Lecture Notes on Differential Equations

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Math 340

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1 Introduction

Goal: Understand what differential equations are, what it means to have a solution to a differential equation, and what conditions are required for solutions to be unique.
Outline:

- Understand what a differential equation *is*. (Section 1.1, 2.1)
- Gain experience with solutions to differential equations. (Section 2.1, 2.9)
- Existence and uniqueness theorems. (Section 2.7)

1.1 What is a differential equation?

Definition 1.1. *A **differential equation** is an equation relating a function and its derivative(s).*

Example 1.2. Convention: $y(t)$.

- $y' = y$
- $y' = ty^2$
- $y'' + 4y' + 4y = 0$
- $$\begin{aligned} x' &= 2x + y \\ y' &= -3y \end{aligned}$$
- $u_t = \nabla u$ The heat equation (a *partial* differential equation).

Some useful adjectives:

Definition 1.3. *An **ordinary** differential equation has a single independent variable. Abbreviation: **ODE**.*

Definition 1.4. *The **order** of a differential equation is the highest order of derivative appearing.*

Definition 1.5. *A* ***solution*** *to a differential equation is a function which satisfies the equation.*

Example 1.6. Solutions to the above equations:

- $y(t) = \exp(t)$ (Here $\exp(t) = e^t$, but more later.)
- $y(t) = \dfrac{-2}{t^2}$
- $y(t) = (1+t)\exp(-2t)$.
- $x(t) = -\exp(-3t) + \exp(2t)$ and $y(t) = 5\exp(-3t)$.
- Involves Fourier series, or similar. (Partial differential equations will not be covered.)

1.2 Solutions to Ordinary Differential Equations

Goal: Understand what it means for a function to be a solution to a differential equation and how to check that.

Example 1.7. Consider $y' = ty^2$. A solution is $y = \frac{-2}{t^2}$:

$$y' = \frac{4}{t^3} = t\frac{4}{t^4} = ty^2.$$

Another solution is $\tilde{y} = \frac{-2}{1+t^2}$:

$$\tilde{y}' = \frac{2}{(1+t^2)^2}(2t) = t\frac{4}{(1+t^2)^2} = t\tilde{y}^2.$$

The solutions in this example are obviously related. They both have the form $y = \frac{2}{C+t^2}$ where in the first $C = 0$ and in the second $C = 1$. In fact, for any value of C this is a solution:

$$y' = \frac{\mathrm{d}}{\mathrm{d}t}\frac{2}{C+t^2} = \frac{2}{(C+t^2)^2}(2t) = t\frac{4}{(C+t^2)^2} = t\tilde{y}^2.$$

The equation with a C is called the *general solution*, while the particular choices of C give *particular solutions.*

Definition 1.8. *The* ***general solution*** *of an ordinary differential equation is a solution containing undetermined constants from which all other solutions can be obtained. A* ***particular solution*** *is a solution containing no undetermined constants.*

Example 1.9.

- $y' = y$ has general solution $C\exp(t)$.
- $y'' + 4y' + 4y = 0$ has general solution $(A+Bt)\exp(-2t)$.

- $$x' = 2x + y$$
  $$y' = -3y$$
  has general solutions $x(t) = -A\exp(-3t) + B\exp(2t)$ and $y(t) = 5A\exp(-3t)$.

Particular solutions arise when solving an initial value problem:

Definition 1.10. *An* ***initial value*** *or* ***initial condition*** *is a requirement of the type $y(t_0) = y_0$, or $y'(t_1) = y_1$, et cetera.*

Example 1.11.
- $y' = y$ with $y(0) = 2$.
- $y' = ty^2$ with $y(1) = 1$.
- $y'' + 4y' + 4y = 0$ with $y(1) = 0$ and $y'(0) = 1$.
- $$x' = 2x + y$$
  $$y' = -3y$$
  with $x(0) = 1$ and $y(0) = 2$.

An ODE of degree k requires k initial conditions, one for each derivative occurring. A system of n equations requires n initial conditions, one for each variable.

Example 1.12. To solve for a particular solution, use the general solution and the initial condition:

- $y' = y$ with $y(0) = 2$ has general solution $y = C\exp(t)$. So $y(0) = C\exp(0) = C = 2$, making the particular solution $y(t) = 2\exp(t)$.
- $y'' + 4y' + 4y = 0$ has general solution $y = (A + Bt)\exp(-2t)$. The condition $y(1) = 0$ means $(A+B)e^{-2} = 0$ so $A + B = 0$ or $A = -B$. The condition $y'(0) = 1$ demands first computing y':
  $$y'(t) = B\exp(-2t) - 2(A + Bt)\exp(-2t)$$
  so $y'(0) = B - 2A = 3B = 1$ so $B = 1/3$ and $A = -1/3$. The particular solution is
  $$y(t) = \frac{-1 + t}{3}\exp(-2t)$$

Exercise 1.13. Solve the other two initial value problems. (Answer: $C = -3$, $A = 2/5$, and $B = 7/5$.)

1.2.1 Geometry of Solutions

Once a solution is known, it can be plotted in the (t, y)-plane as usual. For general solutions, it is useful to plot curves with many different values of the constants to understand the variety of behaviors.

Figure 1: Plots of $y = C\exp(t)$ and $y = \dfrac{1}{C + t^2}$ for different values of C.

Example 1.14.

Without first obtaining solutions, **slope fields** of first order equations can be plotted in the (t, y)-plane. From these, simple estimates of solution curves can be seen.

Example 1.15.

Definition 1.16. *A first-order differential equation is* ***autonomous*** *if it has the form* $y' = g(y)$*. (No* t *terms appear.)*

In this case y and y' can be plotted on the (y, y') plane. The (y, y')-plane is called the **phase plane**, and a plot in it is a phase plane plot. It does not show the solution, but it carries a great deal of information about the solution.

Example 1.17. Phase plots can also be used to sketch solution curves, though not precisely.

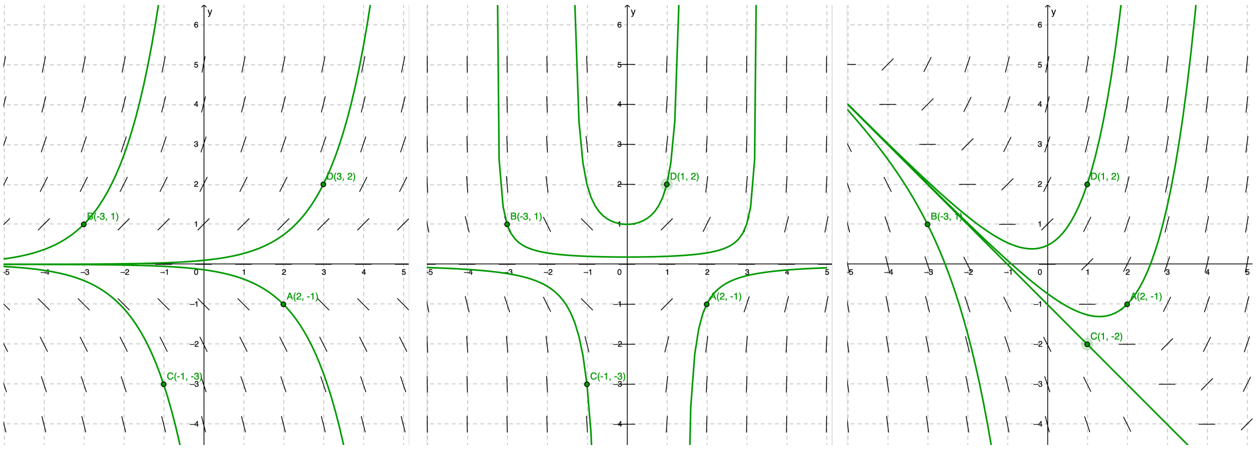


Figure 2: Slope fields for $y' = y$, $y' = ty^2$, and $y' = t + y$. See `https://www.geogebra.org/m/W7dAdgqc`.

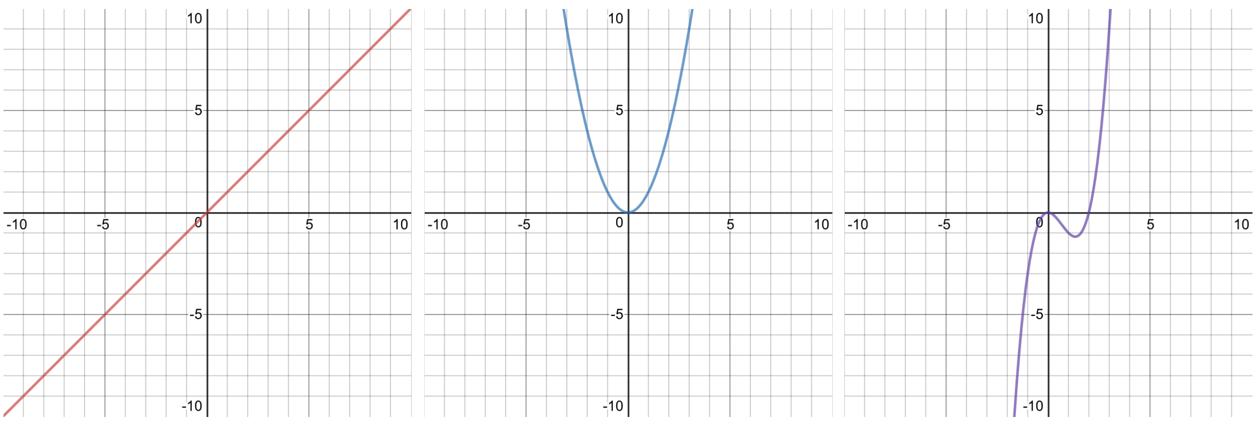


Figure 3: Phase plane plots for $y' = y$, $y' = y^2$, and $y' = y^2(y - 2)$.

1.3 Existence and Uniqueness

Goal: Understand the statement and implications of the existence and uniqueness theorem.

Example 1.18. Consider the ODE $y' = y^{1/3}$. The general solution is $y_1 = \left(\frac{2}{3}t + C\right)^{3/2}$. Check:

$$y_1' = \left(\frac{2}{3}t + C\right)^{1/2} = y^{1/3}$$

However, $y_2 = -\left(\frac{2}{3}t + C\right)^{3/2}$ is also a solution:

$$y_2' = -\left(\frac{2}{3}t + C\right)^{1/2} = y^{1/3}$$

Thus *neither* is really the general solution. In fact, $y_3 := 0$ is also a solution: $y_3' = 0 = 0^{1/3} = y^{1/3}$, and there is no value of C which makes y_2 or y_1 identically zero.

This raises the questions of existence and uniqueness: given an ODE, does a solution (general or particular) exist, and is it unique or are there multiple solutions?

Example 1.19. Adding an initial condition sometimes helps. Suppose $y(0) = 1$ is added to the above ODE. Then y_1 with constant $C = 1$ is a particular solution. For y_2, there is no solution (this is the negative branch of the square root). And of course, $y_3 = 0 \neq 1$, so y_1 is the only particular solution.

However, the initial condition $y(0) = 0$ works for all three solutions (with $C = 0$ in y_1 and y_2).

Theorem 1.20 (Existence and Uniqueness / Picard-Lindelöf). *Let $y' = f(t, y)$ with $y(t_0) = y_0$. If there exists a rectangle R in the (t, y)-plane containing (t_0, y_0) and on which*

1. *f is continuous, and*
2. *$\dfrac{\partial f}{\partial y}$ is continuous,*

then there exists a unique solution $y(t)$ defined on R.

(The proof of this theorem is beyond the scope of this course.)

The theorem is structured as *hypotheses* and *conclusions*. The hypotheses are continuity on the rectangle R, and the conclusions are the existence and uniqueness of a solution.

A logically equivalent statement would be "if it is 10:00am, MTWF, and not the day after an exam, then we are having class." The three hypotheses guarantee the conclusion. Note well that a theorem is an absolute truth. It can never be false; there are no contradictions to it. Therefore, if the conclusion *does not* hold, one of the assumptions must fail. If you walk into the room and we are not having class, either it is not 10:00am, it is not a MTWF, or it is the day after a test. (This is called the **contrapositive**. If a statement is of the form "P implies Q," the contrapositive is the equivalent statement "not Q implies not P.")

Example 1.21. To apply the theorem to the above examples, first identify that $f(t, y) = y^{1/3}$. Next, find a rectangle R containing the initial condition $y(0) = 1$, in other words, the point $(0, 1)$ in the (t, y)-plane. Since f is continuous everywhere, the limiting factor will be $f_y = \frac{1}{3}y^{-2/3}$. This is not continuous when $y = 0$ (division by zero). Any rectangle containing $(0, 1)$ and not containing $y = 0$ is a valid choice, so it is fine to choose the largest such rectangle, $R = (-\infty, +\infty) \times (0, +\infty)$, which is the upper half-plane. The theorem then concludes: there exists a unique solution to $y' = y^{1/3}$ on R. From above, this must be the solution $y_1 = \left(\frac{2}{3}t + C\right)^{3/2}$.

With the second initial condition, $y(0) = 0$, there was not a unique solution. By the contrapositive, this means one of the assumptions of the theorem *must* fail. Again $f(t, y) = y^{1/3}$ is always continuous, but f_y is not continuous when $y = 0$. Thus there is no rectangle containing

$(0,0)$ on which the continuity assumption holds, and so continuity of f_y is the hypothesis which is not satisfied.

1.3.1 Using the existence and uniqueness theorem

Example 1.22. Consider the ODE $y' = \frac{1}{y(1-t)}$. Determine all the initial conditions for which unique solutions are not guaranteed to exist.

For unique solutions to be guaranteed, $f(t,y) = \frac{1}{y(1-t)}$ must be continuous and $f_y = \frac{-1}{y^2(1-t)}$ must be continuous. Both have discontinuities at $t = 1$ and $y = 0$. This divides the (t,y)-plane into four rectangles (picture):

$$
\begin{aligned}
R_1 &= (-\infty, 1) \times (0, +\infty) \\
R_2 &= (-\infty, 1) \times (0, -\infty) \\
R_3 &= (1, \infty) \times (0, +\infty) \\
R_4 &= (1, \infty) \times (0, -\infty)
\end{aligned}
$$

By the existence and uniqueness theorem, unique solutions exist for all initial conditions inside these rectangles, and may not on the lines $y = 0$ and $t = 1$.

An important consequence of uniqueness is that *solution curves cannot cross*, except at points where the hypotheses fail.

Example 1.23. Consider the ODE $y' = 3y/t$. This fails the hypotheses at $t = 0$. Note that all of the solution curves intersect at $(0,0)$. The general solution is $y = Ct^3$ (check!). In this case all of the curves are obtained from different values of C, but this is not always the case.

This also illustrates that existence and uniqueness are separate. There *are* solutions when $t = 0$.

Example 1.24. The uniqueness theorem may be used to obtain bounds on solutions. Consider $y' = (y-1)\cos(yt)$. A solution is $y_1 = 1$ since $y' = 0 = (1-1)\cos(t)$. This is not the general solution.

Note that $f(t,y) = (y-1)\cos(yt)$ satisfies the conditions of existence and uniqueness everywhere. Suppose that y_2 is a solution to $y' = (y-1)\cos(yt)$ with initial condition $y(0) = 2$. Despite lacking a formula for $y_2(t)$, it is possible to conclude that $y_2(t) > 1$ for all t because otherwise y_2 would have to cross y_1, which cannot happen.

Constant solutions like those in the above example are useful for this purpose. They often can be found "out of thin air" by inspecting $f(t,y)$ for roots.

Even when existence and uniqueness are satisfied everywhere, a single solution may not exist for all t.

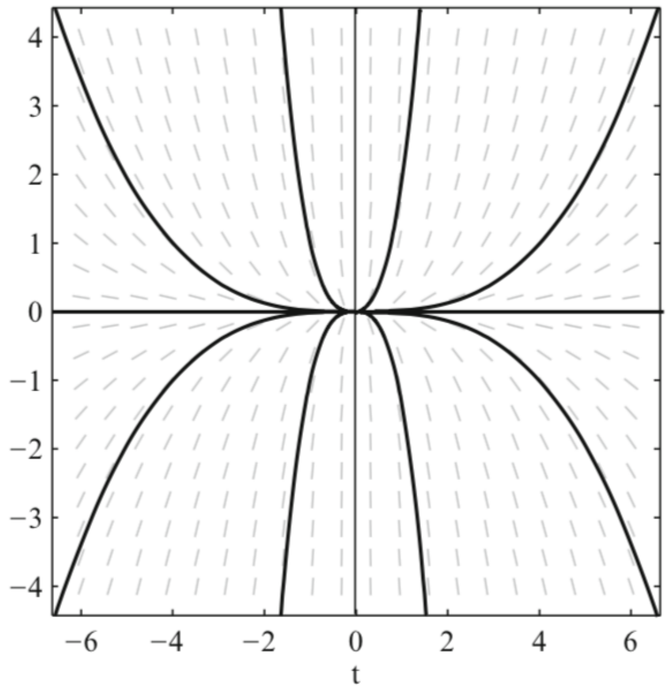


Figure 4: Solution curves for $y' = 3y/t$.

Definition 1.25. *The **interval of existence** of a solution to $y' = f(t, y)$ with $y(t_0) = y_0$ is the largest interval on which a solution is continuous.*

Example 1.26. Consider $y' = y^2$. Obviously, $f(t, y) = y^2$ is continuous everywhere and so is $f_y = 2y$. The general solution is $y = \frac{-1}{t+C}$:

$$y' = \frac{1}{(t+C)^2} = y^2.$$

However, this is not defined when $t = -C$.

In particular, with the initial condition $y(0) = 1$, $C = -1$ and $y = \frac{-1}{t-1}$ which is undefined at $t = 1$. The interval of existence for this solution is $t \in (-\infty, 1)$.

With initial condition $y(0) = -1$, $C = 1$ and $y = \frac{-1}{t+1}$ which has interval of existence $t \in (-1, +\infty)$.

Interestingly, with $y(2) = 1$, $C = -1$ and $y = \frac{-1}{t-1}$. However, the interval of existence is $(1, +\infty)$ since the initial condition is to the right of the discontinuity.

2 Separable and Linear Equations

Goal: Develop a taxonomy of first-order ODEs, then develop a technique for solving each kind.

Outline:

- The exponential function. (No section)
- Classification of first-order ODEs. (No section)
- Separation of variables. (Section 2.2)
- Linear equations and integrating factors. (Section 2.4)

2.1 The Exponential Function

Definition 2.1. *The* ***exponential function*** *is*

$$\exp(t) := 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots + \frac{1}{n!}t^n + \cdots$$

This is the same as e^t. The reason for this definition and notation is that e^t does not mean repeated multiplication of e as the notation naïvely suggests. For example, $e^3 = e \times e \times e$, but what about $e^{3.5}$, or e^{π}?

Proposition 2.2.

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(at) = a\exp(at)$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\exp(at) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(1 + at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 + \cdots + \frac{1}{n!}(at)^n + \cdots\right)\\ &= a + a^2t + \frac{1}{2}a^3t^2 + \cdots + \frac{1}{(n-1)!}a^nt^{n-1} + \cdots\\ &= a\left(1 + at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 + \cdots + \frac{1}{n!}(at)^n + \cdots\right)\\ &= a\exp(at)\end{aligned}$$

□

The simplest, yet most important, ODE, is

$$y' = ay. \tag{1}$$

Exercise 2.3. (2 minutes) Find the general solution.

To be more general, consider solving $y' = f(t)y$. Expecting an exponential, try $y = \exp(g(t))$ for some $g(t)$. Then $y' = g'(t)\exp(g(t)) = g'(t)y$. Choosing $g(t) = \int f(t)\mathrm{d}t$, so $y = \exp\left(\int f(t)\mathrm{d}t\right)$ and $y' = f(t)y$ gives the solution.

Example 2.4. If $y' = ty$, then the solution is

$$y = \exp\left(\frac{1}{2}t^2 + C\right) = \exp(C)\exp\left(\frac{1}{2}t^2\right) = \tilde{C}\exp\left(\frac{1}{2}t^2\right)$$

where $\tilde{C}$ is the constant $\tilde{C} := \exp(C)$. Checking:

$$y' = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{C}\exp\left(\frac{1}{2}t^2\right) = \tilde{C}t\exp\left(\frac{1}{2}t^2\right) = ty.$$

The significance is two-fold:

1. Equation (1) can have "other things" instead of the constant a or the function $f(t)$, say $y' = \bigstar y$. The solution in those cases is still $y = \exp(\int \bigstar)$. (Whatever that means!)
2. Equation (1) is *linear*, *separable*, and *autonomous*, making it the first example for all important classes of first-order ODEs.

2.1.1 Classification for first-order ODEs

Definition 2.5. *An equation is* ***separable*** *if it has the form*

$$y' = f(t)g(y).$$

Definition 2.6. *An equation is* ***linear*** *if it has the form*

$$y' = f(t)y + h(t).$$

Definition 2.7. *An equation is* ***autonomous*** *if it has the form*

$$y' = g(y).$$

Example 2.8. For $y' = ay$, $f(t) = a$ (constant) and $h(t) = 0$.

Example 2.9. Separable:

- $y' = ty$
- $y' = t$
- $y' = 2$
- $y' = t^2y^3$
- $y' = (1 + y^2)$

Linear:

- $y' = ty$
- $y' = y + t$
- $y' = \cos(t)y + t^2$

Autonomous:

- $y' = 2$
- $y' = (1 + y^2)$

- All autonomous equations are separable!
- None of the above:
- $y' = ty^2 + t^2$

2.2 Separation of variables

GOAL: find general method to solve $y' = f(t)g(y)$.

Example 2.10. Motivating case: $y' = f(t)$ (so $g(y) = 1$, if you like).

$$\int y'\mathrm{d}t = \int f(t)\mathrm{d}t$$
$$y(t) = F(t) + C$$

Exercise 2.11. (2 minutes) Solve $y' = 2t$.

Example 2.12. Now one with $g(y) = y$, so both variables appear:

$$y' = ty$$
$$\int y'\mathrm{d}t = \int ty\mathrm{d}t$$

Cannot evaluate right side (y depends on t). Rearrange, *separating* variables, instead:

$$\frac{1}{y}y' = t$$
$$\int \frac{1}{y}y'\mathrm{d}t = \int t\mathrm{d}t$$

Still do not know y, but we can integrate the left side because of the *chain rule*:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(y(t)) = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

(Note how the $\mathrm{d}y$ terms "cancel out".) This means we have

$$\int \frac{1}{y}\mathrm{d}y = \int t\mathrm{d}t$$
$$\log(y) = \frac{1}{2}t^2 + C$$

Solve for y:

$$\begin{aligned}
\exp(\log(y)) &= \exp\left(\frac{1}{2}t^2 + C\right) \\
y &= \exp\left(\frac{1}{2}t^2\right)\exp(C) \\
y &= \tilde{C}\exp\left(\frac{1}{2}t^2\right)
\end{aligned}$$

General case. Starting with $y' = f(t)g(y)$, separate variables:

$$\int \frac{1}{g(y)} y' \mathrm{d}t = \int f(t) \mathrm{d}t$$

The chain rule turns the left side into a y-antiderivative,

$$\int \frac{1}{g(y)} \mathrm{d}y = \int f(t) \mathrm{d}t$$

Compute both antiderivatives to get

$$G(y) = F(t) + C$$

then solve for y, if possible.

Example 2.13 (Extra examples). (Implicit solution) Let

$$x' = 2t\frac{x}{1+x}$$

and apply separation of variables:

$$\begin{aligned}
\frac{1+x}{x}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2t \\
\int \frac{1+x}{x}\mathrm{d}x &= 2t\mathrm{d}t \\
\int \frac{1}{x} + 1\mathrm{d}x &= t^2 + C \\
\log(|x|) + x &= t^2 + C
\end{aligned}$$

The last step is to isolate x. Here that is not possible. Applying exp to get x out of the log would make the other x into an $\exp(x)$. This is called an *implicitly-defined* solution.

Example 2.14 (The exponential). Solve $y' = ay$ using separation of variables,

$$\begin{aligned} \frac{1}{y}y' &= a \\ \int \frac{1}{y}\mathrm{d}y &= \int a\mathrm{d}t \\ \log(y) &= at + C \\ \exp(\log(y)) &= \exp(at + C) \\ y &= \exp(at)\exp(C) \\ y &= \tilde{C}\exp(at) \end{aligned}$$

The constant is $y(0)$, since $y(0) = \tilde{C}\exp(0) = \tilde{C}$.

Example 2.15 (Initial conditions). Initial value problems are solved in the ordinary way:

$$\begin{aligned} y' = 3t^2(1 + y^2) \text{ with } y(0) = 2 \\ \int \frac{1}{1+y^2}\mathrm{d}y &= \int 3t^2\mathrm{d}t \\ \arctan y &= t^3 + C \\ \arctan y(0) &= 0^3 + C \\ 2 &= C \\ \arctan y &= t^3 + 2 \\ y &= \tan\left(t^3 + 2\right) \end{aligned}$$

Summary: method of **separation of variables**:

- Type: $y' = f(t)g(y)$.
- Rewrite as $\frac{1}{g(y)}y' = f(t)$.
- Integrate both sides: $\int \frac{1}{g(y)}\mathrm{d}y = \int f(t)\mathrm{d}t$.
- Isolate y is possible.

2.3 Linear Equations

GOAL: find a method for solving equations of form $y' = f(t)y + h(t)$.

Terminology: $f(t)$ is the **coefficient term**, and $h(t)$ is the **forcing term**. If the forcing term is not just zero, then the equation is **inhomogeneous**, otherwise it is **homogeneous**.

