Physics 112 : Lecture 21

Notes for the Fall 2017 Physics 112 Course taught by Professor Holzapfel
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1 White Dwarfs

For white dwarfs, when the mass isn't too large; we have that:

$$RM^{1/3} = \frac{\gamma \hbar^2}{G m_e m_p^{5/3}}$$

where $\gamma \approx 4.5$ (which is a little different to our simple homogenous density expression we found from last lecture). As we keep increasing the mass, eventually the particles become relativistic, and things start collapsing quickly. This phenomenon occurs roughly when the energy predicted from nonrelativistic theory becomes comparable with the energy in the relativistic limit, i.e.

$$\frac{3}{8}\frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^3} \approx \frac{9}{16}\frac{\hbar c}{R}\left(\frac{M}{m_p}\right)^{4/3}$$

And proceeding, we find that:

$$M^{2/3} = \frac{\delta \hbar c}{m_p^{4/3} G}$$

where $\delta \approx 2.13$ is an order one constant. This mass is known as the Chandrasekhar mass, and if we solve it numerically, it comes out to be roughly 1.45 times the mass of the sun. If we then solve for the radius, we find that

$$R \approx 1.7 \cdot 10^7 \approx 2R_E$$

i.e. roughly twice the radius of the Earth. Now, the fermi energy for the electrons is roughly given by the relativistic limit ($m_e c^2$), which is roughly 10^5 electron Volts. This corresponds to a fermi temperature of roughly 10^9, which is much higher than our observed temperatures of white dwarfs, which are roughly 10^7 Kelvin. Hence, we are justified in dealing with the system as a degenerate fermi gas, since the thermal energy scale is much smaller than the Fermi energy scale.

2 Neutron Stars

In the case of neutron stars, our equiations are pretty similar except we substitute in neutron constants instead of electron constants everywhere:

$$U = \frac{3}{5}\epsilon_F N \approx \frac{3\hbar^3}{m_n}\frac{N^{5/3}}{V^{2/3}}$$

$$V = \frac{4}{3}\pi R^3; \qquad N = \frac{M}{m_n}$$

$$U \approx \frac{\hbar^2}{m_n^{5/3}}\frac{M^{5/3}}{R^2}$$

Now we can do the same thing as before, which is add this energy to the gravitational potential energy, set the derivative of the final expression with respect to R to be equal to 0. Solving this, we find that:

$$RM^{1/3} \approx \frac{10}{3}\frac{\hbar^2}{Gm_n^{5/3}}$$

Now, associated with this final solution, we have many crazy statistics, like

$$M \approx 2M_S; \qquad R \approx 10^4$$

$$\rho \approx 10^{12}\text{kg}m^{-3}; \qquad g \approx 10^{12}\text{m}s^{-2}$$

And, these neutron stars can generate magnetic field, by the spins of the neutron aligning. The magnetic field generated by the neutron star is roughly

$$10^4 - 10^1 1T$$

which is crazy; since 5 Tesla is enough to levitate a frog.

3 Bose-Einstein condensation

We know our energy levels are given by:

$$\epsilon = \frac{\hbar^2 k^2}{2m} = \left(\frac{\hbar^2}{2m}\right)\left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$

Let's consider the energy difference between the ground state and the first excited state. Substituting in $n_x, n_y, n_z = 1, 1, 1$ and $n_x, n_y, n_z = 2, 1, 1$. Now we find that the energy gap is:

$$\Delta\epsilon = \frac{3\hbar^2\pi^2}{2mL^2}$$

Now, with $L = 0.01$ and substituting in the mass of the helium, we find that:

$$T \approx \frac{\Delta\epsilon}{k_b} \approx 1.8 \cdot 10^{-14} K$$

i.e. the temperature we would expect to need to get down to to observe a large proportion of particles in the ground state (so called Bose-Einstein condensation) is very tiny, a temperature we would never expect to find in the laboratory. How then do we observe such a phenomenon? It turns out that Bose-Einstein statistics helps us out, and causes this phenomenon to appear at much higher temperatures that we would naively expect. Recall that the Bose-Einstein distribution is given:

$$f(\epsilon, \tau) = \frac{1}{e^{(\epsilon-\mu)/\tau} - 1}$$

Now, to have a high proportion of particles in our ground state ϵ_0, we would need

$$e^{(\epsilon_0-\mu)/\tau} \approx 1; \qquad \frac{\epsilon_0 - \mu}{\tau} \approx 0$$

Now, taylor expanding our exponential, we find:

$$N_0 = \frac{1}{1 + (\epsilon_0 - \mu)/\tau - 1} \approx \frac{\tau}{\epsilon_0 - \mu}$$

where N_0 is the number of particles in our ground state. Solving for the chemical potential, we find that:

$$\mu = \epsilon_0 - \frac{\tau}{N_0}$$

So we can see that if we have a large number of particles in the ground state (*without regard for how many particles are in excited states*), then μ must be very very close to ϵ_0. Now, substituting our formula for the chemical potentital back into our Bose-Einstein distribution:

$$f(\epsilon, \tau) = \frac{1}{e^{(\epsilon-(\epsilon_0-\tau/N_0))/\tau} - 1} \approx \frac{1}{e^{(\epsilon-\epsilon_0)/\tau}e^{1/N_0} - 1}$$

Now, as we saw, the energy gap $\Delta\epsilon$ is always much smaller than the thermal excitation energy τ. Now, being very wishy-washy and dropping the N_0 term, and doing a taylor expansion in the other exponential, we find that:

$$f(\epsilon, \tau) \approx \frac{\tau}{\epsilon - \epsilon_0}$$

Now, we can find that:

$$\frac{f_1}{f_0} = \frac{\tau}{N_0(\epsilon_1 - \epsilon_0)}$$

so to get comparable amounts of particles in the first excited state compared to the ground state, we want $f_1 \approx f_0$, we need that

$$\tau \approx N_0(\epsilon_1 - \epsilon_0) = N_0\Delta\epsilon$$

Recall that naively, we thought we would just need that $\tau \approx \Delta\epsilon$, which turned out to be very small, but in actuality, we need a temperature N_0 times bigger, which is much bigger. Because of the Bose statistics, the N_0 creeps in, making Bose-Einstein condensates achievable in standard laboratory conditions.