1 Introduction

Raft is a consensus algorithm designed for understandability and ease of implementation compared to the Paxos algorithm, and thus, it is widely utilized in various products to resolve consensus issues.

In distributed systems, Raft requires odd number of servers to always maintain a quorum that remains functional and communicative in case of network partition faults. Practically, this means a Raft cluster size should be at least three to tolerate a single point of failure.

While the requirement for an odd cluster size isn't problematic for large-scale distributed systems, it can pose a challenge for budget-constrained customers who need to reach consensus with fewer servers. One solution to this issue is to substitute one server in the cluster with a cost-effective entity known as a witness. The witness acts as a tie-breaker, ensuring the maintenance of a quorum in case of an even split of servers due to network issues. Witness typically operates on low-configuration hardware to save costs, seldom participates in critical data paths, and only persists a minimal amount of data. In the Paxos algorithm, a witness can be implemented as shared storage, which is not only inexpensive but also widely available to most customers.

Efforts have been undertaken to incorporate witness into Raft algorithm. For example, Diego Ongaro's Ph.D. dissertation explored the reduction of server numbers by using a witness to store log entries for any failed server until recovery or replacement. Jehan-François Pâris and Darrell D. E. Long proposed a type of follower-only server witness that doesn't persist user data. TiKV implements a witness in a similar manner as described by Ongaro: a log-only entity that persists only raft logs without applying them, and supports switching between witness and non-witness servers.

However, all existing research and implementations necessitate a standalone server as a witness, adding to deployment complexity and reducing cost-efficiency compared to shared storage in the Paxos algorithm. Additionally, the witness must participate in log replication when it needs to conform to a quorum with other servers (when some servers are down), implying that its low-configuration hardware could potentially become a performance bottleneck in such scenarios.

2 Algorithm

In this section, we present the extended Raft algorithm, which is a variant of the Raft algorithm. It is designed for cluster with regular servers and at most one witness. The witness functions strictly as a follower server, implying that its role never transitions to candidate or leader. Furthermore, a witness only maintains a minimal set of metadata.

Our newly proposed algorithm minimizes data traffic and reduces the frequency of witness visits while preserving all key properties of the Raft algorithm. This implies that the extended Raft algorithm ensures that each of the following properties is consistently upheld.

- Election Safety
- Leader Append-only
- Log Matching
- Leader Completeness
- State Machine Safety

In the rest of this document, unless otherwise specified, the term 'server' will denote either a regular server or a witness. Consequently, the server set, denoted as *Server*, that constitutes a cluster could either be $RegularServer \cup witness$ or simply *RegularServer*, where *RegularServer* is the set of all regular servers.

2.1 Concepts and Definitions

Definition 1. *A leader's* ***replication set*** *is a subset of the cluster server set, with its cardinality matching that of the regular server set.*

Let *ReplicationSets* represent the set of all replication sets within a cluster. We have:

$$ReplicationSets \triangleq \begin{cases} \{Server\}, & witness \notin Server \\ \{Server \setminus \{x\} : x \in Server\}, & witness \in Server \end{cases}$$

A leader maintains its replication set based on its view of the servers and modifies it under specific conditions. A leader may alter its replication set multiple times within a single term. To uniquely identify each replication set, we introduce the notion of a 'subterm' as detailed below.

Definition 2. *A term is segmented into* SUBTERMS*, each begins with a replication set.*

Subterms are sequentially numbered using consecutive integers, beginning from 0 for each term. The leader retains its current subterm and increments it when the replication set is altered. Consequently, a leader's replication set remains static throughout its subterm.

2.2 States

Figure 2.1 illustrates the additional states introduced by the Extended Raft algorithm to accommodate the witness.

Each leader maintains three additional volatile variables: *replicationSet*, *currentSubterm*, and *witnessSubterm*. The *replicationSet* represents the leader's current replication set, initialized to *RegularServer*, and is modified by *AdjustReplicationSet* action. The *currentSubterm* represents the latest subterm of the leader, initialized to 0 and incremented upon a change in the leader's replication set. The *witnessSubterm* denotes the latest subterm during which the leader replicated its log entry to the witness.

In addition to *currentTerm* and *votedFor*, the witness also includes *witnessReplicationSet*, *witnessLastLogTerm*, and *witnessLastLogSubterm*. These represent the most recent replication set, log entry term, and log entry subterm that the leader sent to the witness. We will further discuss this in Section 2.3.

Beyond the additional states in the leader and witness, each log entry is also associated with the leader's current subterm number when it is appended to the leader's log (Figure 2.2). This subterm number is replicated and persisted alongside the log entry on all regular servers. We now denote a log entry as $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$, which is uniquely identified by *index* and *term*, and associated with *subterm*.

2.3 Log Replication

The leader replicates its log to regular servers in the exact same way as that in the Raft algorithm, and the regular servers handle the received log entries in the same way as well.

However, for the witness, log replication is performed differently, as shown in Figure **??**. In the Extended Raft algorithm, the leader sends an *AppendEntriesToWitnessRequest*

```
 The following variables are used only on leaders:
 The next entry to send to each follower.
VARIABLE nextIndex
 The latest entry that each follower has acknowledged is the same as the
 leader's. This is used to calculate commitIndex on the leader.
VARIABLE matchIndex
 Leader's replication set
VARIABLE replicationSet
 Leader's subterm number
VARIABLE currentSubterm
 The latest subterm that this leader has written to the witness
VARIABLE witnessSubterm
leaderVars ≜ ⟨nextIndex, matchIndex, elections,
              replicationSet, currentSubterm, witnessSubterm⟩

 The following variables are persisted only on witness:
 The latest replication set sent from leader
VARIABLE witnessReplicationSet
 The latest term of replicated log entry
VARIABLE witnessLastLogTerm
 The latest subterm of replicated log entry
VARIABLE witnessLastLogSubterm
witnessVars ≜ ⟨witnessReplicationSet,
               witnessLastLogTerm, witnessLastLogSubterm⟩
```

Figure 2.1: State

```
Leader i receives a client request to add v to the log.
$ClientRequest(i, v) \triangleq$
    $\land state[i] = Leader$
    $\land$ LET $entry \triangleq [term \mapsto currentTerm[i],$
                          $subterm \mapsto currentSubterm[i],$
                          $value \mapsto v]$
            $newLog \triangleq Append(log[i], entry)$
        IN $log' = [log$ EXCEPT $![i] = newLog]$
    $\land$ UNCHANGED $\langle messages, serverVars, candidateVars,$
                      $leaderVars, commitIndex, witnessVars\rangle$
```

Figure 2.2: Client request

to the witness, which includes the current replication and metadata of the log entry that satisfies the following conditions:

1. The log entry's term and subterm are equal to *currentTerm* and *currentSubterm*, respectively.

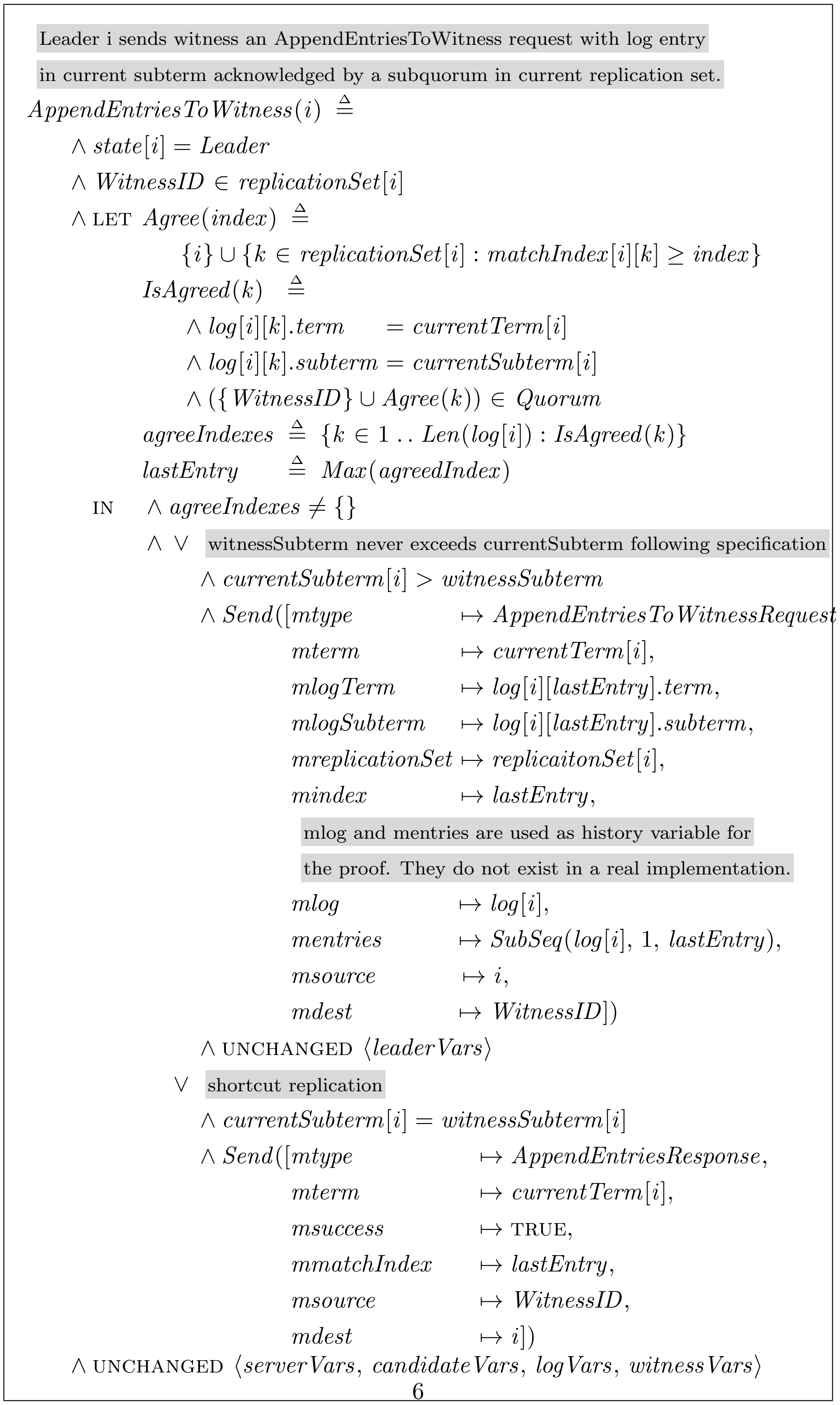
2. The leader's current replication set includes the witness.

