Lambda Calculus

CMPT 331 - Spring 2023 | Dr. Labouseur

Josh Seligman | joshua.seligman1@marist.edu

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1. Beta-reduce the following expressions to their normal form

1.a. $(\lambda a\ \lambda y\ .\ y\ a)(z\ z)$

Expression	Explanation
$(\lambda a\ \lambda y\ .\ y\ a)(z\ z)$	Initial expression
$\lambda y\ .\ y\ a\ [(z\ z)/a]$	Substitute $(z\ z)$ for a
$(\lambda y\ .\ y\ (z\ z))$	β reduction

1.b. $(\lambda x \lambda y.(x\ y))(\lambda z.y)$

Expression	Explanation
$(\lambda x \lambda y.(x\ y))(\lambda z.y)$	Initial expression
$(\lambda x \lambda w.(x\ w))(\lambda z.y)$	Rename y in the left function because it is a bound variable
$(\lambda w.(x\ w))\ [(\lambda z.y)/x]$	Substitute $(\lambda z.y)$ for x
$(\lambda w.((\lambda z.y)\ w))$	β reduction
$(y)\ [w/z]$	Substitute w for z
$(\lambda w.y)$	β reduction

1.c. $(\lambda x.(x\ x))(\lambda y.(y\ y))$

Expression	Explanation
$(\lambda x.(x\ x))(\lambda y.(y\ y))$	Initial expression
$(x\ x)\ [(\lambda y.(y\ y))/x]$	Substitute $(\lambda y.(y\ y))$ for x
$(\lambda y.(y\ y))\ (\lambda y.(y\ y))$	β reduction
$(\lambda y.(y\ y))\ (\lambda z.(z\ z))$	Rename y in the right function because it is a bound variable
$(y\ y)\ [(\lambda z.(z\ z))/y]$	Substitute $(\lambda z.(z\ z))$ for y
$(\lambda z.(z\ z))\ (\lambda z.(z\ z))$	β reduction. The expression is irreducible and there is no normal form because a β reduction will always result in the starting expression.

1.D. $K\ x\ y$

Expression	Explanation
$K\ x\ y$	Initial expression
$(\lambda ab.a)\ x\ y$	α convert K
$(\lambda b.a)\ [x/a]$	Substitute x for a
$(\lambda b.x)\ y$	β reduction
$(x)\ [y/b]$	Substitute y for b
x	β reduction

1.E. $S\ K$

Expression	Explanation
$S\ K$	Initial expression
$(\lambda adc.a\ c\ (d\ c))\ K$	α convert S
$(\lambda dc.a\ c\ (d\ c))\ [K/a]$	Substitute K for a
$(\lambda dc.K\ c\ (d\ c))$	β reduction
$(\lambda dc.((\lambda xy.x)\ c\ (d\ c)))$	α convert K
$(\lambda y.x)\ [c/x]$	Substitute c for x
$(\lambda dc.((\lambda y.c)\ (d\ c))$	β reduction
$(c)\ [(d\ c)/y]$	Substitute $(d\ c)$ for y
$\lambda dc.c$	β reduction. This is equivalent to K' combinator.

1.F. $(S\ K)\ y\ y\ z$

Expression	Explanation
$(S\ K)\ y\ y\ z$	Initial expression
$(\lambda dc.c)\ y\ y\ z$	From Problem 1.e
$(\lambda c.c)\ [y/d]$	Substitute y for d
$(\lambda c.c)\ y\ z$	β reduction
$(c)\ [y/c]$	Substitute y for c
$y\ z$	β reduction

1.G. $K'\ y\ y\ z$

Expression	Explanation
$K'\ y\ y\ z$	Initial expression
$(\lambda dc.c)\ y\ y\ z$	α convert K'
$(\lambda c.c)\ [y/d]$	Substitute y for d
$(\lambda c.c)\ y\ z$	β reduction
$(c)\ [y/c]$	Substitute y for c
$y\ z$	β reduction

2. What is the normal form of $(K\ S)\ (K\ I)$?

Expression	Explanation
$(K\ S)\ (K\ I)$	Initial expression
$((\lambda ab.a)\ S)\ (K\ I)$	α convert first K
$(\lambda b.a)\ [S/a]$	Substitute S for a
$(\lambda b.S)\ (K\ I)$	β reduction
$(S)\ [(K\ I)/b]$	Substitute $(K\ I)$ for b
S	β reduction
$(\lambda pqr.pr(q\ r))$	α convert S

