SUMMER UNDERGRADUATE RESEARCH FELLOWSHIPS

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Investigating Kac-Moody Algebras

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History

Pre-Lie Theory

- 1850s : Development of Group Theory
  (Galois, Cayley - aimed at algebraic applications)
- 1872 : Erlangen Program
  (Klein on using groups to study geometry)

Lie Theory

- 1870s : Introduction of Lie Algebras
  (Sophus Lie on these "infinitesimal transformations")
- 1880s-1890s : Classification
  (Killing and Cartan classify Lie Algebras)
- 1900-1950 : Further Work
  (by Weyl, Chevalley, Harish-Chandra, and others)

Kac-Moody Algebras and Applications

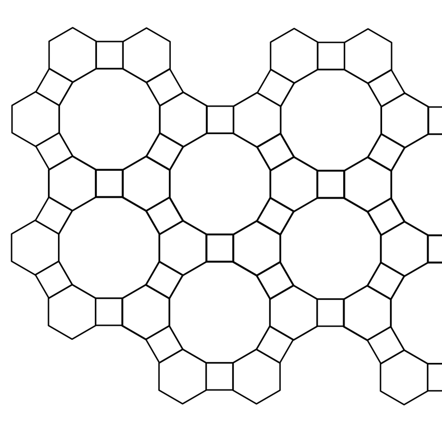
- 1960s : Kac-Moody Algebras
  (Kac and Moody independently define these algebras)
- 1990s : Quantum Knot Invariants
  (work by Drinfeld, Jimbo and Reshetikhin-Turaev)
- 1990s : Vertex String Algbreas
  (Connection with Kac-Moody algebras by Gebert et al.)
- 1990s : Kac-Moody, Modular forms
  (Connection demonstrated by Borcherds)
- 2015 : Umbral Moonshine proved
  (in a paper by Duncan, Griffin and Ono)

Introduction and Definitions

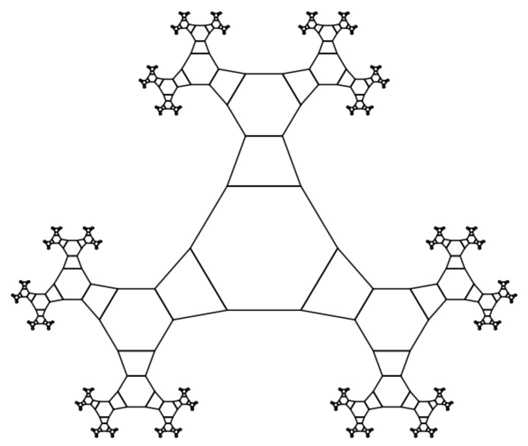
The study of finite-dimensional Lie algebras and groups in mathematics is often motivated by the study of symmetry. One can wonder what happens when we consider the infinite-dimensional analogues. For example, one interesting class are the affine Lie algebras, those who have a $\mathbb{Z}$-gradation, along with the property that they only grow polynomially with respect to the gradation. We can consider an even more general class of Lie algebras, intuitively "those that can be built from a Cartan matrix following a certain procedure". These are the Kac-Moody algebras.

A *realization* of a rank-ℓ matrix $A = (a_{ij})_{i,j=1}^{d}$ is a $(2d-\ell)$-dimensional inner-product space $\mathfrak{h}$ equipped with a linearly independent set $\{\alpha_1, \ldots, \alpha_n\}$, whose elements are called *simple roots*, such that the inner products $\langle \alpha_i, \alpha_j \rangle = a_{ij}$. A *generalized Cartan matrix* is a matrix $A = (a_{ij})_{i,j=1}^{d}$ such that the diagonal entries $a_{ii} = 2$, off-diagonal entries $a_{ij} \le 0$, and $a_{ij} = a_{ji}$.