Example 2.16. Consider $y' = -\frac{1}{t}y + 1$. Separation almost works, $y' + \frac{1}{t}y = 1$, but the left side does not resemble a chain rule. It will help to multiply by t, obtaining $ty' + y = t$. The left side does resemble the *product rule*

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t)u(t) = y'(t)u(t) + y(t)u'(t)$$

So

$$\begin{aligned} \int ty' + y \mathrm{d}t &= \int t \mathrm{d}t \\ ty &= \frac{1}{2}t^2 + C \\ y &= \frac{1}{2}t + \frac{C}{t} \end{aligned}$$

Try a simple one before observing the general method:

Example 2.17. Take $y' = y + t$ and do the same thing:

$$y' - y = t.$$

The left side is *not* a product rule derivative. Multiplying the equation by any nonzero function leaves it unaffected.

$$uy' - uy = ut$$

Now the left side is a product rule if $u = -u'$. This is a separable ODE!

$$u' = -u$$

In fact, it is the most important ODE, so $u = C \exp(-t)$

$$C \exp(-t)y' - C \exp(-t)y = C \exp(-t)t$$

This may not look simpler, but we can cancel the C,

$$\exp(-t)y' - \exp(-t)y = \exp(-t)t$$

and we designed u to make the left side a product rule, so

$$\begin{aligned} \int \exp(-t)y' - \exp(-t)y \mathrm{d}t = \int \exp(-t)t \mathrm{d}t \exp(-t)y &= -\exp(-t)(t+1) + C \\ y &= (t+1) + C \exp(t) \end{aligned}$$

where the right side was integrated by parts.

There is almost never an actual product rule, so this example better illustrates the general method. Start with $y' = f(t)y + h(t)$. Subtract to get the y terms on the same side,

$$y' - f(t)y = h(t)$$

Multiply by an unknown function $u(t)$,

$$u(t)y' - u(t)f(t)y = u(t)h(t)$$

Solve the ODE

$$u'(t) = -u(t)f(t)$$

which can always be separated giving

$$u(t) = \exp\left(-\int f(t)\mathrm{d}t\right)$$

The left is now a product rule, so the antiderivative is

$$u(t)y = \int u(t)h(t)\mathrm{d}t$$

Then we solve for y by dividing out the $u(t)$,

$$y = \frac{1}{u(t)}\int u(t)h(t)\mathrm{d}t.$$

Example 2.18. Consider $y' = y\sin(t) + 2t\exp(-\cos(t))$. Compute

$$u(t) = \exp\left(-\int \sin(t)\mathrm{d}t\right) = \exp(\cos(t))$$

Then plug in

$$y(t) = \frac{1}{\exp(\cos(t))}\int \exp(\cos(t))2t\exp(-\cos(t))\mathrm{d}t$$

and compute the integral:

$$y(t) = \exp(-\cos(t))(t^2 + C) = t^2\exp(-\cos(t)) + C\exp(-\cos(t))$$

Example 2.19 (Initial conditions). Solve $x' = x + \exp(-t)$ with $x(0) = 1$. Compute

$$u(t) = \exp\left(-\int 1\mathrm{d}t\right) = \exp(-t)$$

then

$$\begin{aligned} y(t) &= \frac{1}{\exp(-t)}\int \exp(-t)\exp(-t)\mathrm{d}t = \exp(t)\int \exp(-2t)\mathrm{d}t \\ &= \exp(t)\left(-\frac{1}{2}\exp(-2t) + C\right) = -\frac{1}{2}\exp(-t) + C\exp(t). \end{aligned}$$

Summary: method of **integrating factors**:

- Type: $y' = f(t)y + h(t)$.
- Compute integrating factor $u = \exp\big(-\int f(t)\mathrm{d}t\big)$.
- $y = \frac{1}{u(t)}\big(\int u(t)h(t)\mathrm{d}t\big)$.

3 Exact ODEs and Qualitative Analysis

Goal: Study exact ODEs, the last generally solvable type of first-order equation, and develop qualitative methods to study arbitrary ODEs.
Outline:

- Exact ODEs. (Section 2.6)
- Integrating factors for inexact equations. (Section 2.6)
- Qualitative analysis of autonomous equations. (Section 2.9)

3.1 Exact Equations

The independent variable will be x rather than t for this section. This unseemly and unnecessary change is to maintain consistency with the textbook.

Example 3.1.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y}{x+y}$$

Nonlinear and not separable. Rearrange algebraically,

$$\begin{aligned}(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} &= -y \\ y + (x+y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

Recall the *multivariable chain rule*

$$\frac{\mathrm{d}}{\mathrm{d}x}F(u(x), y(x)) = \frac{\partial F}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

If $u(x) = x$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}$.

The left side of the problem at hand has the form of a multivariable chain rule, where $y = F_x$ and $(x + y) = F_y$. Is there a function F that meets these conditions?

$$\int y \mathrm{d}x = xy + \phi(y)$$
$$\int x + y \mathrm{d}y = xy + \frac{1}{2}y^2 + \psi(x)$$

Equality occurs if $\phi(y) = \frac{1}{2}y^2$ and $\psi(x) = 0$, so $F(x, y) = xy + \frac{1}{2}y^2$. Taking the antiderivative of the right side as well we see that

$$\int y + (x + y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int 0\mathrm{d}x$$
$$xy + \frac{1}{2}y^2 = C$$

is the solution (in implicit form).

General theory: The form of interest is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-P(x, y)}{Q(x, y)}$$

or

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Sometimes these are written in the *standard form*

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = 0$$

for convenience. By the chain rule argument above, finding a function F with $F_x = P$ and $F_y = Q$ provides a solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}F(x, y(x)) &= F_x(x, y(x))\frac{\mathrm{d}x}{\mathrm{d}x} + F_y(x, y(x))\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= F_x(x, y(x)) + F_y(x, y(x))\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0\end{aligned}$$

so

$$\int P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int 0\mathrm{d}x$$

giving

$$F(x, y) = C$$

as the solution.

Definition 3.2. *If $F_x = P$ and $F_y = Q$, then F is a **potential function** for* $P\mathrm{d}x + Q\mathrm{d}y$.

Potential functions do not always exist.

Definition 3.3. *An ODE of the form*

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = 0$$

*is **exact** if $P_y = Q_x$.*

Proposition 3.4. *A potential function exists if and only if the equation is exact.*

Proof. Suppose that a potential function exists, so $F_x = P$ and $F_y = Q$. Then $P_y = F_{xy}$ and $Q_x = F_{yx}$, and $F_{yx} = F_{xy}$, so potential implies exact.

Suppose that the equation is exact. Then a potential function exists as a consequence of *Stokes' theorem* from multivariable calculus. □

To find the potential function, compute

$$\int P\mathrm{d}x \text{ and } \int Q\mathrm{d}y.$$

(Keep in mind that P and Q are multivariable functions, so the "constant" of integration may depend on the other variable.) Comparing the integrals gives $F(x, y)$.

Example 3.5.

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Identify $P(x, y) = 2xy - 9x^2$ and $Q(x, y) = 2y + x^2 + 1$. Check that the equation is exact,

$$P_y = 2x \quad Q_x = 2x$$

so there is a solution. Find F:

$$F(x, y) = \int 2xy - 9x^2\mathrm{d}x = x^2y - 3x^3 + \phi(y)$$
$$\tilde{F}(x, y) = \int (2y + x^2 + 1)\mathrm{d}y = y^2 + x^2y + y + \psi(x)\phi(y) = y + y^2$$

And the solution is

$$x^2y - 3x^3 + y + y^2 = C.$$

Example 3.6. Solve $\frac{dy}{dx} = \frac{3x^2+y^2}{3y-x}$.

Example 3.7. $\sin(x+y)dx + (2y + \sin(x+y)dy = 0$

Method of finding a **potential function**:

- Type: $\frac{dy}{dx} = \frac{-P(x,y)}{Q(x,y)}$.
- Put in standard form: $P(x,y)dx + Q(x,y)dy = 0$.
- Compute $\int P dx$ and $\int Q dy$.
- Compare, to get $F(x,y)$.
- Solution is $F(x,y) = C$.

3.2 Integrating factors for inexact equations

Goal: Solve equations of the form $P dx + Q dy = 0$ which are not exact.

Example 3.8 (Not all forms are exact). Consider $\frac{dy}{dx} = \frac{x+2y^2}{2xy}$, or in standard form, $(x+2y^2)dx - 2xydy = 0$. Applying the method for solving exact equations,

$$\int x + 2y^2 dx = \frac{1}{2}x^2 + 2y^2 x + \phi(y)$$
$$\int -2xydy = -xy^2 + \psi(x)$$

No $\phi(y)$ or $\psi(x)$ can make these equal.

Of course, the equation is not exact since $P_y = 4y$ and $Q_x = 1-2y$. But take the entire equation and multiply by x^{-3}:

$$x^{-3}\left((x+2y^2)dx - 2xydy\right) = x^{-3} \cdot 0$$
$$(x^{-2} + 2x^{-3}y^2)dx - 2x^{-2}ydy = 0$$

Now $\tilde{P}_y = 4x^{-3}y$ and $\tilde{Q}_x = 4x^{-3}y$, so the equation *is* exact. Solving as usual,

$$\int x^{-2} + 2x^{-3}y^2 dx = -x^{-1} - x^{-2}y^2 + \phi(y)$$
$$\int -2x^{-2}ydy = -x^{-2}y^2 + \psi(x)$$
$$F(x,y) = -x^{-1} - x^{-2}y^2$$
$$-x^{-1} - x^{-2}y^2 = C$$

To check that this is a solution, use implicit differentiation:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} - x^{-1} - x^{-2}y^2 &= \frac{\mathrm{d}}{\mathrm{d}x}C \\
x^{-2} + 2x^{-3}y^2 - 2x^{-2}y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
x + 2y^2 - 2xy\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x + 2y^2}{2xy}.
\end{aligned}$$

General theory: For a potential function to exist for $P\mathrm{d}x + Q\mathrm{d}y$, the partials must be equal, $P_y = Q_x$. If they are not equal, there may be a **integrating factor**—a function $u(x, y)$ that *makes* the equation exact. Recall that this form is really fractional, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-P(x,y)}{Q(x,y)} = \frac{-u(x,y)P(x,y)}{u(x,y)Q(x,y)}$$

for any nonzero $u(x, y)$. For the partial to be equal, need

$$\frac{\partial}{\partial y}u(x,y)P(x,y) = \frac{\partial}{\partial x}u(x,y)Q(x,y).$$

This is actually a *partial* differential equation. Sometimes an integrating factor will appear out of the ether. Otherwise, it will help to make the assumption that u is a function of only one variable. Say $u = u(x)$. Then

$$\begin{aligned}
\frac{\partial}{\partial y}u(x)P(x,y) &= \frac{\partial}{\partial x}u(x)Q(x,y). \\
u(x)P_y(x,y) &= u'(x)Q(x,y) + u(x)Q_x(x,y) \\
u(x)\left(P_y - Q_x\right) &= u'(x)Q(x,y) \\
u'(x) &= \frac{1}{Q(x,y)}\left(P_y - Q_x\right)u(x)
\end{aligned}$$

Now, if such a $u(x)$ exists, all of the ys on the right side must cancel out. In that case, this is an example of the most important ODE, $u' = f(x)u$. As such, the solution is (remember?)

$$u(x) = \exp\left(\int \frac{1}{Q(x,y)}\left(P_y - Q_x\right)u(x)\mathrm{d}x\right).$$

By symmetry, if we had assumed $u = u(y)$, the same equation holds swapping P and Q,

$$u(y) = \exp\left(\int \frac{1}{P(x,y)}\left(Q_x - P_y\right)\mathrm{d}y\right).$$

From here, multiply by the integrating factor and find the potential function as normal.

Example 3.9. Taking the equation $(x+2y^2)\mathrm{d}x - 2xy\mathrm{d}y = 0$ from above, and looking for a $u(x)$, get

$$\begin{aligned} u(x) &= \exp\left(\int \frac{1}{-2xy}(4y+2y)\,\mathrm{d}x\right) \\ &= \exp\left(\int \frac{-3}{x}\mathrm{d}x\right) \\ &= \exp(-3\log(x)) \\ &= \exp(\log(x^{-3})) \\ &= \frac{1}{x^3} \end{aligned}$$

which is the same function that arrived *deus ex machina* earlier.

Example 3.10 (Single-variable integrating factors do not always exist.). Consider $(y^2 - xy)\mathrm{d}x + x^2\mathrm{d}y = 0$. Here $P = (y^2 - xy)$ so $P_y = 2y - x$ and $Q = x^2$ so $Q_x 2x$, so it is not exact. Try to find a $u(x)$:

$$u(x) = \exp\left(\int \frac{1}{x^2}(2y - x - 2x)\,\mathrm{d}x\right) = \exp\left(\frac{2y-3x}{x^2}\mathrm{d}x\right)$$

which fails because there are both xs and ys. Try instead to find a $u(y)$:

$$u(y) = \exp\left(\int \frac{1}{y^2 - xy}(2x - 2y + x)\mathrm{d}y\right) = \exp\left(\int \frac{3x - 2y}{y^2 - xy}\mathrm{d}y\right)$$

which also fails. The necessary integrating factor is $u(x,y) = \dfrac{1}{xy^2}$. (Exercise: check!)

Example 3.11 (Integrating factor $u(y)$.). Consider $2y\mathrm{d}x + (x+y)\mathrm{d}y = 0$. Find an integrating factor $u(y)$ and solve.

Method of **integrating factors for exact equations**.

- Type: $P\mathrm{d}x + Q\mathrm{d}y = 0$ with$P_y \neq Q_x$.
- Compute either
  - $u(x) = \exp\left(\int \frac{1}{Q}(P_y - Q_x)\mathrm{d}x\right)$ or
  - $u(y) = \exp\left(\int \frac{1}{P}(Q_x - P_y)\mathrm{d}y\right)$.
- Multiply by u, obtaining $uP\mathrm{d}x + uQ\mathrm{d}y = 0$.
- Solving using method of potential functions.

3.3 Qualitative Analysis

Example 3.12 (Separable does not imply solvable). Consider $y' = \sin(y^2)$. This is autonomous (no t) and thus separable:

$$\int \frac{1}{\sin(y^2)} \mathrm{d}y = \int 1 \mathrm{d}t$$

Exercise: use WolframAlpha to compute the integral on the left.

Example 3.13. Consider $y' = \cos(t)y^2 + t$. This is nonlinear, inseparable, inexact, and does not permit an integrating factor.

Constructing more examples of unsolvable ODEs is fairly simple. As the first example shows, even having a solution method does not mean that an analytic solution is computable. (Analytic means having an explicit equation.)

Definition 3.14. *The **asymptotic behavior** of a solution $y(t)$ to an initial value problem $y' = f(t, y)$ with $y(t_0) = y_0$ is*

$$\lim_{t\to\infty} y(t).$$

(This is the forward *asymptote, the* backwards *asymptote is the limit at $-\infty$.)*

If the limit is $\pm\infty$ the function "blows up". Otherwise, the asymptotic behavior is a horizontal asymptote in the graph of $y(t)$.

Example 3.15. Consider $y' = (y-1)\cos(yt)$ with initial condition $y(0) = 1$. A solution is $y(t) = 1$. Clearly $\lim_{t\to\infty} y(t) = 1$.

Definition 3.16. *A solution of the form $y(t) = c$ for some constant c is an **equilibrium solution**.*

Equilibrium solutions are easy to spot: if $y' = (y - c)f(t, y)$, for any c, then $y(t) = c$ is a solution. More generally, if $f(t, c) = 0$ for all t and $y = c$, then $y(t) = c$ is an equilibrium solution.

Example 3.17.

- For $y' = y$, the only equilibrium solution is $y(t) = 0$.
- For $y' = y(y - 2)$, $y(t) = 0$ and $y(t) = 2$ are equilibrium solutions.
- For $y' = \sin(y^2)$, $y(t) = 0$, $y(t) = \sqrt{\pi}$, $y(t) = \sqrt{2\pi}$, et cetera are equilibrium solutions.

What is the asymptotic behavior of non-equilibrium solutions? This is difficult in general, but can be understood easily for *autonomous* equations. Recall

Definition 3.18. *An ODE of the form $y' = g(y)$ is **autonomous**.*

(All three of the preceding examples are autonomous.)

Exercise 3.19. Sketch slope fields for the first two examples.

Observe that the slope field is *invariant* under left-right translation. There is no t in the equation, so above any point on the t-axis, the slopes look the same. This implies that non-equilibrium solutions either

- tend toward $\pm\infty$,
- increase toward an equilibrium, or
- decrease toward an equilibrium

so all asymptotic behavior is determined by the equilibria.

Definition 3.20. *An equilibrium $y(t) = c$ is (asymptotically)* ***stable*** *if all nearby solutions approach c as $t \to \infty$. An equilibrium is* ***unstable*** *if any nearby solutions do not.*

Example 3.21. For $y' = y(y-2)$, $y = 2$ is unstable and $y = 0$ is stable.

3.3.1 Phase line plots

Recall that autonomous equations $y' = g(y)$ can be plotted on the (y, y')-plane.

Definition 3.22. *An* ***equilibrium point*** *is a point in the (y, y')-plane corresponding to an equilibrium solution.*

In this type of plot

- $g(c) = 0$ implies $y = c$ is an equilibrium,
- $g(y) > 0$ implies that $y(t)$ tends toward the equilibrium point to the right (or $+\infty$),
- $g(y) < 0$ implies that $y(t)$ tends toward the equilibrium point to the left (or $-\infty$).

The type of an equilibrium is characterized by the following theorem:

Theorem 3.23. *If y_0 is an equilibrium point of $y' = g(y)$, then*

- *If* $\dfrac{\mathrm{d}g}{\mathrm{d}y} < 0$ *the equilibrium is* stable.
- *If* $\dfrac{\mathrm{d}g}{\mathrm{d}y} > 0$ *the equilibrium is* unstable.

- *If* $\dfrac{\mathrm{d}g}{\mathrm{d}y} = 0$ *further analysis is required.*

Proof. Draw the phase line plot corresponding to each derivative type. □

Example 3.24. Classify the equilibrium points for the equation $y' = (y^2-4)(y-3)$. The equilibria are of course at $y = \pm 2$ and $y = 3$. Using the theorem,

$$g(y) = (y^2-4)(y-3) = y^3 - 3y^2 - 4y + 12 \implies \frac{\mathrm{d}g}{ddy} = 3y^2 - 6y - 5$$

which has $g'(-2) = 19 > 0$, $g'(2) = -5 < 0$ and $g'(3) = 4 > 0$, so that $y = 2$ is stable and $y = -2$ and $y = 3$ are unstable.

4 Applications

Goal: Understand how modelling problems and physical laws give rise to differential equations, and physically interpret the solutions to those equations.
Outline:

- Mixing problems (Section 2.5)
- Newton's law of cooling (Section 2.2, p. 31)
- Population dynamics (Section 3.1)
- Electric circuits (Section 3.4)
- Mechanics (Section 1.3, p. 14, Section 4.1)

4.1 Mixing problems

Example 4.1. A tank contains 100 gallons of fresh water. A solution of brine with a salt concentration of 2 lb/gallon is poured in at a rate of 3 gallons per minute. A spigot is opened at the same time allowing the mixed solution to drain out at 3 gallons per minute. Find an equation for the amount of salt in the tank after t minutes.

No knowledge of any outside physical laws is required. Make one assumption–the brine and fresh water are always mixed completely. Rather than directly find an expression for the amount of salt, it is easier to find an expression for the *rate of change* in the amount of salt. Let S represent the amount of salt at time t. Then

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \text{ rate in } - \text{ rate out }.$$

The rate at which salt enters is 2 lb/gallon × 3 gallons/minute = 6 lb/minute. The rate at which salt leaves is also concentration of salt × 3 gallons/minute. However, the concentration of salt is not constant. In fact, it is given by the amount of salt, S, divided by the volume of liquid, 100 gallons (this is constant because the rate of *liquid* in and out are equal). Thus

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 6 - 3\frac{S}{100}.$$

This is a linear equation with $f(t) = 3/100$ and $h(t) = 6$, so

$$u(t) = \exp\left(\int 3/100\mathrm{d}t\right) = \exp(3t/100)$$

and the solution is

$$S(t) = \exp\left(\frac{-3t}{100}\right)\int 6\exp\left(\frac{3t}{100}\right)\mathrm{d}t = \exp\left(\frac{-3t}{100}\right)\left(\frac{600}{3}\exp\left(\frac{3t}{100}\right) + C\right) = 200 + C\exp\left(\frac{-3t}{100}\right).$$

This is an example of a **mixing problem**. The key technique is identifying that the problem should be solved indirectly. The given information is about *rates* so it is easy to write an equation for the rate of change. That differential equation can then be solved to give the actual answer.

Example 4.2. Consider the above situation, but now with the solution draining at 2 gallons per minute. Everything is the same except the volume is not constant at 100 gallons. Liquid enters at 3 gallons/minute and leaves at 2 gallons/minute, so the volume is $V(t) = 100 + (3-2)t = 100 + t$. Thus,

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 6 - 2\frac{S}{100+t}.$$

This is linear with $f(t) = 2/(100+t)$ and $h(t) = 6$.

$$u(t) = \exp\left(\int 2/(100+t)\mathrm{d}t\right) = \exp(2\log(100+t)) = (100+t)^2$$

The solution is

$$S(t) = \frac{1}{(100+t)^2}\int 6(100+t)^2\mathrm{d}t = \frac{6}{3(100+t)^2}(100+t)^3 = 200 + 2t$$

Exercise 4.3. What is the maximum amount of salt in the tank in the first example? How about the second?

4.2 Law of cooling

Many physical laws are actual formulas for setting up differential equations. One simple example is **Newton's law of cooling** which says that the rate of change of temperature of an object is proportional to the difference in temperature between that object and the ambient temperature:

$$\frac{dT}{dt} = k(T - A)$$

where A is the ambient temperature and $k < 0$ is a constant depending on the heat capacity of the object, the surface area, and the "heat transfer coefficient".

This is a separable equation:

$$\begin{aligned}
\int \frac{1}{T-A}\, dT &= \int k dt \\
\log(T - A) &= kt + C \\
T &= A \pm \tilde{C} \exp(kt)
\end{aligned}$$

Note that $T(0) = A \pm \tilde{C}$, so $\tilde{C}$ is the initial difference in temperature.

4.3 Population dynamics

Sometimes differential equations arise as models of phenomena. **Population dynamics** refers to how the size of a population (of animals, people, bacteria, et cetera) changes over time.

In 1202, Leonardo of Pisa proposed the following (discrete) model for the size of a population of rabbits: assume that there is a pair of rabbits to begin with, and any pair of rabbits will mate after one month. The population (in pairs of rabbits) follows the pattern

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Of course, this model is quite unrealistic (in particular, no rabbits die).

Thomas Malthus, in 1789, suggested that the (human) population would increase at a rate proportional to its size. If the probability of a person being born is b and of a person dying is d, then

$$P' = rP$$

where $r = b - d$. This gives the equation for population:

$$P(t) = C \exp(rt)$$

(Recall the solution to the most important ODE.) Malthus believed that $r > 0$, so that

$$\lim_{t\to\infty} P(t) = \infty$$

which lead him to worry about a population explosion.

Realistically, when a population increases, its resources become more scarce so b decreases and d increases. Accounting for that in the model, the birth rate is $b - cP$ and the death rate is $d + aP$. Then

$$P'(t) = (b - cP - (d + aP))P = ((b - d) - (a + c)P)P = \left(r - \frac{rP}{k}\right)P = r\left(1 - \frac{P}{k}\right)P$$

where $r = b - d$ as before and $k = r/(a + c)$. This last equation is called the **logistic model** for population growth. It can be solved explicitly (see Section 3.1 of the textbook). More informative is the qualitative analysis via a phase line plot.

Exercise 4.4. Find the equilibrium solutions for the logistic model. Determine what initial populations lead to increasing and decreasing populations.

4.4 Circuits

A simple electric circuit consists of a *voltage source* and some combination of *resistors*, *capacitors*, and *inductors*. The *current* in a circuit is the rate at which charge flows through it. A set of physical laws describe how current depends on the configuration of elements in the circuit.

Denote the voltage, in volts, by $E(t)$, charge by Q, and current by I. Since current is the rate of change of charge,

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

A resistor causes a drop in voltage, E_R, given by **Ohm's law**:

$$E_r = RI$$

where R is the *resistance*, a physical property of the resistor measured in ohms. An inductor produces a magnetic field which opposes change in current. The voltage drop across an inductor is given by **Faraday's law**:

$$E_l = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

where L is the *inductance* measured in henrys. A capacitor allows charge to build up. The voltage drop is given by

$$E_C = \frac{1}{C}Q$$

where C is the capacitance measured in farads and Q is the charge measured in coulombs.

The circuit analyzed here will be simple loops, which obey **Kirchoff's first law**: the sum of the voltage drops around a closed loop is zero.

The voltage source causes an increase in voltage, while all other components cause a decrease, so in general

$$E_R + E_L + E_C - E = 0 \implies E = E_R + E_L + E_C.$$

for a circuit with one of each component type.

Example 4.5. Consider a circuit with a $\frac{1}{2}$-ohm resistor and a 1-henry inductor (and no capacitor), with a constant 1-volt voltage source. Find an equation describing the current as a function of time if the initial current is zero.

