Physics 112 : Lecture 20

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1 Fermi Gas Heat Capacity

Let's consider the heat capacity for a fermi gas. Note that only the particles in the uppermost energy levels (those swimming atop the Fermi sea: at or near the Fermi energy) have the capability of being thermally excited, all the other particles are in some sense "trapped" in a state of energy lower than the fermi energy. In this sense, we expect a great variation in the heat capacity.

Theoretically, we have that the density of states:

$$D(\epsilon) = \frac{\partial N}{\partial \epsilon} = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}$$

With this, we find that:

$$N = \int_0^\infty d\epsilon D(\epsilon) f(\epsilon, \tau, \mu)$$

where f is the Fermi distribution, telling us the likelihood of a state ϵ being filled. We also have:

$$U = \int_0^\infty d\epsilon \epsilon D(\epsilon) f(\epsilon, \tau, \mu)$$

There's a special case, where we don't consider fluctuations of particles about the Fermi energy; i.e. we are "fully denerate". This happens for instance when $\tau = 0$. In this case, our formulas simplify because:

$$f(\epsilon) = 1 \text{ for } \epsilon < \epsilon_F; \qquad f(\epsilon) = 0 \text{ for } \epsilon > \epsilon_F$$

$$N = \int_0^{\epsilon_F} d\epsilon D(\epsilon); \qquad U_0 = \int_0^{\epsilon_F} d\epsilon \epsilon D(\epsilon)$$

Now it might seem a bit strange, but as a computational trick, we can consider the quantity:

$$U' = U - \epsilon_F N = \int_0^\infty d\epsilon (\epsilon - \epsilon_F) D(\epsilon) f(\epsilon, \tau, \mu)$$

We see that we can still use this new quantity to find our heat capacity:

$$C = \frac{\partial U}{\partial \tau} = \frac{\partial U'}{\partial \tau} = \int_0^\infty d\epsilon (\epsilon - \epsilon_F) D(\epsilon) \frac{\partial f}{\partial \tau}(\epsilon, \tau, \mu)$$

Now, in the case where temperature is very low compared to the fermi energy (which is usually the case), our Fermi distribution f looks nearly like a step function at ϵ_F, for all energies less than ϵ_F are almost surely occupied, and energies larger are almost surely not occupied. So when we are looking at $df/d\tau$, the only place this will be significant is near the Fermi energy. So, we can approximate $D(\epsilon)$ in the integral as $D(\epsilon_F)$. Another approximation we can make in the regime where $\tau << \epsilon_F$ is that $\mu(\tau) \approx \mu(0) \approx \epsilon_F$ (we can understand this in that only when $\mu > \epsilon_F$ does a particle from a reservoir fill the state). Putting this all together:

$$C = D(\epsilon_F) \int_0^\infty d\epsilon(\epsilon - \epsilon_F) \frac{\partial}{\partial \tau} \frac{1}{e^{(\epsilon-\epsilon_F)/\tau}+1} = D(\epsilon_F)\tau \int_{-\epsilon_F/\tau}^{\infty} dx x^2 \frac{e^x}{(e^x+1)^2}$$

where $x = (\epsilon - \epsilon_F)/\tau$ is a change into a dimensionless variable. Note that in the final integral, when $\tau << \epsilon_F$, then our bottom limit is very negative, and in this limit our integrand is negligible. So, we can extend our integral's bottom limit all the way down to negative infinity (because beyond that, it's still negligible). Doing this, we find that the definite integral we are left with has a value of $\pi^2/3$. We find also that $D(\epsilon_F) = 3N/2\epsilon_F$. So all together, we have that:

$$C = \frac{\pi^2}{2} N \left(\frac{\tau}{\epsilon_F}\right)$$

Classically, we expect that $C \approx 3N/2$, so with our new model, the τ/ϵ_F makes our heat capacity much less than the classical heat capacity. This reflects the fact that only the particles within τ of the Fermi energy can fluctuate to higher energies. If we want, we can plug in the fermi energy,

$$\epsilon_F = \left(\frac{\hbar^2}{2m}\right)(3\pi^2 n)^{2/3}; \qquad C \approx \frac{m}{\hbar^2} n^{1/3} V \tau$$

Note that all of this has been calculating the heat capacity of a fermi gas, for instance electrons in a metal. If we wanted the actual heat capacity of a metal, we would need to consider both the atoms themselves (the phonons in the Debye model), and the electron gas too (what we just did). So, we end with:

$$C_V = \gamma\tau + A\tau^3; \qquad \gamma = \frac{\pi^2}{2}\frac{N}{\epsilon_F}; \qquad A = \frac{12\pi^4 N}{5T_D^3 k_B^3}$$

We can linearize if we wanted by plotting C/τ as a function of τ^2 to find γ as our y-intercept and A as our slope. We see that there are two regimes, when τ is small, the electronic becomes more important (linear beats cubic), and when τ is large, than the phonon-part becomes more important. There's also a temperature where both contributions are equal (one to one), and solving for this we find:

$$\tau_{\text{eq}} = \left(\frac{\gamma}{A}\right)^{1/2} = \left(\frac{5}{24\pi^2}\frac{\tau_D^3}{\epsilon_F}\right)^{1/2}$$

For ordinary things (like copper), we find that

$$T_{\text{eq}} \approx 4$$

which is very cold! In ordinary lab work, we will only ever see the phonon-part, the electron fermi gas only ever becomes important when temperature is very cold.

2 White Dwarf

Consider a gas of Helium atoms. Let's think about the electrons in this gas.

$$U = \frac{3}{5}\epsilon_F N \approx \frac{3\hbar^2}{m_e}\frac{N^{5/3}}{V^{2/3}}$$

This is the kinetic energy of our electrons. Now, we know that:

$$V = \frac{4}{3}\pi R^3; \qquad N_e = \frac{M}{M_{\text{He}}} \cdot 2 = \frac{M}{2M_p}$$

where m_e is the mass of the electron and m_p is the mass of the proton, and M is the mass of the gas of Helium atoms. Here, we are being loose with constant terms.

$$U \approx \frac{3}{8}\frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2}$$

Now, our gravitational potential energy (assuming uniform density) can be calculated to be:

$$U_p = -\frac{3}{5}\frac{GM^2}{R}$$

Now, our total energy is given by the sum of these two energies:

$$U_T = \frac{3}{8}\frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2} - \frac{3}{5}\frac{GM^2}{R}$$

Now, at some radius, this total energy is minimized. Taking the derivative of U_T against R and setting it equal to zero, and we find that:

$$RM^{1/3} = \frac{5}{4}\frac{\hbar^2}{Gm_e m_p^{5/3}}$$

But the right hand side is all constants! Note that as we add mass, the radius drops. As a real-world example, if we took the sun and turned off it's nuclear fusion, it's equilibrium radius would be about the size of the Earth. Note that at these high energy levels, we really need to take into account the relativistic effects, as the electrons zoom around very quickly. Taking the relativistic limit we find:

$$\epsilon_F = p_F c = \hbar k_F c = \hbar \pi c \left(\frac{3n}{\pi}\right)^{1/3}$$

Now, if we go through the whole derivation again, we find that:

$$U \approx \frac{9}{16}\frac{\hbar c}{R}\left(\frac{m}{m_p}\right)^{4/3}$$

So now our total energy is given by:

$$U_T \approx \frac{9}{16}\frac{\hbar c}{R}\left(\frac{m}{m_p}\right)^{4/3} - \frac{3}{5}\frac{GM^2}{R}$$

So now, there *is no equilibrium*, so in the relativistic case, everything just starts collapsing. We will find that everything turns into neutrons (inverse beta decay) next lecture.