

Assignment Two

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1 SELECTION SORT

1.1 THE ALGORITHM

Selection sort is a sorting algorithm that, for each iteration of the array, selects the smallest (or largest) element of the unsorted part of the array and places the element into its sorted position. As shown in the pseudocode for the sort in Algorithm 1, selection sort works with the subset of the array in the range $[i, n)$ in each iteration because the elements in the indices less than i are already sorted and do not have to be checked. Thus, as more elements get sorted, the quicker each iteration becomes because a smaller portion of the array is compared until $i = n - 2$, which is the final iteration of the algorithm. Selection sort is also very consistent in that it runs in the same amount of time regardless of the order of the elements and has both a best and worst case of n^2 , which will be analyzed in further detail in Section 1.2.

Algorithm 1 Selection Sort Algorithm

```
1: procedure SELECTIONSORT(arr)
2:   for  $i \leftarrow 0, n - 2$  do    // Iterate through the second to last element as an array of size 1 is sorted
3:     smallestIndex  $\leftarrow i$ 
4:     for  $j \leftarrow i + 1, n - 1$  do    // Iterate through the remainder of the array
5:       if  $arr[j] < arr[smallestIndex]$  then
6:         smallestIndex  $\leftarrow j$     // Set the new smallest index if a smaller element is found
7:       end if
8:     end for
9:     swap(arr, i, smallestIndex)    // Place the smallest item in the subarray into its sorted place
10:  end for
11: end procedure
```

1.2 ASYMPTOTIC ANALYSIS AND COMPARISONS

Listing 1 contains the C++ code implementing selection sort on lines 3-24. Line 3 defines a loop that iterates $n - 1$ times and contains 2 assignments and a comparison, all of which operate in constant time for each iteration. Thus, line 3 will take $(n - 1) * C_1$ time, where C_1 is the time needed for each of the operations.

Next, line 5 is an assignment, which takes a constant time and executes $n - 1$ times because it is in the outer loop, resulting in a time of $(n - 1) * C_2$, where C_2 is the constant time needed for the assignment. Line 8, similar to line 3, defines a loop with 3 constant time expressions, which can be marked as C_3 . However, since it is nested inside the loop on line 3, the total number of iterations of the inner loop is more complex. In the first iteration of the outer loop, the inner loop runs $n - 1$ times. From there, each corresponding iteration of the outer loop results in one less iteration of the inner loop with a minimum of 1 pass on the inner loop when $i = n - 2$. Therefore, the total number of times the inner loop on line 8 will be called is $\sum_{k=1}^{n-1} k$, which by the formula for the sum of the first N natural numbers, is equal to $\frac{(n-1)(n-1+1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$. Thus, the total time to execute line 8 is $(\frac{1}{2}n^2 - \frac{1}{2}n) * C_3$. Next, line 10 contains a comparison that, since it is nested inside the inner loop, will run in $(\frac{1}{2}n^2 - \frac{1}{2}n) * C_4$ time, where C_4 is the time needed to make the comparison. Line 11 is a simple assignment and, just like line 10, will run in $(\frac{1}{2}n^2 - \frac{1}{2}n) * C_5$, where C_5 is the time to perform the assignment. The assignment on line 16 is purely for collecting data and not part of the algorithm and, therefore, will be excluded from the asymptotic analysis of selection sort. Line 18 is the end of the inner loop, and represents an unconditional branch back to the top of the loop, which means it runs the same number of iterations as the loop, which is $(\frac{1}{2}n^2 - \frac{1}{2}n) * C_6$, where C_6 is the time needed to execute the branch. Next, lines 22-24 are all assignments, which run in constant time, and are located in the outer loop. Thus, they run in $(n - 1) * C_7$ time, where C_7 is the time needed to perform the swap. Lastly, line 24 is the close and unconditional branch for the outer loop, which will run in $(n - 1) * C_8$ time, where C_8 is the time to execute the unconditional branch. Overall, when adding up the runtimes of each line and dropping the constants, the sum is $4 * (n - 1) + 4 * (\frac{1}{2}n^2 - \frac{1}{2}n) = 2n^2 + 2n - 4 \approx n^2 + n$ is $O(n^2)$.

As shown in Table 4.1, selection sort is very consistent with the number of comparisons made as, regardless of the state of the list, it always makes $\frac{1}{2}n^2 - \frac{1}{2}n$ comparisons. This is no coincidence as it is also the number of times the algorithm's inner loop iterates, which means that the selection sort will run very consistently for all lists, no matter the state of the array prior to running the algorithm.

