b. Show that $\sum_{i=1}^{\infty} \varepsilon_i \prod_{j=1}^{i-1}(1-\varepsilon_j) = 1$ if and only if $\sum_{i=1}^{\infty} \log(1-\varepsilon_i)$ diverges.

The ideal algorithm of optimising choice of g is iterating $g_1, g_2, g_3, \dots$ that yields smallest bound M_i. For $0 < \varepsilon_i = \frac{1}{m_i} < 1$ and $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \dots$, $\sum \varepsilon_i$ is divergent. Also from part a., recall that the probability sample space $\Omega = \{P, FP, FFP, \dots\}$ with the countable collection of pairwise disjoint sets in Ω: $A_i \subseteq \Omega$ s.t. $A_i \cap A_j = \phi$.
$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = P(\Omega) = \sum_{i=1}^{\infty} \varepsilon_i \prod_{j=1}^{i-1}(1-\varepsilon_j) = 1$.

($\Rightarrow$) By Borel-Cantelli Lemma, if $\sum_{i=1}^{\infty} \varepsilon_i \prod_{j=1}^{i-1}(1-\varepsilon_j) = 1 < \infty$, the $P(\lim_{i\to\infty} \sup\{\varepsilon_i \prod_{j=1}^{i-1}(1-\varepsilon_j)\}) = 0$
This implies that the probability infinitely many iterations occuring is 0 and that there exists $N < \infty$ s.t. $\varepsilon_n \prod_{j=1}^{n-1}(1-\varepsilon_j)$ does not happen for $n > N$.

Using the inequality $1 - x \leq e^{-x}$ for $0 \leq x < 1$, the partial products $0 < G_n = \prod_{i=1}^{n}(1-\varepsilon_i) \leq \prod_{i=1}^{n} e^{-\varepsilon_i} = \exp(-\sum_{i=1}^{n} \varepsilon_i)$. $\therefore$ the partial product $\prod_{i=1}^{n}(1-\varepsilon_i)$ is bounded above by the partial sum $\sum_{i=1}^{n} \varepsilon_i$. WLOG $\lim_{n\to\infty} \sum_{i=1}^{n} \varepsilon_i = \sum \varepsilon_i$ is divergent for arbitrarily large n.

Using Limit Comparison Test,
$\lim_{i\to\infty} \frac{\log(1-\varepsilon_i)}{-\varepsilon_i} = \lim_{i\to\infty} \frac{-\log\left(\frac{1}{1-\varepsilon_i}\right)}{-\varepsilon_i} = \lim_{i\to\infty} \frac{\log\left(\frac{1}{1-\varepsilon_i}\right)}{\varepsilon_i} = c > 0$, for $a_i = \log\frac{1}{1-\varepsilon_i}$, $b_i = \varepsilon_i$ s.t. $a_i, b_i \geq 0$ and $0 < \varepsilon_i < 1$, $\sum \log(1-\varepsilon_i)$ and $-\sum \varepsilon_i$ are either both convergent or divergent.

Hence we can establish the Lemma $\sum_{i=1}^{\infty} \log(1-\varepsilon_i)$ and $\sum_{i=1}^{\infty} \varepsilon_i$ diverge if and only if the sequence of partial product $\prod_{i=1}^{n}(1-\varepsilon_i)$ converges to 0 ($\lim_{n\to\infty} G_n = 0$), using Comparison Test from the inequality $1 - x \leq e^{-x}$.
And $\sum_{i=1}^{\infty} \log(1-\varepsilon_i)$ diverges.

($\Leftarrow$) For $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \dots$, $(1-\varepsilon_1) > (1-\varepsilon_1)(1-\varepsilon_2) > \dots$, where $0 < \varepsilon_i < 1$, ε_i is monotone and bounded. By Comparison Test, since $0 < \prod_{j=1}^{i-1}(1-\varepsilon_j) \leq \prod_{j=1}^{i-1} e^{-\varepsilon_j}$ for $\forall j \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \prod_{j=1}^{i-1} e^{-\varepsilon_j}$ converges, $\therefore \sum_{i=1}^{\infty} \prod_{j=1}^{i-1}(1-\varepsilon_j)$ converges. Satisfying Abel's criteria $\sum a_n$ convergent, $\{b_n\}$ is monotone and bounded, by Abel's Test $\sum_{i=1}^{\infty} \varepsilon_i \prod_{j=1}^{i-1}(1-\varepsilon_j) = 1$ is convergent. (shown) *

c. Give examples of sequences ε_i that satisfy, and do not satisfy the requirement of part b.

- For yielding smallest bound M_i, $\varepsilon_i = \frac{1}{m_i}$ for $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \dots \Rightarrow \sum \varepsilon_i$ diverge
- For yielding smallest bound M_i, $\varepsilon_i = 1 - \frac{1}{m_i}$ for $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$. Since $\varepsilon_i = 1 - \frac{1}{m_i} \leq e^{-\frac{1}{m_i}}$ and from power series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ convergent for $\forall x \in \mathbb{R}$, by Comparison Test $\sum \varepsilon_i$ converges & do not satisfy requirement.