Notes on:
Sample Document

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Summer 2025

Contents

1 Introduction to Real Analysis

This chapter covers the foundations of real analysis, beginning with properties of the real number system.

Key Idea 1.1

The completeness property of real numbers states that every non-empty set of real numbers that is bounded above has a least upper bound.

Example 1.1

Consider the set $S = \{x \in \mathbb{R} : x^2 < 2\}$. This set is bounded above, and its least upper bound is $\sqrt{2}$.
We can verify this by noting that:

- If $x^2 < 2$, then $x < \sqrt{2}$, so $\sqrt{2}$ is an upper bound.
- For any $\varepsilon > 0$, $(\sqrt{2} - \varepsilon)^2 < 2$, so $\sqrt{2} - \varepsilon \in S$.

Therefore, $\sqrt{2}$ is the least upper bound of S.

Remark 1.1

The completeness property distinguishes the real numbers from the rational numbers. For instance, the set $\{x \in \mathbb{Q} : x^2 < 2\}$ has no least upper bound in $\mathbb{Q}$.

A proper understanding of completeness[1] is essential for developing the theory of limits and continuity.

[1]This concept was first rigorously formulated by Dedekind in the 19th century.

2 Sequences and Series

In this chapter, we explore infinite sequences and series, their convergence properties, and tests for convergence.

Key Idea 2.1

A sequence $\{a_n\}$ converges to a limit L if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

Example 2.1

Let's prove that the sequence $a_n = \frac{1}{n}$ converges to 0.
Given $\varepsilon > 0$, we need to find N such that $|\frac{1}{n} - 0| < \varepsilon$ for all $n \geq N$.
Since $|\frac{1}{n} - 0| = \frac{1}{n}$, we need $\frac{1}{n} < \varepsilon$, which is equivalent to $n > \frac{1}{\varepsilon}$.
Therefore, we can choose $N = \lceil \frac{1}{\varepsilon} \rceil$, and for all $n \geq N$, we have $|a_n - 0| < \varepsilon$.

Remark 2.1

While many sequences converge, there are important examples of divergent sequences like $a_n = (-1)^n$ and $a_n = n$.