3. Prove the following equivalencies by reducing each side to its normal form.

3.a. $I = S\ K\ K$

Expression	Explanation
$I \stackrel{?}{=} S\ K\ K$	Initial expression
$\lambda x.x \stackrel{?}{=} S\ K\ K$	α convert I
$\lambda x.x \stackrel{?}{=} (\lambda yx.x)\ K$	From Problem 1.e
$(\lambda x.x)\ [K/y]$	Substitute K for y
$\lambda x.x = \lambda x.x$	β reduction

3.b. $S\ K\ K = K\ I\ I$

Expression	Explanation
$S\ K\ K \stackrel{?}{=} K\ I\ I$	Initial expression
$\lambda x.x \stackrel{?}{=} K\ I\ I$	From Problem 3.a
$\lambda x.x \stackrel{?}{=} (\lambda ab.a)\ I\ I$	α convert K
$(\lambda b.a)\ [I/a]$	Substitute I for a
$\lambda x.x \stackrel{?}{=} (\lambda b.I)\ I$	β reduction
$(I)\ [I/b]$	Substitute I for b
$\lambda x.x \stackrel{?}{=} I$	β reduction
$\lambda x.x = \lambda x.x$	α convert I

4. Given the definition of Church numerals, what does $(\bar{m}\ \bar{n})$ do when $\bar{m}$ and $\bar{n}$ are Church numerals?

To determine the output of $(\bar{m}\ \bar{n})$, I will solve several example problems to determine the pattern for the outputs. In the work below, the steps for substitution and β reduction are merged into 1 step for brevity.

4.A. $(\bar{2}\ \bar{3})$

Expression	Explanation
$(\bar{2}\ \bar{3})$	Initial expression
$(\lambda fx.(f\ (f\ x)))(\bar{3})$	Definition of Church numerals
$\lambda x.(\bar{3}\ (\bar{3}\ x))$	β reduction of f
$\lambda x.(\bar{3}\ ((\lambda gy.(g\ (g\ (g\ y))))\ x))$	Definition of Church numerals
$\lambda x.(\bar{3}\ \lambda y.(x\ (x\ (x\ y))))$	β reduction of g
$\lambda x.((\lambda hz.(h\ (h\ (h\ z))))\ (\lambda y.(x\ (x\ (x\ y)))))$	Definition of Church numerals
$\lambda x.\lambda z.(A\ (A\ (A\ z)))$	β reduction of h and let $A = \lambda y.(x\ (x\ (x\ y)))$
$\lambda x.\lambda z.(A\ (A\ ((\lambda y.(x\ (x\ (x\ y))))\ z)))$	Expand A
$\lambda x.\lambda z.(A\ (A\ (x\ (x\ (x\ z)))))$	β reduction of y
$\lambda x.\lambda z.(A\ ((\lambda y.(x\ (x\ (x\ y))))\ (x\ (x\ (x\ z)))))$	Expand A
$\lambda x.\lambda z.(A\ (x\ (x\ (x\ (x\ (x\ (x\ z)))))))$	β reduction of y
$\lambda x.\lambda z.((\lambda y.(x\ (x\ (x\ y))))\ (x\ (x\ (x\ (x\ (x\ (x\ z)))))))$	Expand A
$\lambda x.\lambda z.(x\ (x\ (x\ (x\ (x\ (x\ (x\ (x\ (x\ z)))))))))$	β reduction of y. This is equal to the Church numeral $\bar{9}$, which is $\overline{3^2}$ or $\overline{n^m}$.

4.B. $(\bar{3}\ \bar{2})$

Based on the result of $(\bar{2}\ \bar{3})$, I would expect $(\bar{3}\ \bar{2})$ to be $\bar{8}$, which is $\overline{2^3}$.

