# HW 2 - Graphical and Simplex Solutions

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|---|
| Format Key |
| Notes/Comments |
| Inputs/Givens |
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# Problem 2.3

**Given**

An aqueduct constructed to supply water to industrial users has an excess capacity in the months of June, July, and August of 14,000 ac-ft, 18,000 ac-ft, and 6,000 ac-ft, respectively. It is proposed to develop not more than 10,000 acres of new land by utilizing the excess aqueduct capacity for irrigation water deliveries. Two crops, hay and grain, are to be grown. Their monthly water requirements and expected net returns are given in the following table.

| | Monthly Water Requirement (ac-ft/acre) | | | |
|---|---|---|---|---|
| | June | July | August | Return, $/acre |
| Hay | 2 | 1 | 1 | 100 |
| Grain | 1 | 2 | 0 | 120 |

**Find**

A) Formulate a linear proram to optimize the irrigation development. Clearly define all the variables used and give their units.
B) Solve graphically
C) Solve using a simplex method

**Solution**

**A)** **Formulate the problem**

| Decision variables | | |
|---|---|---|
| Symbol | Units | Description |
| $A_h$ | acre | Area of land allocated to hay crops |
| $A_g$ | acre | Area of land allocated to grain crops |

**Objective Function**

Maximize the return ($) produced by cultivating hay and grain on not more than 10,000 acres of land using only excess water
Or, for simplex methods, minimize the negative return as shown.

$$Max\ Z = \frac{\$100}{acre}(A_h) + \frac{\$120}{acre}(A_g)$$

$$Min\ -Z = -\frac{\$100}{acre}(A_h) - \frac{\$120}{acre}(A_g)$$

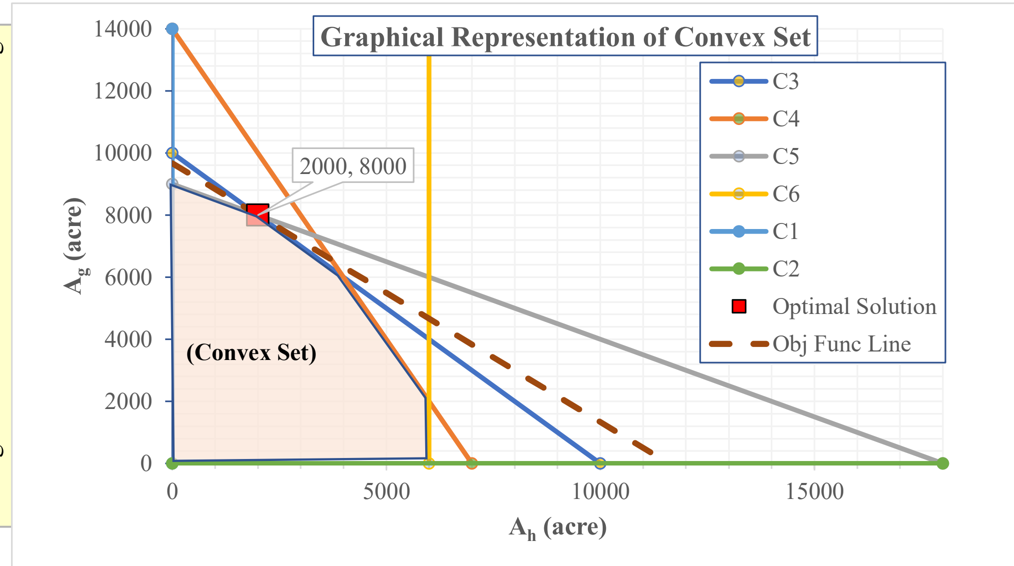
| Constraint Equations | |
|---|---|
| **1** | Area of hay cannot be less than 0 acres |
| **2** | Area of grain cnanot be less than 0 acres |
| **3** | Combined area of hay and grain cannot exceed 10,000 acres |
| **4** | Total water us by crops in June cannot exceed 14,000 acre-ft |
| **5** | Total water us by crops in July cannot exceed 18,000 acre-ft |
| **6** | Total water use by crops in August cannot exceed 6,000 acre-ft |

***Constraint Equations - Graphical***

1. $A_h \geq 0$
2. $A_g \geq 0$
3. $A_h + A_g \leq 10{,}000\ acres$
4. $2\frac{acft}{ac}(A_h) + 1\frac{acft}{ac}(A_g) \leq 14{,}000\ acft$
5. $1\frac{acft}{ac}(A_h) + 2\frac{acft}{ac}(A_g) \leq 18{,}000\ acft$
6. $1\frac{acft}{ac}(A_h) + 0(A_g) \leq 6{,}000\ acft$

| B) | Solve Graphically for extremes (acres) | |
|---|---|---|
| **Constraint** | **$A_h$** | **$A_g$** |
| C1 | 0 | 0 |
| C1 | 0 | 14000 |
| C2 | 0 | 0 |
| C2 | 18000 | 0 |
| C3 | 0 | 10000 |
| C3 | 10000 | 0 |
| C4 | 0 | 14000 |
| C4 | 7000 | 0 |
| C5 | 0 | 9000 |
| C5 | 18000 | 0 |
| C6 | 6000 | 14000 |
| C6 | 6000 | 0 |