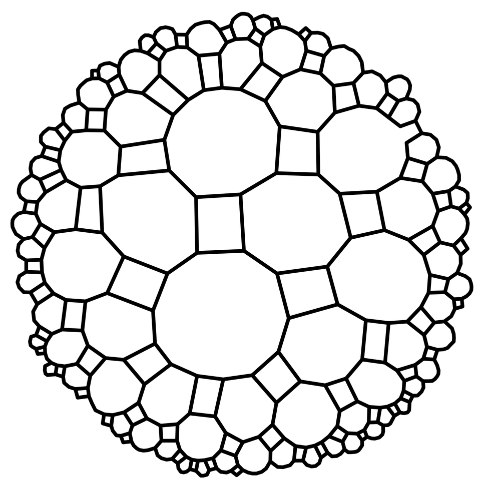
Let A be a generalized Cartan matrix, and let $\mathfrak{h}$ be a realization of A. The *Kac-Moody algebra* $\mathfrak{g}$ defined by A and $\mathfrak{h}$ is the freeest Lie algebra with generators $\mathfrak{h} \cup \{e_1, \ldots, e_d, f_1, \ldots, f_d\}$ and relations $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$, $[h, e_i] = \langle \alpha_i, h \rangle e_i$, $[h, f_i] = -\langle \alpha_i, h \rangle f_i$, $[e_i, f_j] = \delta_{ij}\alpha_i$, and if $c_{ij} \le 0$ then $(\mathrm{ad}\ e_i)^{1-c_{ij}} e_j = 0$ and $(\mathrm{ad}\ f_i)^{1-c_{ij}} f_j = 0$.



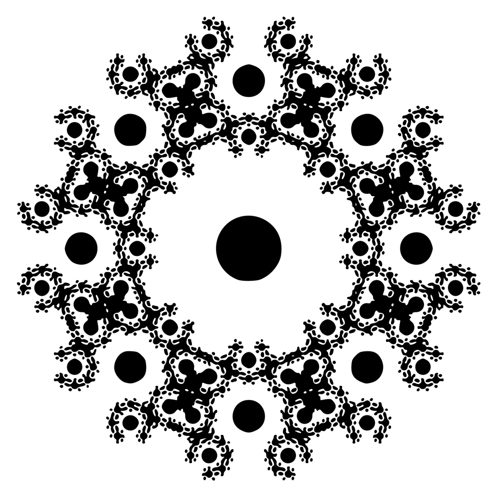
$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & -1 & 2 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -3 \\ -1 & -1 & 2 \end{bmatrix}$$

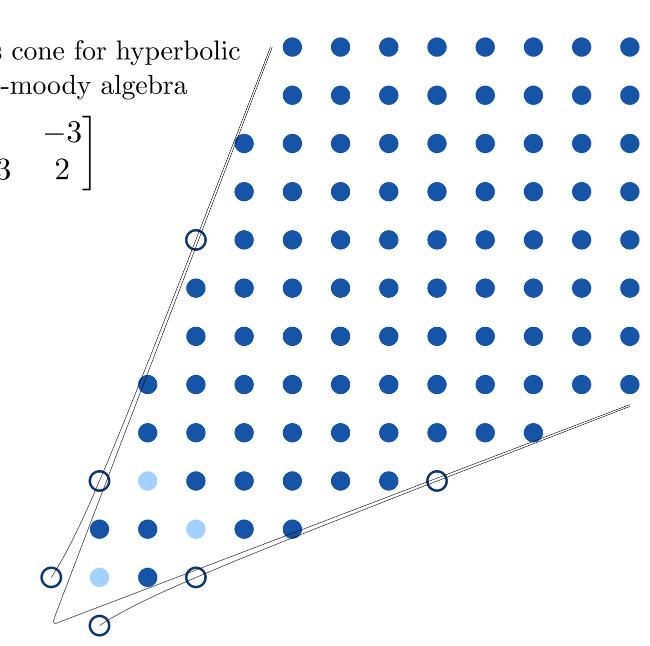


$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Shown here are some geometric representations of the 'strucutre' of Kac-Moody algebras. To be more specific, the left three diagrams are Cayley graphs of the Weyl Groups of the matrices shown below them. The first Kac-Moody algebra is affine - another way to understand this is that the Cayley graph can be realized on the Euclidean plane. The second and third are of hyperbolic Kac-Moody algebras, - realized on the hyperbolic plane. And the last diagram represents some of the structure of the (finite dimensional) E_8.

Tits cone for hyperbolic Kac-moody algebra

$$\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$$



Notice that the real roots are the integer points of a conic section, and their density is logarithmic in height. Given an imaginary root, can you "pingpong" it back down into the semigroup generated by the Hilbert basis?

