Physics 112 : Lecture 10

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1 Photons of constant frequency in a Cavity

Suppose we have some cavity, which has some number of photons contained within it; that is in contact with a thermal reservoir at temperature τ. There are all sorts of questions we might want to ask of our cavity; for instance: What is the average number of photons inside the cavity? What is the distribution of their frequencies? What is the average energy inside the cavity? And so on. The goal of this lecture is to answer some of these questions using the tools we have developed.

First, suppose that we consider photons only of a definite frequency ω inside the cavity. Our energy levels are given by:

$$\epsilon_s = s\hbar\omega$$

where s is the number of photons of frequency ω inside the cavity, and so our partition function is given by:

$$Z = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}}$$

where we evaluated the infinite geometrical sum in the last equality. So, by our theory of partition functions, we have that the probability of finding the system to be in state s is given by:

$$P(s) = \frac{e^{-s\hbar\omega/\tau}}{Z}$$

so the average of s is given:

$$\langle s \rangle = \sum_{s=0}^{\infty} sP(s) = \frac{1}{Z}\sum_{s=0}^{\infty} se^{-s\hbar\omega/\tau} = \frac{1}{e^{\hbar\omega/\tau} - 1}$$

where we can evaluate the final sum by simply noticing that it is the derivative of a geometric series. So, we know that $\langle \epsilon_s \rangle = \langle s\hbar\omega \rangle = \hbar\omega\langle s \rangle$, so for this system of same frequency photons inside a box, we know the average energy.

Perhaps we might want to apply this mathematics to a different system, where there is a zero point energy; so suppose that in a different system,

$$\epsilon_s = (s + 1/2)\hbar\omega$$

The probabilities of each of the states however remain unchanged:

$$P(s) = \frac{e^{-(s+1/2)\hbar\omega/\tau}}{\sum_0^\infty e^{-(s+1/2)\hbar\omega/\tau}} = \frac{e^{-s\hbar\omega/\tau}}{\sum_0^\infty e^{-s\hbar\omega/\tau}}$$

which are the same probabilities we had in our first case, and also we have:

$$\langle\epsilon_s\rangle = \langle s\rangle\hbar\omega + \frac{1}{2}\hbar\omega$$

so the mathematics describing the system doesn't change much; even if we introduce a zero point energy.

Going back to our original system of equal frequency photons inside a cavity, note that there are two limits that we might care about. The first is when $\tau >> \hbar\omega$, what we might call the 'classical limit', where the energy spacings are incredibly small compared to the thermal energy. In this case, our formula for the average energy simplifies:

$$\langle\epsilon\rangle = \hbar\omega\langle s\rangle = \frac{\hbar\omega}{\hbar\omega/\tau} = \tau$$

where we taylor expand the exponential in the denominator of $\langle s\rangle$. This intuitively makes sense; in the limit where 'quantum behaviour' doesn't exist, the energy inside the box is just the thermal energy. In the other limit, where $\tau << \hbar\omega$, we have a situation where the energy spacing dominates the thermal energy (for instance, consider high energy gamma rays inside a cube made of ice), so we would not expect many photons to be inside the box. Indeed, in this case we have:

$$\langle s\rangle \approx e^{-\hbar\omega/\tau}$$

since the -1 term in the denominator becomes negligible compared to $\hbar\omega/\tau$.

2 Photons of arbitrary frequency in a rectangular prism with reflective walls

Now that we have solved for specific frequencies of photons inside a cavity, we want to generalise to arbitrary frequencies. This means that we somehow need to figure out how many photons per frequency are within the cavity. To do this, recall the classical equations governing an electromagnetic wave:

