MH4920 Chp 1 Hw

1.31 Consider estimation in Linear Model $Y = b_1X_1 + b_2X_2 + \varepsilon$, $0 \le b_1, b_2 \le 1$ for sample $(Y_1, X_{11}, X_{21}), \ldots, (Y_n, X_{1n}, X_{2n})$. Errors ε_i are independent and distributed to $N(0,1)$. A noninformative prior is $\pi(b_1, b_2) = \mathbb{I}_{[0,1]}(b_1)\,\mathbb{I}_{[0,1]}(b_2)$.

a. Show that posterior means are given by $i = 1, 2$

$$E^{\pi}(b_i \mid y_1 \ldots y_n) = \frac{\int_0^1\int_0^1 b_i \prod_{j=1}^n \varphi(y_j - b_1X_{1j} - b_2X_{2j})\,db_1\,db_2}{\int_0^1\int_0^1 \prod_{j=1}^n \varphi(y_j - b_1X_{1j} - b_2X_{2j})\,db_1\,db_2}$$

, φ density standard normal.

Soln: Given $\varepsilon_i \sim N(0,1)$, $y_j - b_{1j}x_{1j} - b_{2j}x_{2j} \sim N(0,1)$.

$E^{\pi}(b_i \mid y_1 \ldots y_n) = \int b_i\,\pi(b_1, b_2 \mid (x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}))\,db_1\,db_2$

Since $\pi(b_1, b_2 \mid (x_{11}, x_{21}) \ldots (x_{1n}, x_{2n})) \propto \pi(b_1, b_2)\,f((x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}) \mid b_1, b_2)$, WLOG

$\int b_i\,\pi(b_1, b_2 \mid (x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}))\,db_1\,db_2 = \int_0^1\int_0^1 b_i\,k\,\pi(b_1, b_2)\,f((x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}) \mid b_1, b_2)\,db_1\,db_2$

$= k\int_0^1\int_0^1 b_i\,f((x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}) \mid b_1, b_2)\,\pi(b_1, b_2)\,db_1\,db_2$

$$= \frac{\int_0^1\int_0^1 b_i\,f((x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}) \mid b_1, b_2)\,\pi(b_1, b_2)\,db_1\,db_2}{\int_0^1\int_0^1 f((x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}) \mid b_1, b_2)\,\pi(b_1, b_2)\,db_1\,db_2}$$

, where k is marginal factor.

$$= \frac{\int_0^1\int_0^1 b_i \prod_{j=1}^n \varphi(y_j - b_{1j}x_{1j} - b_{2j}x_{2j})\,\pi(b_1, b_2)\,db_1\,db_2}{\int_0^1\int_0^1 \prod_{j=1}^n \varphi(y_j - b_{1j}x_{1j} - b_{2j}x_{2j})\,\pi(b_1, b_2)\,db_1\,db_2}$$

$$= \frac{\int_0^1\int_0^1 b_i \prod_{j=1}^n \varphi(y_j - b_{1j}x_{1j} - b_{2j}x_{2j})\,db_1\,db_2}{\int_0^1\int_0^1 \prod_{j=1}^n \varphi(y_j - b_{1j}x_{1j} - b_{2j}x_{2j})\,db_1\,db_2}$$

for noninformative priors (Proven).

b. Show an equivalent expression is $\delta_i^{\pi}(y_1 \ldots y_n) = \dfrac{E^{\pi}(b_i\,\mathbb{I}_{[0,1]^2}(b_1, b_2) \mid y_1 \ldots y_n)}{P^{\pi}((b_1, b_2) \in [0,1]^2 \mid y_1 \ldots y_n)}$, where the right-hand term is computed under the distribution $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}, (X^tX)^{-1}\right)$, with $(\hat{b}_1, \hat{b}_2)$ the unconstrained least square estimator of (b_1, b_2) and $X = \begin{pmatrix} x_{11} & x_{21} \\ \vdots & \vdots \\ x_{1n} & x_{2n} \end{pmatrix}$.

Soln: Given $Y = b_1X_1 + b_2X_2 + \varepsilon$, $\delta_i^{\pi}(y_1 \ldots y_n) = \arg\min E(L(\delta, b_1, b_2) \mid y_1 \ldots y_n)$

$= \min_{\delta} \int L(\delta, b_1, b_2)\,\pi(b_1, b_2 \mid y_1 \ldots y_n)\,db_1\,db_2$

For $L(\delta, b_1, b_2)$ with $(\hat{b}_1, \hat{b}_2)$ as the unconstrained Least Square Estimator, we want to minimise ε_i, $\sum_{j=1}^n (y_j - b_{1j}x_{1j} - b_2x_{2j})^2 = \sum_{j=1}^n (\varepsilon_j)^2$. In vector form, minimise $\|y^t - Xb\|^2$, given $y = (y_1 \ldots y_n)$, $X = \begin{pmatrix} x_{11} & x_{21} \\ \vdots & \vdots \\ x_{1n} & x_{2n} \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. That is $L(\delta, b_1, b_2) = \|y^t - Xb\|^2$.

As loss function is quadratic loss, $\delta^{\pi}(y_1 \ldots y_n) = E^{\pi}(b_i \mid y_1 \ldots y_n)$, where b_i is independent parameter consideration given that it is under bivariate normal distribution $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}, (X^TX)^{-1}\right)$.

By Conditional Expectation, $E^{\pi}(b_i \mid y_1 \ldots y_n) = \int b_i * \pi(b_1, b_2 \mid (x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}))\,db_1\,db_2$

$$= \frac{\int b_i * \pi(b_i\,\mathbb{I}_{[0,1]^2}(b_1, b_2) \mid (x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}))\,db_1\,db_2}{\pi(b_1, b_2 \mid (x_{11}, x_{21}) \ldots (x_{1n}, x_{2n}))} = \frac{E^{\pi}(b_i\,\mathbb{I}_{[0,1]^2}(b_1, b_2) \mid y_1 \ldots y_n)}{P^{\pi}((b_1, b_2) \in [0,1]^2 \mid y_1 \ldots y_n)}$$

Generalisation is $E(b_i \mid \text{under joint}) = \dfrac{E(b_i \text{ under joint dist.})}{P(b_i \text{ under joint dist.})}$, $\int \sigma_{b_1}\sigma_{b_2} \neq 0$.

$$\therefore \delta^{\pi}(y_1 \ldots y_n) = \frac{E^{\pi}(b_i\,\mathbb{I}_{[0,1]^2}(b_1, b_2) \mid y_1 \ldots y_n)}{P^{\pi}((b_1, b_2) \in [0,1]^2 \mid y_1 \ldots y_n)} \quad \text{(proven).}$$