The *multiplicity* $m(\beta)$ of a root $\beta \in \Delta^+$ is the dimension of the *root space* $\mathfrak{g}_\beta$ of all $x \in \mathfrak{g}$ such that for every $h \in \mathfrak{h}$, $[h, g] = \beta(h)(g)$. The multiplicity of a real root is always 1. For a classical Lie algebra, every root is real; so multiplicity theory is very specific to Kac-Moody algebras.

Our algorithm to compute $m(\beta)$ for imaginary β is based on the famous Peterson recurrence formula: for every root $\beta \in \Delta^+$,

$$c(\beta) = \frac{1}{(\beta, \beta - 2\rho)} \sum_{\gamma \prec \beta} (\gamma, \beta - \gamma) c(\gamma) c(\beta - \gamma)$$

where $c(\beta) = \sum_\gamma m(\gamma)/(\beta/\gamma)$ where the sum is over all divisors γ of β. Ask us how it works!

We're also interested in how multiplicities $m(\beta)$ grow as the "norm" $\sqrt{-(\beta, \beta)}$ tends to infinity. As an example, under reasonable circumstances, if $\mathfrak{g}$ is a rank-d algebra,

$$m(\beta) \lesssim d^{\sqrt{-(\beta,\beta)}}.$$

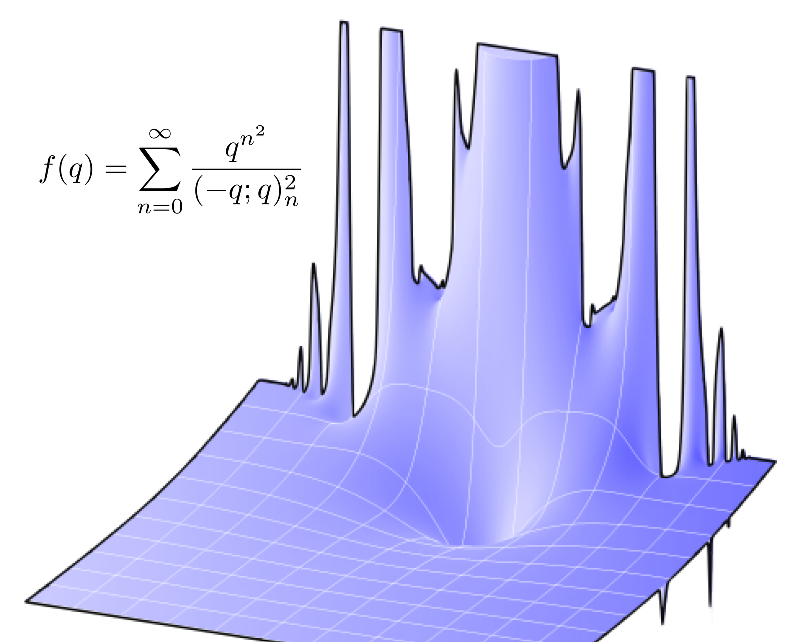
Of particular interest are those Kac-Moody algebras which give denominator identities that can be written in terms of modular forms, since the resulting identities can have number theoretic significance. It is not well understood when there should be such a connection between a Lie algebra and a modular form, but generalizing to Lie superalgebras and multivariable mock modular forms gives a less anomalous connection.

Ramanujan gave the first examples of mock modular forms in 1920 with his mock theta functions, and almost 100 years later Zwegers gave a formal definition for them in his PhD thesis. Kac and Wakimoto then use an analogous definition of multivariable mock modular form to rewrite the (super)character formulae for certain classes of Lie superalgebras in terms of the mock modular forms given by

$$\Theta^{\pm}_{\Lambda,T} = e^{-\frac{(\Lambda|\Lambda)}{2K}\delta} \sum_{\alpha \in L^\#} \varepsilon_{\pm}(t_\alpha)\, t_\alpha \frac{e^{\Lambda}}{\prod_{\beta \in T}(1 \pm e^{-\beta})}$$