Here

$$E = E_R + E_L \implies 1 = \frac{1}{2}I + \frac{ddI}{\mathrm{d}t}$$

with $I(0) = 0$. The equation is both linear and separable. Separating gives

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= 1 - \frac{1}{2}I \\
\frac{1}{1-\frac{1}{2}I}\mathrm{d}I &= \mathrm{d}t \\
\int \frac{1}{1-\frac{1}{2}I}\mathrm{d}I &= \int \mathrm{d}t \\
-2\log\left(1 - \frac{1}{2}I\right) &= t + C \\
\log\left(1 - \frac{1}{2}I\right) &= -\frac{1}{2}t + C' \\
1 - \frac{1}{2}I &= \exp(C')\exp\left(-\frac{1}{2}t\right) \\
I &= 2 - \tilde{C}\exp\left(-\frac{1}{2}t\right)
\end{aligned}$$

The initial condition requires that

$$I(0) = 2 - \tilde{C} = 0$$

so $\tilde{C} = 2$ and

$$I(t) = 2 - 2\exp\left(-\frac{1}{2}t\right).$$

Example 4.6. Suppose that a circuit has a 2-ohm resistor, a 0.25-farad capacitor (and no inductor), and a voltage source $E(t) = \cos(t)$. Find the equation for current as a function of time.

Here

$$E_R + E_C = E \implies 2I + 4Q = \cos(t).$$

Since $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$, this becomes

$$2\frac{\mathrm{d}Q}{\mathrm{d}t} + 4Q = \cos(t) \implies \frac{\mathrm{d}Q}{\mathrm{d}t} = -2Q + \frac{1}{2}\cos(t).$$

This is linear, with

$$u(t) = \exp\left(\int 2\mathrm{d}t\right) = \exp(2t)$$

and

$$Q(t) = \exp(-2t) \int (\frac{1}{2}\cos(t)\exp(2t)\mathrm{d}t.$$

An explicit solution requires repeated integration by parts.

n	sign	derivative	integral
0	$+$	$\exp(2t)$	$\cos(t)$
1	$-$	$2\exp(2t)$	$\sin(t)$
2	$+$	$4\exp(2t)$	$-\cos(t)$

This indicates that

$$\begin{aligned}
\int \exp(2t)\cos(t)\mathrm{d}t &= \exp(2t)\sin(t) - 2\exp(2t)(-\cos(t)) + \int 4\exp(2t)(-\cos(t))\mathrm{d}t \\
&= \exp(2t)\sin(t) + 2\exp(2t)\cos(t) - \int 4\exp(2t)\cos(t)\mathrm{d}t \\
5\int \exp(2t)\cos(t) &= \exp(2t)\sin(t) + 2\exp(2t)\cos(t) \\
\int \exp(2t)\cos(t) &= \frac{1}{5}\exp(2t)\sin(t) + \frac{2}{5}\exp(2t)\cos(t)
\end{aligned}$$

Thus

$$Q(t) = \exp(-2t)\left(\frac{1}{5}\exp(2t)\sin(t) + \frac{2}{5}\exp(2t)\cos(t) + C\right) = \frac{1}{5}\sin(t) + \frac{2}{5}\cos(t) + C\exp(-2t)$$

and

$$I(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{5}\sin(t) + \frac{2}{5}\cos(t) + C\exp(-2t)\right) = \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) - 2C\exp(-2t)$$

Example 4.7. Consider the circuit with a 2-ohm resistor, a 0.25-farad capacitor, a 1-henry inductor, and $E(t) = 1$. Here

$$E_R + E_L + E_C = E \implies 1 = 2I + \frac{\mathrm{d}I}{\mathrm{d}t} + 4Q.$$

To obtain an equation for charge, replace I with Q':

$$1 = 2\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + 4Q.$$

To obtain an equation for current, first differentiate both sides:

$$0 = 2\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + 4\frac{\mathrm{d}Q}{\mathrm{d}t}$$

then replace Q' with I:

$$0 = 2\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + 4I.$$

Observe that these are both *second order* equations.

4.5 Motion

Newton's second law says that $F = ma$. Recognizing that $a(t) = x''(t)$ if x is the position of an object, this gives a second-order differential equation.

Example 4.8. In projectile motion, the only force acting on the object is gravity. Therefore, Newton's second law says $mg = ma$ where g is the acceleration due to gravity. Thus

$$a = g \implies x''(t) = g \implies x'(t) = gt + v_0 \implies x(t) = \frac{1}{2}gt^2 + v_0 t + x_0.$$

where v_0 is the initial velocity of the object and x_0 is the initial position.

Example 4.9. The above is of course a major simplification. In particular, air resistance will cause a force acting opposite the direction of motion. A rough approximation is that the force due to air resistance, $R = rv^2$ where r is some constant. Consider a falling object and add this term to Newton's second law:

$$F = ma \implies mg - rv^2 = mv'.$$

This is nonlinear and separable, though the integration of the separable equation is difficult. Applying qualitative analysis, there are equilibria at $v = \pm\sqrt{mg/r}$. Assuming an object dropped with an initial velocity of zero, the velocity will approach $\sqrt{mg/r}$. This is called the **terminal velocity**.

Example 4.10. Consider a mass suspended on a spring. If it is in equilibrium (completely stationary) the acceleration is zero, and the forces are mg and the restorative force due to the spring, $R(x)$, which depends on the distance, x, that the spring is stretched. Hooke's law says that

$$R(x) = -kx$$

so the differential equation is

$$mx'' = -kx + mg$$

(ignoring friction, air resistance, and that sort of thing.) This can be transformed by defining $y = x - x_0$ (with x_0 the equilibrium point) giving

$$my'' = -k(y + x_0) + mg = -ky$$

because $-kx_0 + mg = 0$. This can be put in the form

$$my'' + ky = 0$$

which is again a second order equation, like the equation obtained in the section on circuits.

5 Higher-Order Linear Equations

Goal: Solve second-order, linear, homogeneous equations with constant coefficients.
Outline:

- Characteristic polynomials and linear equations. (Section 4.1)
- Special case with repeated roots. (Section 4.3)
- Special case with complex roots. (Section 4.3)
- Review of complex numbers. (Appendix, p. 699)

5.1 Linear equations

Definition 5.1. *An ODE is of* ***degree*** *n if it contains an n-th derivative. It is* ***linear*** *if y, y', y'', et cetera appear linearly. It is* ***homogeneous*** *if there are no isolated $f(t)$ terms. It has* ***constant coefficients*** *if the coefficient terms on y, y', y'', et cetera are constants.*

Example 5.2. All of these adjectives apply to equations like $y'' + 3y' + 2y = 0$, or more generally, $y^{(n)} + \cdots + ay' + by = 0$. An equation like $y'' + 3y' + y^2 = 0$ is nonlinear. The equation $y'' + 3y' + 2y = t$ is inhomogeneous, and the equation $y'' + 3ty' + 2y = 0$ does not have constant coefficients.

This section will focus on equations described by *all* these terms, with emphasis on second-order. Homogeneity will be relaxed later.

Example 5.3. The only first order equation of this form is $y' + ay = 0$ (equivalently, $y' = by$). The solution is $y(t) = \exp(at)$.

Example 5.4. Consider $y'' + 3y' + 2y = 0$. The general solution is $y(t) = A\exp(-t) + B\exp(-2t)$.

$$\begin{aligned}
y' &= -A\exp(-t) - 2B\exp(-2t) \\
y'' &= A\exp(-t) + 4B\exp(-2t) \\
y'' + 3y' + 2y &= A\exp(-t) + 4B\exp(-2t) - 3A\exp(-t) - 6B\exp(-2t) + 2A\exp(-t) + 2B\exp(-2t) \\
&= (A - 3A + 2A)\exp(-t) + (4B - 6B + 2B)\exp(-2t) \\
&= 0
\end{aligned}$$

As in the first order case, solutions are exponentials. There are two pieces, with separate constants, because the equation is second order. Why do the exponentials have the arguments $-t$ and $-2t$? Observe that $(x + 1)(x + 2) = x^2 + 3x + 2$ has the same coefficients as the differential equation…

General theory: consider $y'' + py' + qy = 0$. Recall that $\frac{\mathrm{d}}{\mathrm{d}t}\exp(at) = a\exp(at)$. If $y = \exp(at)$, $y^{(n)}(t) = a^n \exp(at) = a^n y$. Therefore, surmising that the solution is an exponential,

$$y(t) = C\exp(\lambda t)$$

$$\begin{aligned} y'' + py' + qy &= \lambda^2 C\exp(\lambda t) + p\lambda C\exp(\lambda t) + qC\exp(\lambda t) \\ &= C\exp(\lambda t)(\lambda^2 + p\lambda + q \end{aligned}$$

Of course, $C\exp(\lambda t) \neq 0$, so this is a solution if and only if

$$\lambda^2 + p\lambda + q = 0$$

or in other words, if λ is a root of this polynomial.

Definition 5.5. *The* ***characteristic polynomial*** *of $y'' + py' + qy = 0$ is*

$$p(\lambda) = \lambda^2 + p\lambda + q.$$

This is a second-order polynomial, so it has exactly two (complex) roots, λ_1 and λ_2. Therefore the solutions are

$$y_1 = C_1\exp(\lambda_1 t) \text{ and } y_2 = C_2\exp(\lambda_2 t).$$

Example 5.6. Consider $y'' - 4y = 0$. The characteristic polynomial is $p(\lambda) = \lambda^2 - 4$. The roots are $\lambda_1 = 2$ and $\lambda_2 = -2$. The solutions are

$$y_1 = C_1\exp(2t) \text{ and } y_2 = C_2\exp(-2t).$$

Definition 5.7. *A* linear combination *of two functions, y_1 and y_2, is any function of the form*

$$y = Ay_1 + By_2$$

where A and B are arbitrary constants.

Theorem 5.8. *If y_1 and y_2 are both solutions to $y'' + py' + qy = 0$, then any linear combination of y_1 and y_2 is also a solution.*

Proof. (Exercise) □

Definition 5.9. *The collection of solutions $y_1(t)$, $y_2(t)$, et cetera, to a linear equation is the* ***fundamental set of solutions****. The* ***general solution*** *is*

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + \cdots$$

Example 5.10 (Higher order). Consider $y''' + 6y'' + 11y' + 6y = 0$. Assuming the same method works, the characteristic polynomial is

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = (\lambda + 1)(\lambda + 2)(\lambda + 3)$$

and the solutions should be

$$y_1 = C_1 \exp(-t), \quad y_2 = C_2 \exp(-2t), \text{ and } y_3 = C_3 \exp(-3t).$$

These indeed work (check!).

5.2 Repeated roots

Example 5.11. Consider $y'' + 2y + 1 = 0$. The characteristic polynomial is $p(\lambda) = \lambda^2 + 2\lambda + \lambda = (\lambda + 1)^2$. So the root is $\lambda = -1$, repeated twice. Clearly a solution is $y_1(t) = C_1 \exp(-t)$.

There is also a second solution, $y_2(t) = C_2 t \exp(-t)$:

$$\begin{aligned}
y_2' &= C_2 \exp(-t) - C_2 t \exp(-t) \\
&= C_2 \exp(-t) - y_2 \\
y_2'' &= -C_2 \exp(-t) - C_2 \exp(-t) + C_2 t \exp(-t) \\
&= -2C_2 \exp(-t) + y_2 \\
y'' + 2y' + y &= -2C_2 \exp(-t) + y_2 + 2C_2 \exp(-t) - 2y_2 + y_2 \\
&= 0
\end{aligned}$$

To understand this, return to the **quadratic formula**:

The two roots of $p(\lambda) = \lambda^2 + p\lambda + q$ are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Typically this gives two different numbers, one from the + and one from the −. However, if $\sqrt{p^2 - 4q} = 0$, then there is only one, repeated, root. Specifically, $\lambda = -p/2$.

Proposition 5.12. *If λ is a repeated root of the characteristic polynomial, then $Ct\exp(\lambda t)$ is a solution.*

Proof. Assuming λ is a root of $\lambda^2 + p\lambda + q$, and $y = Ct\exp(\lambda t)$,

$$\begin{aligned}
y' &= C\exp(\lambda t) + C\lambda t\exp(\lambda t)\\
y'' &= C\lambda\exp(\lambda t) + C\lambda\exp(\lambda t) + C\lambda^2 t\exp(\lambda t)\\
y'' + py' + qy &= 2C\lambda\exp(\lambda t) + C\lambda^2 t\exp(\lambda t) + pC\exp(\lambda t) + pC\lambda t\exp(\lambda t) + qCt\exp(\lambda t)\\
&= C\exp(\lambda t)(2\lambda + t\lambda^2 + p + pt\lambda + qt)\\
&= C\exp(\lambda t)((\lambda^2 + p\lambda + q)t + 2\lambda + p)\\
&= C\exp(\lambda t)(2\lambda + p)
\end{aligned}$$

If λ is a repeated root, then $\lambda = -p/2$, so $2\lambda + p = -p + p = 0$. ◻

Example 5.13. Solve $y'' + 6y' + 9y = 0$. The characteristic polynomial is $\lambda^2 + 6\lambda + 9 = (\lambda+3)(\lambda+3)$. There is one repeated root, $\lambda = -3$, so the general solution is

$$y(t) = C_1\exp(-3t) + C_2 t\exp(-3t).$$

Example 5.14. Solve $y''' - 3y' + 2y = 0$. The characteristic polynomial is $\lambda^3 - 3\lambda + 2 = (\lambda - 2)(\lambda + 1)(\lambda + 1)$. The solution is

$$y(t) = C_1\exp(2t) + C_2\exp(-t) + C_2 t\exp(-t).$$

This in fact works more generally.

Example 5.15. Solve $y''' - 6y'' + 12y' - 8 = 0$. Characteristic polynomial is $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3$. Note the thrice-repeated root $\lambda = 2$. Solution is

$$y'(t) = C_1\exp(2t) + C_2 t\exp(2t) + C_3 t^2\exp(2t).$$

(Check the last term!)

Proposition 5.16. *If λ is an n-times repeated root of the characteristic polynomial, then*

$$y(t) = Ct^{n-1}\exp(\lambda t)$$

is a solution.

Proof. Can be done directly, and tediously. An alternative proof will be done later. ◻

Definition 5.17. *Two functions $u(t)$ and $v(t)$ are* ***linearly independent*** *if the only constants a and b such that*

$$au(t) + bv(t) = 0$$

are $a = b = 0$.

Each piece of the fundamental set of solutions to a linear system must be linearly independent from all of the others. Notice that $C_1 t^j \exp(\lambda t)$ and $C_2 t^k \exp(\lambda t)$ are linearly independent when $j \neq k$.

Proposition 5.18. *Two functions are linearly independent if and only if the Wronskian*

$$W = uv' - u'v$$

is not equal to zero.

5.3 Complex roots

Example 5.19. Solve $y'' + y = 0$. The characteristic polynomial is $\lambda^2 + 1 = 0$. Roots are $\lambda_1 = i$ and $\lambda_2 = -i$. Solution is

$$y(t) = C_1 \exp(it) + C_2 \exp(-it).$$

Returning to the quadratic formula,

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

notice that if $p^2 < 4q$, then the argument of the square root is negative, so a complex number appears. *This does not affect the solution!* It is still $C \exp(\lambda t)$. However, the meaning of the complex exponential deserves elucidation.

5.3.1 Digression: complex numbers

A complex number $z = a + bi$ consists of a real part, $\mathrm{Re}(z) = a$, and an imaginary part, $\mathrm{Im}(z) = b$ (note, not bi). Addition and multiplication are performed as usual, except $i^2 = -1$. The complex numbers are plotted in the **complex plane**, $\mathbb{C}$, with the real part as the horizontal axis and the imaginary part as the vertical. Multiplication of a complex number by a real number is a "stretching." Multiplication by an imaginary number is a "rotation." Multiplication of two complex numbers does both.

Example 5.20. Plot $1 + 2i$, $2(1 + 2i) = 2 + 4i$, and $i(1 + 2i) = -2 + i$. Combining, $(2 + i)(1 + 2i) = 2 + 4i + i - 2 = 5i$.

Definition 5.21. *If z is a complex number, the* ***complex exponential*** *is*

$$\exp(z) := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots .$$

Theorem 5.22 (Euler's Formula).

$$\exp(a + bi) = \exp(a)(\cos(b) + \sin(b)i)$$

The exponential involves powers of z, so it should have both stretching and rotation. The $\exp(a)$ is the stretch, and the cos and sin are rotation.

Proof. First, $\exp(a + bi) = \exp(a)\exp(bi)$ by standard rules. Then,

$$\begin{aligned}
\exp(bi) &= 1 + bi + \frac{(bi)^2}{2!} + \frac{(bi)^3}{3!} + \cdots + \frac{(bi)^n}{n!} + \cdots \\
&= 1 + bi + \frac{-b^2}{2!} + \frac{-b^3 i}{3!} + \cdots + \frac{(bi)^n}{n!} + \cdots \\
&= (1 + \frac{-b^2}{2!} + \frac{b^4}{4!} + \cdots) + (b - \frac{b^3}{3!} + \frac{b^5}{5!} + \cdots \\
&= \cos(b) + \sin(b)i
\end{aligned}$$

recalling the Taylor series for sin and cos. □

Example 5.23. Some complex exponentials:

$$\begin{aligned}
\exp(1 + 2i) &= \exp(1)(\cos(2) + \sin(2)i) = e\cos(2) + e\sin(2)i \\
\exp(\pi i) &= \exp(0)(\cos(\pi) + \sin(\pi)i) = -1 \\
\exp((1 + 2i)t) &= \exp(t)(\cos(2t) + \sin(2t)i)
\end{aligned}$$

This means that complex exponentials as solutions can be interpreted as sin and cos. Even more directly,

Proposition 5.24. *If $z(t) = C\exp((a + bi)t)$ is a solution to $y'' + py' + qy = 0$, then $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are solutions.*

Proof. First, since $z = y_1 + y_2 i$, then $z' = y_1' + y_2' i$ and $z'' = y_1'' + y_2'' i$. So

$$\begin{aligned} y_1'' + y_2'' i + p y_1' + p y_2' i + q y_1 + q y_2 i &= 0 + 0i \\ (y_1'' + p y_1' + q y_1) + (y_2'' + p y_2' + q y_2) i &= 0 + 0i \\ (y_1'' + p y_1' + q y_1) &= 0 \text{ and} \\ (y_2'' + p y_2' + q y_2) &= 0 \end{aligned}$$

□

Explicitly, $y_1 = \operatorname{Re}(z) = \exp(at)\cos(bt)$ and $y_2 = \operatorname{Im}(z) = \exp(at)\sin(bt)$, which are both real-valued functions. That they are solutions can also be checked directly.

Observe that complex roots always appear in complex conjugate pairs.

Definition 5.25. *The* ***complex conjugate*** *of $z = a + bi$ is $\overline{z} = a - bi$.*

Of course, $\operatorname{Re}(z) = \operatorname{Re}(\overline{z})$, and $\operatorname{Im}(z) = -\operatorname{Im}(\overline{z})$.

Example 5.26. Solve $y'' + 2y' + 3y = 0$. The characteristic polynomial is $\lambda^2 + 2\lambda + 3$, which has

$$\lambda = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm \sqrt{-2} = -1 \pm \sqrt{2}i.$$

The solution is

$$y(t) = C_1 \exp\Big((-1 + \sqrt{2}i)t\Big) + C_2 \exp\Big((-1 - \sqrt{-2}i)t\Big).$$

Alternatively, the real and imaginary parts are solutions,

$$y_1(t) = C_1 \exp(-t)\cos\Big(\sqrt{2}t\Big) \text{ and } y_2(t) = C_2 \exp(-t)\sin\Big(\sqrt{2}t\Big)$$

where it does not matter which root's real and imaginary components are taken because of the constant C_2. This gives an real-valued general solution of

$$y(t) = C_1 \exp(-t)\cos\Big(\sqrt{2}t\Big) + C_2 \exp(-t)\sin\Big(\sqrt{2}t\Big).$$

The preceding illustrates that if $\lambda = a + bi$ is a complex root, there are two ways the solution can be written,

$$\begin{aligned} y(t) &= C_1 \exp(\lambda t) + C_2 \exp\Big(\overline{\lambda} t\Big) \text{ or} \\ y(t) &= C_1 \exp(at)\cos(bt) + C_2 \exp(at)\sin(bt) \end{aligned}$$

The former is the **complex-valued solution**, and the latter is the **real-valued solution**.

Example 5.27. The real-valued solution of $y'' + y = 0$ is

$$y(t) = C_1 \cos(t) + C_2 \sin(t).$$

6 Inhomogeneous linear equations

Goal: Solve second-order, linear, inhomogeneous equations with constant coefficients, and understand the basic method of Laplace transforms.
Outline:

- Method of undetermined coefficients for inhomogeneous equations. (Section 4.5)
- Definition of Laplace Transform. (5.1)
- Basic Laplace transforms. (Section 5.2)
- Inverse Laplace transforms. (Section 5.3)

6.1 Undetermined Coefficients

Example 6.1. Solve $y'' + 3y' + 2y = \exp(t)$. Note the existence of a *forcing term*, $\exp(t)$. Ignoring the $\exp(t)$, the solution would be

$$y_h(t) = C_1 \exp(-t) + C_2 \exp(-2t).$$

Note that a solution is

$$y_p(t) = \frac{1}{6}\exp(t).$$

Check:

$$y_p'' + 3y_p' + 2y_p = \frac{1}{6}\exp(t) + \frac{3}{6}\exp(t) + \frac{2}{6}\exp(t) = \exp(t).$$

Claim the actual solution is

$$y(t) = C_1 \exp(-t) + C_2 \exp(-2t) + \frac{1}{6}\exp(t) = y_h + \frac{1}{6}\exp(t).$$

Check:

$$\begin{aligned}
y' &= y_h' + \frac{1}{6}\exp(t)\\
y'' &= y_h'' + \frac{1}{6}\exp(t)\\
y'' + 3y' + 2y &= y_h'' + \frac{1}{6}\exp(t) + 3y_h' + \frac{3}{6}\exp(t) + 2y_h + \frac{2}{6}\exp(t)\\
&= y_h'' + 3y_h' + 2y_h + \exp(t)\\
&= 0 + \exp(t)
\end{aligned}$$

Theorem 6.2. *Suppose that y_p is some solution to $y'' + py' + qy = f(t)$, and that y_1 and y_2 are the usual solutions to $y'' + py' + qy = 0$. Then the general solution to $y'' + py' + qy = f(t)$ is given by $y_p + Ay_1 + By_2$.*

Proof. Suppose that y is any solution to the inhomogeneous equation.

$$\begin{aligned}(y'' + py' + qy) - (y_p'' + py_p' + qy_p) &= f(t) - f(t)\\ (y - y_p)'' + p(y - y_p)' + q(y - y_p) &= 0\end{aligned}$$

so $y - y_p$ is a solution to the homogeneous equation. Thus $y - y_p = Ay_1 + By_2$ for some A, B, and so the general solution is $y = y_p + Ay_1 + By_2$. □

Therefore, to solve an inhomogeneous equation, the only additional data required is a **particular solution**, y_p.[1] For simple forcing terms, the appropriate y_p can often be found by **undetermined coefficients**.

Example 6.3. Assuming it plausible that y_p is an exponential in the preceding example, the coefficient $\frac{1}{6}$ can be found by "guessing" $y_p = C\exp(t)$. Then

$$y'' + 3y' + 2y = C\exp(t) + 3C\exp(t) + 2C\exp(t) = \exp(t)$$

so $6C = 1$ meaning $C = 1/6$.

Standard guesses:

- $f(t) = \exp(at)$ guess $y_p = C\exp(at)$.
- $f(t) = C\cos(\omega t) + D\sin(\omega t)$ guess $y_p = A\cos(at) + B\sin(bt)$ (even if C or D is zero).
- $f(t)$ = polynomial of degree n guess $y_p = A_n t^n + \cdots + A_1 t + A_0$ (even if not all degrees appear in $f(t)$).
- If $f(t)$ is a sum or product of the above, guess a corresponding sum or product.

Example 6.4. For $y'' + 3y' + 2y = \cos(3t)$ guess $y_p = A\cos(3t) + B\sin(3t)$. For $y'' + 3y' + 2y = t^3 + 2t$ guess $y_p = At^3 + Bt^2 + Ct + D$. For $y'' + 3y' + 2y = t^2\sin(t)$ guess $y_p = (At^2 + Bt + C)\sin(t)$. For $y'' + 3y' + 2y = t + \exp(t)$ guess $y_p = At + B\exp(t)$.

[1] Note well that this term is used to mean two different things: either the solution to an IVP, or a solution to an inhomgeneous linear system.

Example 6.5. Solve $y'' + 3y' + 2y = \exp(t) + t$. Here the guess is $y_p = A\exp(t) + Bt + C$. Plugging in gives

$$\begin{aligned} y_p' &= A\exp(t) + B \\ y_p'' &= A\exp(t) y_p'' + 3y_p' + 2y_p \quad = A\exp(t) + 3A\exp(t) + 3B + 2A\exp(t) + 2Bt + 2C6A = 1 \\ 2B &= 1 \\ 3B + 2C &= 0 \end{aligned}$$

So $y_p = \frac{1}{6}\exp(t) + \frac{1}{2}t - \frac{3}{4}$.

Example 6.6. Solve $y''+3y'+2y = t\exp(t)$. Guess is $y_p = A\exp(t)(Bt+C) = ABt\exp(t)+AC\exp(t)$. There are effectively only two constants, so let $\tilde{A} = AB$ and $\tilde{B} = AC$. Then

$$\begin{aligned} y_p' &= \tilde{A}\exp(t) + \tilde{A}t\exp(t) + \tilde{B}\exp(t) \\ y_p'' &= (2\tilde{A} + B)\exp(t) + \tilde{A}t\exp(t) \\ y_p'' + 3y_p' + 2y_p &= (2\tilde{A} + \tilde{B})\exp(t) + \tilde{A}t\exp(t) + 3(\tilde{A} + \tilde{B})\exp(t) + 3\tilde{A}t\exp(t) + 2\tilde{A}t\exp(t) + 2\tilde{B}\exp(t) \\ 2\tilde{A} + \tilde{B} + 3\tilde{A} + 3\tilde{B} + 2\tilde{B} &= 0 \\ \tilde{A} + 3\tilde{A} + 2\tilde{A} &= 1 \end{aligned}$$

So $\tilde{A} = 1/6$ and $\tilde{B} = -5/30$.

