Scientific Computing | Fitchburg State University
Department of Mathematics
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The following project will investigate the Mandelbrot set, an example of a fractal, a geometric structure that has self-similarity at any scale. You will see what how to determine points in the Mandelbrot set and then produce images of them.

Answer all of the questions and upload your results as a PDF or as a .ipynb file

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Project Background

We investigate the Mandelbrot set, a set of number in the complex plane formed by iterating a complex map and determining the set of points that is bounded. Here we give some background on this. Before starting this make sure that you understand basic operations with complex numbers that were given in the course notes (weeks 2 and 3).

Complex Numbers

A few things that are important about complex numbers. Let $z = a + bi$ be a complex number then the number a is the real part and b is the imaginary part. We can think of plotting a point in the complex plane where the horizontal axis is the real part and vertical point is the imaginary axis. To multiply number $x = a + bi$ and $y = c + di$, you get

$$xy = (a + bi)(c + di) = ac + bci + adi + (bi)(di) = ac + (bc + ad)i - bd = (ac - bd)i + (bc + ad)i$$

The magnitude of a complex number is the distance the point is from the origin. It is denoted as $|z|$. If $z = (a+bi)$, then $z = \sqrt{a^2 + b^2}$. See the complex number discussion on purplemath.com for more details about this.

A complex map

A map is a fancy mathematical term for applying a function (usually repeatedly). For example, the repeated iteration of Newton's method can be thought of a map. The iteration

$$x_{n+1} = x_n^2 + c \tag{1}$$

is what we will study in this project where the x and c values are complex. For example, let let $c = i$ and $x_0 = 0$.

$$\begin{aligned}
x_1 &= x_0^2 + c = 0 + i \\
x_2 &= x_1^2 + c = i^2 + i = -1 + i \\
x_3 &= x_2^2 + c = (-1 + i)^2 + i = 1 - 2i - 1 + i = -i \\
x_4 &= x_3^2 + c = (-i)^2 + i = -1 + i
\end{aligned}$$

and since $x_4 = x_2$, then $x_5 = x_3$ and all even will equal $-1 + i$ and all odds will be $-i$.

If we repeat this with $c = 1.01i$

$$\begin{aligned}
x_1 &= x_0^2 + c = 0 + 1.01i = 1.01i \\
x_2 &= (1.01i)^2 + 1.01i = -1.0201 + 1.01i \\
x_3 &= (-1.0201 + 1.01i)^2 + 1.01i = 0.020504010000000017 - 1.0506019999999998i \\
x_4 &= -1.1033441479779194 + 0.96691689217196i \\
&\vdots \\
x_{10} &= -25.934812678057604 + 8.464736394953507i
\end{aligned}$$

Sequences

A sequence is a bunch of numbers in a particular order. We write it as

$$\{a_n\} = a_1, a_2, a_3, \ldots$$

where n is called the index. Typically a sequence does not end. For example, here are a few sequences:

$$\begin{aligned}
&1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \\
&0, 1, 0, -1, 0, 1, 0, -1, \ldots \\
&\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \\
&1, 2, 4, 8, \ldots
\end{aligned}$$

You can see that for any given value of c, the map given above, results in a sequence. The example with $c = i$ and $x_0 = 0$ results in the sequence:

$$0, i, -1+i, -i, -1+i, -i, -1+i, \ldots$$

Often, an interesting question for sequences like this is whether or not the sequence converges or not. The limit of a sequence if it exists is the number that the sequence approaches as n (the index) goes to ∞. This notion is the same as that for limits of functions at infinity.

Just looking at the 5 sequences above, the first sequence goes to 0, the second does not converge, the 3rd goes to 1, the 4th goes to ∞ and the last does not converge. The complex one above appears to bounce between the values $-i$ and $-1+i$ and since it isn't one particular value, this sequence does not converge.

