Simplicial Sets

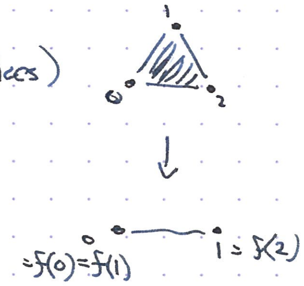
2018·06·20

Motivation: Simplicial Complexes ⟶ Homology

Simplicial Sets ⟶ Homotopy

Basics: Defn: The simplex category, Δ.

- (Nonempty) finite ordered sets, $\{0,\dots,n\} = [n]$ (simplices)
- •→• (Weakly) order-preserving functions

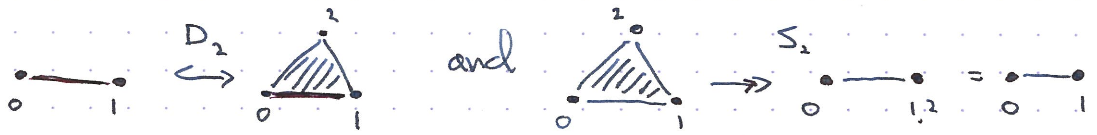


Generating morphisms: there are $n+1$ of each of:

$D_i : [n] \to [n+1]$ by $\{0,\dots,n\} \mapsto \{0,\dots,\hat{i},\dots,n+1\}$

$S_i : [n] \to [n-1]$ by $\{0,\dots,n\} \mapsto \{0,\dots,i,i,\dots,n+1\} = \{0,\dots,n-1\}$

E.g.

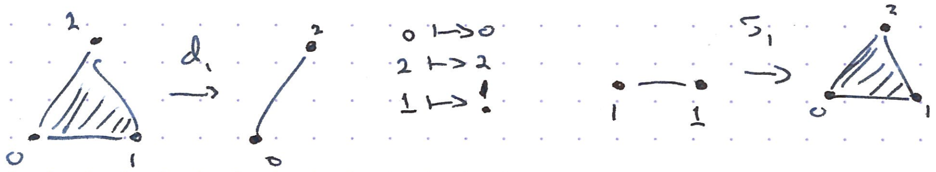


$\{0\ 1\ 2\} \to \{0\ 1\ 1\} \to \{0\ 1\}$

Obvious fact: all morphisms are compositions of S_i and D_i.

- Opposite Category Δ^{op} has morphisms generated by $D_i^{op} = d_i$ and $S_i^{op} = s_i$

E.g.



$0 \mapsto 0$, $2 \mapsto 2$, $1 \mapsto !$

$0 \mapsto 0$, $1 \mapsto 1$ and 2, write as $[0\,1\,1]$

Not set maps. d_i = "fuse" s_i = "degeneracy"

- Follow some rules:

$d_i d_j = d_{j-1} d_i$ if $i<j$

$s_i s_j = s_{j+1} s_i$ if $i \le j$

$d_i s_j = s_{j-1} d_i$ $i<j$

$d_j s_j = d_{j+1} s_j = id$

$d_i s_j = s_j d_{i-1}$ $i > j+1$

$d_0 d_2 [0\,1\,2] = d_0[0\,1] = [1]$

$d_1 d_0 [0\,1\,2] = d_1[1\,2] = [1]$

$s_1 s_2 [0\,1\,2] = s_1[0\,1\,2\,2] = [0\,1\,1\,2\,2]$

$s_3 s_1 [0\,1\,2] = s_3[0\,1\,1\,2] = [0\,1\,1\,2\,2]$

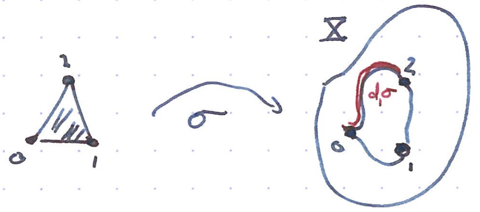
Defn: A *simplicial set* is a functor $X : \Delta^{op} \to Set$. The category *sSet* is the functor category $Set^{\Delta^{op}}$.

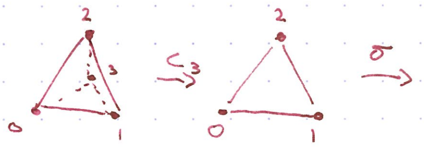
E.g. Write X_i for $X[i]$. The standard 1-simplex as a sSet is

$X_0 = \{[0],[1]\}$ $\quad X_1 = \{[0,0],[0,1],[1,1]\}$ $\quad X_2 = \{[0,0,0],[0,0,1],[0,1,1],[1,1,1]\}$, &c.