2 INSERTION SORT

2.1 THE ALGORITHM

Insertion sort is a sorting algorithm that places an element in its sorted place by sliding previously sorted elements over until the sorted position is found for the element. As shown in Algorithm 2, the element that is being sorted is only compared to elements in positions $[0, i - 1]$ because these are the elements that have been worked with so far and are known to be in order. Therefore, the elements in this area, as described before, are shifted over until one is less than the value being sorted or until $j < 0$, which will break out of the while loop. Unlike selection sort, the performance of insertion sort varies based on the state of the input array, which will be explored more in detail in Section 2.2.

Algorithm 2 Insertion Sort Algorithm

```

1: procedure INSERTIONSORT(arr)
2:   for  $i \leftarrow 1, n - 1$  do    // Start at index 1 because the first element is already sorted
3:     currentVal  $\leftarrow$  arr[ $i$ ]
4:      $j \leftarrow i - 1$ 
5:     while  $j \geq 0$  and currentVal  $<$  arr[ $j$ ] do    // Find the position to place the element
6:       arr[ $j + 1$ ]  $\leftarrow$  arr[ $j$ ]    // Shift the element over because it is greater than the current value
7:        $j \leftarrow j - 1$ 
8:     end while
9:     arr[ $j + 1$ ]  $\leftarrow$  currentVal    // Place the element in its sorted position
10:  end for
11: end procedure

```

2.2 ASYMPTOTIC ANALYSIS AND COMPARISONS

The code implementation of insertion sort is in Listing 2 on lines 4-33. The worst case for insertion sort is when the list is in reverse order. In all cases, the outer loop on line 4 will always run n times. However, when the list is in reverse order, every element that gets compared to in the inner loop will be greater than the element being sorted, which means that the inner loop will run until $j < 0$ so the item gets placed in the front of the array. As explored in Section 1.2, the total number of iterations in the case of a reversed list for insertion sort will be $\sum_{k=1}^{n-1} k$, which solves to be an $O(n^2)$ runtime. However, the best case for insertion sort is when the array is already sorted. In this situation, the outer loop will still be run, but the inner loop will never be entered because the element being sorted is always going to be greater than or equal to the element in the position before it. Therefore, since the array is only being iterated through from the outer loop, the runtime of insertion sort improves to $\Omega(n)$.

In Table 4.1, insertion sort is shown to have 3 very different outcomes for the lists that were used for testing relative to selection sort. First, insertion sort used about half the number of comparisons as selection sort for a list of 666 shuffled magic items. This is because the inner loop of insertion sort may terminate when $arr[j] < currentVal$ (see line 16 in Listing 2) and in a randomly shuffled list, the probability of $arr[j] < currentVal$ will be around 50%. Therefore, insertion sort will on average be about 50% more efficient than selection sort, but is still classified as $O(n^2)$ because it is still running at a function of n^2 . Next, when the list is already sorted, insertion sort only makes $n - 1$ comparisons. As previously mentioned, the best case for insertion sort is $\Omega(n)$ because the inner loop will never be entered as the second condition for $arr[j] < currentVal$ will always return false. This means there will be only 1 comparison made for each iteration of the outer loop, which equates to $n - 1$ comparisons. Lastly, the worst case for insertion sort is when the list is in reverse order because every element will have to compare itself with all of the elements in the sorted portion of the array. This causes insertion sort to have the same number of comparisons as selection sort for a reversed list at $\frac{1}{2}n^2 - \frac{1}{2}n$, which is also the same number of iterations as the inner loop for insertion sort and makes insertion sort $O(n^2)$.

3 MERGE SORT

3.1 THE ALGORITHM

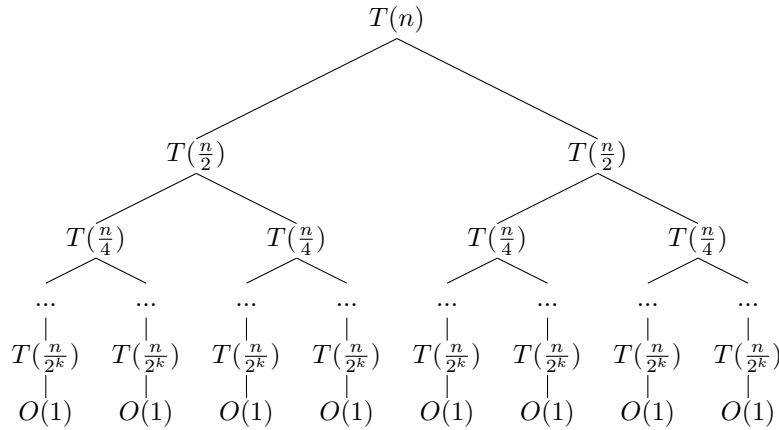
Merge sort is a divide and conquer sorting algorithm that continues to divide an array up until it has n subarrays of size 1, which, by definition, are all sorted. From there, the subarrays are merged together by comparing the elements in each subarray to determine the sorted order of the combined subarrays. Eventually, the full array will be merged back together with all of the elements fully sorted. As displayed in Algorithm 3 in the *MergeSort* procedure, since the sort is a divide and conquer algorithm, merge sort takes advantage of recursion to make the problem smaller until it reaches its base case of $length(arr) \leq 1$, which is shown on line 2. Additionally, on lines 4 and 5, merge sort always divides a given array in half, which makes its performance very predictable and consistent, which will be discussed more in detail in Section 3.2.

