Physics 112 : Lecture 5

Notes for the Fall 2017 Physics 112 Course taught by Professor Holzapfel
prepared by Joshua Lin (email: joshua.z.lin@gmail.com)

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1 Entropy

Consider two systems, one with N_1 particles and U_1 units of energy, and the other with N_2 particles and U_2 units of energy, which are allowed to interact and share their energy. Let $U = U_1 + U_2$ and $N = N_1 + N_2$ so clearly we have:

$$g(N,U) = \sum_{U_1} g_1(N_1,U_1)g_2(N_2,U-U_1)$$

If we wanted to find out which distribution of energy into the two systems (i.e. which value of U_1) is most likely to be measured, then we can calculate:

$$\frac{\partial(g_1 g_2)}{\partial U_1} = \left(\frac{\partial g_1}{\partial U_1}\right)_{N_1} g_2 dU_1 + \left(\frac{\partial g_2}{\partial U_2}\right)_{N_2} g_1 dU_2 = 0$$

Noticing that $U = U_1 + U_2$ so that $\partial U_1 = -\partial U_2$ we have that:

$$\frac{1}{g_1}\left(\frac{\partial g_1}{\partial U_1}\right)_{N_1} = \frac{1}{g_2}\left(\frac{\partial g_2}{\partial U_2}\right)_{N_2}$$

Happily, we notice that this expression is the same thing as:

$$\left(\frac{\partial \ln g_1}{\partial U_1}\right)_{N_1} = \left(\frac{\partial \ln g_2}{\partial U_2}\right)_{N_2}$$

At this point; it feels like we might be interested in the quantity:

$$\sigma(N,U) = \ln g(N,U)$$

because we have shown that in thermal equilibrium, that we have:

$$\left(\frac{\partial \sigma_1}{\partial U_1}\right)_{N_1} = \left(\frac{\partial \sigma_2}{\partial U_2}\right)_{N_2}$$

If you recall results from classical thermodynamics however; we empiracally assigned to every substance a temperature T, and we said that two objects are in thermal equilibrium if $T_1 = T_2$. There was also another quantity, S, called the entropy, such that:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N$$

So in equilibrium, since $T_1 = T_2$ we had (from classical thermodynamics):

$$\left(\frac{\partial S}{\partial U_1}\right)_{N_1} = \left(\frac{\partial S_2}{\partial U_2}\right)_{N_2}$$

Startling! This looks just like our equation from before involving partials of σ! So we have discovered what we classically called the 'entropy' of a system is simply the logarithm of its number of microstates (up to a constant factor):

$$S = k_b \ln g$$

where k_b is the Boltzmann's constant. With this constant, we can also define the 'Fundamental temperature' of a system by:

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_N, \qquad \tau = k_B T$$

which, despite it's name, is actually a measure of the energy of the system in some sense.

2 The Arrow of Time - (Entropy, and other truths about the universe)[1]

Suppose we have two systems, one with energy U_1 and the other with energy U_2, that exchange an amount ΔU of energy (which flows from system 1 to system 2). We might ask whether or not the entropy of the combined system goes up or goes down, and we can calculate this explicitly:

$$\begin{aligned}\Delta\sigma &= \left(\frac{\partial \sigma_1}{\partial U_1}\right)(-\Delta U) + \left(\frac{\partial \sigma_2}{\partial U_2}\right)\Delta U \\ &= \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)\Delta U\end{aligned}$$

so we find that if $T_1 > T_2$, then $\Delta\sigma$ is positive, in other words if the heat flows from the hotter object to the cooler object then the entropy will increase. Since the entropy of the system is monotonic in the multiplicity of the system, we know already that closed systems will tend to increase their entropy; and hence their temperatures will 'even out' as time goes on.

So we are starting to see the idea of entropy become more fleshed. As a thought experiment, we might imagine what would happen if the universe started in a state of maximum entropy: that is that the energy is spread evenly amongst every particle and space is homogenous; then we would find that nothing would ever happen! It is only because (for some

[1] Sorry for being sardonic; this is my usual voice

strange reason) our universe started off with a 'low' entropy level that any sort of complexity could have developed. However, as the arrow of time goes on, and entropy increases and increases, we are ultimately heading towards heat death; where all the energy is eventually spread evenly amongst the universe in the form of weak photons and stray particles; a sad fate for the universe.

3 The Classical Laws of Thermodynamics

The following are the classical laws of thermodynamics:

0th Law: If system A is in thermal equilibrium with system B, and system B is in thermal equilibrium with system C, then system A is in thermal equilibrium with system C.

1st Law: Heat is a form of energy. [2]

2nd Law: If a system is not in a state of equilibrium, then it will evolve so that its entropy increases. [3]

3rd Law: As the temperature of a system approaches zero, its entropy approaches some fixed constant.

4 Measurement Errors

We might wonder whether our calculation for the entropy of a system will depend on how well we can measure the energy of the system. Suppose we have some error δU in our measurement for the energy U of a system. We can define the 'density of states' as follows:

$$D(U) = \frac{\text{Number of states}}{\text{Energy}} \approx \frac{2^N}{U}$$

And so our total entropy is given by:

$$\sigma = \ln(D(U)\delta U) = N\ln 2 - \ln(U) + \ln(\delta U) = N\ln 2 + \ln\left(\frac{\delta U}{U}\right)$$

Suppose we specify the system as best we can, by Heisenberg's uncertainty principle we know that $\delta U/U$ will be on the order of 10^{-34}, so we have that $\ln(\delta U/U)$ is roughly -78, which for large N is negligible compared to the $N\ln 2$ term. All this is to say that our uncertainty in our measurement for U does not matter when calculating entropy.

[2] This was not obvious to the scientists a long time ago!

[3] Once again, this is the 'arrow of time'