"Degenerate simplices". The standard n-simplex is similar: $\Delta^n \simeq [0,\dots,n]$

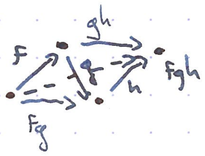
E.g. (Singular Set) $S(X)$ w/ $X \in Top$. The sets $S(X)_i$ = set of cts functions $|\Delta^{i}| \to X$. Morphisms: $d_i\sigma = \sigma|_{\text{ith face of } \Delta^n}$ $\quad s_i\sigma = \sigma c$ $\quad c$ = collapse of ith vertex ~~face~~





E.g. (Nerve) C a small category. The *nerve* of C, $N(C)$ is sSet with

$N_0 = ob(C)$ $\quad N_1 = \bullet \xrightarrow{f} \bullet$ in C. $\quad N_2$ = composable pairs  &c.

N_3: 

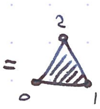
d_i is deletion of i-th arrow in composition.

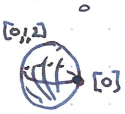
$d_1(f.g.h) = (f.g).h$

s_i inserts the identity $s_0(f.g) = f.id.g$.

Realization: $|-| : sSet \to Top$ by $|X| = \dfrac{\coprod_{n=0}^{\infty} X_n \times |\Delta^n|}{\sim}$ w/ $\sim$ given by face and degeneracy maps.

- In fact, gives a CW complex with an n-cell for each non-degenerate simplex. [No LL.]

E.g. $S^{n-1} = |\partial \Delta^n|$ but also easier! Take $[0,1,2] =$  but with all $d_i = [0,0]$.

Non-degenerates are $[0]$ and $[0,1,2]$. Realization is 

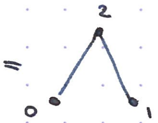
Thm: $|-|$ adjoint to $S(-)$, i.e. $Hom_{Top}(|X|, Y) \cong Hom_{sSet}(X, S(Y))$, and $|S(Y)| \simeq Y$. (for Y CW)

E.g. Let $\mathcal{G}$ be a discrete group as a category. $|N(\mathcal{G})|$ is the classifying space of $\mathcal{G}$.

$G = \mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 = e \rangle$ $\quad N_0 = \{0\}$ $\quad N_1 = \{e, a\}$ $\quad N_2 = \{(e,a), (a,e), (e,e), (a,a)\}$ &c.

One non-degen simplex in each $(a, \dots, a)$. $BG = \mathbb{RP}^\infty$.

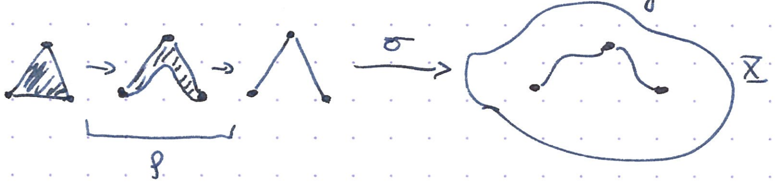
Kan Condition: The k-th horn of Δ^n, Λ^n_k is $\partial\Delta^n \setminus d_k \Delta^n$.

$\Lambda^2_2 =$ 

$= [0],[1],[2]; \ [0,2],[1,2]$; degenerades as standard.

* The Kan condition is satisfied by X if
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$
"every horn has a filler". "Fibrant". not unique

E.g. $S(Y)$ is a Kan complex. Choose any retract $|\Delta^n| \to |\Lambda^n_k|$. This serves as filler.



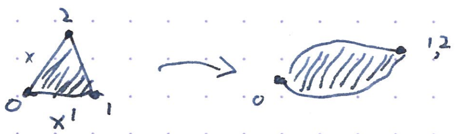
E.g. The n-simplex is not Kan! Take $\Lambda^2_0 =$ 

$\mapsto$ 

by $[0,2] \mapsto [0,0]$, $[0,1] \mapsto [0,1]$.
This does not extend to Δ^2. (Where does $[1,2]$ go?) [Complexes are not Kan]

E.g. $N(C)$ is Kan $\iff$ C is a groupoid. Here all fillers are unique.

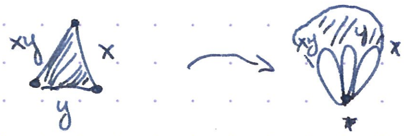
Homotopy: $x, x' \in X$ homotopic if $d_i x = d_i x'$ $\forall i$; $d_n y = x$ and $d_{n+1} y = x'$, and $d_i y = s_{n-1} d_i x = s_{n-1} d_i x'$ for some $y \in X_{n+1}$.

E.g. 

$x \simeq x'$

Defn: X Kan. $\pi_n(X, *)$ is set of homotopy classes of n-simplices with $d_i x = *$ $\forall i$.
$(n > 0)$

Any $x, y \in X_n$ form a horn. Define $x \cdot y$ as the other face of the filler.



Thm: Homotopy theory of Kan complexes is equivalent to that of CW complexes.