3. The leader has received acknowledgments for the log entry from at least a subquorum (one server away from forming a quorum) in the leader's current replication set.

4. The leader has not received any acknowledgment from the witness during the current subterm.

The *AppendEntriesToWitness* can be considered as a special implementation of the *AppendEntries* action with batching of log prefix up to a specific log entry. This log entry belongs to the current subterm and has received subquorum acknowledgements from the current replication set. In addition to the message fields that also exist in *AppendEntriesRequest*, the dispatched message *AppendEntriesToWitnessReques* also contains the current replication set and the metadata of the last batched entries.

Upon receiving the request, the witness persists the replication set, as well as the term and subterm of the log entry, then responds to the leader with an *AppendEntriesResponse* message (Figure 2.4).



```
Leader i sends witness an AppendEntriesToWitness request with log entry
in current subterm acknowledged by a subquorum in current replication set.
$AppendEntriesToWitness(i) \triangleq$
    $\land\ state[i] = Leader$
    $\land\ WitnessID \in replicationSet[i]$
    $\land$ LET $Agree(index) \triangleq$
                $\{i\} \cup \{k \in replicationSet[i] : matchIndex[i][k] \geq index\}$
            $IsAgreed(k) \triangleq$
                $\land\ log[i][k].term \quad = currentTerm[i]$
                $\land\ log[i][k].subterm = currentSubterm[i]$
                $\land\ (\{WitnessID\} \cup Agree(k)) \in Quorum$
            $agreeIndexes \triangleq \{k \in 1\,..\,Len(log[i]) : IsAgreed(k)\}$
            $lastEntry \quad\ \triangleq Max(agreedIndex)$
      IN    $\land\ agreeIndexes \neq \{\}$
            $\land\ \lor$  witnessSubterm never exceeds currentSubterm following specification
                $\land\ currentSubterm[i] > witnessSubterm$
                $\land\ Send([mtype \mapsto AppendEntriesToWitnessRequest,$
                        $mterm \mapsto currentTerm[i],$
                        $mlogTerm \mapsto log[i][lastEntry].term,$
                        $mlogSubterm \mapsto log[i][lastEntry].subterm,$
                        $mreplicationSet \mapsto replicaitonSet[i],$
                        $mindex \mapsto lastEntry,$
                         mlog and mentries are used as history variable for
                         the proof. They do not exist in a real implementation.
                        $mlog \mapsto log[i],$
                        $mentries \mapsto SubSeq(log[i], 1, lastEntry),$
                        $msource \mapsto i,$
                        $mdest \mapsto WitnessID])$
                $\land$ UNCHANGED $\langle leaderVars\rangle$
              $\lor$  shortcut replication
                $\land\ currentSubterm[i] = witnessSubterm[i]$
                $\land\ Send([mtype \mapsto AppendEntriesResponse,$
                        $mterm \mapsto currentTerm[i],$
                        $msuccess \mapsto$ TRUE,
                        $mmatchIndex \mapsto lastEntry,$
                        $msource \mapsto WitnessID,$
                        $mdest \mapsto i])$
    $\land$ UNCHANGED $\langle serverVars, candidateVars, logVars, witnessVars\rangle$
```

Figure 2.3: AppendEntriesToWitness

```
Witness receives an AppendEntriesToWitnessRequest from server j with
m.mterm ¡= currentTerm[WitnessID].
$HandleAppendEntriesToWitnessRequest(j, m) \triangleq$
    $\land m.mterm \leq currentTerm[WitnessID]$
    $\land \lor$ reject request
        $\land m.mterm < currentTerm[WitnessID]$
        $\land Reply([mtype \mapsto AppendEntriesResponse,$
                 $mterm \mapsto currentTerm[WitnessID],$
                 $msuccess \mapsto \text{FALSE},$
                 $mmatchIndex \mapsto 0,$
                 $msource \mapsto WitnessID,$
                 $mdest \mapsto j],$
                 $m)$
        $\land \text{UNCHANGED } \langle serverVars, logVars, witnessVars \rangle$
      $\lor$ accept request, always no conflict.
        $\land m.mterm = currentTerm[WitnessID]$
        $\land \lor m.mlastLogTerm > witnessLastLogTerm[WitnessID]$
          $\lor \land m.mlastLogTerm = witnessLastLogTerm[WitnessID]$
            $\land m.mlastLogSubterm > witnessLastLogSubterm[WitnessID]$
        $\land witnessReplicationSet' =$
          $[witnessReplicationSet \text{ EXCEPT } ![WitnessID] = m.mreplicationSet]$
        $\land witnessLastLogTerm' =$
          $[witnessLastLogTerm \text{ EXCEPT } ![WitnessID] = m.mlastLogTerm]$
        $\land witnessLastLogSubterm' =$
          $[witnessLastLogSubterm \text{ EXCEPT } ![WitnessID] = m.mlastLogSubterm]$
        log will not be modified in real implementation. It is only
        used here for the proof.
        $\land log' = [log \text{ EXCEPT } ![WitnessID] = m.mentries]$
        $\land Reply([mtype \mapsto AppendEntriesResponse,$
                 $mterm \mapsto currentTerm[WitnessID],$
                 $msuccess \mapsto \text{TRUE},$
                 $mmatchIndex \mapsto m.mindex,$
                 $msource \mapsto WitnessID,$
                 $mdest \mapsto j],$
                 $m)$
        $\land \text{UNCHANGED } \langle serverVars, log \rangle$
    $\land \text{UNCHANGED } \langle candidateVars, leaderVars \rangle$
```

Figure 2.4: Handle AppendEntriesToWitness

Note that the *mlog* and *mentries* fields in the message are solely used for proof. They do not exist in the actual implementation, nor will the witness persist the full log prefix in its durable storage. The persistence of *m.mentries* to *log* in *HandleAppendEntriesToWitnessRequest* is exclusively used for proof purposes.

When the leader receives an acknowledgment from the witness, it also updates its *witnessSubterm* to the current subterm if the acknowledged log entry is in the current subterm. Then, according to condition **??**, the leader no longer sends *AppendEntriesToWitnessRequest* in the current subterm. Instead, the leader treats subsequent log entries as already acknowledged by the witness if they fulfill all conditions except condition **??**. This is referred to as **Shortcut Replication**, where the leader sends an *AppendEntriesResponse* to itself on behalf of the witness.

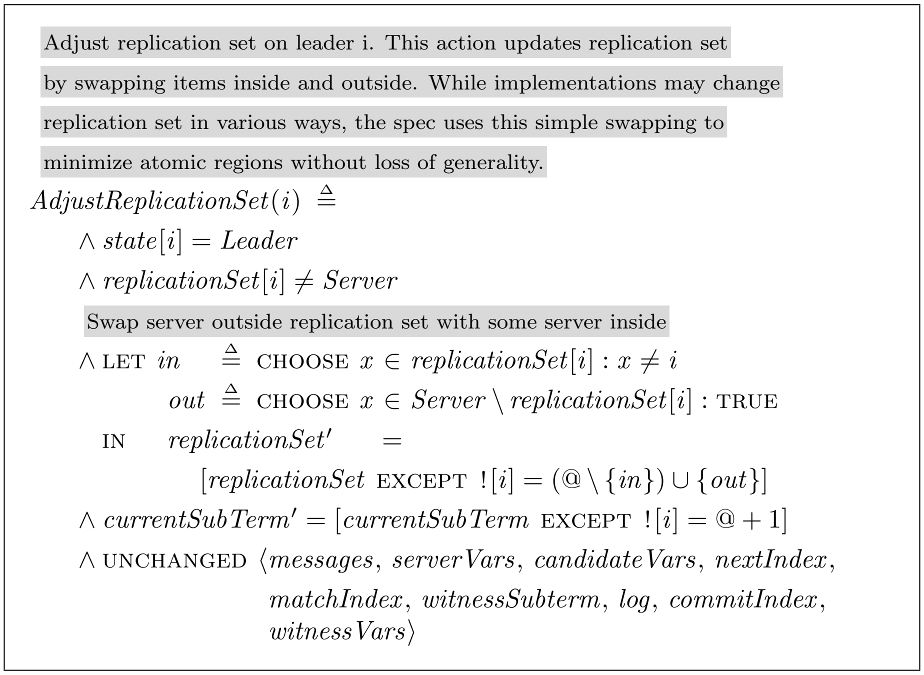
Note that the log entry in *AppendEntriesToWitness* must be within the current term and subterm. So, when the leader advances its subterm (a consequence of replication set adjustment), a new empty log entry must be appended to the leader's log. This is to ensure the leader won't be impeded by condition 1 in committing log entries.

Leader immediately commits a log entry during its term when the log entry is acknowledged by a quorum, in the same way as in Raft. The acknowledgment either comes from a regular server or the witness. Since the leader sends *AppendEntriesToWitnessRequest* to the witness only after it has received subquorum acknowledgments from regular servers, any log entry acknowledged by the witness is immediately committed.

2.4 Replication Set Adjustment

Leader maintains the replication set and modifies it whenever necessary. This process is referred to as **Replication Set Adjustment**. Figure 2.5 illustrates a straightforward method to adjust the replication set. Although there could be multiple ways to adjust a replication set in practical implementations, such a simple method can be used in the algorithm to minimize the atomic regions without loss of generality.

To commit, the leader's replication set must contain at least a subquorum of healthy regular servers if it includes a witness. Therefore, in a practical implementation, replication set adjustment should be performed in a way such that the resulting replication set includes as many healthy regular servers as



Adjust replication set on leader i. This action updates replication set by swapping items inside and outside. While implementations may change replication set in various ways, the spec uses this simple swapping to minimize atomic regions without loss of generality.

```
AdjustReplicationSet(i) ≜
    ∧ state[i] = Leader
    ∧ replicationSet[i] ≠ Server
     Swap server outside replication set with some server inside
    ∧ LET in   ≜ CHOOSE x ∈ replicationSet[i] : x ≠ i
          out  ≜ CHOOSE x ∈ Server \ replicationSet[i] : TRUE
      IN  replicationSet′     =
            [replicationSet EXCEPT ![i] = (@ \ {in}) ∪ {out}]
    ∧ currentSubTerm′ = [currentSubTerm EXCEPT ![i] = @ + 1]
    ∧ UNCHANGED ⟨messages, serverVars, candidateVars, nextIndex,
                  matchIndex, witnessSubterm, log, commitIndex,
                  witnessVars⟩
```

Figure 2.5: Replicaiton set adjustment

possible. To achieve this, leader can track the status of each peer regular server by monitoring their responses. If leader receives no response from a regular peer server over an *electiontimeout*, it assumes the regular server is unreachable and initiates a replication set adjustment. Let *Reachable* and *Unreachable* represent the set of reachable regular servers and unreachable regular servers from the leader's perspective. A practical replication set adjustment can be implemented as below:

1. When a regular server becomes a leader, its replication set is initialized to all regular servers *RegularServer* in the leader's configuration.