Another possible result of a sequence is that it is bounded. We say that a sequence is **bounded above** if there is a number M such that $a_n < M$ for all n and is **bounded below** if there is a number m such that $a_n > m$ for all n. For a complex sequence, we say the sequence is bounded if there is a M such that $|a_n| < M$ for all n.

Note: the complex sequence above is bounded but does not converge.

The Mandelbrot Set

The mandelbrot set is the set of all numbers, c in the complex plane such that the sequence $\{x_n\}$ generated by the map in (1) is bounded. This is difficult to find, however we will find an approximation.

An important result is the following lemma. In short it says that if any point in the sequence is more than 2 units from the origin, then the sequence will not be bounded.

Lemma 1 *If $|x_n| > 2$ for the map in (1), then $|x_{n+1}| > |x_n|$ and as a result $|x_n| \to \infty$*

With this lemma in our pocket, we can determine if the point c is in the Mandelbrot set. The value N is a positive integer that we will use to determine if the sequence is bounded. Below we will use $N = 255$.

1. Let $z_0 = 0 + 0i$, $n = 0$
2. Apply $z_{n+1} = z_n^2 + c$.
3. If $|z_n| > 2$, stop the point is not in the set
4. If $n > N$, then it is probably bounded. stop.
5. goto 2.

A Mandelbrot Module

To answer the questions below, you should write a Module to do at least the following:

- A `MandelbrotView` type. This should contain the min and max of the view for a plot of the Mandelbrot set. You may use 2 complex numbers to store these, 4 reals, or other possibilities. You should also have optional parameters for the resulting size of the image.
- An `iterate` function that iterates a general complex function a certain number of times.
- An `is_in_mbset` function that takes a complex number as input and returns true or false depending on if the input is in the mandelbrot set as explained above. Assume that the initial point is `0+0im`.
- A `leaving_number` function that takes a complex number, c as input and returns the number of iterations to leave defined as $|z_n| > 2$. You should use a optional parameter to determine the maximum number of iterations (use 10 as the default) to take and take another optional parameter for the initial point (use `0+0im` as the default)
- A `plot` function that takes as input a `MandelbrotView` type and produces a plot. It is recommended to create an image similar to that of the complex Newton's method from class. You should also consider setting the aspect ratio to 1 in this case to avoid stretching out the plot.
- For each of the types and functions, you should check that the parameters are valid.
- Write a Unit test (as a separate file) to test that your types/functions are working as expected. (Note: think of the basics like needed parameters are positive, for example).

Questions

1. Find the first ten iterations of the complex map with $x_0 = 0$ and $c = -i, 0.4 + 0i, -0.5 + 0.5i$ and $0.4i$. Which of these values, does the sequence appear to converge? If the sequence doesn't converge, does it appear that the sequence is bounded?
2. For each of the sequences in #1, plot the absolute value of the iterations (recall this means the distance from the origin). (Hint: you may want to use a log scale for any that grow very fast and if you use a log scale, throw out the first point which has a distance of 0 and the log of 0 is not defined.)
3. Find the leaving numbers for $c = -i, -1.01i, 0.4 + 0i, -0.5 + 0.5i$ and $2i$. If it appears that the sequence is in the Mandelbrot set, return 255.
4. Create a 400 by 600 array of complex numbers with real values -2 to 1 and imaginary numbers -1 to 1. For each number in the array, return the result from `leaving_number`. Convert the array to unsigned 8-bit integers (UInt8), apply a color map and view the image. This should be the standard view of the Mandelbrot set. (Google it if you don't know what it looks like). Play with the various built-in colormaps or trying creating some of your own. For this problem, you can built up the plot (image).
5. Create the above image using the `plot` function from your module. It should be called like `plot(MandelbrotView(-2-im,1+im))`.
6. Plot the Mandelbrot set for $z = x + iy$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

   Note: make sure that you have the plot oriented correctly.
7. Once you have #5 and #6 finished, try finding at least 4 interesting regions of the Mandelbrot set to plot. Zooming in around the edges of the set usually produce interesting results.