Example 6.7 (Exceptional Case). For $y'' + 3y' + 2y = \exp(-t)$, guess $y_p = C\exp(-t)$ fails:

$$\begin{aligned} y_p' &= -y_p \\ y_p'' &= y_p \\ y_p'' + 3y_p' + 2y_p &= y_p - 3y_p + 2y_p = 0 \end{aligned}$$

which is not $\exp(-2t)$ no matter what C is. Of course, this is because $y_h = C\exp(-t)$ already. In this case, the correct guess is $y_p = Ct\exp(-t)$. This now works:

$$\begin{aligned} y_p' &= C\exp(-t) - Ct\exp(-t) \\ y_p'' &= -C\exp(-t) - C\exp(-t) + Ct\exp(-t) = -2C\exp(-t) + Ct\exp(-t) \\ y_p'' + 3y_p' + 2y_p &= -2C\exp(-t) + Ct\exp(-t) + 3(C\exp(-t) - Ct\exp(-t)) + 2Ct\exp(-t) \\ &= C\exp(-t) \end{aligned}$$

as desired.

The addition of a t makes the solution linearly independent from the homogeneous solution, while still being a function that is "close enough" to the forcing term to work (most of the time).

Method of **undetermined coefficients**:

- Type: $y'' + py' + qy = f(t)$.
- Solve the homogeneous equation for y_h.
- Guess that $y_p = Cf(t)$, unless $Cf(t)$ is already a solution, in which case add a t term.
- Plug in and solve for the correct value of C.

Example 6.8. Find the particular solution for

$$y'' + 2y' - 3y = 5\sin(3t).$$

Make a guess of the form $y_p = A\cos(3t) + B\sin(3t)$. Then

$$\begin{aligned} y_p' &= -3A\sin(3t) + 3B\cos(3t) \\ y_p'' &= -9A\cos(3t) - 9B\sin(3t) \end{aligned}$$

and so

$$\begin{aligned} -9A\cos(3t) - 9B\sin(3t) + 2(-3A\sin(3t) + 3B\cos(3t)) - 3(A\cos(3t) + B\sin(3t)) &= 5\sin(3t) \\ -9A + 6B - 3A &= 0 \\ -9B - 6A - 3B &= 5 \\ B &= 2A \\ -30A &= 5 \\ A &= -\frac{1}{6} \\ B &= -\frac{1}{3} \\ y) =_p &= -\frac{1}{3}\cos(3t) - \frac{1}{6}\sin(3t) \end{aligned}$$

6.2 The Laplace Transform

The Laplace transform is a very powerful method for solving ODEs. Here it is used to solve the initial value problem $ay'' + by' + cy = f(t)$ with $y(0) = y_0$ and $y'(0) = y_1$ even for unusual functions $f(t)$.

Definition 6.9. *If $f(t)$ is any function defined for $t > 0$, the **Laplace transform** of f is*

$$\mathscr{L}\{f\} := \int_0^\infty f(t)\exp(-st)\mathrm{d}t.$$

Example 6.10. If $f(t) = t$, then

$$\mathscr{L}\{f\} = \int_0^\infty t\exp(-st)\mathrm{d}t = -\frac{t}{s}\exp(-st) - \frac{1}{s^2}\exp(-st)\Big|_0^\infty = \frac{1}{s^2}$$

Observe that $\mathscr{L}\{\}$ transforms a function of t into a function of the new variable s.

Laplace transforms of different functions are not often computed by hand. The transforms of many common functions can be found in the table.

Example 6.11. If $f(t) = \exp(4t)$, then $\mathscr{L}\{f\} = \frac{1}{s-4}$. If $f(t) = \exp(4t)\cos(t)$ then $\mathscr{L}\{f\} = \frac{s-4}{(s-4)^2+1}$.

Example 6.12. Suppose that $F(s) = \frac{1}{s^2+4}$. Then the $f(t)$ with $\mathscr{L}\{f(t)\} = F(s)$ must be $f(t) = \sin(2t)$. If $F(s) = \frac{2}{(s-3)^3}$ then $f(t) = t^2\exp(3t)$.

It is conventional for the Laplace transform of $f(t)$ to be $F(s)$, or $g(t)$ to be $G(s)$. Alternatively, $\mathscr{L}^{-1}\{F(s)\} = f(t)$.

Proposition 6.13. *If $\mathscr{L}\{f\} = \mathscr{L}\{g\}$, then $f = g$ almost everywhere.*

Proof. Omitted. Rigorously defining *almost everywhere* is beyond the scope of this course. □

This means that the Laplace transform is *one-to-one*. Therefore, the inverses computed above are in fact the only possibilities.

6.2.1 Properties of the Laplace transform

Theorem 6.14. *The Laplace transform is a linear operator: $\mathscr{L}\{af + bg\} = a\mathscr{L}\{f\} + b\mathscr{L}\{g\}$.*

Proof. Let f and g be functions, a and b constants. Then

$$\begin{aligned}\mathscr{L}\{af + bf\} &= \int_0^\infty \exp(-st)(af(t) + bg(t))\mathrm{d}t \\ &= \int_0^\infty a\exp(-st)f(t)\mathrm{d}t + \int_0^\infty b\exp(-st)g(t)\mathrm{d}t \\ &= a\mathscr{L}\{f\} + b\mathscr{L}\{g\}\end{aligned}$$

□

Example 6.15. Compute $f(t) = \cos(2t) + 2\exp(4t)$. This is

$$F(s) = \frac{s}{s^2+4} + \frac{2}{s-4}.$$

Table of Laplace Transforms

$f(t)$	$\mathscr{L}\{f\} = F(s)$	Domain
1	$1/s$	$s > 0$
t	$1/s^2$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\exp(at)$	$\frac{1}{s-a}$	$s > a$
$\exp(at)\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$\exp(at)\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n \exp(at)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
δ_p	$\exp(-sp)$	
$H(t-c)$	$\frac{\exp(-cs)}{s}$	$s > 0$

	$f(t)$	$\mathscr{L}\{f\}$
Definition:	$f(t)$	$\int_0^\infty \exp(-st)f(t)\mathrm{d}t$
Linearity:	$af(t) + bg(t)$	$aF(s) + bG(s)$
Dilation:	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First Translation:	$\exp(at)f(t)$	$F(s-a)$
Second Translation:	$\mathrm{H}(t-c)f(t-c)$	$\exp(-cs)F(s)$
Input derivative:	y'	$sY(s) - y(0)$
Input derivative second order:	y''	$s^2Y(s) - sy(0) - y'(0)$
Input derivative n-th order:	$y^{(n)}$	$s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{n-1}(0)$
Transform derivative:	$tf(t)$	$-F'(s)$
Transform derivative second order:	$t^2f(t)$	$F''(s)$
Transform derivative n-th order:	$t^nf(t)$	$(-1)^nF^{(n)}(s)$
Convolution:	$(f * g)(t)$	$F(s) \cdot G(s)$

The theorem also implies that $\mathscr{L}^{-1}\{\}$ is a linear operator: $\mathscr{L}^{-1}\{aF(s)+bG(s)\} = af(t)+bg(t)$.

Example 6.16.

$$\mathscr{L}^{-1}\left\{\frac{2}{s-3}+\frac{6}{s^2+4}\right\} = 2\exp(3t)+3\sin(2t)$$

Warning: $\mathscr{L}\{fg\} \neq \mathscr{L}\{f\}\mathscr{L}\{G\}$.

Theorem 6.17 (First Translation Principle).

$$\mathscr{L}\{\exp(at)f(t)\} = F(s-a).$$

Proof.

$$\begin{aligned}
\mathscr{L}\{\exp(at)f(t)\} &= \int_0^\infty \exp((a-s)t)f(t)\mathrm{d}t \\
&= \int_0^\infty \exp(-(s-a)t)f(t)\mathrm{d}t \\
&= F(s-a)
\end{aligned}$$

□

Example 6.18. Given that $\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$,

$$\mathscr{L}\{\exp(3t)\cos(\pi t)\} = \frac{s-3}{(s-3)^2+pi^2}$$

Theorem 6.19 (Transform Derivative Principle).

$$\mathscr{L}\{-tf(t)\} = \frac{\mathrm{d}}{\mathrm{d}s}F(s).$$

Proof. The proof comes from doing the computation in reverse.

$$\begin{aligned}
\mathscr{L}\{-F'(s)\} &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \exp(-st)f(t)\mathrm{d}t \\
&= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\exp(-st)f(t)\mathrm{d}t \\
&= \int_0^\infty -t\exp(-st)f(t)\mathrm{d}t \\
&= \mathscr{L}\{-tf(t)\}
\end{aligned}$$

□

Repeating the above proof gives the:

Theorem 6.20 (Transform n-th derivative principle).

$$(-1)^n \mathscr{L}\left\{t^n f(t)\right\} = \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$$

Example 6.21.

$$\mathscr{L}\left\{t\cos(t)\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2}$$

Theorem 6.22 (Dilation Principle).

$$\mathscr{L}\left\{f(bt)\right\} = \frac{1}{b} F\left(\frac{s}{b}\right).$$

Proof.

$$\begin{aligned}\mathscr{L}\left\{f(bt)\right\} &= \int_0^\infty \exp(-st) f(bt)\mathrm{d}t \\ &= \int_0^\infty \frac{1}{b}\exp((-s/b)r) f(r)\mathrm{d}r\end{aligned}$$

by letting $r = bt$. Then

$$= \frac{1}{b} F\left(\frac{s}{b}\right)$$

□

Example 6.23. Using the dilation principle:

$$\mathscr{L}\left\{\sin(2t)\right\} = \frac{1}{2}\mathscr{L}\left\{\sin(t)\right\}(s/2) = \frac{1}{2}\frac{1}{(s/2)^2+1} = \frac{1}{2(s^2/4)+2} = \frac{1}{s^2/2+2} = \frac{2}{s^2+4}$$

which is the known answer.

6.3 Differential Equations with Laplace

Proposition 6.24.

$$\mathscr{L}\left\{y'\right\} = sY(s) - y(0)$$

Proof.

$$\begin{aligned}\mathscr{L}\left\{y'(t)\right\} &= \int_0^\infty \exp(-st) y'(t)\mathrm{d}t \\ &= \exp(-st)y(t)\big|_0^\infty + s\int_0^\infty \exp(-st)y(t)\mathrm{d}t \\ &= \lim_{t\to\infty}\exp(-st)y(t) - y(0) + sY(s) \\ &= sY(s) - y(0)\end{aligned}$$

$\square$

Proposition 6.25.

$$\mathscr{L}\left\{y''\right\} = s^2 Y(s) - sy(0) - y'(0)$$

Proof.

$$\begin{aligned}\mathscr{L}\left\{y''(t)\right\} &= s\mathscr{L}\left\{y'\right\} - y'(0)\\ &= s\left(s\mathscr{L}\left\{y\right\} - y(0)\right) - y'(0)\\ &= s^2 Y(s) - sy(0) - y'(0)\end{aligned}$$

$\square$

Example 6.26. Solve the initial value problem $y' + 2y = \exp(-2t)$ with $y(0) = 0$.

$$\begin{aligned}\mathscr{L}\left\{y' + 2y\right\} &= \mathscr{L}\left\{\exp(-2t)\right\}\\ \mathscr{L}\left\{y'\right\} + \mathscr{L}\left\{2y\right\} &= \frac{1}{s+2}\\ sY - y(0) + 2Y &= \frac{1}{s+2}\\ (s+2)Y &= \frac{1}{s+2}\\ Y &= \frac{1}{(s+2)^2}\end{aligned}$$

Recognize that the right hand side is the Laplace transform of $t\exp(at)$ with $a = -2$. So $y(t) = t\exp(-2t)$.

The Method of Laplace Transforms:

- Type: linear equations with constant coefficients and IVP $y(0) = y_0$, $y'(0) = y_1$.
- Apply $\mathscr{L}\left\{\right\}$ to both sides of the equation.
- Solve the result for Y in terms of s (there will be no ts).
- Identify the function $y(t)$ which gives $Y(s)$ when transformed.

Example 6.27. Solve $y'' + 4y' + 4y = 2t\exp(-2t)$ with $y(0) = 1$ and $y'(0) = 1$.

$$\begin{aligned}
\mathscr{L}\{y'' + 4y' + 4y\} &= \mathscr{L}\{2t\exp(-2t)\} \\
\mathscr{L}\{y''\} + \mathscr{L}\{4y'\} + \mathscr{L}\{4y\} &= 2\mathscr{L}\{2t\exp(-2t)\} \\
s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) &= 2\frac{1!}{s-(-2)^{1+1}} \\
Y(s)(s^2 + 4s + 4) - (s+4)y(0) - y'(0) &= \frac{2}{(s+2)^2} \\
Y(s)(s^2 + 4s + 4) - s - 1 &= \frac{2}{(s+2)^2} \\
Y(s)(s+2)^2 &= s + 1 + \frac{2}{(s+2)^2} \\
Y(s) &= \frac{s+1}{(s+2)^2} + \frac{2}{(s+2)^4} \\
Y(s) &= \frac{s+1}{(s+2)^2} + \frac{2}{(s+2)^4} \\
\mathscr{L}^{-1}\left\{\frac{2}{(s+2)^4}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{3}\frac{3!}{(s+2)^4}\right\} \\
&= \frac{1}{3}t^3\exp(-2t) \\
\frac{s+1}{(s+2)^2} &= \frac{(s+2)-1}{(s+2)^2} \\
&= \frac{s+2}{(s+2)^2} - \frac{1}{(s+2)^2} \\
&= \frac{1}{s+2} - \frac{1}{(s+2)^2} \\
\mathscr{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{(s+2)^2}\right\} &= \exp(-2t) - t\exp(-2t) \\
\mathscr{L}^{-1}\{Y(s)\} &= \mathscr{L}^{-1}\left\{\frac{s+1}{(s+2)^2} + \frac{2}{(s+2)^4}\right\} \\
y(t) &= \exp(-2t) - t\exp(-2t) + \frac{1}{3}t^3\exp(-2t)
\end{aligned}$$

7 Laplace Transforms

Goal: Use Laplace transforms to solve ODEs.
Outline:

- Partial fractions. (Section 5.3)
- Laplace transform of discontinuous functions. (Section 5.5)
- Delta functions, and convolutions. (Sections 5.6 and 5.7)

7.1 Partial Fractions

Example 7.1. Solve $y'' - 9y = -2\exp(t)$ with $y(0) = 0$ and $y'(0) = 1$.

$$\begin{aligned}
\mathscr{L}\{y'' - 9y\} &= \mathscr{L}\{-2\exp(t)\} \\
s^2 Y - sy(0) - y'(0) - 9Y &= \frac{-2}{s-1} \\
(s^2 - 9)Y - 1 &= \frac{-2}{s-1} \\
Y &= \frac{1}{s^2-9}\left(\frac{-2}{s-1} + 1\right) \\
Y &= \frac{1}{(s-3)(s+3)}\left(\frac{s-3}{s-1}\right) \\
Y &= \frac{1}{s-1)(s+3)}
\end{aligned}$$

$$\begin{aligned}
\frac{A}{s-1} + \frac{B}{s+3} &= \frac{1}{s-1)(s+3)} \\
A(s+3) + B(s-1) &= 1 \\
A &= 1/4 \\
B = -1/4Y &= \frac{1/4}{s-1} - \frac{1/4}{s+3} \\
y &= \frac{1}{4}\exp(t) - \frac{1}{4}\exp(-3t)
\end{aligned}$$

Laplace transforms typically produce rational functions.

Definition 7.2. *A **rational function** is a function of the form $f(x) = p(x)/q(x)$ where p and q are both polynomials.*

To invert the Laplace transform, those function must be put into special forms, and this nearly always requires a **partial fractions decomposition**.

Example 7.3 (Distinct real roots).

$$\frac{s-2}{(s-3)(s-4)(s-5)} = \frac{A}{s-3} + \frac{B}{s-4} + \frac{C}{s-5}$$

Clearing denominators,

$$s - 2 = A(s-4)(s-5) + B(s-3)(s-5) + C(s-3)(s-4)$$

expanding out the functions, we have

$$s - 2 = A(s^2 - 9s + 20) + B(s^2 - 8s + 15) + C(s^2 - 7s + 12)$$

which gives rise to a system of three equations

$$\begin{aligned} 0 &= A + B + C \\ 1 &= -9A - 8B - 7C \\ -2 &= 20A + 15B + 12C \end{aligned}$$

by setting terms of like degree equal. (The first equation comes from the s^2 terms, the second from the s terms, and the last from the constant terms.) This can be solved for A, B, and C. Alternatively, since first equation must be true for all values of s, it must be true at $s = 4$, so

$$\begin{aligned} s - 2 &= A(s-4)(s-5) + B(s-3)(s-5) + C(s-3)(s-4) \\ 4 - 2 &= A(4-4)(4-5) + B(4-3)(4-5) + C(4-3)(4-4) \\ 2 &= 0 + B(1)(-1) + 0 \\ 2 &= -B \\ B &= -2 \end{aligned}$$

It also must be true at $s = 5$, so

$$\begin{aligned} 5 - 2 &= A(5-4)(5-5) + B(5-3)(5-5) + C(5-3)(5-4) \\ 3 &= 0 + 0 + C(2)(1) \\ 3 &= 2C \\ C &= \frac{3}{2} \end{aligned}$$

Lastly, at $s = 3$,

$$\begin{aligned} 3 - 2 &= A(3-4)(3-5) + B(3-3)(3-5) + C(3-3)(3-4) \\ 1 &= A(-1)(-2) + 0 + 0 \\ 1 &= 2A \\ A &= \frac{1}{2} \end{aligned}$$

Thus

$$\frac{s-2}{(s-3)(s-4)(s-5)} = \frac{1/2}{s-3} + \frac{-2}{s-4} + \frac{3/2}{s-5}$$

Example 7.4 (Repeated real roots).

$$\frac{s}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{s+2}$$

Then

$$\begin{aligned} s &= As + 2A + Bs + 2B \\ A + B &= 1 \\ 2A + 2B &= 0 \end{aligned}$$

but the second equation implies $A + B = 0$, not 1!

Instead, set up as follows:

$$\frac{s}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

Then

$$\begin{aligned} s &= As + 2A + B \\ A &= 1 \\ 2A + B &= 0 \\ B &= -2 \end{aligned}$$

Example 7.5 (Irreducible polynomials).

$$\frac{s}{(s^2+2)(s-1)} = \frac{A}{s^2+2} + \frac{B}{s-1}$$

Then

$$\begin{aligned} s &= As - A + Bs^2 + 2B \\ A &= 1 \\ B &= 0 \\ -A + 2B &= 0 \end{aligned}$$

and these equations are inconsistent.

Instead, set up as follows:

$$\frac{s}{(s^2+2)(s-1)} = \frac{A}{s^2+2} + \frac{Bs}{s^2+2} + \frac{C}{s-1}$$

Then

$$\begin{aligned} & s = As - A + Bs^2 - Bs + Cs^2 + 2C \\ & C = 3 \text{ by evaluating at } s = 1 \\ & B = -3 \text{ by comparing the } s^2 \\ & A = -2. \end{aligned}$$

Note that the "trick" of evaluating at the roots of the polynomial still applies here.

The three techniques sometimes must be mixed:

Example 7.6. Simplify

$$\frac{s+1}{s(s^2+1)^2}$$

The irreducible term is also repeated, so the setup is

$$\begin{aligned}
\frac{s+1}{s(s^2+1)^2} &= \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2} + \frac{E}{s} \\
s+1 &= As^4 + As^2 + Bs^3 + Bs + Cs^2 + Ds + Es^2 + E \\
E &= 1 \\
B+D &= 1 \\
A+C+E &= 0 \\
B &= 0 \\
A &= 0 \\
D &= 1 \\
C &= -1
\end{aligned}$$

by equating terms of like degree.

7.2 Discontinuous Forcing Terms

Definition 7.7. *The* ***Heaviside function*** *is defined to be*

$$H(t) := \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

(The behavior at zero is unimportant.)

Piecewise-continuous functions can be written in terms of the Heaviside.

Example 7.8.

$$f(t) = \begin{cases} 1 & 0 \le t < 2 \\ 3t & 2 \le t < 4 \\ 5\exp(t) & 4 \le t \end{cases} = H(t) + (3t-1)H(t-2) + (5\exp(t) - 3t)H(t-4)$$

Notice that $H(t-c)$ "turns on" at $t = c$.

$$f(t) = \begin{cases} t & 0 < t \le 2 \\ -t^2 & 2 < t \end{cases} = tH(t) + (t^2 - t)H(t-2)$$

The Laplace transform of the Heaviside function is given in the table. It was also assigned as a homework problem!

Proposition 7.9.

$$\mathscr{L}\left\{H(t-c)\right\} = \frac{\exp(-cs)}{s}$$

Example 7.10. Solve $y'' + 4y = f(t)$ with $y(0) = 0$, $y'(0) = 0$, and

$$f(t) = \begin{cases} 5 & t > 3 \\ 0 & t < 3 \end{cases}.$$

The right side can be written as $5H(t-3)$. Therefore,

$$\begin{aligned} y'' + 4y &= 5H(t-3) \\ \mathscr{L}\left\{y''\right\} + \mathscr{L}\left\{4y\right\} &= \mathscr{L}\left\{5H(t-3)\right\} \\ s^2 Y + 4Y &= 5\frac{\exp(3s)}{s} \end{aligned}$$

Proposition 7.11 (Second Translation Principle).

$$\mathscr{L}\left\{H(t-c)f(t-c)\right\} = \exp(-cs)F(s)$$

Proof.

$$\begin{aligned} \mathscr{L}\left\{H(t-c)f(t-c)\right\} &= \int_0^\infty H(t-c)f(t-c)\exp(-st)\mathrm{d}t \\ &= \int_c^\infty f(t-c)\exp(-st)\mathrm{d}t \\ (u &:= t-c) \\ &= \int_0^\infty f(u)\exp(-s(t+c))\mathrm{d}u \\ &= \exp(-cs)\int_0^\infty f(u)\exp(-st)\mathrm{d}u \\ &= \exp(-cs)F(s) \end{aligned}$$

□

Example 7.12. Find the inverse Laplace transform of

$$Y(s) = \frac{\exp(-2s)}{s(s^2+9)}.$$

Partial fractions gives:

$$\frac{1}{s(s^2+9)} = \frac{1}{9}\left(\frac{1}{s} - \frac{s}{s^2+9}\right) =: F(s).$$

Then $f(t) = \frac{1}{9}(1 - cos(3t))$. Apply the second translation principle:

$$y(t) = H(t-2)f(t-2) = \frac{1}{9}H(t-2)(1-\cos(3(t-2))).$$

Example 7.13. Find the Laplace transform of $f(t) = t^2H(t-2)$. Note that the t^2 does not contain a shift. However,

$$t^2 = (t-2)^2 - 4t + 4 = (t-2)^2 + 2(t-2)$$

so $f(t) = (t-2)^2H(t-2) + 2(t-2)H(t-2)$. Both parts contain a shift, giving,

$$F(s) = \exp(-2s)\frac{2}{s^3} + 2\exp(-2s)\frac{1}{s^2}.$$

7.3 Deltas and Convolutions

Definition 7.14. *The* ***Dirac delta*** *at p is the distribution*

$$\delta(t) = \begin{cases} +\infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

with the property that $\int_a^b \delta_p(t)\mathrm{d}t = 1$ if $a \leq p \leq b$, and $\int_a^b \delta_p(t)\mathrm{d}t = 0$ otherwise.

Observe that $\int_a^b f(t)\delta(t)\mathrm{d}t = f(0)$ so long as $0 \in [a, b]$.

Proposition 7.15.

$$\mathscr{L}\{\delta(t-c)\} = \exp(-cs)$$

In particular, $\mathscr{L}\{\delta(t)\} = \exp(0) = 1$.

Proof.

$$\begin{aligned}\mathscr{L}\{\delta(t-c)\} &= \int_0^\infty \delta(t-c)\exp(-st)\mathrm{d}t \\ &= \exp(-sc)\end{aligned}$$

□

Definition 7.16. *The* ***unit impulse response function*** *is the solution to*

$$ay'' + by' + cy' = \delta(t)$$

with $y(0) = 0$ and $y'(0) = 0$. It is denoted $e(t)$.

Example 7.17. Find the unit impulse response function to $y'' + 2y' + 2y$. This means solving

$$y'' + 2y' + 2y = \delta(t) \text{ with } y(0) = 0, y'(0) = 0.$$

By Laplace, and calling the solution e,

$$\begin{aligned} s^2E(s) + 2sE(s) + 2E(s) &= 1 \\ E(s) &= \frac{1}{s^2 + 2s + 2} \\ &= \frac{1}{(s+1)^2 + 1} \\ e(t) &= \exp(-t)\sin(t) \end{aligned}$$

using the second translation principle.