2. If all servers in *RegularServer* are reachable from the leader, leader changes its replication set to *RegularServer*, i.e.,

$$\forall s \in \mathit{RegularServer} : s \in \mathit{Reachable} \implies \mathit{ReplicationSet}' = \mathit{RegularServer}$$

3. Leader swaps one unreachable regular server inside its replication set with the reachable regular server or witness outside, i.e.,

$$\begin{aligned} \exists x \in \mathit{ReplicationSet}, y \in \mathit{Server} \setminus \mathit{ReplicationSet} : \ & \land\, x \in \mathit{Unreachable} \\ & \land\, \lor\, y \in \mathit{Reachable} \\ & \quad\ \lor\, y = \mathit{witness} \\ \implies \mathit{ReplicationSet}' = \mathit{Server} \setminus \{x\} \end{aligned}$$

4. Leader increments its *currentSubterm* and appends a new empty log entry if its replication set changes.

2.5 Leader Election

When a regular server transitions into candidate, it requests votes from all peer servers including witness. The extended Raft algorithm adheres to the exact same rules as those in Raft when it comes to requesting and granting votes between a candidate and a regular server. However, the candidate does not request a vote from the witness until it has received subquorum votes from regular servers. Figure 2.6 describes the new *RequestWitnessVote* action for a candidate to request vote from a witness.

Considering that the witness does not persist the full log prefix, the extended Raft algorithm employs a new set of rules for the witness in leader elections.

```
Candidate i sends witness a RequestWitnessVote request.
$RequestWitnessVote(i) \triangleq$
    $\land\ state[i] = Candidate$
    $\land\ WitnessID \notin votesResponded[i]$
    $\land\ (votesGranted[i] \cup \{WitnessID\}) \in Quorum$
    $\land\ Send([mtype \mapsto RequestWitnessVoteRequest,$
            $mterm \mapsto currentTerm[i],$
            $mlastLogTerm \mapsto LastTerm(log[i]),$
            $mlastLogSubterm \mapsto LastSubterm(log[i]),$
            $mvotesGranted \mapsto votesGranted[i],$
            $msource \mapsto i,$
            $mdest \mapsto WitnessID])$
    $\land$ UNCHANGED $\langle serverVars,\ candidateVars,\ leaderVars,\ logVars,$
                    $witnessVars\rangle$
```

Figure 2.6: RequestWitnessVote

Figure 2.7 describes the new action for the witness to handle a vote request from a candidate. The *RequestWitnessVoteRequest* message includes the *term* and *subterm* of the last log entry in the candidate, as well as the votes the candidate has already received in the current election. The witness votes for a candidate if it has not casted vote in the current term and any of the following conditions is true:

- Candidate's last log entry has larger term, i.e. $m.mlastLogTerm > witnessLastLogTerm$.
- Candidate's last log entry has same term but larger subterm, i.e. $m.mlastLogTerm = witnessLastLogTerm \land m.lastLogSubterm > witnessLastLogSubterm$.
- Candidate's last log entry has same term and subterm comparing to witness, and all subqorum votes it got are from regular servers in witness's replication set, i.e.

$$m.mlastLogTerm = witnessLastLogTerm$$
$$m.mlastLogSubterm = witnessLastLogSubterm$$
$$m.mvotesGranted \subseteq witnessReplicationSet$$

```
Witness receives a RequestWitnessVote request from server j with
m.mterm ¡= currentTerm[WitnessID].
$HandleRequestWitnessVoteRequest(j, m) \triangleq$
    LET $logOk \triangleq$ $\lor m.mlastLogTerm > witnessLastLogTerm$
                     $\lor \land m.mlastLogTerm = witnessLastLogTerm$
                       $\land m.mlastLogSubterm > witnessLastLogSubterm$
                     $\lor \land m.mlastLogTerm = witnessLastLogTerm$
                       $\land m.mlastLogSubterm = witnessLastLogSubterm$
                       $\land m.mvotesGranted \subseteq witnessReplicationSet$
        $grant \triangleq$ $\land m.mterm = currentTerm[WitnessID]$
                     $\land logOk$
                     $\land votedFor[WitnessID] \in \{Nil, j\}$
    IN  $\land m.mterm \leq currentTerm[WitnessID]$
        $\land \lor grant \land votedFor' = [votedFor$ EXCEPT $![WitnessID] = j]$
          $\lor \neg grant \land$ UNCHANGED $votedFor$
        $\land Reply([mtype \mapsto RequestVoteResponse,$
                 $mterm \mapsto currentTerm[WitnessID],$
                 $mvoteGranted \mapsto grant,$
                  mlog is used just for the ‘elections’ history variable for
                  the proof. It would not exist in a real implementation.
                 $mlog \mapsto log[WitnessID],$
                 $msource \mapsto WitnessID,$
                 $mdest \mapsto j],$
                 $m)$
        $\land$ UNCHANGED $\langle state, currentTerm, candidateVars, leaderVars,$
                          $logVars, witnessVars\rangle$
```

Figure 2.7: HandleRequestVoteToWitness

3 Witness Implementation

With the extended Raft algorithm, the witness possesses the following properties that make it particularly suitable for implementation as a storage object:

- Witness does not have volatile variables, implying that all states within the witness are persisted in durable storage.
- Witness only persists a small amount of metadata. Neither user data nor the full log prefix metadata are stored in witness. This results in a trivial and predictable capacity requirement.
- Witness is rarely accessed. The shortcut replication ensures that leader only needs to write to witness when its replication set changes. Therefore, witness may only need to be accessed when faults change (e.g. a server shuts down or recovers) or a candidate requests its vote. In a relatively stable cluster, neither of these events occur frequently.

The absence of volatile variables allows the synchronization of witness states to be distributed across all regular servers. For each log replication between leader and witness, the *HandleAppendEntriesToWitnessRequest* action can be implemented in the leader as a single operation that loads states from the witness's durable storage, make new states according to *HandleAppendEntriesToWitnessRequest*, and writes back new states to the witness's durable storage. Similarly for voting between candidate and witness, the *HandleRequestWitnessVoteRequest* action can be implemented in the candidate as a single operation that loads states from the witness's durable storage, make new states according to *HandleRequestVoteToWitness*, and writes back the new states to the witness's durable storage. These two implementations can be synchronized using an optimistic concurrency control scheme since they both occur infrequently.

Figure 3.1 illustrates an implementation using a share (NFS or SMB) as a witness. We associate a version number with the states, starting from 0. Each modification to the states increases the version by 1. We can then persist each version of the witness states in a file uniquely named after the version number (e.g. <version>.st). Whenever *HandleAppendEntriesToWitnessRequest* or *HandleRequestVoteToWitness* are invoked, a regular server (leader or candidate) loads the witness states from the file with the largest version number, calculates the new states, and saves the new states back to a distinct temporary file (e.g. <server id>.<version>.st). Then, the regular server tries to create a hard link

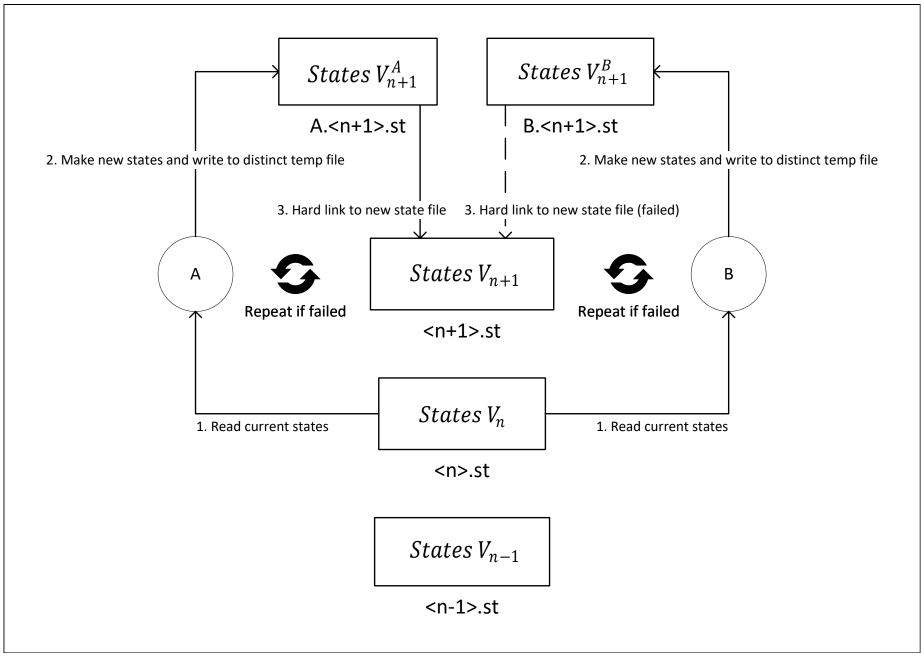


Figure 3.1: Implement witness as a share

from the temporary file to the desired versioned state file (i.e. <version+1>.st). As long as the file system guarantees atomicity in creating hardlinked files, the regular server will only succeed if the target versioned state file does not exist. If any other server has finished update states in the same version, the process needs to be repeated from the beginning. The optimistic concurrency control eliminates the need for locking overhead, which is challenging to implement in practical applications. The infrequency of witness visits makes such a scheme very suitable.

In addition to share, the witness functionality can also be incorporated into cloud storage, such as Azure Storage or AWS S3. Many cloud storage vendors provide optimistic concurrency mechanism for updating object data. For instance, Azure Blob Storage allows clients to update a blob object using the original ETag, combined with a conditional header, to ensure that updates only occur if the ETag remains unchanged. This guarantees that no other client has updated the blob object concurrently. The infrequent access requirements and trivial payload size for witness make cloud storage an ideal solution for this function. Consequently, it provides a broader range of implementation options for end-users.

