Appendix A: Calculations

Decision Variables

Name	Variable	Units
Allocated water to urban users	Q_u	ac-ft
Allocated water to wetlands	Q_w	ac-ft

Constraints

Name	Value	Units
Available water volume in river	320	ac-ft

$V_{available} = Q_u + Q_w = 320$ acre feet

Objective Maximize total benefits

Objective Function

Total benefits include the sum of financial benefits from the wetland water use and payments from urban water users.

$B(Q_u, Q_w) = (800Q_u) + (2200Q_w^{0.8})$

Formulate the Lagrangian and find the partials of each decision variable

$$L(Q_u, Q_w, \lambda) = B + \lambda(Constraints)$$

$$L(Q_u, Q_w, \lambda) = 800Q_u + 2200Q_w^{0.8} + \lambda(Q_u + Q_w - 320)$$

$$\nabla L = \begin{bmatrix} \frac{\partial L}{\partial Q_u} = 800 + 0 + \lambda \\ \frac{\partial L}{\partial Q_w} = 0 + 0.8\left(2200Q_w^{-0.2}\right) + \lambda \\ \frac{\partial L}{\partial \lambda} = 0 + 0 + Q_u + Q_w - 320 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for first order conditions

Excel Solver

Parameter	Value	Units
Q_u	268.46	ac-ft
Q_w	51.54	ac-ft
λ	-800	
Benefits ($)	$266,307.26	

First Order Condition Equations	Value
Partial Qu	0.00000
Partial Qw	0.00000
Partial λ	0.00000

Analytical Hand Solution

- $800 + \lambda = 0 \Rightarrow \lambda = -800$
- $1760\, Q_w^{-0.2} + \lambda = 0$

  $(Q_w^{-0.2})^{-5} = \left(\frac{-\lambda}{1760}\right)^{-5}$

  $Q_w = \left(\frac{800}{1760}\right)^{-5} = \frac{161051}{3125} \approx 51.54$ ac-ft
- $Q_u + Q_w - 320 = 0$

  $Q_u + 51.54 - 320 = 0$

  $Q_u = 268.46$ ac-ft

What if 10 more ac-ft of water were available in the river?

Parameter	Value	Units
Q_u	278.46	ac-ft
Q_w	51.54	ac-ft
λ	-800	
Benefits ($)	$274,307.26	

First Order Condition Equations	Value
Partial Qu	0.00000
Partial Qw	0.00000
Partial λ	0.00000

$V_{available} = Q_u + Q_w = 330$ acre feet

$$L(Q_u, Q_w, \lambda) = B + \lambda(Constraints)$$
$$L(Q_u, Q_w, \lambda) = 800Q_u + 2200Q_w^{0.8} + \lambda(Q_u + Q_w - 330)$$

$$\nabla L = \begin{bmatrix} \frac{\partial L}{\partial Q_u} = 800 + 0 + \lambda \\ \frac{\partial L}{\partial Q_w} = 0 + 0.8\left(2200Q_w^{-0.2}\right) + \lambda \\ \frac{\partial L}{\partial \lambda} = 0 + 0 + Q_u + Q_w - 330 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- $800 + \lambda = 0 \Rightarrow \lambda = -800$
- $1760\, Q_w^{-0.2} + \lambda = 0$
  $(Q_w^{-0.2})^{-5} = \left(\frac{-\lambda}{1760}\right)^{-5}$
  $Q_w = \left(\frac{800}{1760}\right)^{-5} = \frac{161051}{3125} \approx 51.54$ ac-ft
- $Q_u + Q_w - 330 = 0$
  $Q_u + 51.54 - 330 = 0$
  $Q_u = 278.46$ ac-ft