3.2 ASYMPTOTIC ANALYSIS AND COMPARISONS

The C++ implementation of merge sort can be found in Listing 3, more specifically lines 6-24. As shown on lines 17 and 20, the array is always split in half for each successive call of merge sort until the subarray is of length 1. The recursion tree for merge sort is displayed below.

Algorithm 3 Merge Sort Algorithm

```
1: procedure MERGESORT(arr)
2:   if length(arr) > 1 then    // An array of size 1 is already sorted
3:     mid = floor((length(arr))/2)  // Get the middle index of the array for splitting it in half
4:     MergeSort(arr[0 : mid])    // Perform merge sort on the first half of the array (index 0 - mid,
    inclusive)
5:     MergeSort(arr[mid + 1 : length(arr) - 1])  // Perform merge sort on the second half of the
    array
6:     Merge(arr, mid)    // Merge the 2 subarrays together in order
7:   end if
8: end procedure
9:
10: procedure MERGE(arr, mid)
11:   leftIndex ← 0    // Index for the left subarray
12:   rightIndex ← mid + 1    // Index for the right subarray
13:   newArr ← []
14:   for i ← 0, length(arr) - 1 do    // Iterate through all elements
15:     if rightIndex ≥ length(arr) then    // All the right subarray items are already in newArr
16:       newArr[i] = arr[leftIndex]    // Add the next item from the left subarray
17:       leftIndex ++
18:     else if leftIndex > mid then    // All the right subarray items are already in newArr
19:       newArr[i] = arr[rightIndex]    // Add the next item from the right subarray
20:       rightIndex ++
21:     else if arr[leftIndex] < arr[rightIndex] then    // The next element from the left subarray is
    less than the next element from the right subarray
22:       newArr[i] = arr[leftIndex]    // Add the next item from the left subarray
23:       leftIndex ++
24:     else
25:       newArr[i] = arr[rightIndex]    // Add the next item from the right subarray
26:       rightIndex ++
27:     end if
28:   end for
29:   for j ← 0, length(arr) - 1 do
30:     arr[j] ← newArr[j]    // Transfer the sorted elements to the original array
31:   end for
32: end procedure
```



4 APPENDIX

4.1 COMPARISONS TABLE

Algorithm	List	Comparisons	Time
Selection Sort	666 magic items, shuffled	221445	3625271 ns
	20 Yankees greats, sorted	190	3673 ns
	20 Yankees greats, reversed	190	3636 ns
Insertion Sort	666 magic items, shuffled	104628	2523161 ns
	20 Yankees greats, sorted	19	795 ns
	20 Yankees greats, reversed	190	2966 ns
Merge Sort	666 magic items, shuffled	5417	1006113 ns
	20 Yankees greats, sorted	48	6751 ns
	20 Yankees greats, reversed	40	6745 ns

Table 4.1: A table of the number of comparisons made and time to complete each sort on a variety of lists.

4.2 SELECTION SORT

```
1 void selectionSort(StringArr* data, int* comparisons) {
2     // Iterate through the second to last element because the last element will already be
   sorted as is
3     for (int i = 0; i < data->length - 1; i++) {
4         // The smallest index is going to start as the start of the subset of the list
5         int smallestIndex = i;
6
7         // Iterate through the rest of the list
8         for (int j = i + 1; j < data->length; j++) {
9             // Compare the current element to the current smallest element in the subset
10            if (data->arr[smallestIndex].compare(data->arr[j]) > 0) {
11                // If the current element comes first, make it the new smallest element
12                smallestIndex = j;
13            }
14            // Increment comparisons
15            if (comparisons != nullptr) {
16                (*comparisons)++;
17            }
18        }
19
20        // Put the smallest index in its respective place
21        std::string temp = data->arr[i];
22        data->arr[i] = data->arr[smallestIndex];
23        data->arr[smallestIndex] = temp;
24    }
25 }
```