3.1 Compatibility

In a cluster without witness, the extended Raft algorithm functions identically to the Raft algorithm. This compatibility allows for an existing cluster based on the Raft algorithm to upgrade seamlessly to the extended Raft algorithm in a practical application. A binary upgrade can be applied to the current cluster to adopt the extended Raft algorithm. Subsequently, the upgraded cluster behaves just like a typical Raft-based cluster and does not include a witness.

Once all servers have been upgraded to accommodate the extended Raft algorithm, a membership reconfiguration operation can be conducted to add a witness into the cluster. Conversely, a cluster operating with the extended Raft algorithm can revert to the standard Raft algorithm through a binary downgrade after the removal of the witness from the cluster.

This flexibility is crucial for real-world applications because it negates the need to set up a new cluster and migrate data from the existing cluster to leverage the benefits of having a witness. Consequently, this minimizes disruptions and enhances the operational efficiency of these applications.

4 Specification

```
------------------------------ MODULE extendedraft ------------------------------
This is the formal specification for the extended Raft consensus algorithm.
< TODO: Copyright Info >

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International License https://creativecommons.org/licenses/by/4.0/

EXTENDS Naturals, FiniteSets, Sequences, TLC

The set of server IDs
CONSTANTS Server

The set of requests that can go into the log
CONSTANTS Value

ID of the witness
```

```
CONSTANTS WitnessID

 Server states.
CONSTANTS Follower, Candidate, Leader

 A reserved value.
CONSTANTS Nil

 Message types:
CONSTANTS RequestVoteRequest, RequestVoteResponse,
          AppendEntriesRequest, AppendEntriesResponse,
          RequestWitnessVoteRequest, AppendEntriesToWitnessRequest

 Global variables

 A bag of records representing requests and responses sent from one server
 to another. TLAPS doesn't support the Bags module, so this is a function
 mapping Message to Nat.
VARIABLE messages

 A history variable used in the proof. This would not be present in an
 implementation.
 Keeps track of successful elections, including the initial logs of the
 leader and voters' logs. Set of functions containing various things about
 successful elections (see BecomeLeader).
VARIABLE elections

 A history variable used in the proof. This would not be present in an
 implementation.
 Keeps track of every log ever in the system (set of logs).
VARIABLE allLogs

 The following variables are all per server (functions with domain Server).

 The server's term number.
VARIABLE currentTerm
 The server's state (Follower, Candidate, or Leader).
VARIABLE state
 The candidate the server voted for in its current term, or
 Nil if it hasn't voted for any.
VARIABLE votedFor
serverVars ≜ ⟨currentTerm, state, votedFor⟩

 A Sequence of log entries. The index into this sequence is the index of the
 log entry. Unfortunately, the Sequence module defines Head(s) as the entry
 with index 1, so be careful not to use that!
VARIABLE log
 The index of the latest entry in the log the state machine may apply.
VARIABLE commitIndex
logVars ≜ ⟨log, commitIndex⟩

 The following variables are used only on candidates:
 The set of servers from which the candidate has received a RequestVote
 response in its currentTerm.
VARIABLE votesResponded
 The set of servers from which the candidate has received a vote in its
 currentTerm.
VARIABLE votesGranted
 A history variable used in the proof. This would not be present in an
 implementation.
 Function from each server that voted for this candidate in its currentTerm
 to that voter's log.
VARIABLE voterLog
candidateVars ≜ ⟨votesResponded, votesGranted, voterLog⟩

 The following variables are used only on leaders:
 The next entry to send to each follower.
VARIABLE nextIndex
 The latest entry that each follower has acknowledged is the same as the
 leader's. This is used to calculate commitIndex on the leader.
VARIABLE matchIndex
 Leader's replication set
VARIABLE replicationSet
 Leader's subterm number
VARIABLE currentSubterm
 The latest subterm that this leader has written to the witness
VARIABLE witnessSubterm
```

```
leaderVars ≜ ⟨nextIndex, matchIndex, elections,
                     replicationSet, currentSubterm, witnessSubterm⟩

 The following variables are persisted only on witness:
 The latest replication set sent from leader
VARIABLE witnessReplicationSet
 The latest term of replicated log entry
VARIABLE witnessLastLogTerm
 The latest subterm of replicated log entry
VARIABLE witnessLastLogSubterm
witnessVars ≜ ⟨witnessReplicationSet,
                      witnessLastLogTerm, witnessLastLogSubterm⟩


 End of per server variables.
├──────────────────────────────────────────────────────────────┤

 All variables; used for stuttering (asserting state hasn't changed).
vars ≜ ⟨messages, allLogs, serverVars, candidateVars, leaderVars, logVars, witnessVars⟩

├──────────────────────────────────────────────────────────────┤
 Helpers

 The set of all quorums. This just calculates simple majorities, but the only
 important property is that every quorum overlaps with every other.
Quorum ≜ {i ∈ SUBSET (Server) : Cardinality(i) ∗ 2 > Cardinality(Server)}

 The term of the last entry in a log, or 0 if the log is empty.
LastTerm(xlog) ≜ IF Len(xlog) = 0 THEN 0 ELSE xlog[Len(xlog)].term

 The subterm of the last entry in a log, or 0 if the log is empty.
LastSubTerm(xlog) ≜ IF Len(xlog) = 0 THEN 0 ELSE xlog[Len(xlog)].subterm

 Helper for Send and Reply. Given a message m and bag of messages, return a
 new bag of messages with one more m in it.
WithMessage(m, msgs) ≜
    IF m ∈ DOMAIN msgs THEN
        [msgs EXCEPT ![m] = msgs[m] + 1]
     ELSE
        msgs @@ (m :> 1)
```

```
Helper for Discard and Reply. Given a message m and bag of messages, return
a new bag of messages with one less m in it.
WithoutMessage(m, msgs) ≜
    IF m ∈ DOMAIN msgs THEN
        [msgs EXCEPT ![m] = msgs[m] − 1]
     ELSE
        msgs

Add a message to the bag of messages.
Send(m) ≜ messages′ = WithMessage(m, messages)

Remove a message from the bag of messages. Used when a server is done
processing a message.
Discard(m) ≜ messages′ = WithoutMessage(m, messages)

Combination of Send and Discard
Reply(response, request) ≜
    messages′ = WithoutMessage(request, WithMessage(response, messages))

Return the minimum value from a set, or undefined if the set is empty.
Min(s) ≜ CHOOSE x ∈ s : ∀ y ∈ s : x ≤ y
Return the maximum value from a set, or undefined if the set is empty.
Max(s) ≜ CHOOSE x ∈ s : ∀ y ∈ s : x ≥ y

├──────────────────────────────────────────────────────────────────────┤
Define initial values for all variables

InitHistoryVars ≜ ∧ elections = {}
                  ∧ allLogs   = {}
                  ∧ voterLog = [i ∈ Server ↦ [j ∈ {} ↦ ⟨⟩]]
InitServerVars ≜ ∧ currentTerm = [i ∈ Server ↦ 1]
                 ∧ state       = [i ∈ Server ↦ Follower]
                 ∧ votedFor    = [i ∈ Server ↦ Nil]
InitCandidateVars ≜ ∧ votesResponded = [i ∈ Server ↦ {}]
                    ∧ votesGranted   = [i ∈ Server ↦ {}]
The values nextIndex[i][i] and matchIndex[i][i] are never read, since the
leader does not send itself messages. It's still easier to include these
in the functions.
InitLeaderVars ≜ ∧ nextIndex  = [i ∈ Server ↦ [j ∈ Server ↦ 1]]
```

```
                         ∧ matchIndex = [i ∈ Server ↦ [j ∈ Server ↦ 0]]
                         ∧ replicationSet    = Server \ {WitnessID}
                         ∧ currentSubterm = 0
InitLogVars ≜  ∧ log              = [i ∈ Server ↦ ⟨⟩]
               ∧ commitIndex  = [i ∈ Server ↦ 0]
InitWitnessVars ≜  ∧ witnessReplicationSet = {}
                   ∧ witnessLastLogTerm = 0
                   ∧ witnessLastLogSubterm = 0
Init ≜  ∧ messages = [m ∈ {} ↦ 0]
        ∧ InitHistoryVars
        ∧ InitServerVars
        ∧ InitCandidateVars
        ∧ InitLeaderVars
        ∧ InitLogVars
        ∧ InitWitnessVars

├────────────────────────────────────────────────────────────────────────┤
  Define state transitions

  Server i restarts from stable storage.
  It loses everything but its currentTerm, votedFor, and log.
Restart(i) ≜
    ∧  i                ≠ WitnessID
    ∧  state′              = [state EXCEPT ![i] = Follower]
    ∧  votesResponded′ = [votesResponded EXCEPT ![i] = {}]
    ∧  votesGranted′    = [votesGranted EXCEPT ![i] = {}]
    ∧  voterLog′          = [voterLog EXCEPT ![i] = [j ∈ {} ↦ ⟨⟩]]
    ∧  nextIndex′         = [nextIndex EXCEPT ![i] = [j ∈ Server ↦ 1]]
    ∧  matchIndex′       = [matchIndex EXCEPT ![i] = [j ∈ Server ↦ 0]]
    ∧  commitIndex′     = [commitIndex EXCEPT ![i] = 0]
    ∧  currentSubterm′ = [currentSubterm EXCEPT ![i] = 0]
    ∧  replicationSet′   = [replicationSet EXCEPT ![i]    = Server]
    ∧  witnessSubterm′ = [witnessSubterm EXCEPT ![i] = 0]
    ∧  UNCHANGED ⟨messages, currentTerm, votedFor, log, elections, witnessVars⟩

  Server i times out and starts a new election.
Timeout(i) ≜  ∧ i ≠ WitnessID
              ∧ state[i] ∈ {Follower, Candidate}
```

```
                    ∧ state′ = [state EXCEPT ![i] = Candidate]
                    ∧ currentTerm′ = [currentTerm EXCEPT ![i] = currentTerm[i] + 1]
                     Most implementations would probably just set the local vote
                     atomically, but messaging localhost for it is weaker.
                    ∧ votedFor′ = [votedFor EXCEPT ![i] = Nil]
                    ∧ votesResponded′ = [votesResponded EXCEPT ![i] = {}]
                    ∧ votesGranted′   = [votesGranted EXCEPT ![i] = {}]
                    ∧ voterLog′       = [voterLog EXCEPT ![i] = [j ∈ {} ↦ ⟨⟩]]
                    ∧ UNCHANGED ⟨messages, leaderVars, logVars, witnessVars⟩

 Candidate i sends j a RequestVote request.