Definition 7.18. *The* ***convolution*** *of $f(t)$ and $g(t)$ is*

$$(f * g)(t) := \int_0^t f(u)g(t-u)\mathrm{d}u$$

Convolution behaves like multiplication:

$$\begin{aligned} f * g &= g * f \\ f * (g + h) &= f * g + f * h \\ (f * g) * h &= f * (g * h) \\ f * 0 &= 0 \\ f * \delta &= f \end{aligned}$$

In fact, the Laplace transform turns convolution into multiplication:

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$$

Example 7.19. Find the inverse Laplace transform of

$$Y(s) = \frac{\exp(-2s)}{s^2}.$$

Writing $F(s) = \exp(-2s)$ and $G(s) = \frac{1}{s^2}$ gives

$$y(t) = \mathscr{L}^{-1}\{F\} * \mathscr{L}^{-1}\{G\} = \delta(t-2) * t$$

By definition

$$\delta(t-2) * t = \int_0^t \delta(u-2)(t-u)\mathrm{d}u = t-2$$

8 Interlude - Linear Algebra

Goal: Understand matrix and vector operations and how they relate to systems of linear equations.
Outline:

- Vectors, matrices, and multiplication. (Section 7.1)
- Solving $Ax = b$ and linear systems. (Section 7.2)
- Row reduction algorithm. (Section 7.3)
- Free variables and inconsistent systems. (Section 7.3)

8.1 Vectors and Matrices

Definition 8.1. *A **vector** of size n is an ordered list of n numbers (real or complex). Vectors will be denoted like so:* $\underline{v}$.

Example 8.2. Some vectors of size 2:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} \quad \begin{bmatrix} 2 \\ \frac{-1}{2} \end{bmatrix} \quad \begin{bmatrix} 1+i \\ 3 \end{bmatrix}$$

Some vectors of size 3:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} e \\ \pi \\ \sqrt{2} \end{bmatrix} \quad \begin{bmatrix} 2 \\ \frac{-1}{2} \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1+i \\ 3 \\ 1-i \end{bmatrix}$$

Typically the n is unstated. The number n may also be called the **size** of the vector. Vectors of the same size may be added.

Example 8.3. Adding vectors of size 2:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1+\pi \\ \sqrt{2} \end{bmatrix} \quad \begin{bmatrix} 2 \\ \frac{-1}{2} \end{bmatrix} + \begin{bmatrix} 1+i \\ 3 \end{bmatrix} = \begin{bmatrix} 3+i \\ \frac{5}{2} \end{bmatrix}$$

Adding vectors of size 3:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} e \\ \pi \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1+e \\ \pi \\ \sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} 2 \\ \frac{-1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 1+i \\ 3 \\ 1-i \end{bmatrix} = \begin{bmatrix} 3+i \\ \frac{5}{2} \\ 1-i \end{bmatrix}$$

Addition is component-wise so the vectors to be added must have the same size.

Vectors can also be multiplied by **scalars**—individual numbers.

Example 8.4.

$$2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \sqrt{2}\begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2}\pi \\ 2 \end{bmatrix} \qquad 4\begin{bmatrix} 2 \\ \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \qquad (1-i)\begin{bmatrix} 1+i \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3-3i \end{bmatrix}$$

(The vector dot and cross products will not be used.)

Definition 8.5. *A **matrix** is an ordered array of numbers. Matrices will be denoted like so: $\underline{\underline{M}}$.*

Example 8.6. Some 2×2 matrices:

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1+i & 1 \\ 0 & 3 \end{bmatrix}$$

Some 3×3 matrices:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3×2 and a 2×3 matrix:

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

By convention, and $m \times n$ matrix is one with m rows and n columns. Matrices of the same size can be added together:

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 10 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Matrices also can be multiplied by scalars:

$$2\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 4 & 16 \end{bmatrix}$$

Vectors can be multiplied by matrices of the appropriate size:

Example 8.7. 2×2 matrix times 2-vector:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{-1}{2} \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

3×3 matrix times 3-vector:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e \\ \pi \\ \sqrt{2} \end{bmatrix} = e \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \pi \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e + 2\pi \\ \pi \\ \sqrt{2} \end{bmatrix}$$

3×2 matrix times 2-vector:

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{-1}{2} \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-1}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ \frac{-1}{2} \end{bmatrix}$$

2×3 matrix times 3-vector:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Definition 8.8. *The* ***matrix-vector product*** *is defined by*

$$\begin{bmatrix} | & | & \cdots & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := x_1 \underline{v}_1 + x_2 \underline{v}_2 + \cdots + x_n \underline{v}_n$$

Notice that the size of the vector must equal the number of columns of the matrix, and that the resulting vector is not always the same size as the original.

An $m \times n$-matrix multiplied by an n-vector produces and m-vector.

Matrices can also be multiplied by matrices of the appropriate size:

Example 8.9.

$$\begin{bmatrix}1 & 4\\2 & 8\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} = \left[\begin{bmatrix}1 & 4\\2 & 8\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 4\\2 & 8\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}\right] = \begin{bmatrix}1 & 8\\2 & 16\end{bmatrix}$$

$$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1+i & 1\\0 & 3\end{bmatrix} = \begin{bmatrix}0 & -3\\1+i & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 4\\2 & 8\end{bmatrix} = \begin{bmatrix}1 & 4\\4 & 16\end{bmatrix}$$

In general,

An $m \times n$ and a $n \times k$ matrix can be multiplied (in that order).

Example 8.10.

$$\begin{bmatrix}1 & 1\\0 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\1 & 0 & 0\end{bmatrix} = \begin{bmatrix}2 & 2 & 1\\2 & 0 & 0\\1 & 0 & 0\end{bmatrix}$$

However,

$$\begin{bmatrix}1 & 2 & 1\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 6\\0 & 1\end{bmatrix}$$

Example 8.11. The matrix with ones on the diagonal and zeros elsewhere is called the **identity matrix**, and denoted $\underline{\underline{I}}$. It has an important property:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

and likewise for other sizes: $\underline{\underline{AI}} = \underline{\underline{IA}} = \underline{\underline{A}}$ and $\underline{\underline{I}}\underline{x} = \underline{x}$ for any $\underline{\underline{A}}$ or $\underline{x}$.

8.2 Linear Systems

Linear *algebra* is about solving for unknowns. Given a matrix $\underline{\underline{A}}$ and a vector $\underline{b}$, find an unknown vector $\underline{x}$ such that

$$\underline{\underline{A}}\underline{x} = \underline{b}.$$

Example 8.12. Let

$$\underline{\underline{A}} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \text{ and } \underline{b} = \begin{bmatrix}4\\10\end{bmatrix}.$$

Call the unknown vector $\underline{x} = [x_1, x_2]^T$.[2] Multiplying $\underline{\underline{A}}$ by $\underline{x}$ gives:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 & 4 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

Setting equal to $\underline{b}$ gives

$$x_1 + 2x_2 = 4$$
$$3x_1 + 4x_2 = 10$$

This is a system of linear equations. To solve, multiply the first equation by 2 and subtract from the second:

$$3x_1 + 4x_2 - 2x_1 - 4x_2 = 10 - 8$$
$$x_1 = 2$$

Substituting this into either equation gives $x_2 = 1$. The solution is the vector

$$\underline{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Example 8.13. Conversely, a linear system corresponds to a matrix system. Solving

$$x_1 + x_2 + x_3 = 2$$
$$2x_1 - x_3 = 1$$
$$x_2 + x_3 = 0$$

is equivalent to solving

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

8.2.1 Gaussian Elimination

The matrix form of a linear system of equations permits a simple algorithm for solving them, called **Guassian elimination**.

Example 8.14. The system from the first example:

$$x_1 + 2x_2 = 4$$
$$3x_1 + 4x_2 = 10$$

[2] For notational convenience these notes use the *transpose* of a vector, meaning that the column is written as a row.

Can be turned into

$$
\begin{aligned}
2x_1 + 4x_2 &= 8 \\
3x_1 + 4x_2 &= 10
\end{aligned}
$$

without changing the solution set (since both sides of the first equation were multiplied by the same number). Likewise, by changing the order of the equations,

$$
\begin{aligned}
3x_1 + 4x_2 &= 10 \\
2x_1 + 4x_2 &= 8
\end{aligned}
$$

is obtained, again without affecting the solution. Finally, subtracting one equation from the other does not change the solutions.

$$
\begin{aligned}
x_1 &= 2 \\
2x_1 + 4x_2 &= 8
\end{aligned}
$$

Now notice that subtracting twice the first equation from the second gives

$$
\begin{aligned}
x_1 &= 2 \\
4x_2 &= 4
\end{aligned}
$$

and dividing the second row by 4 gives

$$
\begin{aligned}
x_1 &= 2 \\
x_2 &= 1
\end{aligned}
$$

This same procedure can be carried out on the matrix version:

Example 8.15. First, *augment* the matrix by placing the $\underline{b}$ vector as an extra column, separated by a line:

$$
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & | & 4 \\ 3 & 4 & | & 10 \end{bmatrix}
$$

Then

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 4 & 10 \end{array}\right] \implies \left[\begin{array}{cc|c} 2 & 4 & 8 \\ 3 & 4 & 10 \end{array}\right]$$
$$\implies \left[\begin{array}{cc|c} 3 & 4 & 10 \\ 2 & 4 & 8 \end{array}\right]$$
$$\implies \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 4 & 8 \end{array}\right]$$
$$\implies \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 4 & 4 \end{array}\right]$$
$$\implies \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

The first column corresponds to x_1 and the second to x_2, so the solution $x_1 = 2$ and $x_2 = 1$ can be immediately determined.

Every linear system of equations can be solved with these three operations.

Definition 8.16. *The* ***elementary row operations*** *are*

- *Multiplication of a row by a scalar.*
- *Addition/subtraction of one row from another.*
- *Swapping the order of rows.*

Definition 8.17. *A matrix is in* ***row echelon form*** *if the first nonzero entry of each row is a* 1 *and is strictly to the right of those in above rows. These entries are called* ***pivots***

Example 8.18. The matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

are in row echelon form. The matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

are not.

Definition 8.19. *A matrix is in **reduced row echelon form** (RREF) if it is in row echelon form and in addition each pivot is the only nonzero entry in its column.*

Exercise 8.20. Which of the preceding matrices is in RREF? (Hint: there is only one.)

The elementary row operations can be used to put any matrix into RREF. When an augmented matrix is in RREF, the solutions are immediate.

Example 8.21. Solving

$$\begin{bmatrix} 1 & 1 & 1 \\ -9 & -8 & -7 \\ 20 & 15 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ -9 & -8 & -7 & | & 1 \\ 20 & 15 & 12 & | & -2 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 1 \\ 0 & -5 & -8 & | & -2 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & -1 & | & -1 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 2 & | & 3 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & -1 & | & -1 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 1 & | & 3/2 \end{bmatrix}$$

Here $x_3 = 3/2$, and it can be seen that $x_2 + 2x_3 = 1$ and $x_1 - x_3 = -1$, which can be solved. Or the RREF can be found:

$$\begin{bmatrix} 1 & 0 & -1 & | & -1 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 3/2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 & | & 1/2 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 3/2 \end{bmatrix}$$

Example 8.22. Solve the system of equations

$$\begin{aligned} -x + y + 3z &= 0 \\ x - 2y + z &= 5 \\ 3x + y - z &= 4 \end{aligned}$$

Setting up as a matrix:

$$\begin{bmatrix} -1 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$

And then performing Gaussian elimination:

$$\begin{bmatrix} -1 & 1 & 3 & | & 0 \\ 1 & -2 & 1 & | & 5 \\ 3 & 1 & -1 & | & 4 \end{bmatrix} \implies \begin{bmatrix} 1 & -2 & 1 & | & 5 \\ -1 & 1 & 3 & | & 0 \\ 3 & 1 & -1 & | & 4 \end{bmatrix} \implies \begin{bmatrix} 1 & -2 & 1 & | & 5 \\ 0 & -1 & 4 & | & 5 \\ 0 & 7 & -4 & | & -11 \end{bmatrix} \implies \begin{bmatrix} 1 & -2 & 1 & | & 5 \\ 0 & 1 & -4 & | & -5 \\ 0 & 7 & -4 & | & -11 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & -7 & | & -5 \\ 0 & 1 & -4 & | & -5 \\ 0 & 0 & 24 & | & 24 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & -7 & | & -5 \\ 0 & 1 & -4 & | & -5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

So $x = 2$, $y = -1$, and $z = 1$.

8.2.2 Free variables and inconsistent systems

Example 8.23. Solve $\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \underline{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Using Gaussian elimination,

$$\begin{bmatrix} 1 & 4 & | & 2 \\ 2 & 8| & 4 \end{bmatrix} \implies \begin{bmatrix} 1 & 4 & | & 2 \\ 1 & 4| & 2 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 4 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$$

This is the RREF. The system reduces to $x_1 + 4x_2 = 2$, but no further. Solving for x_1 gives $x_2 = 2 - 4x_2$, but there is no constraint on x_2. It is a free variable, say $x_2 = t$. Then the solution is

$$\underline{x} = \begin{bmatrix} 2 - 4t \\ 2 \end{bmatrix}.$$

Definition 8.24. *The **rank** of a matrix is the number of pivots when in reduced row echelon form.*

Note that in the example the matrix has rank 1. The rank describes the number of unique equations in the system. The number of columns in a matrix corresponds to the number of variables in the equation $\underline{\underline{A}}\underline{x} = \underline{b}$. If the rank is equal to the number of columns, there will be a solution. Otherwise, if the rank is less than the number of columns, there is either a free variable, or the system is inconsistent.

Definition 8.25. *If* $\mathrm{rank}(\underline{\underline{A}})$ = *number of columns of* $\underline{\underline{A}}$, *then* $\underline{\underline{A}}$ *is **full rank**. If* $\mathrm{rank}(\underline{\underline{A}}) <$ *number of columns of* $\underline{\underline{A}}$, *then* $\underline{\underline{A}}$ *is **rank deficient**.*

Example 8.26. Solve $\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Using Gaussian elimination,

$$\begin{bmatrix} 1 & 4 & | & 1 \\ 2 & 8| & 4 \end{bmatrix} \implies \begin{bmatrix} 1 & 4 & | & 1 \\ 1 & 4| & 2 \end{bmatrix}$$
$$\implies \begin{bmatrix} 1 & 4 & | & 1 \\ 0 & 0 & | & 1 \end{bmatrix}$$

Clearly, $0x_1 + 0x_2 \neq 1$. Thus, there are no solutions.

Non-square matrices often indicate inconsistent systems, or systems with free variables. Short, wide matrices are always rank deficient.

Example 8.27 (Short, wide). Solve

$$\begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 0 & -1 & -2 & | & 2 \end{bmatrix}.$$

Row reducing:

$$\begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 0 & -1 & -2 & | & 2 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 0 & 1 & 2 & | & -2 \end{bmatrix}$$
$$\implies \begin{bmatrix} 1 & 0 & -3 & | & 5 \\ 0 & 1 & 2 & | & -2 \end{bmatrix}$$

This is RREF. The extra column leaves room for many nonzero entries. Since there is no pivot for x_3, it is free, $x_3 = t$. Then $x_2 = -2 - 2t$ and $x_1 = 5 + 3t$. The solution is

$$\underline{x} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

Note the separation into two vectors, one constant, one depending on t. This is called the **parametric form** of the solution.

Example 8.28 (Tall, narrow). Solve

$$\begin{bmatrix} 1 & 2 & | & 1 \\ 2 & 1 & | & 1 \\ 1 & 1 & | & 3 \end{bmatrix}$$

Row reducing:

$$\begin{bmatrix} 1 & 2 & | & 1 \\ 2 & 1 & | & 1 \\ 1 & 1 & | & 3 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & | & 1 \\ 2 & 1 & | & 1 \\ 0 & -1 & | & 2 \end{bmatrix}$$
$$\implies \begin{bmatrix} 1 & 2 & | & 1 \\ 0 & -3 & | & -1 \\ 0 & -1 & | & 2 \end{bmatrix}$$
$$\implies \begin{bmatrix} 1 & 2 & | & 1 \\ 0 & 1 & | & 1/3 \\ 0 & 1 & | & -2 \end{bmatrix}$$

There is now an inconsistency, since $x_2 = 1/3 \neq -2$.

Proposition 8.29. *A square matrix is full rank if and only if its RREF is the identity matrix.*

Proof. Obvious. □

9 Theory of Matrices

GOAL: Understand the properties of matrices as linear transformations – in particular, when, how, and why $\underline{\underline{A}}\underline{x} = \underline{b}$ can be solved.
OUTLINE:

- Image and span. (Section 7.5)
- The nullspace of a matrix. (Section 7.4)
- Linear independence, dimension, and Rank-nullity theorem. (Section 7.5)
- Invertible matrices. (Section 7.6)

9.1 Image and Span

Example 9.1. Consider the matrix $\underline{\underline{A}} = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ and the vector $\underline{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Row reducing to solve $\underline{\underline{A}}\underline{x} = \underline{b}$:

$$\begin{bmatrix} 1 & 4 & | & 1 \\ 2 & 8 & | & 2 \end{bmatrix} \implies \begin{bmatrix} 1 & 4 & | & 1 \\ 0 & 0 & | & 0 \end{bmatrix}$$

So that $x_2 = t$ and $x_1 = 1 - 4t$, so $\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \end{bmatrix}$. However, if $\underline{b}' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, solving $\underline{\underline{A}}\underline{x} = \underline{b}'$:

$$\left[\begin{array}{cc|c} 1 & 4 & 1 \\ 2 & 8 & 3 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

which has no solution.

Question: Given a matrix $\underline{\underline{A}}$, for what vectors $\underline{b}$ can $\underline{\underline{A}}\underline{x} = \underline{b}$ be solved?

The columns of $\underline{\underline{A}}$ can be thought of as vectors, $\underline{a}_1, \dots, \underline{a}_n$, so the question is equivalent to "when does

$$x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n = \underline{b}$$

have a solution?"

Definition 9.2. *A* ***linear combination*** *of vectors is any sum of scalar multiples of the vectors.*

Example 9.3. The vector

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is a linear combination of $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Definition 9.4. *The set of all linear combinations of the vectors* $\underline{v}_1, \dots, \underline{v}_n$ *is the* ***span*** *of* $\underline{v}_1, \dots, \underline{v}_n$*, written* $\operatorname{span}\left(\underline{v}_1, \dots, \underline{v}_n\right)$.

Definition 9.5. *The span of the columns of the matrix* $\underline{\underline{A}}$ *is the* ***image*** *of* $\underline{\underline{A}}$*, written* $\operatorname{im}(\underline{\underline{A}})$.

Answer: If $\underline{b}$ is in the image of $\underline{\underline{A}}$, or equivalently, if $\underline{b}$ is in the span of the columns of $\underline{\underline{A}}$.

Example 9.6. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is in the image of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the vector $[1, 2, 3]^T$ is in the image of

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Example 9.7. Find the image of $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since the image is all linear combinations of columns, it is any vector of the form

$$\underline{v} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}.$$

So any 2-vector is in $\mathrm{im}(\underline{\underline{A}})$. Similarly, any 3-vector is in the image of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 9.8. Find the image of

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

By row reducing,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

which has only one pivot. This indicates that the second column is a multiple of the first, and so the image of $\underline{\underline{A}}$ is $\mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ as the second column is unnecessary.

The image of a matrix is the span of the columns which contain pivots in RREF.

Proposition 9.9. *If $\underline{u}$ and $\underline{v}$ are in the image of $\underline{\underline{A}}$, then so is any linear combination $a\underline{u} + b\underline{v}$.*

Proof. Obvious. □

9.2 The Nullspace

Example 9.10. Compute the product

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Question: What are the solutions to $\underline{\underline{A}}\underline{x} = \underline{0}$?

Example 9.11.

$$\begin{bmatrix} 1 & 4 & | & 0 \\ 2 & 8 & | & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$
$$\implies \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$$

so $x_2 = t$ and $x_1 + 4t = 0$ so $x_1 = -4t$. In other words,

$$\operatorname{span}\left(\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right)$$

are the solutions.

Definition 9.12. *The set of solutions to* $\underline{\underline{A}}\underline{x} = \underline{0}$ *is the* ***nullspace*** *of* $\underline{\underline{A}}$*, written* $\operatorname{null}\left(\underline{\underline{A}}\right)$.

Example 9.13. Find the nullspace of

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & 0 \\ 0 & -3 & 3 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

from which the solution is $x_3 = t$, $x_2 = x_3 = t$, and $x_1 = -x_3 = -t$, so the nullspace is

$$\operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right).$$

Example 9.14. Find the nullspace of

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so the solution is $x_1 = 0$ and $x_2 = 0$.

Proposition 9.15. *For any matrix, $\underline{0}$ is in the nullspace.*

Proof. Obvious. □

If $\underline{0}$ is the only element of $\operatorname{null}\left(\underline{\underline{A}}\right)$, then the nullspace is said to be **trivial**.

Proposition 9.16. *The nullspace of a square matrix is trivial if and only if the RREF is the identity matrix.*

Proof. Obvious. □

Proposition 9.17. *If $\underline{u}$ and $\underline{v}$ are in $\operatorname{null}\left(\underline{\underline{A}}\right)$, then so is $a\underline{u} + b\underline{v}$.*

Proof.

$$\underline{\underline{A}}(a\underline{u} + b\underline{v}) = a\underline{\underline{A}}\underline{u} + b\underline{\underline{A}}\underline{v} = 0 + 0.$$

□

9.3 Linear Independence, Dimension, and Rank-Nullity

Definition 9.18. *A set of vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ is **linearly independent** if*

$$a_1\underline{v}_1 + a_2\underline{v}_2 + \ldots + a_n\underline{v}_n = 0$$

only *when $a_1 = a_2 = \cdots = a_n = 0$. Otherwise the set of vectors is **linearly dependent**.*

Example 9.19. Show that the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are linearly independent. This means solving

$$a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \underline{0}$$

which is the same as

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, the usual Gaussian elimination gives:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A set of vectors is linearly independent if they form a matrix which row reduces to a form with a pivot in each column.

Definition 9.20. *A **subspace** is the span of a set of vectors.*

Example 9.21. The vector $\underline{0}$ is a subspace (it is the span of itself). The vectors of the form $\begin{bmatrix} t \\ 2t \end{bmatrix}$ form a subspace as they are the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The vectors of the form

$$\begin{bmatrix} t+s \\ 2t \\ 3s \end{bmatrix}$$

are a subspace, as they are the span of

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

The image and kernel of a matrix are subspaces.

Definition 9.22. *The **dimension** of a subspace is the number of nonzero vectors in a minimal spanning set.*

Example 9.23. The subspace $\operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ is 1-dimensional because it is the span of one vector. The subspace

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}\right)$$

is 2-dimensional. However, the subspace

$$\operatorname{span}\left(\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\3\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix}\right)$$

is 2-dimensional, *not* 3-dimensional because

$$\begin{bmatrix}2\\2\\3\end{bmatrix}=\begin{bmatrix}1\\2\\0\end{bmatrix}+\begin{bmatrix}1\\0\\3\end{bmatrix}$$

so there are only two linearly independent vectors here. The subspace $\operatorname{span}\left(\underline{0}\right)$ is 0-dimensional because there are no nonzero vectors in it.

Example 9.24. Find the dimension of the image and kernel of

$$\underline{\underline{A}}=\begin{bmatrix}1&1&0\\1&2&1\\1&0&-1\end{bmatrix}.$$

Row reducing:

$$\begin{bmatrix}1&1&0\\1&2&1\\1&0&-1\end{bmatrix}\Longrightarrow\begin{bmatrix}1&1&0\\1&2&1\\0&-1&-1\end{bmatrix}\Longrightarrow\begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix}\Longrightarrow\begin{bmatrix}1&1&0\\0&1&1\\0&1&1\end{bmatrix}\Longrightarrow\begin{bmatrix}1&1&0\\0&1&1\\0&0&0\end{bmatrix}\Longrightarrow\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$$

$$\operatorname{null}\left(\underline{\underline{A}}\right)=\operatorname{span}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)\text{ and }\operatorname{im}\underline{\underline{A}}=\operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\0\end{bmatrix}\right)$$

so the nullspace is 1-dimensional and the image is 2-dimensional.

Proposition 9.25. *The dimension of the image of $\underline{\underline{A}}$ is the same as the rank of $\underline{\underline{A}}$.*

Proof. The pivot columns are the ones in the image, and the number of pivots is the rank. □

Note that the dimension of a span of n-vectors is at most n, but otherwise the dimension of a subspace and the size of the vectors are unrelated.

Theorem 9.26 (Rank-Nullity). *Dimension of image + dimension of nullspace = number of columns.*

Proof. Pivot columns go in the image. Non-pivot columns are in the nullspace. All columns are either pivots or non-pivots. □

Example 9.27. Find the dimension of the image and kernel of

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

From the rank-nullity theorem, it suffices to find the number of pivots in RREF.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \implies \begin{bmatrix} 2 & 2 & 1 \\ 0 & -2 & 1 \end{bmatrix} \implies \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1/2 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & -1/2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \end{bmatrix}$$

which has 2 pivots. Therefore, the dimension of the image is 2, and the dimension of the nullspace is 1.

9.4 Invertible Matrices

Example 9.28. Compute the product $\underline{\underline{AB}}$, where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \underline{\underline{B}} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 2 & -2 & -1 \end{bmatrix}.$$

Definition 9.29. *A matrix $\underline{\underline{A}}$ is **invertible** (or **nonsingular**) if there exists a matrix (denoted $\underline{\underline{A}}^{-1}$) such that $\underline{\underline{AA}}^{-1} = \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$.*

Note that only square matrices can be invertible (for rectangular matrices only one order of produces is defined).