$(\bar{3}\ \bar{2})$	Initial expression
$((\lambda fx.(f\ (f\ (f\ x))))\ \bar{2})$	Definition of Church numerals
$\lambda x.(\bar{2}\ (\bar{2}\ (\bar{2}\ x)))$	β reduction of f
$\lambda x.(\bar{2}\ (\bar{2}\ ((\lambda gy.(g\ (g\ y)))\ x)))$	Definition of Church numerals
$\lambda x.(\bar{2}\ (\bar{2}\ (\lambda y.(x\ (x\ y)))))$	β reduction of g
$\lambda x.(\bar{2}\ ((\lambda hz.(h\ (h\ z)))\ (\lambda y.(x\ (x\ y)))))$	Definition of Church numerals
$\lambda x.(\bar{2}\ (\lambda z.(A\ (A\ z))))$	β reduction of h and let $A = \lambda y.(x\ (x\ y))$
$\lambda x.(\bar{2}\ (\lambda z.(A\ ((\lambda y.(x\ (x\ y)))\ z))))$	Expand A
$\lambda x.(\bar{2}\ (\lambda z.(A\ (x\ (x\ z)))))$	β reduction of y
$\lambda x.(\bar{2}\ (\lambda z.((\lambda y.(x\ (x\ y)))\ (x\ (x\ z)))))$	Expand A
$\lambda x.(\bar{2}\ (\lambda z.(x\ (x\ (x\ (x\ z))))))$	β reduction of y
$\lambda x.((\lambda gy.(g\ (g\ y)))\ (\lambda z.(x\ (x\ (x\ (x\ z))))))$	Definition of Church numerals
$\lambda x.(\lambda y.(B\ (B\ y)))$	β reduction of g and let $B = \lambda z.(x\ (x\ (x\ (x\ z))))$
$\lambda x.(\lambda y.(B\ ((\lambda z.(x\ (x\ (x\ (x\ z)))))\ y)))$	Expand B
$\lambda x.(\lambda y.(B\ (x\ (x\ (x\ (x\ y))))))$	β reduction of z
$\lambda x.\lambda y.((\lambda z.(x\ (x\ (x\ (x\ z)))))\ ((x\ (x\ (x\ (x\ y))))))$	Expand B
$\lambda x.\lambda y.(x\ (x\ (x\ (x\ (x\ (x\ (x\ (x\ y))))))))$	β reduction of z. This is equal to the Church numeral $\bar{8}$, which is $\overline{2^3}$ or $\overline{n^m}$.

4.C. $(\bar{1}\ \bar{2})$

Since both examples thus far have matched the pattern of $\overline{n^m}$, I would expect math to not be broken here as $(\bar{1}\ \bar{2})$ should equal $\bar{2}$.

$(\bar{1}\ \bar{2})$	Initial expression
$((\lambda fx.(f\ x))\ \bar{2})$	Definition of Church numerals
$\lambda x.(\bar{2}\ x)$	β reduction of f
$\lambda x.((\lambda gy.(g\ (g\ y)))\ x)$	Definition of Church numerals
$\lambda x.(\lambda y.(x\ (x\ y)))$	β reduction of g. This is equal to the Church numeral $\bar{2}$, which is $\overline{2^1}$ or $\overline{n^m}$.

4.D. $(\bar{0}\ \bar{2})$

The final check to ensure $(\bar{m}\ \bar{n})$ is rasing n to the power of m is to make sure that anything raised to the power of 0 is equal to 1. Thus, I would expect $(\bar{0}\ \bar{2})$ to equal $\bar{1}$.

$(\bar{0}\ \bar{2})$	Initial expression
$((\lambda fx.x)\ \bar{2})$	Definition of Church numerals
$\lambda x.x$	β reduction of f. There are no further reductions that can be done as this is the I combinator. Thus, the pattern of $\overline{n^m}$ is not satisfied when $m = 0$.

4.E. Final Results

Overall, given 2 Church numerals, $\bar{m}$ and $\bar{n}$, the expression $(\bar{m}\ \bar{n})$ will evaluate as one of the following expressions based on the value of m. Note: Subscripts for f denote count as they are all representing the same variable f.

$$(\bar{m}\ \bar{n}) = \begin{cases} \overline{n^m} = \lambda fx.(f_1\ (f_2\ (f_3\ \cdots\ (f_{n^m-1}\ (f_{n^m}\ x))\cdots))), & m > 0 \\ I = \lambda x.x, & m = 0 \end{cases}$$