RequestVote(i, j) ≜
    ∧ state[i] = Candidate
    ∧ j ≠ WitnessID
    ∧ j ∉ votesResponded[i]
    ∧ Send([mtype          ↦ RequestVoteRequest,
            mterm          ↦ currentTerm[i],
            mlastLogTerm   ↦ LastTerm(log[i]),
            mlastLogIndex  ↦ Len(log[i]),
            msource        ↦ i,
            mdest          ↦ j])
    ∧ UNCHANGED ⟨serverVars, candidateVars, leaderVars, logVars, witnessVars⟩

 Candidate i sends witness a RequestWitnessVote request.
RequestWitnessVote(i) ≜
    ∧ state[i] = Candidate
    ∧ WitnessID ∉ votesResponded[i]
    ∧ (votesGranted[i] ∪ {WitnessID}) ∈ Quorum
    ∧ Send([mtype            ↦ RequestWitnessVoteRequest,
            mterm            ↦ currentTerm[i],
            mlastLogTerm     ↦ LastTerm(log[i]),
            mlastLogSubterm  ↦ LastSubterm(log[i]),
            mvotesGranted    ↦ votesGranted[i],
            msource          ↦ i,
            mdest            ↦ WitnessID])
    ∧ UNCHANGED ⟨serverVars, candidateVars, leaderVars, logVars,
                   witnessVars⟩
```

```
Leader i sends j an AppendEntries request containing up to 1 entry.
While implementations may want to send more than 1 at a time, this spec uses
just 1 because it minimizes atomic regions without loss of generality.
AppendEntries(i, j) ≜
    ∧ i ≠ j
    ∧ state[i] = Leader
    ∧ LET prevLogIndex ≜ nextIndex[i][j] − 1
          prevLogTerm ≜ IF prevLogIndex > 0 THEN
                              log[i][prevLogIndex].term
                          ELSE
                              0
           Send up to 1 entry, constrained by the end of the log.
          lastEntry ≜ Min({Len(log[i]), nextIndex[i][j]})
          entries ≜ SubSeq(log[i], nextIndex[i][j], lastEntry)
      IN  Send([mtype          ↦ AppendEntriesRequest,
                mterm          ↦ currentTerm[i],
                mprevLogIndex  ↦ prevLogIndex,
                mprevLogTerm   ↦ prevLogTerm,
                mentries       ↦ entries,
                 mlog is used as a history variable for the proof.
                 It would not exist in a real implementation.
                mlog           ↦ log[i],
                mcommitIndex   ↦ Min({commitIndex[i], lastEntry}),
                msource        ↦ i,
                mdest          ↦ j])
    ∧ UNCHANGED ⟨serverVars, candidateVars, leaderVars, logVars,
                 witnessVars⟩

Leader i sends witness an AppendEntriesToWitness request with log entry
in current subterm acknowledged by a subquorum in current replication set.
AppendEntriesToWitness(i) ≜
    ∧ state[i] = Leader
    ∧ WitnessID ∈ replicationSet[i]
    ∧ LET Agree(index) ≜
              {i} ∪ {k ∈ replicationSet[i] : matchIndex[i][k] ≥ index}
          IsAgreed(k) ≜
              ∧ log[i][k].term      = currentTerm[i]
```

```
                  ∧ log[i][k].subterm = currentSubterm[i]
                  ∧ ({WitnessID} ∪ Agree(k)) ∈ Quorum
              agreeIndexes ≜ {k ∈ 1 .. Len(log[i]) : IsAgreed(k)}
              lastEntry    ≜ Max(agreedIndex)
          IN  ∧ agreeIndexes ≠ {}
              ∧ ∨ witnessSubterm never exceeds currentSubterm following specification
                  ∧ currentSubterm[i] > witnessSubterm
                  ∧ Send([mtype           ↦ AppendEntriesToWitnessRequest,
                          mterm           ↦ currentTerm[i],
                          mlogTerm        ↦ log[i][lastEntry].term,
                          mlogSubterm     ↦ log[i][lastEntry].subterm,
                          mreplicationSet ↦ replicaitonSet[i],
                          mindex          ↦ lastEntry,
                           mlog and mentries are used as history variable for
                           the proof. They do not exist in a real implementation.
                          mlog            ↦ log[i],
                          mentries        ↦ SubSeq(log[i], 1, lastEntry),
                          msource         ↦ i,
                          mdest           ↦ WitnessID])
                  ∧ UNCHANGED ⟨leaderVars⟩
                ∨ shortcut replication
                  ∧ currentSubterm[i] = witnessSubterm[i]
                  ∧ Send([mtype             ↦ AppendEntriesResponse,
                          mterm             ↦ currentTerm[i],
                          msuccess          ↦ TRUE,
                          mmatchIndex       ↦ lastEntry,
                          msource           ↦ WitnessID,
                          mdest             ↦ i])
    ∧ UNCHANGED ⟨serverVars, candidateVars, logVars, witnessVars⟩

 Candidate i transitions to leader.
BecomeLeader(i) ≜
    ∧ state[i] = Candidate
    ∧ votesGranted[i] ∈ Quorum
    ∧ state'       = [state EXCEPT ![i] = Leader]
    ∧ nextIndex'   = [nextIndex EXCEPT ![i] =
                          [j ∈ Server ↦ Len(log[i]) + 1]]
```

```
     ∧ matchIndex′ = [matchIndex EXCEPT ![i] =
                         [j ∈ Server ↦ 0]]
     ∧ elections′    = elections ∪
                         {[eterm      ↦ currentTerm[i],
                           eleader    ↦ i,
                           elog       ↦ log[i],
                           evotes     ↦ votesGranted[i],
                           evoterLog ↦ voterLog[i]]}
     ∧ UNCHANGED ⟨messages, currentTerm, votedFor, candidateVars, logVars, witnessVars⟩

 Leader i receives a client request to add v to the log.
ClientRequest(i, v) ≜
     ∧ state[i] = Leader
     ∧ LET entry      ≜ [term          ↦ currentTerm[i],
                          subterm       ↦ currentSubterm[i],
                          value         ↦ v]
            newLog ≜ Append(log[i], entry)
        IN  log′ = [log EXCEPT ![i] = newLog]
     ∧ UNCHANGED ⟨messages, serverVars, candidateVars,
                   leaderVars, commitIndex, witnessVars⟩

 Leader i advances its commitIndex.
 This is done as a separate step from handling AppendEntries responses,
 in part to minimize atomic regions, and in part so that leaders of
 single-server clusters are able to mark entries committed.
AdvanceCommitIndex(i) ≜
     ∧ state[i] = Leader
     ∧ LET  The set of servers that agree up through index.
           Agree(index) ≜ {i} ∪ {k ∈ Server :
                                    matchIndex[i][k] ≥ index}
            The maximum indexes for which a quorum agrees
           agreeIndexes ≜ {index ∈ 1 . . Len(log[i]) :
                                 Agree(index) ∈ Quorum}
            New value for commitIndex′[i]
           newCommitIndex ≜
              IF  ∧ agreeIndexes ≠ {}
                  ∧ log[i][Max(agreeIndexes)].term = currentTerm[i]
```

```
                    THEN
                        Max(agreeIndexes)
                    ELSE
                        commitIndex[i]
          IN    commitIndex′ = [commitIndex EXCEPT ![i] = newCommitIndex]
        ∧ UNCHANGED ⟨messages, serverVars, candidateVars, leaderVars, log, witnessVars⟩

  Adjust replication set on leader i. This action updates replication set
  by swapping items inside and outside. While implementations may change
  replication set in various ways, the spec uses this simple swapping to
  minimize atomic regions without loss of generality.
AdjustReplicationSet(i) ≜
        ∧ state[i] = Leader
        ∧ replicationSet[i] ≠ Server
          Swap server outside replication set with some server inside
        ∧ LET in    ≜ CHOOSE x ∈ replicationSet[i] : x ≠ i
                out ≜ CHOOSE x ∈ Server \ replicationSet[i] : TRUE
          IN    replicationSet′    =
                    [replicationSet EXCEPT ![i] = (@ \ {in}) ∪ {out}]
        ∧ currentSubTerm′ = [currentSubTerm EXCEPT ![i] = @ + 1]
        ∧ UNCHANGED ⟨messages, serverVars, candidateVars, nextIndex,
                        matchIndex, witnessSubterm, log, commitIndex,
                        witnessVars⟩

├─────────────────────────────────────────────────────────────────────────┤
  Message handlers
  i = recipient, j = sender, m = message

  Server i receives a RequestVote request from server j with
  m.mterm ≤ currentTerm[i].
HandleRequestVoteRequest(i, j, m) ≜
      LET logOk ≜ ∨ m.mlastLogTerm > LastTerm(log[i])
                  ∨ ∧ m.mlastLogTerm = LastTerm(log[i])
                    ∧ m.mlastLogIndex ≥ Len(log[i])
          grant ≜ ∧ m.mterm = currentTerm[i]
                  ∧ logOk
                  ∧ votedFor[i] ∈ {Nil, j}
      IN    ∧ m.mterm ≤ currentTerm[i]
```

```
         ∧ ∨ grant   ∧ votedFor′ = [votedFor EXCEPT ![i] = j]
           ∨ ¬grant ∧ UNCHANGED votedFor
         ∧ Reply([mtype          ↦ RequestVoteResponse,
                  mterm          ↦ currentTerm[i],
                  mvoteGranted   ↦ grant,
                   mlog is used just for the elections history variable for
                   the proof. It would not exist in a real implementation.
                  mlog          ↦ log[i],
                  msource       ↦ i,
                  mdest         ↦ j],
                  m)
         ∧ UNCHANGED ⟨state, currentTerm, candidateVars, leaderVars, logVars, witnessVars⟩

 Witness receives a RequestWitnessVote request from server j with
 m.mterm ≤ currentTerm[WitnessID].
HandleRequestWitnessVoteRequest(j, m) ≜
    LET logOk ≜ ∨ m.mlastLogTerm > witnessLastLogTerm
                ∨ ∧ m.mlastLogTerm = witnessLastLogTerm
                  ∧ m.mlastLogSubterm > witnessLastLogSubterm
                ∨ ∧ m.mlastLogTerm = witnessLastLogTerm
                  ∧ m.mlastLogSubterm = witnessLastLogSubterm
                  ∧ m.mvotesGranted ⊆ witnessReplicationSet
        grant ≜ ∧ m.mterm = currentTerm[WitnessID]
                ∧ logOk
                ∧ votedFor[WitnessID] ∈ {Nil, j}
    IN   ∧ m.mterm ≤ currentTerm[WitnessID]
         ∧ ∨ grant   ∧ votedFor′ = [votedFor EXCEPT ![WitnessID] = j]
           ∨ ¬grant ∧ UNCHANGED votedFor
         ∧ Reply([mtype          ↦ RequestVoteResponse,
                  mterm          ↦ currentTerm[WitnessID],
                  mvoteGranted   ↦ grant,