Proposition 9.30. *If $\underline{\underline{A}}$ is invertible, then the RREF of $\underline{\underline{A}}$ is the identity.*

Proof. If $\underline{\underline{A}}^{-1}$ exists, then

$$\underline{\underline{A}}\underline{x} = \underline{b} \implies \underline{\underline{A}}^{-1}\underline{\underline{A}}\underline{x} = \underline{\underline{A}}^{-1}\underline{b} \implies \underline{\underline{I}}\underline{x} = \underline{\underline{A}}^{-1}\underline{b} \implies \underline{x} = \underline{\underline{A}}^{-1}\underline{b}.$$

Therefore, there is a solution $\underline{x}$ to $\underline{\underline{A}}\underline{x} = \underline{b}$, for any $\underline{b}$. This means that the dimension of the image is n if $\underline{\underline{A}}$ is $n \times n$, so the RREF is the identity. □

Proposition 9.31. *For an $n \times n$ square matrix $\underline{\underline{A}}$, the following are equivalent:*

1. *The RREF of $\underline{\underline{A}}$ is the identity matrix*
2. *The rank of $\underline{\underline{A}}$ is n.*
3. *The nullspace of $\underline{\underline{A}}$ is trivial.*
4. *$\underline{\underline{A}}\underline{x} = \underline{b}$ has a solution for every $\underline{b}$.*
5. *The columns of $\underline{\underline{A}}$ are linearly independent vectors.*
6. *$\underline{\underline{A}}$ is invertible (nonsingular).*

For 2×2 matrices, the inverse can be computed with the formula

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \underline{\underline{A}}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Exercise 9.32. Show that this is the correct inverse.

In general, the inverse is computed by Gaussian elimination:

Example 9.33. Find the inverse of

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Augmenting the matrix with $\underline{\underline{I}}$, then row reducing:

$$\begin{bmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 1 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -1 & 1 & 0 \\ 0 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & 1 & 1/2 & | & 0 & 0 & 1/2 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & -1 & | & -1 & 2 & 0 \\ 0 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & 0 & -1/2 & | & -1 & 1 & 1/2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & -1 & | & -1 & 2 & 0 \\ 0 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & 0 & 1 & | & 2 & -2 & -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 1 & 0 & | & -1 & 1 & 1 \\ 0 & 0 & 1 & | & 2 & -2 & -1 \end{bmatrix}$$

the augmented part is the inverse matrix, as determined earlier.

10 Eigenvalues and Eigenvectors

Goal: Be able to compute the eigenvalues and eigenvectors of a matrix, and understand their meaning geometrically and algebraically.
Outline:

- Determinants. (Section 7.7)
- Eigenvalues and the characteristic polynomial. (Section 9.1)
- Eigenvectors. (Section 9.1)
- Algebraic and geometric multiplicity. (Section 9.5, pp. 413)

10.1 Determinants

Definition 10.1. *The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant of a matrix $\underline{\underline{A}}$ is denoted $|\underline{\underline{A}}|$.

Example 10.2. The determinant of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

The determinant of larger matrices is defined recursively:

Definition 10.3.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Example 10.4.

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 1 - 2 + 2 = 1$$

Proposition 10.5. *Determinants and row operations:*

- *Scaling a row scales the determinant by the same amount.*
- *Switching rows changes the sign of the determinant.*
- *Adding/subtracting rows does not affect the determinant.*

Proof. For the 2×2 case, where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc),$$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc),$$

$$\begin{vmatrix} a + c & b + d \\ c & d \end{vmatrix} = (a + c)d - (b + d)c = ad - bc + cd - cd = ad - cd.$$

Larger matrices inherit the same properties. □

Example 10.6. Compute the determinants:

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix} \quad \begin{vmatrix} 2 & -1 & 3 \\ 0 & 0 & 6 \\ 0 & 4 & 1 \end{vmatrix} \quad \begin{vmatrix} 2 & -1 & 3 \\ 2 & 3 & 4 \\ 0 & 4 & 3 \end{vmatrix}$$

The first is simple:

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 4 & 1 \\ 0 & 2 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 3 \begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} = 16.$$

The second is the first matrix with one swap and a multiplication, so

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 0 & 6 \\ 0 & 4 & 1 \end{vmatrix} = (-1)(3)(16) = -48.$$

The third is obtained by row additions, so

$$\begin{vmatrix} 2 & -1 & 3 \\ 2 & 3 & 4 \\ 0 & 4 & 3 \end{vmatrix} = 16.$$

10.1.1 Determinants of special matrices

Definition 10.7. *The transpose of a matrix, $\underline{\underline{A}}^T$ is the matrix $\underline{\underline{A}}$ with rows and columns switched.*

Example 10.8.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Proposition 10.9. $|\underline{\underline{A}}| = |\underline{\underline{A}}^T|$.

Proof.

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb = ad - bc$$

□

Corollary 10.10. *The operations on rows have the same affect on the determinant when applied to columns.*

Proposition 10.11. *If the columns of $\underline{\underline{A}}$ are linearly dependent, then $|\underline{\underline{A}}| = 0$.*

Proof. If the columns are linearly dependent, the row operations can be performed so that a column is repeated. If a column is repeated, then the swapping rule says the determinant should change by a negative sign, but the matrix itself does not change. Therefore $|\underline{\underline{A}}| = -|\underline{\underline{A}}|$, so $\underline{\underline{A}} = 0$. □

Corollary 10.12. *A matrix $\underline{\underline{A}}$ is invertible if and only if $|\underline{\underline{A}}| \neq 0$.*

Definition 10.13. *A matrix is **diagonal** if all entries of the diagonal are zero. A matrix is upper-triangular if all entries below the diagonal are zero. A matrix is lower-triangular if all entries above the diagonal are zero. A matrix is **triangular** if it is either upper- or lower- triangular.*

Example 10.14. Diagonal, upper-, and lower- triangular matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}.$$

Proposition 10.15. *The determinant of a diagonal or triangular matrix is the product of the diagonal entries.*

Proof. For a diagonal or upper-triangular matrix, only one entry in the first row is nonzero, so the rest of the terms in the determinant are zero. The transpose of a lower-triangular matrix is an upper triangular matrix and transposing does not change the determinant, so it also holds in this case. □

Example 10.16. The determinants of the matrices in the preceding example are 6, 24, and 18.

10.2 Eigenvalues and Eigenvectors

Definition 10.17. *For a square matrix $\underline{\underline{A}}$, an* ***eigenvalue*** *and* ***eigenvector*** *are a number, λ, and a (nonzero) vector $\underline{v}$ such that*

$$\underline{\underline{A}}\underline{v} = \lambda \underline{v}.$$

Example 10.18. For the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. The eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

However, not all vectors are eigenvectors. For example,

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

which is not a scalar multiple of the vector.

Definition 10.19. *The characteristic polynomial of a matrix $\underline{\underline{A}}$ is the determinant:*

$$p_A(\lambda) := |\underline{\underline{A}} - \lambda \underline{\underline{I}}|$$

Example 10.20. The characteristic polynomial of $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda).$$

Proposition 10.21. *The eigenvalues of a matrix are the roots of the characteristic polynomial. The eigenvectors of $\underline{\underline{A}}$ for eigenvalue λ are the elements of the nullspace of $\underline{\underline{A}} - \lambda\underline{\underline{I}}$.*

Proof. If $\underline{\underline{A}}\underline{v} = \lambda\underline{v}$, then $(\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{v} = \underline{0}$, so $\underline{v}$ is in the nullspace of $\underline{\underline{A}} - \lambda\underline{\underline{I}}$. Therefore, the matrix $\underline{\underline{A}} - \lambda\underline{\underline{I}}$ must not be invertible, so λ must be a number such that $|\underline{\underline{A}} - \lambda\underline{\underline{I}}| = 0$. □

Example 10.22. Find the eigenvalues and vectors for the matrices:

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \underline{\underline{B}} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} \quad \underline{\underline{C}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

A:

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) \quad \text{eigenvalues: } 1, 2$$
$$\begin{bmatrix} 1-1 & 0 \\ 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1-2 & 0 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

B:

$$\begin{vmatrix} 2-\lambda & 0 \\ 3 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) \quad \text{eigenvalues: } 2, -1$$
$$\begin{bmatrix} 2-2 & 0 \\ 3 & -1-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2+1 & 0 \\ 3 & -1+1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

C:

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) \quad \text{eigenvalues: } 0, 5$$
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$$
$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} \implies \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \implies \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Proposition 10.23. *The eigenvalues of a diagonal or triangular matrix are the entries on the diagonal.*

Proof. If $\underline{\underline{A}}$ is triangular, then so is $\underline{\underline{A}} - \lambda \underline{\underline{I}}$, so its determinant is a product of the form $(a_i - \lambda)$ where a_i are the diagonal entries. Thus the diagonal entries are the roots. □

Proposition 10.24. *If $\underline{v}$ is an eigenvector for λ, then so is $a\underline{v}$ for any scalar a.*

Proof.

$$\underline{\underline{A}}a\underline{v} = a\underline{\underline{A}}\underline{v} = a\lambda\underline{v} = \lambda a\underline{v}$$

□

Example 10.25. For $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$ and eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, as are any scalar multiples. For example:

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

10.3 Algebraic and Geometric Multiplicity

Example 10.26. Find the eigenvalues and eigenvectors of the following matrices:

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad \underline{\underline{B}} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \quad \underline{\underline{C}} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

A:

$$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) \quad \text{eigenvalues: } 3, 2$$

$$\begin{bmatrix} 3-3 & 1 \\ 0 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

B:

$$\begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda)+4 = 25-10\lambda+\lambda^2 = (\lambda-5)^2 \quad \text{eigenvalues: } 5$$

$$\begin{bmatrix} 7-5 & 1 \\ -4 & 3-5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \implies \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

C:

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) \quad \text{eigenvalues: } 3$$

$$\begin{bmatrix} 3-3 & 0 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{eigenvector: anything! } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Definition 10.27. *The* ***algebraic multiplicity****, a_λ, of an eigenvalue is the number of times it occurs as a root of the characteristic polynomial. The* ***geometric multiplicity****, g_λ of an eigenvalue is the number of (linearly independent) eigenvectors it has.*

Note that g_λ is the same as the dimension of the nullspace of $\underline{\underline{A}} - \lambda \underline{\underline{I}}$.

Example 10.28. In the preceding example, A has two eigenvectors, each with algebraic and geometric multiplicity 1, B has one eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1, and C has one eigenvalue with algebraic and geometric multiplicity 2.

Proposition 10.29. *For any eigenvalue λ of an $n \times n$ matrix, $1 \leq g_\lambda \leq a_\lambda \leq n$, and the sum of the a_λs, for all eigenvalues is n.*

Proof. The characteristic polynomial will have degree n, and by the fundamental theorem of algebra, it has n roots. The roots are the eigenvalues, so the total algebraic multiplicity is n.

By definition, $g_\lambda \geq 1$. The proof that $g_\lambda \leq a_\lambda$ will be saved for later. □

Example 10.30. Observe that these relations are true in the example.

The span of the eigenvectors associated with the eigenvalue λ is the **eigenspace** of λ. Since g_λ is the number of linearly independent eigenvectors for λ, the dimension of the eigenspace of λ is always g_λ.

Example 10.31 (Complex eigenvalues). Find the eigenvalues and eigenvectors for

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 6 \\ -3 & -3 \end{bmatrix}.$$

$$\begin{vmatrix} 3-\lambda & 6 \\ -3 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) + 18 = 9 + \lambda^2 \quad \text{eigenvalues: } \pm 3i$$

$$\begin{bmatrix} 3-3i & 6 \\ -3 & -3-3i \end{bmatrix} = \begin{bmatrix} 18 & 18+18i \\ -3 & -3-3i \end{bmatrix} \implies \begin{bmatrix} 1 & 1+i \\ -1 & -1-i \end{bmatrix} \implies \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3+3i & 6 \\ -3 & -3+3i \end{bmatrix} = \begin{bmatrix} 18 & 18-18i \\ -3 & -3+3i \end{bmatrix} \implies \begin{bmatrix} 1 & 1-i \\ -1 & -1+i \end{bmatrix} \implies \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

Proposition 10.32. *If $\underline{\underline{A}}$ is a real-valued matrix and λ and $\underline{v}$ are an eigenvalue and eigenvector, so are $\overline{\lambda}$ and $\overline{\underline{v}}$.*

Proof.

$$\underline{\underline{A}}\underline{v} = \lambda\underline{v} \implies \overline{\underline{\underline{A}}\underline{v}} = \overline{\lambda\underline{v}} \implies \underline{\underline{A}}\overline{\underline{v}} = \overline{\lambda}\overline{\underline{v}}$$

□

Example 10.33.

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 \quad \text{eigenvalues: } 1 \pm 2i$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \implies \begin{bmatrix} -2i & 2 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \implies \begin{bmatrix} 2i & 2 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

11 Systems of ODEs

Goal: Be able to qualitatively analyze systems of differential equations, identify linear systems, and relate linear systems to linear algebra.
Outline:

- Systems of ODEs. (Section 8.1)
- Qualitative Analysis – nullclines, equilibria, phase plane. (Sections 8.2 and 8.3)
- The matrix exponential and solutions. (No section)

11.1 Systems of ODEs

Example 11.1. Verify that $x(t) = \exp(2t) + \exp(3t)$ and $y(t) = -\exp(2t)$ is a solution to:

$$\begin{aligned} x' &= 3x + y \\ y' &= 2y \end{aligned}$$

By evaluating directly,

$$x'(t) = 2\exp(2t) + 3\exp(3t) = 3\exp(2t) + 3\exp(3t) - \exp(2t) = 3x + y$$

and

$$y'(t) = -2\exp(2t) = 2y.$$

Definition 11.2. *A **system** of differential equations is a set of equations relating several variables and their derivatives. A system is **autonomous** if the independent t variable does not appear. It is **linear** if all dependent variables only appear linearly.*

Example 11.3. A linear, autonomous, equation:

$$\begin{aligned} x' &= x + 2y \\ y' &= 2x + y \end{aligned}$$

A linear, non-autonomous system:

$$\begin{aligned} x' &= 3x + y + t^2 \\ y' &= -x - y - \cos(t) \end{aligned}$$

A nonlinear, autonomous system:

$$\begin{aligned} x' &= 3x + xy \\ y' &= y^2 - x \end{aligned}$$

Exercise 11.4. Write down a nonlinear, non-autonomous system.

11.1.1 Higher-order equations

Any system or equation involving higher derivatives can be interpreted as a system of first-order ODEs.

Example 11.5. For the equation $x'' + 2x' + x = 0$, introduce a new variable $u = x'$. Then $u' = x''$. Replacing derivatives with u whenever possible,

$$u' + 2u + x = 0 \text{ and } x' = u$$

is an equivalent system of equations.

Example 11.6. The equation $y''' + 3y'' + 3y' + y = 0$ with substitutions $u = y'$ and $v = u'$ becomes

$$\begin{aligned} v' &= -y - 3u - 3v \\ u' &= v \\ y' &= u \end{aligned}$$

because $v = y''$ and $v' = y'''$.

Proposition 11.7. *An n-th order ODE is equivalent to a system of n first-order equations.*

Proof. If x is the unknown variable, apply the substitution $u_1 = x'$, $u_2 = x''$, and so on. □

Example 11.8. The equation $x''' + xx'' = \cos(t)$ reduces to a third-order system. Define $u = x'$, $v = x''$. Then $x''' = v'$, so

$$\begin{aligned} v' &= \cos(t) - xv \\ u' &= x \\ x' &= u \end{aligned}$$

is the system of three first-order equations.

11.1.2 Linear and Matrix Systems

A linear system can be written as a matrix.

Example 11.9. Consider the system:

$$\begin{aligned} x' &= x + 2y \\ y' &= 2x + y \end{aligned}$$

Define $\underline{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $\underline{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$. The system is then equivalent to

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y \end{bmatrix}.$$

The system:

$$x' = 3x + y + t^2$$
$$y' = -x - y - \cos(t)$$

can be written as

$$\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t^2 \\ \cos(t) \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Proposition 11.10. *A linear, autonomous equation can be written in the form $\underline{\underline{A}}\underline{x} = \underline{x}'$. A linear, non-autonomous equation where the t terms occur in isolation can be written in the form $\underline{\underline{A}}\underline{x} + \underline{f}(t) = \underline{x}'$.*

The latter type of equation is called **inhomogeneous**.

11.2 Qualitative analysis of Autonomous Systems

For this section, only autonomous, **planar** systems will be covered; that is, systems with two unknowns, $x(t)$ and $y(t)$, two equations, and no occurrences of the t variable.

Definition 11.11. *The **phase plane** of an autonomous planar system is the (x, y)-plane.*

Definition 11.12. *An **equilibrium point** is a point in the phase plane where $x' = y' = 0$.*

Example 11.13. The system

$$x' = x + 2y$$
$$y' = 2x + y$$

has an equilibrium point at $(0, 0)$. The system

$$x' = 3x + xy$$
$$y' = y^2 - x$$

has an equilibrium point at $(0, 0)$. Are there others?

Definition 11.14. *The x-nullclines of an autonomous planar system are the sets of points where $x' = 0$. The y-nullclines are the sets of points where $y' = 0$.*

Example 11.15. Find the nullclines of

$$
\begin{aligned}
x' &= 3x + xy \\
y' &= y^2 - x.
\end{aligned}
$$

Setting $x' = 0$ gives $0 = 3x + xy$ so $x = 0$ or $y = -3$ are the x-nullclines. Setting $y' = 0$ gives $0 = y^2 - x$ so $x = y^2$ is the y-nullclines.

Since equilibria are the points where both x' and y' are zero, they occur exactly at the intersections of the nullclines. Therefore in this system the equilibria are at the intersection of $x = 0$ and $x = y^2$ and at the intersection of $y = -3$ and $x = y^2$. In particular, $(0, 0)$ and $(9, -3)$.

Example 11.16. Find the equilibrium points of

$$
\begin{aligned}
x' &= (1 - x - y)x \\
y' &= (4 - 2x - 7y)y.
\end{aligned}
$$

First find the nullclines:

- x-nullclines:
  - $x = 0$
  - $y = 1 - x$
- y-nullclines:
  - $y = 0$
  - $y = \dfrac{4 - 2x}{7}$

Then intersect each of the x-nullclines with each of the y-nullclines, giving $(0, 0)$, $(0, 4/7)$, $(1, 0)$, and $(3/5, 2/5)$.

Example 11.17 (Linear systems). Find the equilibria and nullclines of

$$
\begin{aligned}
x' &= x + 2y \\
y' &= 2x + y
\end{aligned}
$$

The nullclines are at $y = x/2$ and at $y = -2x$. The only intersection is $(0, 0)$.

More generally, find the equilibira of

$$x' = ax + by$$
$$y' = cx + dy$$

Here the nullclines are $y = (a/b)x$ and $y = (c/d)x$. (If $b = 0$ or $d = 0$ interpret these as vertical lines.) Assuming that $a/b \neq c/d$ the only equilibrium is at the origin. Otherwise the equilibrium is the entire nullcline $y = (a/b)x = (c/d)x$.

Example 11.18 (Lotka-Volterra). The dynamics of two interacting populations, one a predator and one prey, can be modeled by the Lotka-Volterra model, given by the system

$$F' = (a - bS)F$$
$$S' = (-c + dF)S$$

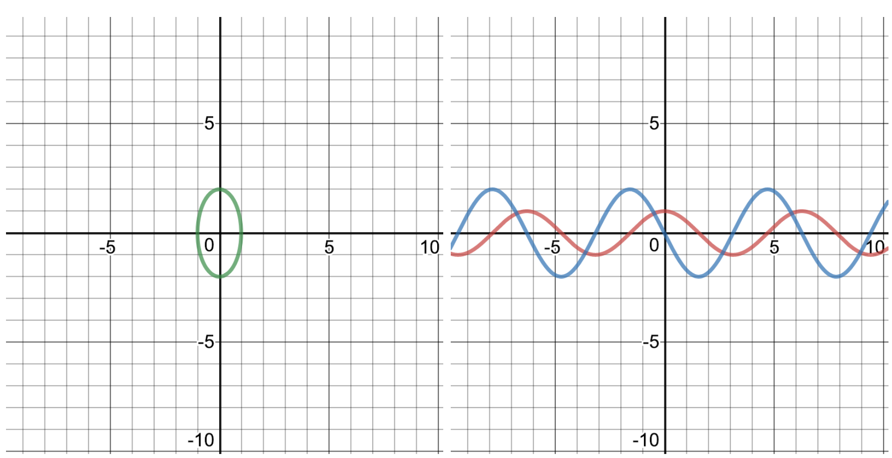
where F are the prey, S are the predators, and a, b, c, d are positive constants. (In more detail, the population F should have a birth rate of aF, and a death rate that increases with both S and F. Conversely, the predator population S should have a death rate of $-cS$ and a birth rate that increases with S and F. Compare this system to the population dynamics in week 4.)

The F-nullclines are at $(a-bS)F = 0$, so $F = 0$ or $S = a/b$. The S nullclines are at $(-c+dF)S = 0$ so $S = 0$ or $F = c/d$. Thus there are two equilibria, the trivial one where both populations are zero, and the stable population at $(c/d, a/b$.

11.2.1 Phase Plane Solutions

Phase plane behavior can be translated into plot of $x(t)$ and $y(t)$, and vice-versa.

Example 11.19. Suppose that $x(t) = \cos(t)$ and $y(t) = -2\sin(t)$ are the solution to a system of ODEs. Below are the phase plane plot, and the plots of x and y in the (t, y)-plane.



11.3 The Matrix Exponential

Example 11.20. Write the system

$$x' = x + 2y \quad y' = 3y$$

as a matrix system: $\underline{\underline{A}}\underline{v} = \underline{v}'$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Exercise 11.21. Recall the solution to $y' = ay$.

Definition 11.22. *The* ***matrix exponential*** *is the matrix*

$$\exp\left(\underline{\underline{A}}\right) := \underline{\underline{I}} + \underline{\underline{A}} + \frac{1}{2!}\underline{\underline{A}}^2 + \frac{1}{3!}\underline{\underline{A}}^3 + \cdots + \frac{1}{n!}\underline{\underline{A}}^n + \cdots$$

Example 11.23. Compute the matrix exponential of

$$\underline{\underline{A}} = \begin{bmatrix} 1 & -2 \\ 1/2 & -1 \end{bmatrix}$$

Directly, it is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 1/2 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1/2 & 0 \end{bmatrix}$$

This example is "lucky" in that $A^2 = 0$. In general no such phenomenon occurs.

Note that

$$\exp\left(\underline{\underline{A}}t\right) = \underline{\underline{I}} + \underline{\underline{A}}t + \frac{t^2}{2!}\underline{\underline{A}}^2 + \cdots + \frac{t^n}{n!}\underline{\underline{A}}^n + \cdots$$

Proposition 11.24.

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp\left(\underline{\underline{A}}t\right) = \underline{\underline{A}}\exp\left(\underline{\underline{A}}t\right)$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\exp\left(\underline{\underline{A}}t\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{\underline{I}} + \underline{\underline{A}}t + \frac{t^2}{2!}\underline{\underline{A}}^2 + \frac{t^3}{3!}\underline{\underline{A}}^3 + \cdots + \frac{t^n}{n!}\underline{\underline{A}}^n + \cdots\right) \\
&= \underline{\underline{0}} + \underline{\underline{A}} + t\underline{\underline{A}}^2 + \frac{t^2}{2}\underline{\underline{A}}^3 + \cdots + \frac{t^{n-1}}{(n-1)!} + \cdots \\
&= \underline{\underline{A}}\left(\underline{\underline{I}} + \underline{\underline{A}}t + \frac{t^2}{2!}\underline{\underline{A}}^2 + \frac{t^3}{3!}\underline{\underline{A}}^3 + \cdots + \frac{t^n}{n!}\underline{\underline{A}}^n + \cdots\right) \\
&= \underline{\underline{A}}\exp\left(\underline{\underline{A}}t\right)
\end{aligned}$$

□

Corollary 11.25. *Let $\underline{c}$ be any vector. Then $\underline{v} = \exp\left(\underline{\underline{A}}t\right)\underline{c}$ satisfies $\underline{\underline{A}}\underline{v} = \underline{v}'$.*

Proof.

$$\underline{v}' = \underline{\underline{A}}\exp\left(\underline{\underline{A}}t\right)\underline{c} = \underline{\underline{A}}\underline{v}$$

□

11.3.1 Diagonal matrices

Example 11.26. Find the exponential $\exp\left(\underline{\underline{D}}t\right)$ where

$$\underline{\underline{D}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Computing directly,

$$\begin{aligned} \exp\left(\underline{\underline{D}}t\right) &= \underline{\underline{I}} + \begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} t^2 & 0 \\ 0 & 4t^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} t^3 & 0 \\ 0 & 16t^3 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 + t + t^2/2 + t^3/6 + \cdots & 0 \\ 0 & 1 + \frac{1}{2}(2t)^2 + \frac{1}{6}(2t)^3 + \cdots \end{bmatrix} \\ &= \begin{bmatrix} \exp(t) & 0 \\ 0 & \exp(2t) \end{bmatrix} \end{aligned}$$

Proposition 11.27. *If $\underline{\underline{D}}$ is a diagonal matrix, $\underline{\underline{D}} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ then*

$$\exp\left(\underline{\underline{D}}t\right) = \begin{bmatrix} \exp(d_1 t) & 0 \\ 0 & \exp(d_2 t) \end{bmatrix}.$$

Proof. The same computation as the example, but with generic entries. □

Example 11.28. Find the solution to

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \exp\left(\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} t\right)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Computing the exponential,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \exp(2t) & 0 \\ 0 & \exp(-3t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} C_1 \exp(2t) \\ C_2 \exp(-3t) \end{bmatrix}.$$

Or, component-wise, $x = C_1 \exp(2t)$ and $y = C_2 \exp(-3t)$.

