $c \in \mathcal{C}$, there is a natural bijection $\mathrm{Nat}(\mathcal{C}(c,-), F) \cong Fc$, or $\mathrm{Nat}(\mathcal{C}(-,c), F) \cong Fc$.

<u>Note</u>: Cat is not locally small, so $\mathrm{Nat}(-,-)$ need not be a set! But we will show it is.

<u>Proof</u>: (Sketch) Define $\Phi : \mathrm{Nat}(\mathcal{C}(c,-), F) \to Fc$ by $\Phi(\alpha) = \alpha_c(1_c)$.
$\Psi : Fc \to \mathrm{Nat}(\mathcal{C}(c,-), F)$ by $\Psi(x)_d(f) := F(f)(x)$, $f \in \mathcal{C}(c,d)$.

This is more than a generalization...

Corollary (Yoneda Embedding) The functor $y: \mathcal{C} \to Psh(\mathcal{C})$ is fully faithful embedding.

Explanation: $Psh(\mathcal{C}) = Set^{\mathcal{C}^{op}}$ = functors (contravariant) from $\mathcal{C} \to Set$.

- $y(c) = \mathcal{C}(-,c)$ and $f: c \to d \mapsto f_*: \mathcal{C}(-,c) \to \mathcal{C}(-,d)$ (left comp)
- Full, Faithful = surjective & injective on hom sets, so $\mathcal{C}(a,c) \cong Psh(\mathcal{C}(-,a), \mathcal{C}(-,c))$

  (B/c $Nat(\mathcal{C}(-,a), \mathcal{C}(-,c)) \cong \mathcal{C}(a,c)$ by Yoneda.)
- Embedding = injective on objects. $\mathcal{C}(-,c) \neq \mathcal{C}(-,d)$ unless $c = d$.

Covariant also holds.

E.g. Yoneda lets us "expand the universe". Knowing an object = knowing all maps into it.

Take $(\mathbb{Q}, \leq)$. For any $q \in \mathbb{Q}$, $\mathbb{Q}(p,q)$ is $\emptyset$ if $p > q$, $\{*\}$ if $p \leq q$.

"Knowing q = knowing all rationals $\leq q$." $\mathbb{Q}(-,q)$ "picks out" q.

What about the presheaf $F: \mathbb{Q}^{op} \to Set$ by $F(p) = \{*\}$ if $p^2 \leq 2$, $\emptyset$ if $p^2 \geq 2$?

Not representable. But "picks out" $\sqrt{2}$.

$Psh(\mathbb{Q}) \approx$ Dedekind cuts $= \mathbb{R}$! //

The "extended universe" is the "free cocompletion" of $\mathcal{C}$ (modulo details, size).

Threw in all small colimits. (Think of $\mathbb{R}$ as $\mathbb{Q}$ w/ all limits.)

Research: Met is a bad category. Does $Psh(Met)$ work better? What is it?

$M_1 \oplus M_2$ does not exist. The "right" answer is $M_1 \sqcup M_2$ w/ $d(m_1, m_2) = \infty$ if $m_1 \in M_1$, $m_2 \in M_2$. See Lawvere metric spaces.