Listing 1: Selection Sort (C++)

4.3 INSERTION SORT

```
1 void insertionSort(StringArr* data, int* comparisons) {
2     // We begin with the second element because an array of size 1 is already sorted
3     // So no need to check on the first element
4     for (int i = 1; i < data->length; i++) {
5         // Save the current element for later use
6         std::string currentVal = data->arr[i];
7
8         // Comparisons are going to start with the previous index
9         int j = i - 1;
10    }
```

```

11         // Continue until j is a valid index (< 0) or until we found an element that is less
12         // than the
13         // current element that is being sorted
14         while (j >= 0 && currentVal.compare(data->arr[j]) < 0) {
15             // We made a comparison so increment it
16             if (comparisons != nullptr) {
17                 (*comparisons)++;
18             }
19             // Shift the compared element over 1 to make room for the element being sorted
20             data->arr[j + 1] = data->arr[j];
21             j--;
22         }
23
24         // After the loop, we want to increment comparisons only if j >= 0 because
25         // if j < 0, then the boolean expression would have immediately returned false
26         // without making
27         // a comparison
28         if (j >= 0 && comparisons != nullptr) {
29             (*comparisons)++;
30         }
31         // Place the value in its proper place
32         data->arr[j + 1] = currentVal;
33     }
34 }

```

Listing 2: Insertion Sort (C++)

4.4 MERGE SORT

```

1 void mergeSort(StringArr* data, int* comparisons) {
2     // Sort the entire array
3     mergeSortWithIndices(data, 0, data->length - 1, comparisons);
4 }
5
6 void mergeSortWithIndices(StringArr* data, int start, int end, int* comparisons) {
7     // Base case is array of size 1 or size 0 (if the list is completely empty)
8     if (start >= end) {
9         // No work is needed
10        return;
11    }
12
13    // Get the midpoint for the sections
14    int mid = (start + end) / 2;
15
16    // Sort the first half
17    mergeSortWithIndices(data, start, mid, comparisons);
18
19    // Sort the second half
20    mergeSortWithIndices(data, mid + 1, end, comparisons);
21
22    // Merge both halves
23    merge(data, start, end, mid, comparisons);
24 }
25
26 void merge(StringArr* data, int start, int end, int mid, int* comparisons) {
27     // The left half is at the start
28     int leftIndex = start;
29
30     // The right half starts at the midpoint + 1
31     int rightIndex = mid + 1;
32
33     // Get the size of the array that the 2 halves will merge into
34     // and create the merged sub array

```

```

35     int subArrLength = end - start + 1;
36     std::string newSubArr[subArrLength];
37
38     // Iterate through the entire merged subarray
39     for (int i = 0; i < subArrLength; i++) {
40         // If the rightIndex > end, then the entire right half is already merged
41         if (rightIndex > end) {
42             // Add the next element from the left half
43             newSubArr[i] = data->arr[leftIndex];
44             leftIndex++;
45         } else if (leftIndex > mid) { // If the leftIndex > mid, then the entire left half
            // is already merged
46             // Add the next element from the right half
47             newSubArr[i] = data->arr[rightIndex];
48             rightIndex++;
49         } else if (data->arr[leftIndex].compare(data->arr[rightIndex]) < 0) { // Compare the
            // 2 elements from each half
50             // Add the next element from the left half
51             newSubArr[i] = data->arr[leftIndex];
52             leftIndex++;
53
54             // Increment the number of comparisons made
55             if (comparisons != nullptr) {
56                 (*comparisons)++;
57             }
58         } else {
59             // Add the next element from the right half
60             newSubArr[i] = data->arr[rightIndex];
61             rightIndex++;
62
63             // Make sure to increment comparisons because a comparison was made in the last
            // else-if condition
64             if (comparisons != nullptr) {
65                 (*comparisons)++;
66             }
67         }
68     }
69
70     // Transfer the merged subarray to the actual array
71     for (int j = 0; j < subArrLength; j++) {
72         data->arr[start + j] = newSubArr[j];
73     }
74 }

```

Listing 3: Merge Sort (C++)

4.5 QUICKSORT

```

1 void quickSort(StringArr* data, int* comparisons) {
2     // Run the helper function for the entire array
3     quickSortWithIndices(data, 0, data->length - 1, comparisons);
4 }
5
6 void quickSortWithIndices(StringArr* data, int start, int end, int* comparisons) {
7     // Base case for arrays of size 1 or 0
8     if (start >= end) {
9         // No work is needed
10        return;
11    }
12
13    // The variable used for the pivot index
14    int pivotIndex;
15
16    if (end - start < 3) {