These points are dummy points to produce the graph shown at the right. They were found by setting one decision variable to 0 and solving.




$$\frac{100}{120}A_h + A_g = return$$
$$A_g = return - \frac{5}{6}A_h$$

**For optimal solution, objective function lies along the line**

$$A_g = \frac{\$1.16 \times 10^6}{\$120/acre} - \frac{5}{6}A_h = \frac{29000}{3} - \frac{5}{6}A_h$$

Identify all of the vertices of the feasible decision space and compute the return for each of the feasible options

Solve each vertex point by linear algebra (MMULT & MINVERSE functions; see formulas below)

| Equations | A Matrix Coefficients $A_h$ | A Matrix Coefficients $A_g$ | B Vector (acft or acre) | X Vector (Acres) | Point |
|---|---|---|---|---|---|
| C5 | 1 | 2 | 18000 | 2000 | 3 |
| C3 | 1 | 1 | 10000 | 8000 | |
| C3 | 1 | 1 | 10000 | 4000 | 4 |
| C4 | 2 | 1 | 14000 | 6000 | |
| C4 | 2 | 1 | 14000 | 6000 | 5 |
| C6 | 1 | 0 | 6000 | 2000 | |

| Coeff / Point | \$100.00 / $A_h$ | \$120.00 / $A_g$ | Total Return (\$) |
|---|---|---|---|
| 1 | 0 | 0 | \$0.00 |
| 2 | 0 | 9000 | \$1,080,000.00 |
| **3** | **2000** | **8000** | **\$1,160,000.00** |
| 4 | 4000 | 6000 | \$1,120,000.00 |
| 5 | 6000 | 2000 | \$840,000.00 |
| 6 | 6000 | 0 | \$600,000.00 |

=MMULT(MINVERSE(B58:C59),D58:D59)

=MMULT(MINVERSE(B60:C61),D60:D61)

=MMULT(MINVERSE(B62:C63),D62:D63)

| Plot Objective Function Slope through optimal solution | | | | |
|---|---|---|---|---|
| $A_h$ | 0 | 2000 | 5000 | 11600 |
| $A_g$ | 9666.667 | 8000 | 5500 | 0 |

**The optimum solution is to plant 2000 acres of hay and 8000 acres of grain for a profit of \$1.16 million.**

**C) Solve using the Simplex Method**

The simplex method requires equity constraints, positive variable values,and a function to minimize. Modify the inequality constraints to equity consraints with slack variables.

$$Max\ Z = \frac{\$100}{acre}(A_h) + \frac{\$120}{acre}(A_g)$$

$$Min\ -Z = -\frac{\$100}{acre}(A_h) - \frac{\$120}{acre}(A_g)$$

***Constraint Equations - Simplex***

1. $A_h \geq 0: Implied\ in\ simplex$
2. $A_g \geq 0: Implied\ in\ simplex$
3. $A_h + A_g + S_3 = 10{,}000\ acres$
4. $2\frac{acft}{ac}(A_h) + 1\frac{acft}{ac}(A_g) + S_4 = 14{,}000\ acft$
5. $1\frac{acft}{ac}(A_h) + 2\frac{acft}{ac}(A_g) + S_5 = 18{,}000\ acft$
6. $1\frac{acft}{ac}(A_h) + 0(A_g) + S_6 = 6{,}000\ acft$

| **Constraint Equations** | |
|---|---|
| **1** | Area of hay cannot be less than 0 acres |
| **2** | Area of grain cnanot be less than 0 acres |
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| **4** | Total water us by crops in June cannot exceed 14,000 acre-ft |
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| **6** | Total water use by crops in August cannot exceed 6,000 acre-ft |

**Slack Variables**

| Symbol | Units | Description |
|---|---|---|
| $S_3$ | acres | Unused land area |
| $S_4$ | acre-ft | Unused available water from aquedcut excess in June |
| $S_5$ | acre-ft | Unused available water from aquedcut excess in July |
| $S_6$ | acre-ft | Unused available water from aqueduct excess in August |

Set up the first Tableau with the basic feasible solution containing the **initial basis. Initial basis sets all decision variables to 0, slack values at maximum**.

