Physics 112 : Lecture 3

Notes for the Fall 2017 Physics 112 Course taught by Professor Holzapfel
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August 31, 2017

1 More Magnetism

Recall from lecture 2 that we defined $g(N,s)$ to be the multiplicity of the macrostate with spin excess s when there are N particles in total. Since all the microstates are equally likely, we know that:

$$P(N,s) = \frac{g(N,s)}{2^N}$$

where $P(N,s)$ is the probability of finding the system with spin excess s when measured. Recalling our approximations from last time (using Stirling's approximation assuming $N >> 1$ and using taylor series approximation for log using $N >> s$), we derived that:

$$g(N,s) \approx \left(\frac{2}{\pi N}\right)^{1/2} 2^N e^{-2s^2/N}$$

As a first check to see whether or not this expression for the number of microstates is correct, we know that the total number of microstates is given by 2^N, so we'd expect that the sum of $g(N,s)$ over all possible values of s gives 2^N. Since $N >> 1$ however, we can approximate this sum with an integral, and so we have:

$$\int_{-\infty}^{+\infty} g(N,s)ds = 2^N \left(\frac{2}{\pi N}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-2s^2/N} ds = 2^N$$

where we can evaluate the final integral by the substitution $u = \sqrt{2/N}s$ and noticing that the resulting integral is just the Gaussian integral. [1] Hence, our multiplicity function $g(N,s)$ has the correct normalization (though it should be noted that it is still only an approximation). Evidently, we have then that:

$$\int_{-\infty}^{+\infty} P(N,s)ds = 1$$

which we should expect.

[1] For the Gaussian integral, if $I = \int_{-\infty}^{+\infty} e^{-x^2}dx$, notice that $I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2}e^{-y^2}dxdy = \int_0^{+\infty}\int_0^{2\pi} e^{-r^2} r d\theta dr = \pi$, so $I = \sqrt{\pi}$.

Now, notice that:

$$g(N,0) = \frac{N!}{[(\frac{1}{2}N)!]^2} \approx \left(\frac{2}{\pi N}\right)^{1/2} 2^N$$

where the first expression is exact, and the second is our approximation. Note that at $N = 50$, our approximation is already correct to about 1%, for larger systems this error would vanish quickly.

To get a better understanding of how sharply peaked the $g(N,s)$ function is at $s = 0$, note that the function scales with $e^{-2s^2/N}$, so when $s = \sqrt{N/2}$ the multiplicity has dropped by a factor of e. Crucially, as N gets larger and larger, $\sqrt{N/2}$ becomes a smaller and smaller fraction of N, so for extremely large N the multiplicity is extremeley peaked at $s = 0$.

Suppose that we have a function $f(s)$, and we want to find $\langle f \rangle$. Note that this is given exactly by:

$$\langle f \rangle = \sum_s f(s) P(N,s)$$

and approximately by:

$$\langle f \rangle = \left(\frac{2}{\pi N}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-2s^2/N} f(s) ds$$

so for example, if $f(s) = s$ then:

$$\langle s \rangle = \left(\frac{2}{\pi N}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-2s^2/N} s ds = 0$$

since $e^{-2s^2/N}$ is an even function of s and $f(s) = s$ is an odd function of s, an even function multiplied by an odd function is an odd function, and integrating an odd function over a symmetric domain yields 0. On the contrary, if $f(s) = s^2$ we find:

$$\langle s^2 \rangle = \left(\frac{2}{\pi N}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-2s^2/N} s^2 ds = \frac{N}{4}$$

where we can evaluate the last integral by the same substitution as before, $u = \sqrt{2/N}$. Now, we can calculate:

$$\sigma_s = \sqrt{\langle s^2 \rangle - \langle s \rangle^2} = \sqrt{\frac{N}{2}}$$

2 Einstein Solids

Suppose that we model simple solids as consisting of a lattice of quantum oscillators, each of which at any moment of time has a certain number of quanta of energy. You can imagine the system as an array of buckets, each of which has some number of balls inside them. If there are N oscillators and q quantas of energy (that are shared between all of the oscillators), then a specific method of distributing the quantas of energy amongst all the buckets is a microstate of the system, and the macrostate is given by the total number, q, of quantas of energy.

To find the multiplicity, $g(N, q)$ of this system, we use a combinatorial trick known affectionately as stars and bars. To count the number of ways of putting q units of energy into N buckets, we place all q units of energy in a line, along with $N-1$ dividers (that divide the line into N different regions) and ask the equivalent question of how many ways there are of permuting all these elements. For example, suppose that $N = 3$ and $q = 4$. In the following table, we list 4 different ways of distributing the 4 units of energy into 3 different buckets, (the first way being all 4 into the third bucket, then one in the second and three in the third, then two in the first, one in the second and one in the third, and finally one in the first and three in the third) and the corresponding stars and bars diagram. Clearly, counting the number of permutations for the stars and bars problem is equivalent to counting the number of ways of distributing units of energy amongst oscillators.

$(0,0,4)$			$\star\star\star\star$
$(0,1,3)$		$\star$	$\star\star\star$
$(2,1,1)$	$\star\star$	$\star$	$\star$
$(1,0,3)$	$\star$		$\star\star\star$
...	...		

The stars and bars problem is relatively easy to solve, for N buckets and q units of energy then:

$$\text{Number of permutations } = {}^{q+N-1}C_q = \frac{(q+N-1)!}{q!(N-1)!}$$

(since there are $(q+N-1)$ elements in total and we want to choose q of them to be stars rather than bars). In the limit where $N >> 1$, we can simplify this down to $(q+N)!/(q!N!)$. This model for a solid that we have described is known as an Einstein Solid.

Suppose now that we have two different Einstein Solids, the first with N_A oscillators and q_A units of energy, and the second with N_B oscillators and q_B units of energy. When they don't interact with each other, but we consider them as a single system, then the total multiplicity of the system is given by:

$$g_T = g_A(N_A, q_A) g_B(N_B, q_B)$$

Suppose now that we place them in thermal contact, which means that they can now share the energy that they both own. Now, we have many more microstates, since previously impossible microstates (such as all $N_A + N_B$ units of energy being placed in a single oscillator) are now possible. Let $q = q_A + q_B$ and $N = N_A + N_B$, then for two Einstein Solids in thermal contact the multiplicity is given by:

$$g = {}^{q+N-1}C_q = \sum_{q_A=0}^{q_A=q} g_A(N_A, q_A) g_B(N_B, q_B)$$

where the sum reflects the fact that we can count the total number of microstates by counting the number of microstates where all the energy is in solid A, plus the number of microstates where a single unit is in B, plus the number of microstates where two units are in B, and so on. It turns out that for any two systems in thermal contact, they will most probably be found in the macrostate (where the macrostates are specific partitions of energy into solid A and solid B) which maximises the number of microstates (and soon, Entropy).