Physics 112 : Lecture 17

Notes for the Fall 2017 Physics 112 Course taught by Professor Holzapfel
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1 Fermi-Dirac, versus Bose-Einstein

From quantum mechanics, we know that all particles fall into two categories, bosons (with integer spin) and fermions (with half-integer spin). Mathematically, this corresponds to whether states of identical particles are symmetric or antisymmetric under permutation operators. We have normal examples, like an electron (spin half) is a fermion, a photon (spin 1) is a boson, Helium 4 (as a composite particle, where we combine the spins of each of its component particles) is a boson, and Helium 3 is a fermion.

Fermions, by nature, obey the Pauli Exclusion principle, which says that if you have a group of identical fermions, each must be in a distinct state. So, we can now begin deriving the Fermi-Dirac distribution, which holds for fermions. Each state is either occupied or not (by Pauli Exclusion), so the Gibbs sum for a single energy level:

$$\zeta = \sum e^{(N\mu - \epsilon_s)/\tau} = 1 + e^{(\mu-\epsilon)/\tau} = 1 + \lambda e^{-\epsilon/\tau}$$

[1] Now, the average number of particles occupying a particular energy is:

$$f(\epsilon) = \langle N(\epsilon) \rangle = \frac{\lambda e^{-\epsilon/\tau}}{1 + \lambda e^{-\epsilon/\tau}} = \frac{1}{\lambda^{-1}e^{\epsilon/\tau} + 1}$$

$$f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/\tau} + 1}$$

which is the Fermi Diract distribution. As we see, the probability a state of energy ϵ is occupied depends on the size of ϵ with respect to μ, which makes sense, as μ describes the chemical potential energy (energy required in moving particles around).

Now, moving onto the Bose-Einstein distribution. In this case (when we are dealing with bosons), each energy level can be filled with any number of bosons. So the Gibbs Sum is:

$$\zeta = \sum_{N=0}^{\infty} e^{N(\mu-\epsilon)/\tau} = \sum_{N=0}^{\infty} \left[e^{(\mu-\epsilon)/\tau} \right]^N$$

Doing this infinite sum, we find that:

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[1]Note that I've changed notation in my notes, chemical potentital is now denoted by μ, absolute activity by λ

$$\zeta = \frac{1}{1 - \lambda e^{-\epsilon/\tau}}$$

Performing an analogous sum to find the average number of particles occupying the energy state, we have:

$$f(\epsilon) = \lambda \frac{\partial \ln \zeta}{\partial \lambda} = \frac{1}{\lambda^{-1} e^{\epsilon/\tau} - 1}$$

where we have, as opposed to the Fermi distribution, a minus sign instead of a plus sign in the denominator! Note that at high energies, the two distributions are equal, since in the boson case, we are only ever going to get zero or one particle in the state (since the state is so large energy). The distributions become very different when $\epsilon \approx \mu$, since when this occurs, in the bose case, the number of particles that want to be in the energy state will explode (to infinity). So, in the limit where $\epsilon >> \mu$, we are in the classical limit, where both distributions are equal.

For this classical limit, we want our ϵ to be such that

$$e^{(\epsilon - \mu)/\tau} >> 1$$

for all our energy levels ϵ. But this requires that:

$$\lambda = e^{\mu/\tau} << 1$$

which, recalling from previous lectures, means that:

$$\frac{n}{n_Q} << 1$$

subtituting our formula for $\lambda = n/n_Q$ into our classical limit distribution, we have:

$$f(\epsilon) = \frac{n}{n_Q} e^{-\epsilon/\tau} = \frac{N}{n_Q V} e^{-\epsilon/\tau}$$

Note that the $N/n_Q V$ is just a 'uniform distribution' where we distribute the N particles amongst all the energy states uniformly, and the exponential term is the 'weighting term'. Now, another thing we can do is look at the explicit formula for the chemical potentital:

$$\mu = \tau \ln \left(\frac{n}{n_Q} \right) = \tau \left[\ln N - \ln V - \frac{3}{2} \ln \tau + \frac{3}{2} \ln \left(\frac{2\pi \hbar^2}{m} \right) \right]$$

Now, a few caveats for dealing with the real world. If we had an external potential, recall that we just add that term to the chemical potentital, so nothing really changes. Another situation is when we consider when atoms have spin, and we have extra degeneracy. In this case, we can use the 'trick' that $n_Q' = (2s + 1)n_Q$, i.e. we just multiply the quantum concentration by the degeneracy, since we have 'more ways' of putting particles in the system. Finally, we can consider internal degrees of freedom. What we do then is that the new partition function is the old one, multiplied by the partition function given by the internal degrees of freedom.

Note that:

$$\mu(N) = \left(\frac{\partial F}{\partial N}\right)_{\tau,V} = F(N, \tau, V) - F(N-1, \tau, V)$$

$$\sum_{N=1}^{N} F(N, \tau, V) - \sum_{N=1}^{N} F(N-1, \tau, V) = \sum_{N=1}^{N} \mu(N)$$

$$F(N, \tau, V) = \sum_{1} N\mu(N); \qquad \mu(N) = \tau \ln\left(\frac{N}{n_Q V}\right)$$

$$F(N, \tau, V) = \tau\left[\ln\frac{N}{V n_Q} + \ln\frac{N-1}{V n_Q} + \cdots\right] = \tau \ln\left(\frac{N!}{(V n_Q)^N}\right)$$

$$F(N, \tau, V) = N\tau\left[\ln(n/n_Q) - 1\right] = N(\mu - \tau)$$

where we use the Stirling's approximation. Physically, what this says is that if $\tau = \mu$, it doesn't take any energy to build the system, it comes for free. We can keep squeezing this relation for what its got,

$$p = -\left(\frac{\partial F}{\partial V}\right)_{\tau,N} = \frac{\tau N}{V}$$