12 Solutions to Systems

GOAL: Give the solution to 2×2 linear systems and be able to classify the solution.
OUTLINE:

- Diagonalizable Matrices (no section).
- Solutions to $\underline{\underline{A}}\underline{x} = \underline{x}'$ (Section 9.2).
- Special case: complex roots (Section 9.2).
- Classification of equilibrium type (Sections 9.3 and 9.4).

12.1 Diagonalization

Example 12.1. Find the eigenvalues and vectors of

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}.$$

Since triangular, the eigenvalues are 2 and 4. The eigenvector associated to $\lambda = 4$ is $\underline{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The other is in the nullspace of

$$\begin{bmatrix} 2-2 & 0 \\ 3 & 4-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}$$

so $\underline{v}_2 = \begin{bmatrix} 1 \\ -3/2 \end{bmatrix}$.

Let $\underline{\underline{P}} = [\underline{v}_1, \underline{v}_2]$. Compute $\underline{\underline{P}}^{-1}$:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & -3/2 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|cc} 1 & -3/2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 3/2 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Let $\underline{\underline{D}} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ be the diagonal matrix of eigenvalues. Compute $\underline{\underline{PDP}}^{-1}$:

$$\begin{bmatrix} 0 & 1 \\ 1 & -3/2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3/2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3/2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$$

Definition 12.2. *A matrix $\underline{\underline{A}}$ is **diagonalizable** if there exists an invertible matrix $\underline{\underline{P}}$ and a* *diagonal matrix $\underline{\underline{D}}$ such that*

$$\underline{\underline{A}} = \underline{\underline{PDP^{-1}}}.$$

Example 12.3. The matrix $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is *not* diagonalizable. It has only one eigenvalue, $\lambda = 0$, and only one eigenvector, $\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Theorem 12.4. *Almost every matrix is diagonalizable. If a matrix is diagonalizable, then $\underline{\underline{D}}$ is the matrix of eigenvalues and $\underline{\underline{P}}$ is the matrix of eigenvectors.*

Proof. Sadly beyond the scope of this course. □

Lemma 12.5. $(\underline{\underline{PDP^{-1}}})^n = \underline{\underline{PD}}^n\underline{\underline{P}}^{-1}$.

Proof. Obvious when $n = 1$. When $n = 2$, then $\underline{\underline{PDP^{-1}}}\,\underline{\underline{PDP^{-1}}} = \underline{\underline{PDIDP^{-1}}} = \underline{\underline{PD}}^2\underline{\underline{P}}^{-1}$. The "inside" $\underline{\underline{P}}$ and $\underline{\underline{P}}^{-1}$ terms always cancel out. More formally, by induction, if $(\underline{\underline{PDP^{-1}}})^{n-1} = \underline{\underline{PD}}^{n-1}\underline{\underline{P}}^{-1}$, then $(\underline{\underline{PDP^{-1}}})^n = (\underline{\underline{PDP^{-1}}})^{n-1}\underline{\underline{PDP^{-1}}} = \underline{\underline{PD}}^{n-1}\underline{\underline{P}}^{-1}\underline{\underline{PDP^{-1}}} = \underline{\underline{PD}}^{n-1}\underline{\underline{IDP^{-1}}} = \underline{\underline{PD}}^n\underline{\underline{P}}^{-1}$. □

Proposition 12.6. *If $\underline{\underline{A}}$ is diagonalizable, then* $\exp\left(\underline{\underline{A}}\right) = \underline{\underline{P}}\exp\left(\underline{\underline{D}}\right)\underline{\underline{P}}^{-1}$.

Proof. By the lemma, $(\underline{\underline{PDP^{-1}}})^n = \underline{\underline{PD}}^n\underline{\underline{P}}^{-1}$. Therefore

$$\begin{aligned}
\exp\left(\underline{\underline{A}}\right) &= \exp\left(\underline{\underline{PDP^{-1}}}\right) \\
&= \underline{\underline{I}} + \underline{\underline{PDP^{-1}}} + \frac{1}{2!}(\underline{\underline{PDP^{-1}}})^2 + \cdots + \frac{1}{n!}(\underline{\underline{PDP^{-1}}})^n \\
&= \underline{\underline{PIP^{-1}}} + \underline{\underline{PDP^{-1}}} + \frac{1}{2!}(\underline{\underline{PDP^{-1}}})^2 + \cdots + \underline{\underline{P}}\frac{1}{n!}\underline{\underline{D}}^n\underline{\underline{P}}^{-1} \\
&= \underline{\underline{P}}\exp\left(\underline{\underline{D}}\right)\underline{\underline{P}}^{-1}
\end{aligned}$$

□

12.2 Solutions to systems

Theorem 12.7. *If $\underline{\underline{A}}$ is diagonalizable, with eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\underline{v}_1$ and $\underline{v}_2$, then the general solution to*

$$\underline{\underline{A}}\underline{x} = \underline{x}'$$

is

$$\underline{x} = C_1 \exp(\lambda_1 t)\underline{v}_1 + C_2 \exp(\lambda_2 t)\underline{v}_2.$$

Example 12.8. The general solution to

$$\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \underline{x} = \underline{x}'$$

is

$$\underline{x} = C_1 \exp(4t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 \exp(2t) \begin{bmatrix} 1 \\ -3/2 \end{bmatrix}$$

Checking that this is correct:

$$\underline{x}' = 4C_1 \exp(4t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2C_2 \exp(2t) \begin{bmatrix} 1 \\ -3/2 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \left(C_1 \exp(4t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 \exp(2t) \begin{bmatrix} 1 \\ -3/2 \end{bmatrix} \right) &= \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \left(C_1 \exp(4t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \left(C_2 \exp(2t) \begin{bmatrix} 1 \\ -3/2 \end{bmatrix} \right) \\ &= C_1 \exp(4t) \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 \exp(2t) \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -3/2 \end{bmatrix} \\ &= C_1 \exp(4t)(4) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 \exp(2t)(2) \begin{bmatrix} 1 \\ -3/2 \end{bmatrix} \end{aligned}$$

Proof of the theorem. From a previous theorem, the solution is

$$\underline{x} = \exp\left(\underline{\underline{A}}t\right)\underline{c}$$

for any arbitrary $\underline{c}$. If $\underline{\underline{A}}$ is diagonalizable, then

$$\underline{x} = \underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\underline{P}}^{-1}\underline{c}$$

Since $\underline{c}$ is arbitrary, so is $\underline{\underline{P}}^{-1}\underline{c} = \underline{\tilde{c}}$. Additionally, because $\exp\left(\underline{\underline{D}}t\right)$ is diagonal,

$$\underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right) = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 \end{bmatrix} \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2)t \end{bmatrix} = \exp(\lambda_1 t)\underline{v}_1 + \exp(\lambda_2 t)\underline{v}_2.$$

And so letting $\underline{\tilde{c}} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$,

$$\underline{x} = C_1 \exp(\lambda_1 t)\underline{v}_1 + C_2 \exp(\lambda_2 t)\underline{v}_2.$$

□

Example 12.9.

$$\underline{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underline{x}$$

From earlier, eigenvalues are $\lambda = 1$ and $\lambda = 2$. Eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus the solution is

$$\underline{x} = C_1 \exp(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \exp(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 \exp(t) \\ C_2 \exp(2t) \end{bmatrix}.$$

Note that writing $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the system is really

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= 2x_2 \end{aligned}$$

which are completely independent. Both can be solved as first-order equations, and the solutions are exactly the components of $\underline{x}$.

Example 12.10.

$$\underline{y}' = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} \underline{y}$$

Again, from a previous computation the eigenvalues are $\lambda = 2$ and $\lambda = -1$. The eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus

$$\underline{y} = C_1 \exp(2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \exp(-t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Example 12.11.

$$\underline{z}' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \underline{z}$$

The eigenvalues are $\lambda = 0$ and $\lambda = 5$ with eigenvectors $\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$. Solution is

$$\underline{z} = C_1 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} + C_2 \exp(5t) \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

12.2.1 Complex eigenvalues

Example 12.12. Solve the system

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \underline{x} = \underline{x}'.$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 \quad \text{eigenvalues: } 1 \pm 2i$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \implies \begin{bmatrix} -2i & 2 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \implies \begin{bmatrix} 2i & 2 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The solution is

$$\underline{x} = C_1 \exp((1+2i)t) \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 \exp((1-2i)t) \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Proposition 12.13. *If $\underline{z}(t)$ is a solution to $\underline{\underline{A}}\underline{x}' = \underline{x}$, then so are* $\mathrm{Re}(\underline{z})$ *and* $\mathrm{Im}(\underline{z})$.

Recall that the same proposition was true of the solutions to a second-order linear equation with complex roots.

Proof. Say $\underline{z}(t) = \underline{a}(t) + i\underline{b}(t)$. Then

$$\underline{\underline{A}}\underline{z}'(t) = \underline{\underline{A}}(\underline{a}'(t) + i\underline{b}'(t)) = \underline{\underline{A}}\underline{a}'(t) + i\underline{\underline{A}}\underline{b}'(t) = \underline{a}(t) + i\underline{b}(t).$$

Therefore $\underline{\underline{A}}\underline{a}'(t) = \underline{a}(t)$ and $\underline{\underline{A}}\underline{b}'(t) = \underline{b}(t)$, so that the real part, $\underline{a}(t)$, and the imaginary part, $\underline{b}(t)$, are each (real-valued) solutions. □

Example 12.14. To apply the proposition to the preceding example, split into real and imaginary parts:

$$
\begin{aligned}
\underline{x} &= C_1 \exp(t)(\cos(2t) + i\sin(2t))\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right) + C_2\exp(t)(\cos(-2t) + i\sin(-2t))\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\-1\end{bmatrix}\right)\\
&= C_1 \exp(t)(\cos(2t) + i\sin(2t))\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right) + C_2\exp(t)(\cos(2t) - i\sin(2t))\left(\begin{bmatrix}1\\0\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= C_1\exp(t)\left(\cos(2t)\begin{bmatrix}1\\0\end{bmatrix} + i\cos(2t)\begin{bmatrix}0\\1\end{bmatrix} + i\sin(2t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&\quad + C_2\exp(t)\left(\cos(2t)\begin{bmatrix}1\\0\end{bmatrix} - i\cos(2t)\begin{bmatrix}0\\1\end{bmatrix} - i\sin(2t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= C_1\exp(t)\left(\cos(2t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right) + iC_1\exp(t)\left(\cos(2t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(2t)\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&\quad + C_2\exp(t)\left(\cos(2t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right) - iC_2\exp(t)\left(\cos(2t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(2t)\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= (C_1 + C_2)\exp(t)\left(\cos(2t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right) + i(C_1 - C_2)\exp(t)\left(\cos(2t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(2t)\begin{bmatrix}1\\0\end{bmatrix}\right)
\end{aligned}
$$

From which the real part of the solution is

$$\underline{x}_R(t) = \tilde{C}_1\exp(t)\left(\cos(2t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right)$$

and the imaginary part of the solution is

$$\underline{x}_I(t) = \tilde{C}_2\exp(t)\left(\cos(2t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(2t)\begin{bmatrix}1\\0\end{bmatrix}\right).$$

The most general real-valued solution is $\underline{x}(t) = \underline{x}_R(t) + \underline{x}_I(t)$.

Proposition 12.15. *If $\lambda = a + ib$ is a complex eigenvalue to $\underline{\underline{A}}$ with complex eigenvector $\underline{v} = \underline{u} + i\underline{w}$, then the real-valued solution to $\underline{\underline{A}}\underline{x}' = \underline{x}$ is given by*

$$\underline{x}(t) = C_1\exp(at)\left(\cos(bt)\underline{u} - \sin(bt)\underline{w}\right) + C_2\exp(at)\left(\cos(bt)\underline{w} + \sin(bt)\underline{u}\right)$$

Proof. Repeat the example computation with generic numbers. □

Example 12.16. Solve $\underline{\underline{A}}\underline{v} = \underline{v}'$ with

$$\underline{\underline{A}} = \begin{bmatrix}3 & 6\\-3 & -3\end{bmatrix}.$$

$$\begin{vmatrix} 3-\lambda & 6 \\ -3 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) + 18 = 9 + \lambda^2 \quad \text{eigenvalues: } \pm 3i$$

$$\begin{bmatrix} 3-3i & 6 \\ -3 & -3-3i \end{bmatrix} = \begin{bmatrix} 18 & 18+18i \\ -3 & -3-3i \end{bmatrix} \implies \begin{bmatrix} 1 & 1+i \\ -1 & -1-i \end{bmatrix} \implies \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3+3i & 6 \\ -3 & -3+3i \end{bmatrix} = \begin{bmatrix} 18 & 18-18i \\ -3 & -3+3i \end{bmatrix} \implies \begin{bmatrix} 1 & 1-i \\ -1 & -1+i \end{bmatrix} \implies \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

So the complex-valued solution is

$$\underline{v}(t) = C_1 \exp(3it) \begin{bmatrix} -1-i \\ 1 \end{bmatrix} + C_2 \exp(-3it) \begin{bmatrix} -1+i \\ 1 \end{bmatrix}.$$

Identifying that $a = 0$, $b = 3$, $\underline{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, the real-valued solution is

$$\underline{v}(t) = \tilde{C}_1 \left(\cos(3t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin(3t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) + \tilde{C}_2 \left(\cos(3t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(3t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right).$$

12.3 Classification of Solutions

Recall that a linear system

$$\begin{aligned} x' &= ax + by \\ y' &+ cx + dy \end{aligned}$$

has exactly one equilibrium: the origin. However, not all linear systems behave in the same qualitative manner.

Example 12.17. The system

$$\underline{x}' = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} \underline{x}$$

has solution

$$\underline{x} = C_1 \exp(2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \exp(-t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

while the system

$$\underline{x}' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \underline{x}$$

has (real-valued) solution

$$\underline{x}(t) = \tilde{C}_1 \exp(t) \left(\cos(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + \tilde{C}_2 \exp(t) \left(\cos(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

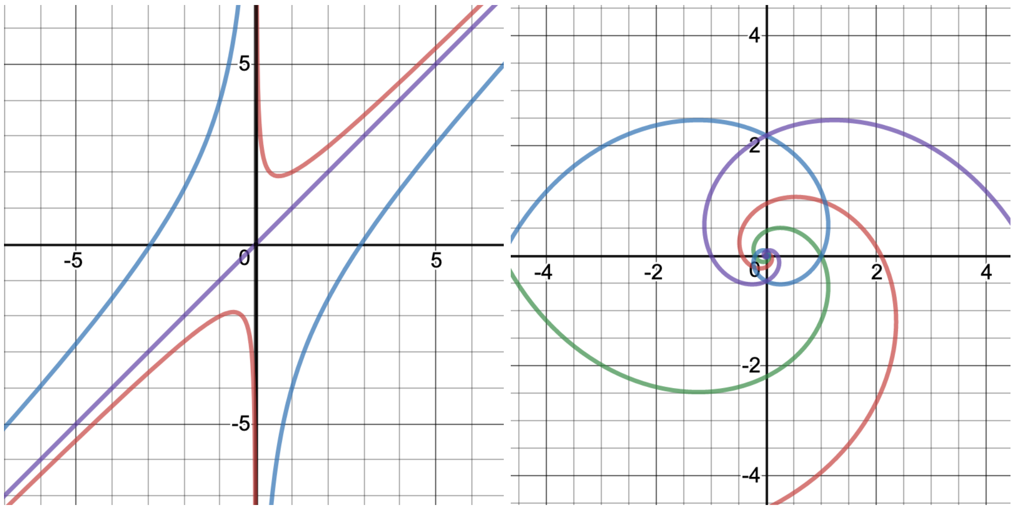


Figure 5: Various solutions to the two systems.

Plotting these shows that the solution curves are very different. Note in particular that the straight-line solutions point exactly along the eigenvectors.

There are a total of six qualitatively different solution types to a linear system.

Definition 12.18. *The origin is a*

1. ***nodal source*** *if both eigenvalues are positive,*
2. *a* ***nodal sink*** *if both eigenvalues are negative,*
3. *a* ***saddle point*** *if one eigenvalue is positive and one is negative,*
4. *a* ***center*** *if both eigenvalues are purely imaginary,*
5. *a* ***spiral source*** *if the eigenvalues are complex with positive real part, and*
6. *a* ***spiral sink*** *if the eigenvalues are complex with negative real part.*

Exercise 12.19. If both eigenvalues are negative, what is $\lim_{t\to\infty} \underline{x}(t)$? What if both eigenvalues are positive? What changes if there is an imaginary part as well?

Sinks (of either type) are asymptotically stable, while sources are unstable. A center is stable, but not asymptotically stable.

Example 12.20. A **phase-plane portrait** is a sketch of all types of solution curves for a given system near the origin, including direction of motion. For the system

$$\underline{v}' = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \underline{v}$$

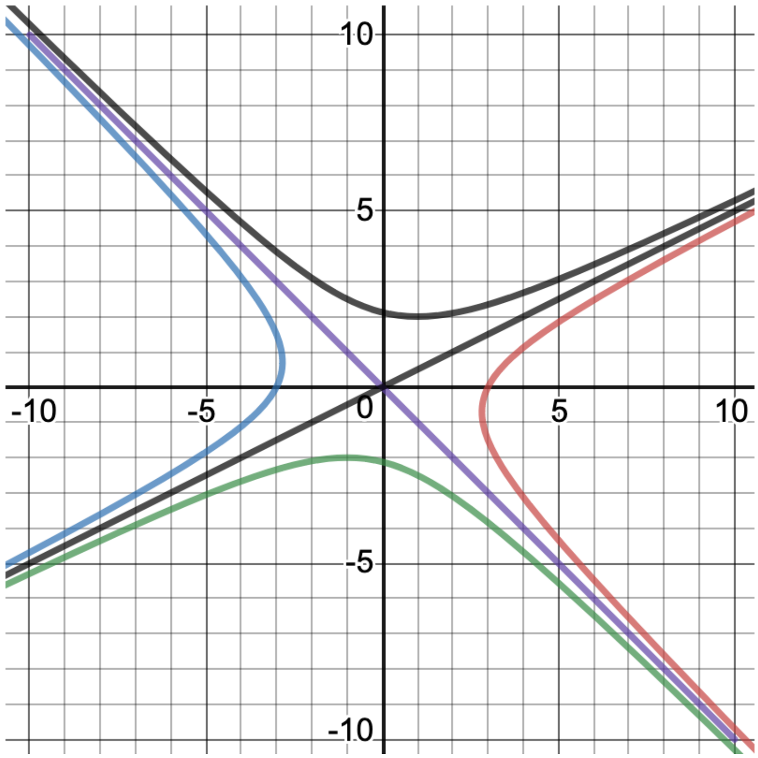


Figure 6: Phase-plane portrait (without direction arrows).

the eigenvalues are -3 and 3, with eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The phase-plane portrait is

12.3.1 The Trace-Determinant Plane

Given a generic linear system

$$\underline{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{x}$$

the characteristic polynomial is

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc$$

which has eigenvalues

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

Note that $(ad - bc)$ is the determinant, D. The term $a + d$ is called the **trace**, T, of the 2×2 matrix. Thus,

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

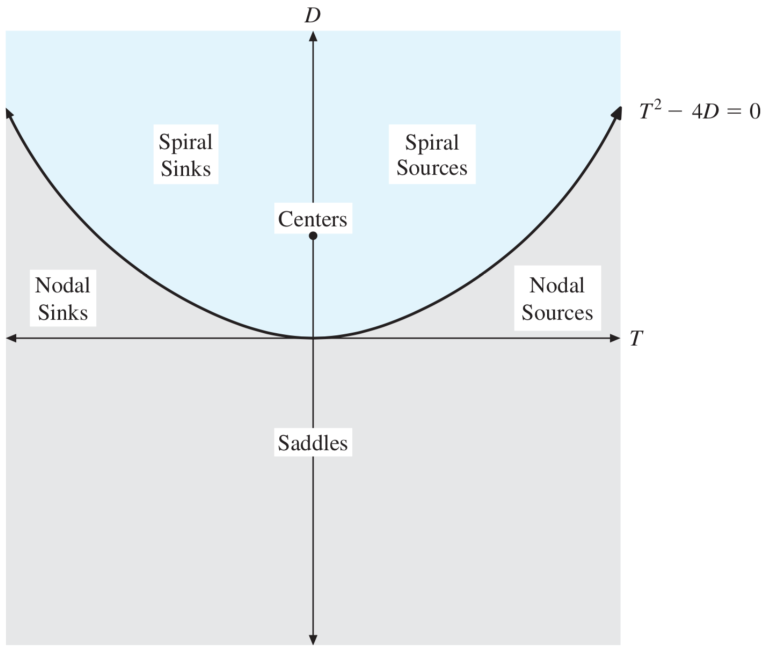


Figure 7: Classification of equilibria in the trace-determinant plane.

The type of eigenvalue is determined by the sign of T and $T^2 - 4D$.

13 Non-diagonalizable Matrices and Inhomogeneous Systems

Outline:

- Higher-dimensional systems
- Non-diagonalizable matrices and generalized eigenvectors
- Solutions with non-diagonalizable matrices
- Solutions to inhomogeneous systems.

13.1 Higher-Dimensional Systems

Example 13.1. Find the general solution to

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_3 = -1$, with eigenvectors $\underline{v}_1 = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\underline{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$. Therefore the general solution is

$$\underline{x} = C_1 \exp(3t) \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} + C_2 \exp(t) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 \exp(-t) \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

None of the theory changes for 3×3 systems, or in fact for $n \times n$ systems.

13.2 Non-diagonalizable matrices and generalized eigenvectors

Example 13.2. Solve

$$\underline{x}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \underline{x}.$$

$$\begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 4 = 25 - 10\lambda + \lambda^2 = (\lambda - 5)^2 \quad \text{eigenvalues: } 5$$

$$\begin{bmatrix} 7-5 & 1 \\ -4 & 3-5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \implies \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \text{eigenvector: } \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

One solution is then

$$\underline{x}(t) = \exp(5t) \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

However, this is not the general solution, and because there is only one eigenvector, it is impossible to write:

$$\underline{\underline{A}} = \underline{\underline{PDP}}^{-1}$$

because the matrix $\underline{\underline{P}}$ would not be square!

Definition 13.3. *An $n \times n$ matrix is* ***non-diagonalizable*** *or* ***defective*** *if the total number of eigenvectors is strictly less than n. (Equivalently, there is an eigenvalue with $g_\lambda < a_\lambda$ called the defective eigenvalue.)*

The solutions to $\underline{x}' = \underline{\underline{A}}\underline{x}$ when $\underline{\underline{A}}$ is defective are still $\underline{x} = \exp\left(\underline{\underline{A}}t\right)\underline{c}$ for some arbitrary $\underline{c}$. The difficulty is in computing the matrix exponential.

Definition 13.4. *A **generalized eigenvector** for a matrix $\underline{\underline{A}}$ with eigenvalue λ is a nonzero vector $\underline{w}$ in the nullspace of $(\underline{\underline{A}} - \lambda \underline{\underline{I}})^p$ for some $p \geq 1$. The number p is called the **rank** of the generalized eigenvector.*

Note that rank-1 generalized eigenvectors are just ordinary eigenvectors.

Example 13.5. In the above example,

$$\underline{\underline{A}} - 5\underline{\underline{I}} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \implies (\underline{\underline{A}} - 5\underline{\underline{I}})^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore, every vector in $\mathbb{R}^2$ is a generalized eigenvector. The rank-2 generalized eigenvectors are those not in $(\underline{\underline{A}} - 5\underline{\underline{I}})$. For example, $\underline{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

A common trick can be used to express the exponential matrix in a simpler form when it is multiplied by a vector:

$$\begin{aligned} \exp\left(\underline{\underline{A}}t\right)\underline{w} &= \exp\left((\underline{\underline{A}} - \lambda\underline{\underline{I}} + \lambda\underline{\underline{I}})t\right)\underline{w} \\ &= \exp\left((\underline{\underline{A}} - \lambda\underline{\underline{I}})t\right)\exp\left(\lambda\underline{\underline{I}}t\right)\underline{w} \\ &= \exp(\lambda t)\exp\left((\underline{\underline{A}} - \lambda\underline{\underline{I}})t\right)\underline{w} \end{aligned}$$

Warning: $\exp\left(\underline{\underline{A}} + \underline{\underline{D}}\right) = \exp\left(\underline{\underline{A}}\right)\exp\left(\underline{\underline{D}}\right)$ for diagonal matrices, but not in general.

$$\begin{aligned} \exp\left(\underline{\underline{A}}t\right)\underline{w} &= \exp(\lambda t)\left(\underline{\underline{I}} + (\underline{\underline{A}} - \lambda\underline{\underline{I}})t + \frac{1}{2}(\underline{\underline{A}} - \lambda\underline{\underline{I}})^2 t^2 + \cdots + \frac{1}{n!}(\underline{\underline{A}} - \lambda\underline{\underline{I}})^n t^n + \cdots\right)\underline{w} \\ &= \exp(\lambda t)\left(\underline{w} + (\underline{\underline{A}} - \lambda\underline{\underline{I}})t\underline{w} + \frac{1}{2}(\underline{\underline{A}} - \lambda\underline{\underline{I}})^2 t^2\underline{w} + \cdots + \frac{1}{n!}(\underline{\underline{A}} - \lambda\underline{\underline{I}})^n t^n\underline{w} + \cdots\right) \end{aligned}$$

Observe that if $\underline{w}$ is an eigenvector $\underline{v}$ for λ, then

$$\begin{aligned} \exp\left(\underline{\underline{A}}t\right)\underline{v} &= \exp(\lambda t)\left(\underline{v} + (\underline{\underline{A}} - \lambda\underline{\underline{I}})t\underline{v} + \frac{1}{2}(\underline{\underline{A}} - \lambda\underline{\underline{I}})^2 t^2\underline{v} + \cdots + \frac{1}{n!}(\underline{\underline{A}} - \lambda\underline{\underline{I}})^n t^n\underline{v} + \cdots\right) \\ &= \exp(\lambda t)\underline{v} \end{aligned}$$

because $\underline{v}$ is in the nullspace of $(\underline{\underline{A}} - \lambda\underline{\underline{I}})$, by definition.