                   mlog is used just for the elections history variable for
                   the proof. It would not exist in a real implementation.
                  mlog          ↦ log[WitnessID],
                  msource       ↦ WitnessID,
                  mdest         ↦ j],
                  m)
```

```
          ∧ UNCHANGED ⟨state, currentTerm, candidateVars, leaderVars,
                          logVars, witnessVars⟩

 Server i receives a RequestVote response from server j with
 m.mterm = currentTerm[i].
HandleRequestVoteResponse(i, j, m) ≜
     This tallies votes even when the current state is not Candidate, but
     they won't be looked at, so it doesn't matter.
    ∧ m.mterm = currentTerm[i]
    ∧ votesResponded′ = [votesResponded EXCEPT ![i] =
                              votesResponded[i] ∪ {j}]
    ∧ ∨ ∧ m.mvoteGranted
          ∧ votesGranted′ = [votesGranted EXCEPT ![i] =
                                  votesGranted[i] ∪ {j}]
          ∧ voterLog′ = [voterLog EXCEPT ![i] =
                              voterLog[i] @@ (j :> m.mlog)]
       ∨ ∧ ¬m.mvoteGranted
          ∧ UNCHANGED ⟨votesGranted, voterLog⟩
    ∧ Discard(m)
    ∧ UNCHANGED ⟨serverVars, votedFor, leaderVars, logVars, witnessVars⟩

 Server i receives an AppendEntries request from server j with
 m.mterm ≤ currentTerm[i]. This just handles m.entries of length 0 or 1, but
 implementations could safely accept more by treating them the same as
 multiple independent requests of 1 entry.
HandleAppendEntriesRequest(i, j, m) ≜
    LET logOk ≜ ∨ m.mprevLogIndex = 0
                ∨ ∧ m.mprevLogIndex > 0
                  ∧ m.mprevLogIndex ≤ Len(log[i])
                  ∧ m.mprevLogTerm = log[i][m.mprevLogIndex].term
    IN  ∧ m.mterm ≤ currentTerm[i]
          ∧ ∨ ∧  reject request
                ∨ m.mterm < currentTerm[i]
                ∨ ∧ m.mterm = currentTerm[i]
                  ∧ state[i] = Follower
                  ∧ ¬logOk
              ∧ Reply([mtype          ↦ AppendEntriesResponse,
                  mterm          ↦ currentTerm[i],
                  msuccess       ↦ FALSE,
                  mmatchIndex    ↦ 0,
                  msource        ↦ i,
                  mdest          ↦ j],
                  m)
        ∧ UNCHANGED ⟨serverVars, logVars⟩
      ∨  return to follower state
        ∧ m.mterm = currentTerm[i]
        ∧ state[i] = Candidate
        ∧ state′ = [state EXCEPT ![i] = Follower]
        ∧ UNCHANGED ⟨currentTerm, votedFor, logVars, messages⟩
      ∨  accept request
        ∧ m.mterm = currentTerm[i]
        ∧ state[i] = Follower
        ∧ logOk
        ∧ LET index ≜ m.mprevLogIndex + 1
           IN   ∨  already done with request
                   ∧ ∨ m.mentries = ⟨⟩
                     ∨ ∧ m.mentries ≠ ⟨⟩
                       ∧ Len(log[i]) ≥ index
                       ∧ log[i][index].term = m.mentries[1].term
                     This could make our commitIndex decrease (for
                     example if we process an old, duplicated request),
                     but that doesn't really affect anything.
                   ∧ commitIndex′ = [commitIndex EXCEPT ![i] =
                                              m.mcommitIndex]
                   ∧ Reply([mtype          ↦ AppendEntriesResponse,
                            mterm          ↦ currentTerm[i],
                            msuccess       ↦ TRUE,
                            mmatchIndex    ↦ m.mprevLogIndex +
                                               Len(m.mentries),
                            msource        ↦ i,
                            mdest          ↦ j],
                            m)
                   ∧ UNCHANGED ⟨serverVars, log⟩
                ∨  conflict: remove 1 entry
```

```
                     ∧ m.mentries ≠ ⟨⟩
                     ∧ Len(log[i]) ≥ index
                     ∧ log[i][index].term     ≠ m.mentries[1].term
                     ∧ LET new ≜ [index2 ∈ 1 . . (Len(log[i]) − 1) ↦
                                       log[i][index2]]
                       IN   log′ = [log EXCEPT ![i] = new]
                     ∧ UNCHANGED ⟨serverVars, commitIndex, messages⟩
                  ∨  no conflict: append entry
                     ∧ m.mentries ≠ ⟨⟩
                     ∧ Len(log[i]) = m.mprevLogIndex
                     ∧ log′ = [log EXCEPT ![i] =
                                  Append(log[i], m.mentries[1])]
                     ∧ UNCHANGED ⟨serverVars, commitIndex, messages⟩
        ∧ UNCHANGED ⟨candidateVars, leaderVars, witnessVars⟩

 Witness receives an AppendEntriesToWitnessRequest from server j with
 m.mterm ≤ currentTerm[WitnessID].
HandleAppendEntriesToWitnessRequest(j, m) ≜
    ∧ m.mterm ≤ currentTerm[WitnessID]
    ∧ ∨  reject request
        ∧ m.mterm < currentTerm[WitnessID]
        ∧ Reply([mtype          ↦ AppendEntriesResponse,
                 mterm          ↦ currentTerm[WitnessID],
                 msuccess       ↦ FALSE,
                 mmatchIndex    ↦ 0,
                 msource        ↦ WitnessID,
                 mdest          ↦ j],
                 m)
        ∧ UNCHANGED ⟨serverVars, logVars, witnessVars⟩
      ∨  accept request, always no conflict.
        ∧ m.mterm = currentTerm[WitnessID]
        ∧ ∨ m.mlastLogTerm > witnessLastLogTerm[WitnessID]
          ∨ ∧ m.mlastLogTerm = witnessLastLogTerm[WitnessID]
            ∧ m.mlastLogSubterm > witnessLastLogSubterm[WitnessID]
        ∧ witnessReplicationSet′ =
            [witnessReplicationSet EXCEPT ![WitnessID] = m.mreplicationSet]
        ∧ witnessLastLogTerm′ =
                 [witnessLastLogTerm EXCEPT ![WitnessID] = m.mlastLogTerm]
              ∧ witnessLastLogSubterm′ =
                 [witnessLastLogSubterm EXCEPT ![WitnessID] = m.mlastLogSubterm]
              log will not be modified in real implementation. It is only
              used here for the proof.
              ∧ log′ = [log EXCEPT ![WitnessID] = m.mentries]
              ∧ Reply([mtype          ↦ AppendEntriesResponse,
                       mterm          ↦ currentTerm[WitnessID],
                       msuccess       ↦ TRUE,
                       mmatchIndex    ↦ m.mindex,
                       msource        ↦ WitnessID,
                       mdest          ↦ j],
                       m)
              ∧ UNCHANGED ⟨serverVars, log⟩
        ∧ UNCHANGED ⟨candidateVars, leaderVars⟩

 Server i receives an AppendEntries response from server j with
 m.mterm = currentTerm[i].
HandleAppendEntriesResponse(i, j, m) ≜
        ∧ m.mterm = currentTerm[i]
        ∧ ∨ ∧ m.msuccess  successful
              ∧ nextIndex′   = [nextIndex  EXCEPT ![i][j]  = m.mmatchIndex + 1]
              ∧ matchIndex′ = [matchIndex EXCEPT ![i][j] = m.mmatchIndex]
           ∨ ∧ ¬m.msuccess  not successful
              ∧ nextIndex′ = [nextIndex EXCEPT ![i][j] =
                                    Max({nextIndex[i][j] − 1, 1})]
              ∧ UNCHANGED ⟨matchIndex⟩
        ∧ Discard(m)
        ∧ UNCHANGED ⟨serverVars, candidateVars, logVars, elections, witnessVars⟩

 Any RPC with a newer term causes the recipient to advance its term first.
UpdateTerm(i, j, m) ≜
        ∧ m.mterm > currentTerm[i]
        ∧ currentTerm′    = [currentTerm EXCEPT ![i] = m.mterm]
        ∧ state′          = [state       EXCEPT ![i]   = Follower]
        ∧ votedFor′       = [votedFor     EXCEPT ![i]  = Nil]
           messages is unchanged so m can be processed further.
```

```
     ∧ UNCHANGED ⟨messages, candidateVars, leaderVars, logVars⟩

 Responses with stale terms are ignored.
DropStaleResponse(i, j, m) ≜
     ∧ m.mterm < currentTerm[i]
     ∧ Discard(m)
     ∧ UNCHANGED ⟨serverVars, candidateVars, leaderVars, logVars⟩

 Receive a message.
Receive(m) ≜
    LET i ≜ m.mdest
        j ≜ m.msource
    IN    Any RPC with a newer term causes the recipient to advance
          its term first. Responses with stale terms are ignored.
         ∨ UpdateTerm(i, j, m)
         ∨ ∧ m.mtype = RequestVoteRequest
           ∧ HandleRequestVoteRequest(i, j, m)
         ∨ ∧ m.mtype = RequestWitnessVoteRequest
           ∧ HandleRequestWitnessVoteRequest(j, m)
         ∨ ∧ m.mtype = RequestVoteResponse
           ∧ ∨ DropStaleResponse(i, j, m)
             ∨ HandleRequestVoteResponse(i, j, m)
         ∨ ∧ m.mtype = AppendEntriesRequest
           ∧ HandleAppendEntriesRequest(i, j, m)
         ∨ ∧ m.mtype = AppendEntriesToWitnessReques
           ∧ HandleAppendEntriesToWitnessRequest(j, m)
         ∨ ∧ m.mtype = AppendEntriesResponse
           ∧ ∨ DropStaleResponse(i, j, m)
             ∨ HandleAppendEntriesResponse(i, j, m)

 End of message handlers.
--------------------------------------------------------------------------------
 Network state transitions

 The network duplicates a message
DuplicateMessage(m) ≜
     ∧ Send(m)
     ∧ UNCHANGED ⟨serverVars, candidateVars, leaderVars, logVars⟩
 The network drops a message
DropMessage(m) ≜
     ∧ Discard(m)
     ∧ UNCHANGED ⟨serverVars, candidateVars, leaderVars, logVars⟩

 Defines how the variables may transition.
Next ≜  ∧ ∨ ∃ i ∈ Server : Restart(i)
          ∨ ∃ i ∈ Server : Timeout(i)
          ∨ ∃ i, j ∈ Server : RequestVote(i, j)
          ∨ ∃ i ∈ Server : BecomeLeader(i)
          ∨ ∃ i ∈ Server, v ∈ Value : ClientRequest(i, v)
          ∨ ∃ i ∈ Server : AdvanceCommitIndex(i)
          ∨ ∃ i, j ∈ Server : AppendEntries(i, j)
          ∨ ∃ i  ∈ Server : AdjustReplicationSet(i)
          ∨ ∃ i  ∈ Server : RequestWitnessVote(i)
          ∨ ∃ i  ∈ Server : AppendEntriesToWitness(i)
          ∨ ∃ m ∈ DOMAIN messages : Receive(m)
          ∨ ∃ m ∈ DOMAIN messages : DuplicateMessage(m)
          ∨ ∃ m ∈ DOMAIN messages : DropMessage(m)
           History variable that tracks every log ever:
        ∧ allLogs' = allLogs ∪ {log[i] : i ∈ Server}

 The specification must start with the initial state and transition according
 to Next.
Spec ≜ Init ∧ □[Next]_vars
```

