2.35 There is a direct generalisation Corollary 2.17 that allows the proposal density to change at each iteration.

Algorithm A.16 — Generalised Accept-Reject

At iteration $i\ (i \geq 1)$
1. Generate $X_i \sim g_i$ and $U_i \sim U_{[0,1]}$, independently
2. If $U_i \leq \varepsilon_i f(X_i)/g_i(X_i)$, accept $X_i \sim f$;
3. otherwise, move to iteration $i+1$.

a. Let Z denote the random variable that is output by this algorithm. Show that Z has the cdf $P(Z \leq z) = \sum_{i=1}^{\infty} \varepsilon_i \prod_{j=1}^{i-1} (1-\varepsilon_j) \int_{-\infty}^{z} f(x)\,dx$

From Corollary 2.17 from Qn 2.34 part a, one inference is that $P(\text{accept}) = \frac{1}{M}$ for the Random Variable Z generated. Thus a comparison between different simulations based on different normalised instrumental densities $g_1, g_2, g_3, \dots$ can be undertaken through the comparison of the respective bounds $M_1, M_2, M_3, \dots$. The ideal algorithm of optimising the choice of g is iterating $g_1, g_2, g_3, \dots$ that yields smallest bound M_i.

From Algorithm A.16, accept $Z = X$ if $U_i \leq f(X_i)/M_i g_i(X_i) = \varepsilon_i f(X_i)/g_i(X_i)$ for $M_i \geq 1$, $0 < \varepsilon_i = \frac{1}{M_i} \leq 1$ $\left(\lim_{M \to \infty} \frac{1}{M} = 0\right)$. For some $U_i \sim U_{[0,1]}$, $\varepsilon_i \propto U_i$ wrt. acceptance ratio $\frac{f}{g}$. Hence $\varepsilon_i \sim U_{[0,1]}$ and WLOG, at iteration i^{th} $P(\text{accept}) = \varepsilon_i$ while $P(\text{reject}) = 1 - \varepsilon_i$ which makes up the Accept-Reject Algorithm probability outcomes.

Consider the outcomes Pass — P and Fail — F and the probability sample space: $\Omega = \{P, FP, FFP, \dots\}$ for all iterations in A.16 algorithm. The σ-algebra $\mathcal{F} = \{\phi, \{P, FP\}, \{P, FP, FFP\}, \dots, \Omega\}$ be the event space with $A_i \subseteq \Omega$ be a countable collection of pairwise disjoint sets in Ω. With the set function $p: \Omega \mapsto [0,1]$, by Countable Addivity $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \varepsilon_1 + \varepsilon_2(1-\varepsilon_1) + \varepsilon_3(1-\varepsilon_2)(1-\varepsilon_1) + \dots = \sum_{i=1}^{\infty} \varepsilon_i \prod_{j=1}^{i-1} (1-\varepsilon_j)$.
A simple function on the cdf of RV. Z can be constructed $H = \sum_{i=1}^{\infty} \int_{-\infty}^{z} f(x)\,dx \cdot \mathbb{I}_{A_i}$, where $\mathbb{I}_{A_i}$ is indicator function of successful outcome at iteration i^{th}.

By Lebesgue Integral over the set function p,

$$\int \left(\sum_{i=1}^{\infty} \int_{-\infty}^{z} f(x)\,dx \cdot \mathbb{I}_{A_i} \right) dp = \sum_{i=1}^{\infty} \int_{-\infty}^{z} f(x)\,dx \int \mathbb{I}_{A_i}\, dp = \sum_{i=1}^{\infty} \int_{-\infty}^{z} f(x)\,dx\, P(A_i) = \sum_{i=1}^{\infty} P(A_i) \int_{-\infty}^{z} f(x)\,dx$$

$$\therefore P(Z \leq z) = \int H\, dp = \int \left(\sum_{i=1}^{\infty} \int_{-\infty}^{z} f(x)\,dx \cdot \mathbb{I}_{A_i} \right) dp = \sum_{i=1}^{\infty} P(A_i) \int_{-\infty}^{z} f(x)\,dx = \sum_{i=1}^{\infty} \varepsilon_i \prod_{j=1}^{i-1} (1-\varepsilon_j) \int_{-\infty}^{z} f(x)\,dx$$

(shown) *