```



```

17     // Array of size 2 should take either element as it will need exactly 1 more divide
18     // regardless of the pivot
19     pivotIndex = start;
20 } else {
21     // Initialize random seed
22     srand(time(NULL));
23
24     // Generate 3 random indices that are all unique to use to pick a pivot
25     int pivotChoice1 = rand() % (end - start) + start;
26     int pivotChoice2 = rand() % (end - start) + start;
27     while (pivotChoice2 != pivotChoice1) {
28         pivotChoice2 = rand() % (end - start) + start;
29     }
30     int pivotChoice3 = rand() % (end - start) + start;
31     while (pivotChoice3 != pivotChoice1 && pivotChoice3 != pivotChoice2) {
32         pivotChoice3 = rand() % (end - start) + start;
33     }
34
35     // Find the median of the 3 indices picked and set the pivot index appropriately
36     // This will hopefully create balanced partitions regardless of the status of the
37     // array
38     if (data->arr[pivotChoice1] <= data->arr[pivotChoice2] && data->arr[pivotChoice1] >=
39         data->arr[pivotChoice3]) {
40         pivotIndex = pivotChoice1;
41
42         // We made 2 comparisons to choose the pivot
43         if (comparisons != nullptr) {
44             *comparisons += 2;
45         }
46     } else if (data->arr[pivotChoice1] <= data->arr[pivotChoice3] && data->arr[
47         pivotChoice1] >= data->arr[pivotChoice2]) {
48         pivotIndex = pivotChoice1;
49
50         // Estimated 4 comparisons to choose the pivot
51         if (comparisons != nullptr) {
52             *comparisons += 4;
53         }
54     } else if (data->arr[pivotChoice2] <= data->arr[pivotChoice1] && data->arr[
55         pivotChoice2] >= data->arr[pivotChoice3]) {
56         pivotIndex = pivotChoice2;
57
58         // Estimated 6 comparisons to choose the pivot
59         if (comparisons != nullptr) {
60             *comparisons += 6;
61         }
62     } else if (data->arr[pivotChoice2] >= data->arr[pivotChoice1] && data->arr[
63         pivotChoice2] <= data->arr[pivotChoice3]) {
64         pivotIndex = pivotChoice2;
65
66         // Estimated 8 comparisons to choose the pivot
67         if (comparisons != nullptr) {
68             *comparisons += 8;
69         }
70     } else {
71         pivotIndex = pivotChoice3;
72
73         // Estimated 8 comparisons to choose the pivot
74         if (comparisons != nullptr) {
75             *comparisons += 8;
76         }
77     }
78 }
79
80 // Partition the data around the pivot
81 int partitionOut = partition(data, start, end, pivotIndex, comparisons);

```

```

76
77 // Sort each of the partitions
78 quickSortWithIndices(data, start, partitionOut - 1, comparisons);
79 quickSortWithIndices(data, partitionOut + 1, end, comparisons);
80 }
81
82 int partition(StringArr* data, int start, int end, int pivotIndex, int* comparisons) {
83 // Move the pivot to the end of the subarray
84 std::string pivot = data->arr[pivotIndex];
85 data->arr[pivotIndex] = data->arr[end];
86 data->arr[end] = pivot;
87
88 // We initially do not have any items in the low partition, so make it less than the
89 // start
90 int lastLowPartitonIndex = start - 1;
91
92 // Iterate through the subarray, excluding the pivot
93 for (int i = start; i <= end - 1; i++) {
94 // Check if the element is less than the pivot
95 if (data->arr[i].compare(pivot) < 0) {
96 // We have an element for the low partition
97 lastLowPartitonIndex++;
98
99 // Move the element to the end of the low partition
100 std::string temp = data->arr[i];
101 data->arr[i] = data->arr[lastLowPartitonIndex];
102 data->arr[lastLowPartitonIndex] = temp;
103 }
104 // Incement comparisons
105 if (comparisons != nullptr) {
106 (*comparisons)++;
107 }
108 }
109
110 // Move the pivot into its appropriate place between the partitions
111 data->arr[end] = data->arr[lastLowPartitonIndex + 1];
112 data->arr[lastLowPartitonIndex + 1] = pivot;
113
114 return lastLowPartitonIndex + 1;
115 }

```

Listing 4: Quicksort (C++)