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}; \qquad \vec{E} = Ae^{i\vec{k}\cdot\vec{r}}e^{-i\omega t} = \vec{E}(\vec{r})e^{-i\omega t}$$

where $\vec{k}$ is the vector wavenumber for the wave, which satisfies $|k| = \omega/c$. This is the solution of a wave in free space, but we want to somehow introduce a rectangular prism; so the idea is that we tile all of space with tessellating copies of a rectangular prism (that has length L_x in the x direction, L_y in the y direction, and L_z in the z direction), and impose spatial symmetry on the electric field:

$$\vec{E}(x,y,z) = \vec{E}(x+L_x,y,z) = \vec{E}(x,y+L_y,z) = \vec{E}(x,y,z+L_z)$$

we can think of this condition as requiring the electric field to form standing waves inside any single rectangular prism; which makes sense if the walls of the prism are reflective. Note that we have:

$$\vec{E}(x,y,z) = Ae^{i\vec{k}\cdot\vec{r}} = Ae^{i(k_x x + k_y y + k_z z}$$

so the spatial symmetry condition enforces the relationships:

$$k_x = \frac{2\pi}{L_x}n_x; \qquad k_y = \frac{2\pi}{L_y}n_y; \qquad k_z = \frac{2\pi}{L_z}n_z$$

where n_x, n_y, n_z are arbitrary positive integers. Note that since $|k| = \omega/c$, we have:

$$\omega = 2\pi c\left[\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2\right]^{1/2}$$

hence we have:

$$\epsilon = \hbar\omega = hc\left[\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2\right]^{1/2}$$

This is good news, because now we have the energy of the system as a function of its quantum numbers, n_x, n_y, n_z! We can define a quantity ρ_k which is the density of states in k-space, i.e. we have:

$$\rho_k d^3k = \Delta n_x \Delta n_y \Delta n_z = \left(\frac{L_x L_y L_z}{8\pi^3}\right) dk_x dk_y dk_z$$

which can be thought of as the 'number of states per unit volume in k-space', where k-space is the space of 'all possible photons'. However, we don't actually care about the direction of k, since we have $|k| = \omega/c$, all we really care about is the magnitude. So we have the related quantity:

$$\rho_{|k|} d|k| = \rho_k d^3k = \left(\frac{L_x L_y L_z}{8\pi^3}\right)(4\pi k^2) d|k|$$

where we use spherical coordinates to convert our cartesian differentials dk_i to our radial differential $d|k|$. Now, using $|k| = \omega/c$, we finally reach:

$$\rho_\omega d\omega = \frac{V}{2c^3\pi^2}\omega^2 d\omega$$

where V is the volume of the box, and $\rho_o mega$ is our density of states in frequency space, *with one small caveat*! Throughout this entire derivation; we never considered the polarizations of the photons; but it turns out that photons have two orthogonal polarizations;

and so we have to *double* the density of states that we derived. All together, we can now find the energy of the system:

$$U = \sum_n \langle \epsilon_n \rangle = \sum_n \langle s_n \rangle \hbar\omega_n = \sum_n \frac{\hbar\omega_n}{e^{\hbar\omega/\tau} - 1} = \int_0^\infty \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1} \cdot \frac{V}{\pi^2 c^3}\omega^2 d\omega$$

where we can turn the sum over states into an integral provided we include the appropriate density of states factor into our integral. We can actually evaluate this integral, and we find that:

$$u = \frac{U}{V} = \frac{\pi^2}{(\hbar c)^3 \cdot 15}\tau^4$$

where u is the energy density. Note that u scales with τ^4, this is known as Stefan-Boltzmann's Law.

3 Specific Intensity and Specific Flux

[1] We have two related notions of Specific Intensity and Specific Flux. Consider a stream of photons; impinging on some surface A. We can write the energy flow through A as:

$$dE = I_\nu dA dt d\Omega d\nu$$

where dE is a small unit of energy, I_ν is the specific intensity at frequency ν, dt is a small element of time, $d\Omega$ is a small element of solid angle to integrate over, and $d\nu$ is a small change in frequency. A related quantity is the specific flux:

$$dF_\nu = I_v \cos\theta d_\Omega$$

where θ is the angle at which the photons are impinging the plate. You can think of this as maximum when the photons come in perpendicular.

[1]This section is unrelated to the previous sections