5 Formal proof

In this section, we aim to prove the key properties of the Raft algorithm (namely, Election Safety, Leader Append-Only, Log Matching, Leader Completeness, and State Machine Safety) within the context of the extended Raft algorithm. These properties will be proven to hold true consistently in the extended Raft algorithm. Our proof will build upon the existing proof of the Raft algorithm, and we will reference lemmas from the Raft proof in the format of "Lemma R.n," where 'n' represents the lemma's index in the Raft proof.

Lemma 1. *Following lemmas are true in the extended Raft algorithm.*

1. *Lemma R.1. Each server's currentTerm monotonically increases.*
2. *Lemma R.2. There is at most one leader per term.*
3. *Lemma R.3. A leader's log monotonically grows during its term.*
4. *Lemma R.4. An* $\langle < \rangle index, term >$ *pair identifies a log prefix.*
5. *Lemma R.5. When a follower appends an entry to its log, its log after the append is a prefix of the leader's log at the time the leader sent the AppendEntries request.*
6. *Lemma R.6. A server's current term is always at least as large as the terms in its log.*
7. *Lemma R.7. The terms of entries grow monotonically in each log.*

Proof.

1. Witness, similar to other regular servers, monotonically increases its term upon receiving a message with a larger *mterm* (action *UpdateTerm*). This immediately validates Lemma R.1 as per the specification.
2. In the extended Raft algorithm, leader is elected by a quorum. Neither regular servers nor the witness vote for different servers within the same term, thus maintaining the truth of Lemma R.2 according to its proof.
3. Leader in the extended Raft algorithm manages its log identically to the Raft algorithm, validating Lemma R.3 since
   (a) Actions *BecomeLeader*, *ClientRequest*, and *AppendEntries* behave same as those in the Raft algorithm. Although the log entry is associated with the current subterm, this is trivial to the proof.
   (b) *AppendEntriesToWitness* is an implementation of the *AppendEntries* action, with batching log prefix to a log entry (that is associated with the current subterm and has received subquorum acknowledgments from the current replication set).
4. Lemma R.4 and Lemma R.5 are valid in the extended Raft algorithm given that:

(a) Regular servers replicate logs identically to the Raft algorithm. The existence of a subterm in a log entry is trivial to the proof.

(b) Leader always sends its log's prefix to the witness.

(c) Witness replaces its log with the log prefix from the leader, hence $log'[WitnessID]$ remains a prefix of $m.mlog$ for m sent to the witness.

5. The validation of Lemma R.6 in the Raft algorithm relies on the servers' behavior in log replication. Therefore, Lemma R.6 holds true in the extended Raft algorithm as:

   (a) Regular servers replicate logs in an identical manner.

   (b) Witness accepts the request with the necessary condition $currentTerm[WitnessID] = m.mterm$.

   (c) Witness always replaces its log with the log prefix sent from the leader.

6. The proof of Lemma R.7 depends on Lemma R.5 and Lemma R.6. Thus, Lemma R.7 is also valid in the extended Raft algorithm, as both Lemma R.5 and Lemma R.6 are proven true.

□

Now we have Election Safety property (Lemma R.2), Leader Append-Only property (Lemma R.3), and Log Matching property (Lemma R.4) hold in the extended Raft algorithm.
Similarly, we have following lemmas specific for the extended Raft algorithm.

Lemma 2. *Each leader's currentSubterm monotonically increases during each term:*

$$\begin{aligned}
&\forall i \in Server : \\
&\quad currentTerm[i] = currentTerm'[i] \\
&\quad\quad \implies currentSubterm[i] \leq currentSubterm'[i]
\end{aligned}$$

Proof. This follows immediately from the specification. □

Lemma 3. *The subterms of entries grow monotonically during its termin each* *log*

$$\begin{aligned}
&\forall l \in \mathit{allLogs} : \\
&\quad \forall \mathit{index} \in 1..(\mathit{Len}(l) - 1) : \\
&\quad\quad l[\mathit{index}].\mathit{term} = l[\mathit{index} + 1].\mathit{term} \\
&\quad\quad\quad \implies l[\mathit{index}].\mathit{subterm} \leq l[\mathit{index} + 1].\mathit{subterm}
\end{aligned}$$

Proof.

1. Initial state: all logs are empty, so the invariant holds.
2. Inductive step: logs change in one of the following ways:
   - (a) Case: a leader adds one entry (client request)
     - i. The new entry's subterm is $\mathit{currentSubterm}(\mathit{leader})$.
     - ii. *currentSubterm* monotonically increases during *currentTerm*.
     - iii. Thus, the new entry's subterm is at least as larger as the subterms during *currentTerm* in this log, since leader's log monotonically grows during its term.
   - (b) Case: a follower removes one entry (AppendEntries request)
     - i. The invariant still holds, since only the length of the log decreased.
   - (c) Case: a follower adds one entry (AppendEntries request), or witness replaces its log by log prefix from leader
     - i. $\mathit{log}'[\mathit{follower}]$ is a prefix of $m.\mathit{mlog}$ (by Lemma R.5)
     - ii. $m.\mathit{mlog} \in \mathit{allLogs}$
     - iii. By the inductive hypothesis, the subterms in $m.\mathit{mlog}$ monotonically grow during its term, so the subterms in $\mathit{log}'[\mathit{follower}]$ monotonically grow during its term.

□

*Lemma R.8 is part of the Leader Completeness property in Raft algorithm. Given the existence of shortcut replication, and the fact that witness uses different rules to cast vote, we present proof to the equivalent **Lemma** 4 following the same idea as that in the Raft algorithm.*

Lemma 4. *Immediately committed entries are committed*

$$\begin{aligned}
&\forall \langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle \in \mathit{immediatelyCommitted} : \\
&\quad \langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle \in \mathit{committed}(\mathit{term})
\end{aligned}$$

Proof. 1. Consider an entry $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ that is *immediately committed*.

2. Define

$$\begin{aligned} \mathit{Contradicting} \triangleq \{ & \mathit{election} \in \mathit{elections} : \\ & \wedge\ \mathit{election.eterm} > \mathit{term} \\ & \wedge\ \langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle \notin \mathit{election.elog} \} \end{aligned}$$

3. Let *election* be an element in *Contradicting* with a minimal *term* field. That is,

$$\forall\ e \in \mathit{Contradicting} : \mathit{election.eterm} \leq e.\mathit{eterm}.$$

   If more than one election has the same term, choose the earliest one. (The specification does not allow this to happen, but it is safe for a leader to step down and become leader again in the same term)

4. It suffices to show a contradiction, which implies $\mathit{Contradicting} = \phi$.

5. Let $\langle \mathit{index}', \mathit{term}, \mathit{subterm} \rangle$ be any entry that exists in logs of a quorum during *term*, where $\mathit{index}' \leq \mathit{index}$. Such entry must exist if the subterm's replication set contains the witness, since:

   (a) Case $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ is immediately committed with no shortcut replication: $\langle \mathit{index}', \mathit{term}, \mathit{subterm} \rangle$ exists in logs of a quorum during *term*, following the specification.

   (b) Case $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ is immediately committed with shortcut replication: Following the specification, shortcut replication can only be executed after a log entry in the same subterm being replicated to the witness. Thus $\langle \mathit{index}', \mathit{term}, \mathit{subterm} \rangle$ exists for some $\mathit{index}' < \mathit{index}$.

6. Let *voter* be either

   - any regular server that both votes in *election* and contains $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ in its log during *term*
   - witness that both votes in *election* and contains $\langle \mathit{index}', \mathit{term}, \mathit{subterm} \rangle$ in its log during *term*

Such server must exist, since:

(a) Case: $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ is immediately committed with no shortcut replication.

voter must exists, since:

i. $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ exists in logs of a quorum during *term*.

ii. A quorum of servers voted in *election* to make it succeed.

iii. Two quorums always overlap.

iv. *voter* can be either regular server or witness.