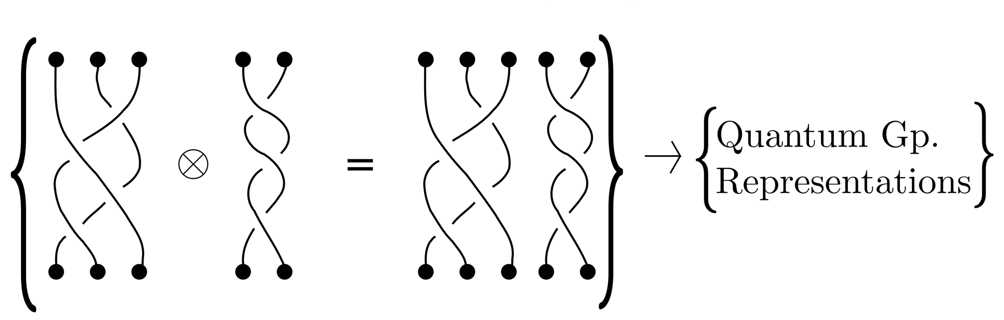
(arxiv: 1308.1261)

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$



Mock theta functions are given by a q-expansion on the unit disk, with poles along the unit circle. Expanding $q = e^{2\pi i \tau}$ gives the mock theta function on the upper half plane, with poles along the real axis.

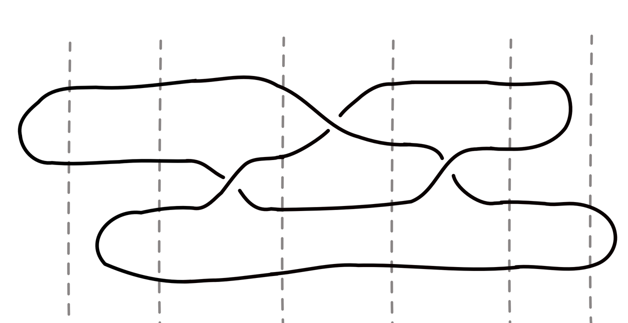
We can describe link invariants in modern language as a functor out of a suitable 'Category of Tangles' (example shown on right) to a suitable Algebraic category. In the case of Quantum Invariants (Reshetikhin-Turaev), we choose the target category to be the representations of a Quantum Group, a particular algebraic structure which can be built from a Kac-Moody algebra.



(left): Monoidal structure in the Tangle category

(right): Target category, which has compatible structure.

Research Question: We can construct Quantum Groups through Kac-Moody algebras by the Drinfeld-Jimbo construction. It is known that certain classical knot invariants come from this construction, e.g. the Jones Polynomial comes from the quantum knot invariant associated to $U_q(SL(2))$. What can we say about the knot invariants associated to related kac-moody algebras, such as the affinisation of SL(2)? Will we have twisted versions of our classical invariants?



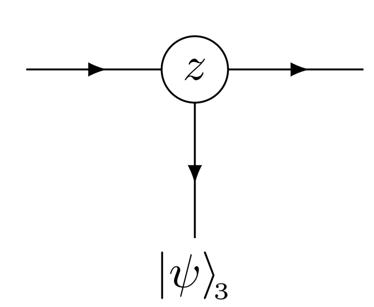
Expressing a trefoil knot in an 'easy basis', where we interpret the knot as a movie from left to write, with dotted lines representing specific vector spaces and pieces of knots representing particular linear maps.

More generally, we can think of this construction similarly to a TQFT.

In theoretical physics, String Theory refers to the proposition that particles can be described as vibrational modes of a string. Decomposing the motion of such a string into normal modes gives it the structure of a vector space, and imposing quantum mechanics on it essentially converts that vector space into an operator algebra, which we call a *Fock* Space.

Once such a Fock Space is constructed, a new object which can be considered is the *vertex algebra*, referring to formal Laurent series with coefficients in the Fock space algebra, which was originally developed to handle scattering processes in string theory. Another object which we can construct is the multi-string vertex, which is defined as a linear map from N- copies of the Fock space to the complex plane.

$\mathcal{V}(\psi, z) \quad \longleftrightarrow$



(arxiv: 9505106)

Given a vertex operator it is possible to construct a 3-string vertex operator, which we formally define as a map from three tensored copies of our Fock space to the complex plane. Pictorially, these can be visualized as a vertex with three strings coming off, and we can describe string scattering processes by "tying" these 3-vertices together to make N-vertices.