Phys 112 : Lecture 22

Notes for the Fall 2017 Physics 112 Course taught by Professor Holzapfel
prepared by Joshua Lin (email: joshua.z.lin@gmail.com)

November 9, 2017

1 Bose-Einstein condensation

Last lecture, we discussed the Bose-Einstein condensation (for boson gases), the phenomenon where we find a much larger proportion of particles in the ground state energy than expected at low temperatures. Now, as a verification check, we can check the proportion of the number of particles in the ground state vs the number of particles in all the other states:

$$N = \sum_n f_n = N_0(\tau) + N_e(\tau)$$

where N_0 is the number in ground state, and N_e is the number in excited states. Now, we have that:

$$N_e(\tau) = \int_{2\epsilon_0}^{\infty} d\epsilon D(\epsilon) f(\epsilon, \tau)$$

where we integrate over all the other excited states. Turns out that the bottom limit doesn't matter too much, it will always be small compared to N_0. Now, we know that:

$$D(\epsilon) = g_s \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2} = D_0 \epsilon^{1/2}$$

where we are considering for simplicity the case where the spin multiplicity $g_s = 1$, i.e. they are spin 0 particles. Note that also we are using the approximation we found from last lecture, which is that $\mu \approx \epsilon_0$. Now, plugging this into the integral, we find that:

$$N_e(\tau) = \int_{2\epsilon_0}^{\infty} D_0 \epsilon^{1/2} \frac{1}{e^{1/N_0} e^{(\epsilon - \epsilon_0)/\tau} - 1}$$

Now, we can consider the limit where $N_0 >> 1$, and make the usual looking dimensionless constant change of parameter $x = (\epsilon - \epsilon_0)/\tau$, and we find that:

$$N_e(\tau) = \int_{\epsilon_0/\tau}^{\infty} \tau dx D_0 (\epsilon_0 + \tau x)^{1/2} \frac{1}{e^x - 1} = D_0 \tau^{3/2} \int_{\epsilon_0/\tau}^{\infty} \tau dx (1 + episilon_0/(\tau x))^{1/2} \frac{x^{1/2}}{e^x - 1}$$

Now if we consider the function $(1+\epsilon_0/(\tau x))^{1/2}$, within the integral limits, we have that this expression is roughly of order 1, so we have that

$$N_e(\tau) \approx C \int_{\epsilon_0/\tau}^{\infty} \frac{x^{1/2} dx}{e^x - 1}$$

Now, if we look at the part of the integral that is close to $x = 0$, i.e. for $x << 1$, we have that:

$$N_e(\tau) \approx C \int_{x_0}^{x_1} \frac{dx}{\sqrt{x}} \approx C(x_1^{1/2} - x_0^{1/2})$$

so we see that the term from the bottom limit, $x_0^{1/2}$, is really small and doesn't actually contribute much to the overall integral, which is why it doesn't really matter. Now, we are justified in only considering the tail behaviour of the integral, so we can assert that:

$$x >> \epsilon_0/\tau; \qquad \left(1 + \frac{\epsilon_0}{x\tau}\right)^{1/2} = 1$$

So, we have that:

$$N_e(\tau) = D_0 \tau^{3/2} \int_0^{\infty} \frac{dx x^{1/2}}{e^x - 1}$$

which we can solve numerically, to find that:

$$N_e = 2.61 n_Q V$$

Or, rewriting this in a more suggestive way, we have that:

$$\frac{N_e(\tau)}{N} = 2.61 \frac{n_Q}{n}$$

So we can see that if we want the fraction of particles being excited to be small, then we want the density to be very large compared to the quantum concentration, i.e. we want 'quantum effects' to be important, as expected. Note that as we lower and lower the temperature, the quantum concentration goes down and down, so it becomes 'easier' in some sense to reach this condensation limit. Now you might think that this 2.61 constant is a bit loose, but it is a bit tighter than you think, since we made a rigorous argument to show that $C \approx 1$. Now, we can ask the question of what temperature is required so that almost all the particles are excited:

$$N \approx 2.61 \left(\frac{m\tau_E}{2\pi\hbar^2}\right)^{3/2} V$$

where T_E is the Einstein temperature, and we can solve for it. For Helium, we have that the Einstein temperature is roughly 3 Kelvin. You might be wondering, since we assumed before that $N_0 >> 1$, how we can now consider the case when all the particles are excited, but note that we can just consider 90% of the particles to be excited, and 10% of particles in the ground state still gives a large number in the ground state. So we have:

$$\frac{N_e}{N} = \left(\frac{\tau}{\tau_E}\right)^{3/2}$$

So that we have:

$$N_0(\tau) = N\left[1 - \left(\frac{\tau}{\tau_E}\right)^{3/2}\right]$$

This is a good model, but not great. For instance, we have been treating the particles as an ideal gas, and have not been taking into account any of the intermolecular forces. To treat this, we can introduce an effective mass for Helium due to its interactions with the other particles, and doing the calculation we get a slightly different Einstein temperature.

As a counterexample, we can use Helium 3 instead, and since it has 3 spin halves, it's actually a fermion, so it won't exhibit Bose-Einstein condensation at the limit of low temperature, like Helium 4 would (which exhibits superfluidity). Even wackier, if you keep cooling Helium 3, at roughly 3 millikelvin, pairs of Helium 3 team up to become a boson, and Helium 3 starts exhibiting Bose-Einstein condensation. It's the same deal for superconductors; electrons pair up to form integer-spin particles, so that they act as a Boson-gas, not a fermion-gas.

Now, we can think about the heat capacity. We know that the energy must look like:

$$U \approx N_e \tau \approx \frac{N\tau^{5/2}}{\tau_E^{3/2}}$$

$$C_V = \frac{\partial U}{\partial \tau} \approx \frac{5}{2} N \frac{\tau^{3/2}}{\tau_E^{3/2}}$$

We know that in the classical limit, we have $C_V = (3/2)N\tau$, so if we plot C_V/N against τ/τ_E, the heat capacity initially increases with a 3/2 power law up to $\tau = \tau_E$, where it takes a value of 5/2, then it has to drop down back to the classical limit where it has a value of 3/2, so there is a 'hump' in the heat capacity near τ_E.