**Tableau 1**

| **Item** | **$A_h$** | **$A_g$** | **$S_3$** | **$S_4$** | **$S_5$** | **$S_6$** | **b** | **-Z** | **Most - Obj Func. Coeff** | **Pivot Variable** | **Binding: $b_i/a_{ij}$** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Value | **0** | **0** | **10000** | **14000** | **18000** | **6000** | | | | | |
| Obj. Z Coeff | -100 | -120 | 0 | 0 | 0 | 0 | | $0.00 | -120 | 2 | |
| C3 | 1 | 1 | 1 | 0 | 0 | 0 | 10000 | | Continue | | 10000 |
| C4 | 2 | 1 | 0 | 1 | 0 | 0 | 14000 | | | | 14000 |
| C5 | 1 | 2 | 0 | 0 | 1 | 0 | 18000 | | | | 9000 |
| C6 | 1 | 0 | 0 | 0 | 0 | 1 | 6000 | | | | #DIV/0! |

The basic solution shows $S_5$, the unused available water from the aqueduct excess in July, is most binding. $S_5$ will leave the basis.

The most negative objective function coefficient is -120 for the area planted with grain ($A_g$). Pivot the next tableau on this variable column in constraint 5's row.

Operations: R[C5]=R[C5]/2; R[Obj]=R[Obj]+120*R[C5]; R[C3]=R[C3]-R[C5]; R[C4]=R[C4]-R[C5]; R[C6]=R[C6]

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 108 | **Tableau 2** | | | | | | | | | | | |
| 109 | **Item** | **$A_h$** | **$A_g$** | **$S_3$** | **$S_4$** | **$S_5$** | **$S_6$** | **b** | **-Z** | **Most - Obj Func. Coeff** | **Pivot Variable** | **Binding: $b_i/a_{ij}$** |
| 110 | Value | 0 | 9000 | 1000 | 14000 | 0 | 6000 | | | | | |
| 111 | Obj. Z Coeff | -40 | 0 | 0 | 0 | 60 | 0 | | -$1,080,000.00 | -40 | 1 | |
| 112 | C3 | 0.5 | 0 | 1 | 0 | -0.5 | 0 | 1000 | | Continue | | 2000 |
| 113 | C4 | 1.5 | 0 | 0 | 1 | -0.5 | 0 | 5000 | | | | 3333.3333 |
| 114 | C5 | 0.5 | 1 | 0 | 0 | 0.5 | 0 | 9000 | | | | 18000 |
| 115 | C6 | 1 | 0 | 0 | 0 | 0 | 1 | 6000 | | | | 6000 |
| 116 | This iteration shows $S_3$, the unused land area, is next most binding. $S_3$ will leave the basis. | | | | | | | | | | | |
| 117 | The most negative objective function coefficient is -40 for the area planted with hay (Ah). Pivot the next tableau on this variable column in constraint 3's row. | | | | | | | | | | | |
| 118 | Operations: R[C3]=R[C3]/0.5; R[Obj]=R[Obj]+40*R[C3]; R[C4]=R[C4]-1.5*R[C3]; R[C5]=R[C5]-R[C3]; R[C6]=R[C6]-R[C3] | | | | | | | | | | | |
| 119 | **Tableau 3** | | | | | | | | | | | |
| 120 | **Item** | **$A_h$** | **$A_g$** | **$S_3$** | **$S_4$** | **$S_5$** | **$S_6$** | **b** | **-Z** | **Most - Obj Func. Coeff** | **Pivot Variable** | **Binding: $b_i/a_{ij}$** |
| 121 | Value | 2000 | 8000 | 0 | 2000 | 0 | 4000 | | | | | |
| 122 | Obj. Z Coeff | 0 | 0 | 80 | 0 | 20 | 0 | | -$1,160,000.00 | 0 | 1 | |
| 123 | C3 | 1 | 0 | 2 | 0 | -1 | 0 | 2000 | | STOP | | #DIV/0! |
| 124 | C4 | 0 | 0 | -3 | 1 | 1 | 0 | 2000 | | | | #DIV/0! |
| 125 | C5 | 0 | 1 | -1 | 0 | 1 | 0 | 8000 | | | | 8000 |
| 126 | C6 | 0 | 0 | -2 | 0 | 1 | 1 | 4000 | | | | #DIV/0! |
| 127 | The identity matrix is found in the real decision variables and no objective function cefficients are negative. Simplex algorithm has arrived at a solution. | | | | | | | | | | | |
| 128 | **Slack Variabl** | **Status** | **Slack Value** | **Shadow Valu** | **Interpretation** | | | | | | | |
| 129 | $S_3$ | Binding | 0 | 80 | Profit will increase $80/add. Acre of land available | | | | | **The optimum solution is to plant 2000 acres of hay and 8000 acres of grain for a profit of $1.16 million.** | | |
| 130 | $S_4$ | Non-binding | 2000 | 0 | 2000 ac-ft of excess water will remain in June, no profit increas | | | | | | | |
| 131 | $S_5$ | Binding | 0 | 20 | Profit increases $20/add. ac-ft of excess water from aqueduct | | | | | | | |
| 132 | $S_6$ | Non-binding | 4000 | 0 | 4000 ac-ft of excess water will remain in August, no profit incre | | | | | | | |