Thus if $\underline{w}$ is a generalized eigenvector of rank p,

$$\exp\left(\underline{\underline{A}}t\right)\underline{w} = \exp(\lambda t)\left(\underline{w} + (\underline{\underline{A}} - \lambda\underline{\underline{I}})t\underline{w} + \frac{1}{2}(\underline{\underline{A}} - \lambda\underline{\underline{I}})^2 t^2\underline{w} + \cdots + \frac{1}{p!}(\underline{\underline{A}} - \lambda\underline{\underline{I}})^p t^p\underline{w}\right)$$

Example 13.6. Continuing from above,

$$\begin{aligned}
\exp\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \exp(5t)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2t & t \\ -4t & -2t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\
&= \exp(5t)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3t \\ -6t \end{bmatrix}\right) \\
&= \exp(5t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\exp(5t)\begin{bmatrix} 3 \\ -6 \end{bmatrix}
\end{aligned}$$

This is the missing second solution:

$$\begin{aligned}
\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}\exp(5t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\exp(5t)\begin{bmatrix} 3 \\ -6 \end{bmatrix} &= \exp(5t)\begin{bmatrix} 8 \\ -1 \end{bmatrix} + t\exp(5t)\begin{bmatrix} 15 \\ -30 \end{bmatrix} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(5t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\exp(5t)\begin{bmatrix} 3 \\ -6 \end{bmatrix}\right) &= \exp(5t)\begin{bmatrix} 5 \\ 5 \end{bmatrix} + \exp(5t)\begin{bmatrix} 3 \\ -6 \end{bmatrix} + 5t\exp(5t)\begin{bmatrix} 3 \\ -6 \end{bmatrix} \\
&= \exp(5t)\begin{bmatrix} 8 \\ -1 \end{bmatrix} + t\exp(5t)\begin{bmatrix} 15 \\ -30 \end{bmatrix}
\end{aligned}$$

Lemma 13.7. *If λ is an eigenvalue of $\underline{\underline{A}}$ with algebraic multiplicity a_λ, then there is a p such that $1 \leq p \leq a_\lambda$ such that the dimension of $(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = a_\lambda$.*

Therefore, the following method can be used to solve $\underline{x}' = \underline{\underline{A}}\underline{x}$ when $\underline{\underline{A}}$ is defective:

1. Find the eigenvalues and eigenvectors of $\underline{\underline{A}}$.
2. Determine the algebraic and geometric multiplicity of each eigenvalue.
3. For any defective eigenvalues, compute $(\underline{\underline{A}} - \lambda\underline{\underline{I}})^p$ for each $1 \leq p \leq a_\lambda$ to find the generalized eigenvectors $\underline{w}$.
4. Compute $\exp\left(\underline{\underline{A}}t\right)\underline{w}$ for each generalized eigenvector using the method above. This gives the complete fundamental set of solutions.

Example 13.8. Solve $\underline{\underline{A}}\underline{x} = \underline{x}'$ where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}.$$

The only eigenvalue is $\lambda = 1$. The eigenvector is given by

$$\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 6 & 3 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ so } \underline{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\lambda = 1$ is defective because $a_\lambda = 3$ while $g_\lambda = 1$. Next, compute

$$(\underline{\underline{A}} - \underline{\underline{I}})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus a generalized eigenvector of rank-2 is $\underline{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Finally, $(\underline{\underline{A}} - \underline{\underline{I}})^3 = \underline{\underline{0}}$ so the last generalized eigenvector is any vector not in $\operatorname{span}(\underline{v}, \underline{w}_1)$. A natural choice is $\underline{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

The solution for the true eigenvector is

$$\underline{x}_1(t) = \exp(t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The solution for the first generalized eigenvector is

$$\begin{aligned} \underline{x}_2(t) = \exp\left(\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} t \right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \exp(t) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 6 & 3 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \exp(t) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \exp(t) \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \end{aligned}$$

The solution for the second generalized eigenvector is

$$\begin{aligned} \underline{x}_3(t) = \exp\left(\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} t \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \exp(t) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 6 & 3 & 0 \end{bmatrix} + t^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \exp(t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \exp(t) \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + t^2 \exp(t) \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} \end{aligned}$$

13.3 Inhomogeneous Systems

Example 13.9. Write the system of equations

$$\begin{aligned} x' &= x + 2y + \exp(-t) \\ y' &= 2x + y \end{aligned}$$

with $x(0) = 0$ and $y(0) = 0$ in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -t \\ 0 \end{bmatrix} \text{ with } \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Definition 13.10. *A system is **inhomogeneous** if it can be written in the form*

$$\underline{y}' = \underline{\underline{A}}\underline{y} + \underline{f}(t).$$

An autonomous linear system is also called homogeneous. Inhomgeneous is a particular type of non-autonomous system, namely one where all the t terms can be separated into their own vector.

If $\underline{y}' = \underline{\underline{A}}\underline{y} + \underline{f}(t)$ is an inhomogenous system, then the associated homogeneous system is $\underline{y}' = \underline{\underline{A}}\underline{y}$. The solution to the homogeneous part, $\underline{y}_h = \exp\left(\underline{\underline{A}}t\right)\underline{c}$. As with second-order equations, all that is needed is *one* solution to

$$\underline{y}' = \underline{\underline{A}}\underline{y} + \underline{f}(t).$$

Proof. Suppose that $\underline{y}_1$ and $\underline{y}_2$ are both solutions to $\underline{y}' = \underline{\underline{A}}\underline{y} + \underline{f}(t)$. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\underline{y}_2 - \underline{y}_1) &= \underline{y}_2' - \underline{y}_1' \\ &= \underline{\underline{A}}\underline{y}_2 + \underline{f}(t) - \underline{\underline{A}}\underline{y}_1 - \underline{f}(t) \\ &= \underline{\underline{A}}\underline{y}_2 - \underline{\underline{A}}\underline{y}_1 \\ &= \underline{\underline{A}}(\underline{y}_2 - \underline{y}_1) \end{aligned}$$

so $\underline{y}_2 - \underline{y}_1$ is a solution to the homogeneous system. Thus it has the form $\exp\left(\underline{\underline{A}}t\right)\underline{c}$ for some constant vector $\underline{c}$. □

To find a particular solution $\underline{y}_p$, make a guess that

$$\underline{y}_p = \exp\left(\underline{\underline{A}}t\right)\underline{\psi}(t)$$

for some non-constant vector $\underline{\psi}(t)$. For simplicity, assume that $\underline{\underline{A}}$ is diagonalizable. (The non-diagonalizable case can also be solved, it simply requires a bit more care.) With this assumption,

$$\underline{y}_p = \underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\underline{P}}^{-1}\underline{\psi}(t).$$

As usual, note that $\underline{\underline{P}}^{-1}$ and $\underline{\psi}$ can be multiplied, giving some undetermined vector, $\underline{\underline{P}}^{-1}\underline{\psi} = \underline{\phi}(t)$. Now compute the derivative

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\underline{y}_p &= \frac{\mathrm{d}}{\mathrm{d}t}\underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(C_1\phi_1(t)\exp(\lambda_1 t) + C_2\phi_2(t)\exp(\lambda_2 t) + \cdots + C_3\phi_3(t)\exp(\lambda_3 t)\right) \\
&= \underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}'(t) + \underline{\underline{PD}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}(t) \\
&= \underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}'(t) + \underline{\underline{PDI}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}(t) \\
&= \underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}'(t) + \underline{\underline{PDP^{-1}}}\,\underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}(t) \\
&= \underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}'(t) + \underline{\underline{A}}\underline{y}_p
\end{aligned}$$

(The product rule works here!) Therefore

$$\underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right)\underline{\phi}'(t) = \underline{f}(t).$$

Consequently,

$$\underline{\phi}'(t) = \exp\left(-\underline{\underline{D}}t\right)\underline{\underline{P}}^{-1}\underline{f}(t),$$

and so,

$$\underline{\phi}(t) = \int \exp\left(-\underline{\underline{D}}t\right)\underline{\underline{P}}^{-1}\underline{f}(t)\mathrm{d}t.$$

This proves the following theorem

Theorem 13.11. *The solution to* $\underline{y}' = \underline{\underline{A}}\underline{y} + \underline{f}(t)$ *is given by* $y_h + y_p$ *where* $y_h = \exp\left(\underline{\underline{A}}t\right)\underline{c}$ *an*

$$y_p = \underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right) \int \exp\left(-\underline{\underline{D}}t\right)\underline{\underline{P}}^{-1}\underline{f}(t)\mathrm{d}t.$$

Example 13.12. Solving the system from the beginning:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underline{\underline{A}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} \exp(-t) \\ 0 \end{bmatrix}$$

Begin by computing the eigenvalues and vectors: $\lambda_1 = -1$, $\lambda_2 = 3$,

$$\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The homogeneous solution is

$$\begin{bmatrix} x_h \\ y_h \end{bmatrix}(t) = C_1 \exp(-t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \exp(3t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The particular solution is

$$\begin{aligned}
\begin{bmatrix} x_p \\ y_p \end{bmatrix}(t) &= \underline{\underline{P}} \exp\left(\underline{\underline{D}}t\right) \int \exp\left(-\underline{\underline{D}}t\right) \underline{\underline{P}}^{-1} \underline{f}(t) \mathrm{d}t \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \exp(-t) & 0 \\ 0 & \exp(3t) \end{bmatrix} \int \begin{bmatrix} \exp(t) & 0 \\ 0 & \exp(-3t) \end{bmatrix} \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \exp(-t) \\ 0 \end{bmatrix} \mathrm{d}t \\
&= \frac{1}{-2} \begin{bmatrix} -\exp(-t) & \exp(3t) \\ \exp(-t) & \exp(3t) \end{bmatrix} \int \begin{bmatrix} \exp(t) & -\exp(t) \\ -\exp(-3t) & -\exp(-3t) \end{bmatrix} \begin{bmatrix} \exp(-t) \\ 0 \end{bmatrix} \mathrm{d}t \\
&= \frac{1}{-2} \begin{bmatrix} -\exp(-t) & \exp(3t) \\ \exp(-t) & \exp(3t) \end{bmatrix} \int \begin{bmatrix} 1 \\ -\exp(-4t) \end{bmatrix} \mathrm{d}t \\
&= \frac{1}{-2} \begin{bmatrix} -\exp(-t) & \exp(3t) \\ \exp(-t) & \exp(3t) \end{bmatrix} \begin{bmatrix} t \\ \frac{1}{4}\exp(-4t) \end{bmatrix} \\
&= \frac{1}{-2} \begin{bmatrix} -t\exp(-t) + \frac{1}{4}\exp(-t) \\ t\exp(-t) + \frac{1}{4}\exp(-t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{t}{2}\exp(-t) - \frac{1}{8}\exp(-t) \\ \frac{-t}{2}\exp(-t) - \frac{1}{8}\exp(-t) \end{bmatrix}
\end{aligned}$$

Then the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_h \\ y_h \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix} = C_1 \exp(-t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \exp(3t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{t}{2}\exp(-t) - \frac{1}{8}\exp(-t) \\ \frac{-t}{2}\exp(-t) - \frac{1}{8}\exp(-t) \end{bmatrix}.$$

The particular solution is found by

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ -\frac{1}{8} \end{bmatrix} \\
\begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \end{bmatrix} &= C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} -1 & 1 & \frac{1}{8} \\ 1 & 1 & \frac{1}{8} \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{1}{8} \\ -1 & 1 & \frac{1}{8} \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{1}{8} \\ 0 & 2 & \frac{1}{4} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{8} \end{bmatrix}$$

So $C_2 = \frac{1}{8}$ and $C_1 = 0$.

14 Nonlinear Systems Revisited

Outline:

- Linearization
- Review

14.1 Linearization

Example 14.1. Consider the system:

$$x' = (1 - x - y)x$$
$$y' = (4 - 2x - 7y)y.$$

This has equilibria at $(0, 0)$, $(0, 4/7)$, $(1, 0)$, and $(3/5, 2/5)$. Defining

$$F(x, y) = \begin{bmatrix} (1 - x - y)x \\ (4 - 2x - 7y)y \end{bmatrix}$$

the system can be thought of as the vector field F.

Definition 14.2. *The* ***Jacobian*** *of a system $\underline{x}' = F(\underline{x})$ is the matrix*

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \dots \\ \frac{\partial F_2}{\partial x_1} & \dots \end{bmatrix}$$

Definition 14.3. *The* ***linearization*** *of a system at an equilibrium point $\underline{u}$ is $J(\underline{u})$, the Jacobian evaluated at that point.*

A nonlinear system can be approximated at an equilibrium point by the linearization.

Example 14.4. For the example above,

$$\underline{\underline{J}} = \begin{bmatrix} 1 - 2x - y & x \\ 2y & 4 - 2x - 14y \end{bmatrix}.$$

Thus,

$$J(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

This is a nodal source. Sketching the nullclines and vector field reveals that all solution curves near the origin move away from $(0, 0)$.

Likewise, at $(1, 0)$,

$$J(1, 0) = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

which is a saddle point, again matching the vector field.

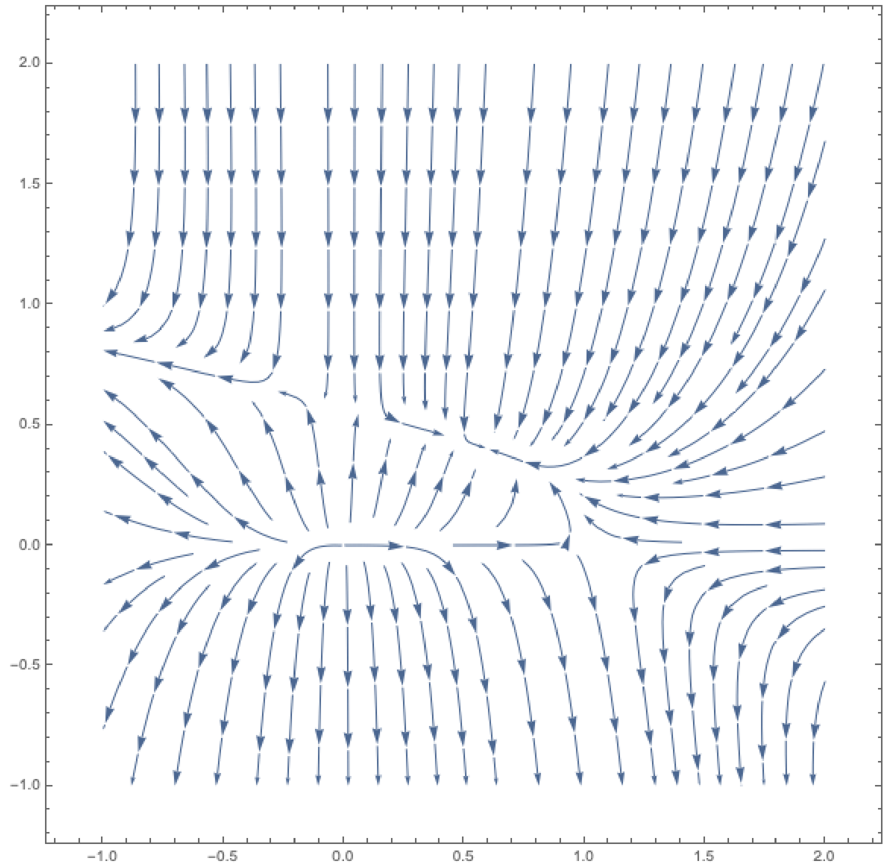


Figure 8: The vector field for the above system.

Example 14.5. Convert the nonlinear equation $y'' = y^2 + 2yy' + 3y$ into a system of first-order equations:

$$\begin{aligned} y' &= u \\ u' &= y^2 + 2yu + 4y \end{aligned}$$

The equilibrium points are $(0,0)$ and $(-4,0)$. Note that the y-nullcline is $u = 0$, so all equilibrium points fall on the horizontal axis, just as they do for the phase line in a first-order equation. Here

$$J = \begin{bmatrix} 0 & 1 \\ 2y + 2u + 4 & 2y \end{bmatrix}$$

so

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

which is a saddle point, and

$$J(-4,0) = \begin{bmatrix} 0 & 1 \\ -4 & -8 \end{bmatrix}$$

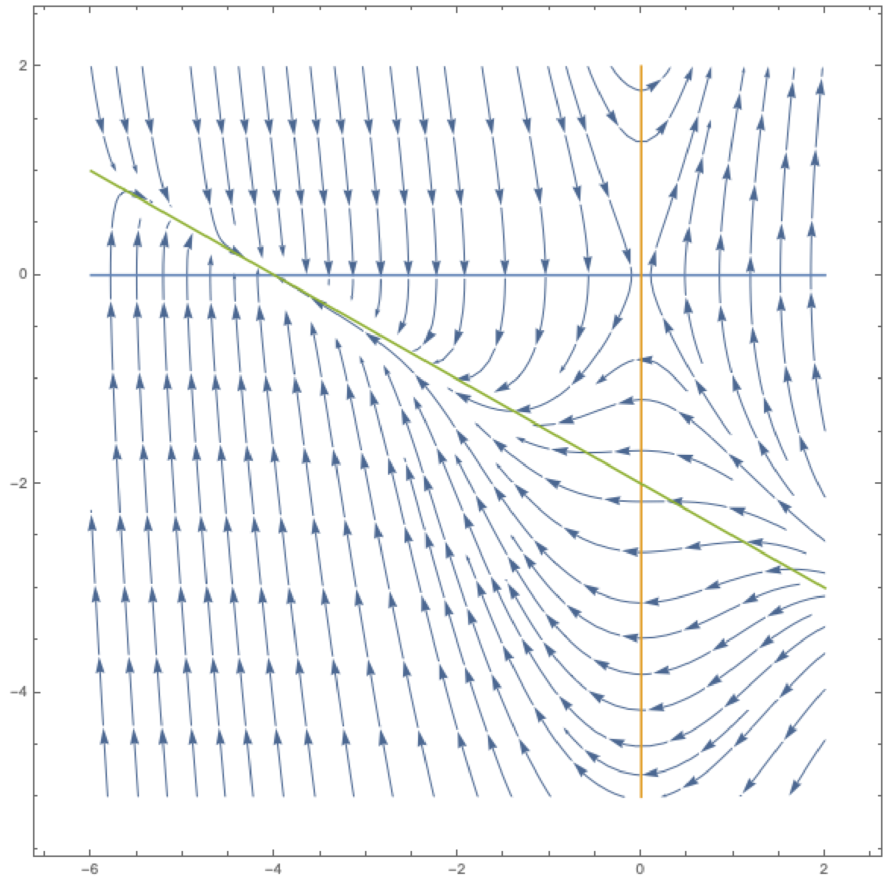


Figure 9: The vector field and nullclines for the above system.

which is a nodal sink.

Definition 14.6. *A **phase portrait** is a sketch of the local behavior at an equilibrium point in the phase plane.*

14.2 Limit Cycles and Higher Dimensions

Example 14.7. Equilibria need not always be points. The system

$$x' = -y + x(1 - x^2 - y^2)$$
$$y' = x + y(1 - x^2 - y^2)$$

has a stable *circle* in the phase plane. Points on this circle are considered **periodically stable.** This is also characterizes the stability of centers: they are periodically stable but not asymptotically stable (since no solution curves tend toward a center).

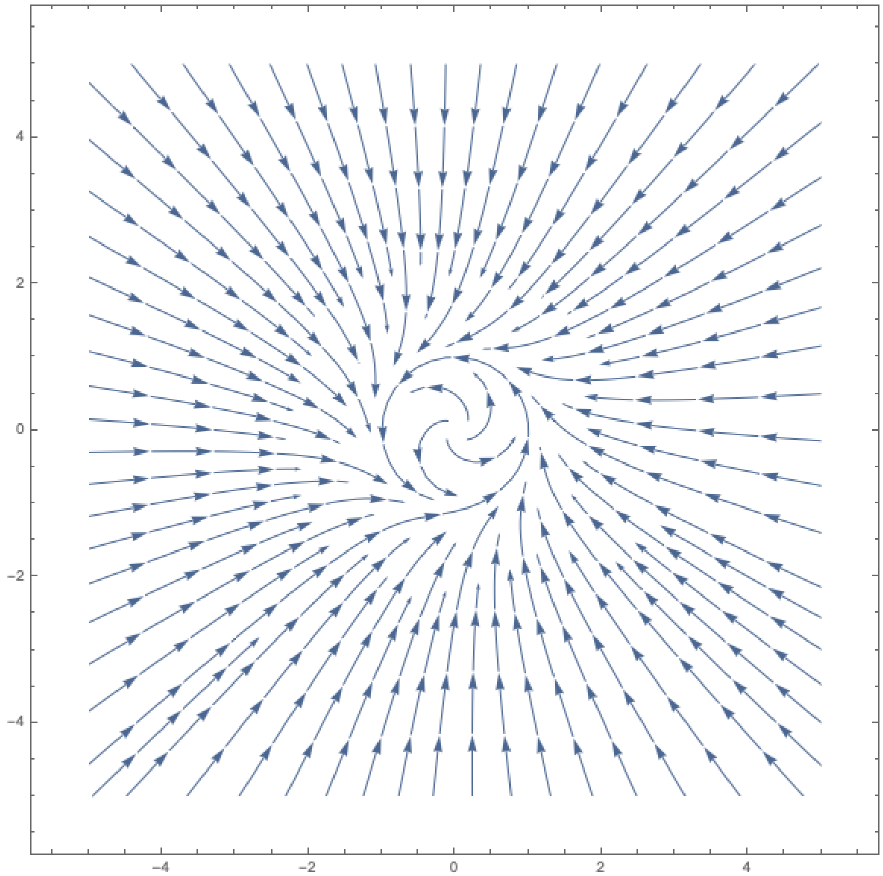


Figure 10: The unit circle attracts all solution curves.

Example 14.8. In higher dimensions there are a multitude of possible stable shapes. A famous example is the Lorenz System:

$$\begin{aligned} x' &= -3(x-y) \\ y' &= -xz + 28x - y \\ z' &= xy - z \end{aligned}$$

Here the attracting set is a butterfly-like fractal: The curve never exactly repeats the same path as it moves around the two loops.

A great resource for more on these systems is the book *Nonlinear Dynamics and Chaos* by Steven Strogatz, `http://www.stevenstrogatz.com/books/nonlinear-dynamics-and-chaos-with-applica`

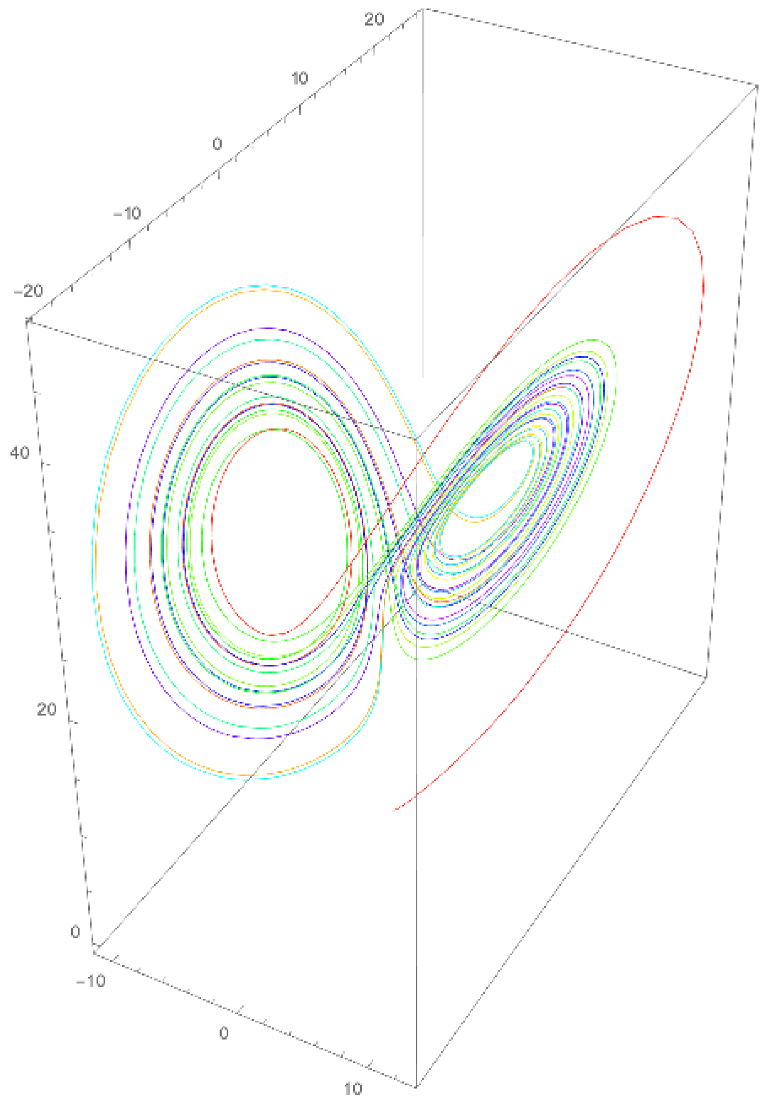


Figure 11: The Lorenz attractor.

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