(b) Case: $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ is immediately committed with shortcut replication. *voter* must exists, since:

i. A subquorum of regular servers contains $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ in its log during *term*. And the witness server contains $\langle \mathit{index}', \mathit{term}, \mathit{subterm} \rangle$ in its log during *term*. These make a quorum.

ii. A quorum of servers voted in *election* to make it succeed. The server is either regular server or witness.

iii. Two quorums always overlap.

7. Let $\mathit{voterLog} \triangleq \mathit{election.evoterLog}[\mathit{voter}]$, the voter's log at the time it casts its vote.

8. For any entry $\langle k, \mathit{term}, \mathit{subterm} \rangle (k \leq \mathit{index})$ that the voter contains during *term*, it is also contained by the voter when it casts its vote during $\mathit{election.eterm}$. That is, $\langle k, \mathit{term}, \mathit{subterm} \rangle \in \mathit{voterLog}$, where $k \leq \mathit{index}$:

(a) $\langle k, \mathit{term}, \mathit{subterm} \rangle$ was in the *voter*'s log during *term*.

(b) The *voter* must have stored the entry in *term* before voting in $\mathit{election.eterm}$, since:

i. $\mathit{election.eterm} > \mathit{term}$.

ii. The *voter* rejects requests with terms smaller than its current term, and its current term monotonically increases.

(c) The *voter* couldn't have removed the entry before casting its vote:

i. Case: No *AppendEntriesRequest* with $\mathit{mterm} < \mathit{term}$ removes the entry from the voter's log, since $\mathit{currentTerm}[\mathit{voter}] \geq \mathit{term}$ upon storing the entry, and the voter rejects requests with terms smaller than $\mathit{currentTerm}[\mathit{voter}]$.

ii. Case: No *AppendEntriesRequest* with $mterm = term$ removes the entry from the voter's log, since:

A. There is only one leader of *term*.

B. The leader of *term* created and therefore contains the entry.

C. The leader would not send any conflicting requests to *voter* during *term*.

iii. Case: No *AppendEntriesRequest* with $mterm > term$ removes the entry from the voter's log, since:

A. Case: $mterm > election.eterm$:
This can't happen, since $currentTerm[voter] > election.eterm$ would have presented the voter from voting in *term*.

B. Case: $mterm = election.eterm$:
Since there is at most one leader per term, this request would have to come from *election.eleader* as a result of an earlier election in the same term ($election.eterm$).
Because a leader's log grows monotonically during its term, the leader could not have had $\langle k, term, subterm \rangle$ in its log at the start of its term, hence it could not have had $\langle index, term, subterm \rangle$ because $index \geq k$. Then there exists an earlier election with the same term in *Contradicting*; this is a contradiction.

C. Case $mterm < election.eterm$:
The leader of *mterm* must have contained $\langle index, term, subterm \rangle$ (otherwise its election would also be *Contradicting* but have a smaller term than *election*, which is a contradiction). Thus, the leader of *mterm* could not send any conflicting entries to the voter for this index, nor could it send any conflicting entries for prior indexes: that it has this entry implies that it has the entrire prefix before it (Raft lemma 4).

9. Thus, we have $\langle index, term, subterm \rangle \in voterLog$ for non-witness voter, and $\langle index', term, subterm \rangle \in voterLog$ for witness voter.

   We show contradiction in following cases which will then be used to show contradiction in each case of voting rules.

10. Case: $LastTerm(election.elog) = LastTerm(voterLog) \land Len(election.elog) \geq index$

(a) The leader of $\mathit{LastTerm}(\mathit{voterLog})$ monotonically grew its log during its term (by Lemma R.3).

(b) The same leader must have had $\mathit{election.elog}$ as its log at some point, since it created the last entry.

(c) Thus, $\mathit{voterLog}[1..\mathit{index}]$ is a prefix of $\mathit{election.elog}$.

(d) Then $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle \in \mathit{election.elog}$, since $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle \in \mathit{voterLog}[1..\mathit{index}]$.

(e) But $\mathit{election} \in \mathit{Contradicting}$ implies that $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle \notin \mathit{election.elog}$.

11. Case: $\mathit{LastTerm}(\mathit{election.elog}) > \mathit{term}$

(a) $\mathit{election.eterm} > \mathit{LastTerm}(\mathit{election.elog})$ since servers increment their $\mathit{currentTerm}$ when starting an election, and Lemma R.6 states that a server's $\mathit{currentTerm}$ is at least as large as the terms in its log.

(b) Let prior be the election in $\mathit{elections}$ with $\mathit{prior.eterm} = \mathit{LastTerm}(\mathit{election.elog})$. Such an election must exist since $\mathit{LastTerm}(\mathit{election.elog}) > 0$ and a server must win an election before creating an entry.

(c) By transitivity, we now have the following inequalities:

$$\begin{aligned}&\mathit{term} \leq \\ &\qquad \mathit{LastTerm}(\mathit{election.elog}) = \mathit{prior.eterm} < \\ &\qquad\qquad \mathit{election.eterm}\end{aligned}$$

(d) $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle \in \mathit{prior.elog}$, since $\mathit{prior} \notin \mathit{Contradicting}$ ($\mathit{election}$ was assumed to have the lowest term of any election in $\mathit{Contradicting}$, and $\mathit{prior.eterm} < \mathit{election.eterm}$).

(e) $\mathit{prior.elog}$ is a prefix of $\mathit{election.elog}$ since:

i. $\mathit{prior.eleader}$ creates entries with $\mathit{prior.eterm}$ by appending them to its log, which monotonicallygrows during $\mathit{prior.eterm}$ from $\mathit{prior.elog}$.

ii. Thus, any entry with term $\mathit{prior.eterm}$ must follow $\mathit{prior.elog}$ in all logs (by Lemma R.4).

iii. $\mathit{LastTerm}(\mathit{election.elog}) = \mathit{prior.eterm}$

(f) $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle \in \mathit{election.elog}$. Note that this is true no matter $\langle \mathit{index}, \mathit{term}, \mathit{subterm} \rangle$ existing in $\mathit{voterLog}$ or not.

(g) This is a contradiction, since $\mathit{election.elog}$ was assumed to not contain the committed entry ($\mathit{election} \in \mathit{Contradicting}$).

12. The log comparison during elections states the following, since *voter* granted its vote during *election*:

$$
\begin{aligned}
&\lor \land voter \neq WitnessID \\
&\quad \land \lor LastTerm(election.elog) > LastTerm(voterLog) \\
&\quad\quad \lor \land LastTerm(election.elog) = LastTerm(voterLog) \\
&\quad\quad\quad \land Len(election.elog) \geq Len(voterLog) \\
&\lor \land voter = WitnessID \\
&\quad \land \lor LastTerm(election.elog) > LastTerm(voterLog) \\
&\quad\quad \lor \land LastTerm(election.elog) = LastTerm(voterLog) \\
&\quad\quad\quad \land LastSubterm(election.elog) > LastSubterm(voterLog) \\
&\quad\quad \lor \land LastTerm(election.elog) = LastTerm(voterLog) \\
&\quad\quad\quad \land LastSubterm(election.elog) = LastSubterm(voterLog) \\
&\quad\quad\quad \land election.evoters \setminus WitnessID \subset ReplicationSet
\end{aligned}
$$

ReplicationSet is replication set of $LastSubterm(voterLog)$ in $LastTerm(voterLog)$ if *voter* is witness.

We now investigate each voting rule in following cases, and use the result of above two cases to show contradiction.

13. Case: *voter* is a regular server, and $LastTerm(election.elog) = LastTerm(voterLog) \land Len(election.elog) \geq Len(voterLog)$
    (a) $Len(election.elog) \geq Len(voterLog) \implies Len(election.elog) \geq index$
    (b) This leads to contradiction following case 10

14. Case: $LastTerm(election.elog) > LastTerm(voterLog)$, no matter *voter* being a regular server or witness
    (a) $LastTerm(election.elog) > LastTerm(voterLog) \implies LastTerm(election.elog) > term$
    (b) This leads to contradiction following case 11

15. Case: *voter* is witness, and $LastTerm(election.elog) = LastTerm(voterLog) \land LastSubterm(election.elog) > LastSubterm(voterLog)$
    (a) $LastSubterm(election.elog) > LastSubterm(voterLog) \implies Len(election.elog) > Len(voterLog) > term$, following Lemma 3

   (b) This leads to contradiction following case 10

16. Case: *voter* is witness, and

$$\begin{aligned}&\wedge \mathit{LastTerm}(\mathit{election}.\mathit{elog}) = \mathit{LastTerm}(\mathit{voterLog})\\ &\wedge \mathit{LastSubterm}(\mathit{election}.\mathit{elog}) = \mathit{LastSubterm}(\mathit{voterLog})\\ &\wedge \mathit{election}.\mathit{evoters} \cup \mathit{WitnessID} \subset \mathit{ReplicationSet}\end{aligned}$$

   (a) $\mathit{LastTerm}(\mathit{voterLog}) = \mathit{term} \wedge \mathit{LastSubterm}(\mathit{voterLog}) = \mathit{subterm}$

      i. There must be voter' which is a regular server that contains $\langle \mathit{index}, \mathit{term}, \mathit{subterm}\rangle$ and votes in *election*, since

         A. $\langle \mathit{index}, \mathit{term}, \mathit{subterm}\rangle$ exists in at least subquorum regular servers in replication set of *term* and *subterm*, no matter if it is shortcut replicated or not.

         B. A subquorum in replication set of *term* and *subterm* vote in *election*.

         C. Subquorums of regular servers overlap in a replication set.

      ii. This leads to contradiction following case 13 and 14

   (b) $\mathit{LastTerm}(\mathit{voterLog}) = \mathit{term} \wedge \mathit{LastSubterm}(\mathit{voterLog}) > \mathit{subterm}$

      i. $\mathit{LastTerm}(\mathit{voterLog}) = \mathit{term} \wedge \mathit{LastSubterm}(\mathit{voterLog}) > \mathit{subterm} \implies \mathit{Len}(\mathit{voterLog}) > \mathit{index}$, per Lemma 3

      ii. This leads to contradiction following case 10

   (c) $\mathit{LastTerm}(\mathit{voterLog}) > \mathit{term}$. This leads to contradiction following case 11

Having Lemma 4 proved, Lemma R.9 and Theorem R.1 are true, following the same proof in Raft algorithm. Now we have Leader Completeness and State Machine Safety proved for the extended Raft algorithm. □