

# What is fuzzy logic?



- A super set of Boolean logic
- Builds upon fuzzy set theory
- Graded truth. Truth values between True and False. Not everything is *either/or, true/false, black/white, on/off* etc.
- Grades of membership. Class of tall men, class of far cities, class of expensive things, etc.
- Lotfi Zadeh, UC/Berkely 1965. Introduced **FL to model uncertainty in natural language**. *Tall, far, nice, large, hot, ...*
- Reasoning using linguistic terms. Natural to express expert knowledge.  
*If the weather is *cold* then wear *warm* clothing*

# Why use fuzzy logic?



## Pros:

- Conceptually easy to understand w/ “natural” maths
- Tolerant of imprecise data – and therefore more usable perhaps
- Universal approximation: can model arbitrary nonlinear functions
- Intuitive
- Based on linguistic terms
- Convenient way to express **expert and common sense** knowledge

## Cons:

- Not a cure-all
- Crisp/precise models can be more efficient and even convenient
- Other approaches might be formally verified to work

# Tipping example

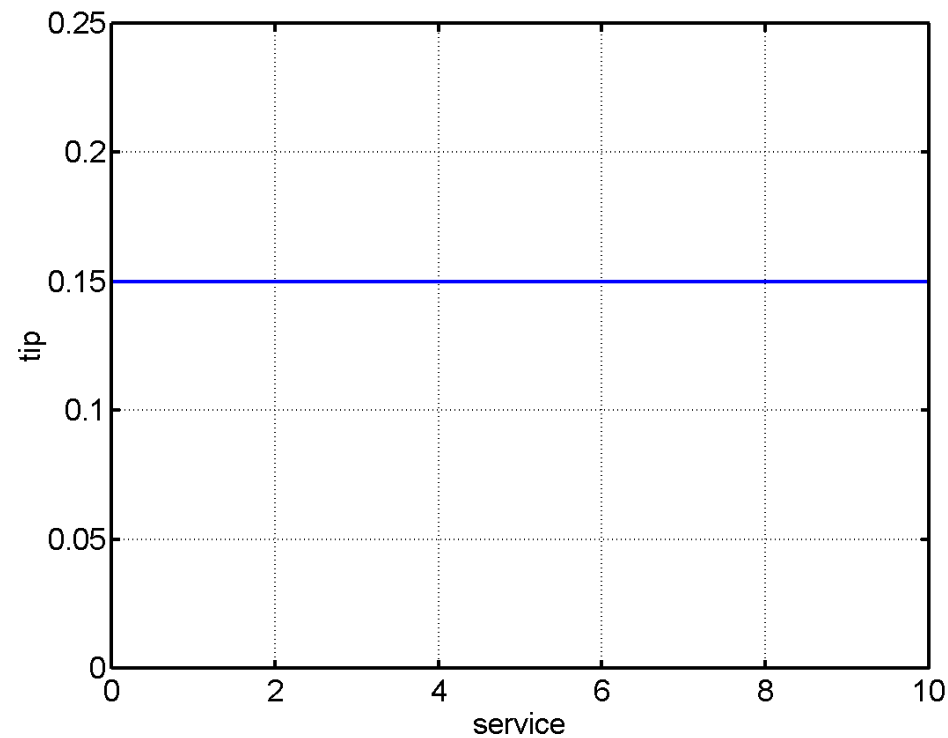


- **The Basic Tipping Problem:** Given a number between 0 and 10 that represents the quality of service at a restaurant what should the tip be?

Cultural footnote: An average tip for a meal in the U.S. is 15%, which may vary depending on the quality of the service provided.

# Tipping example: The non-fuzzy approach

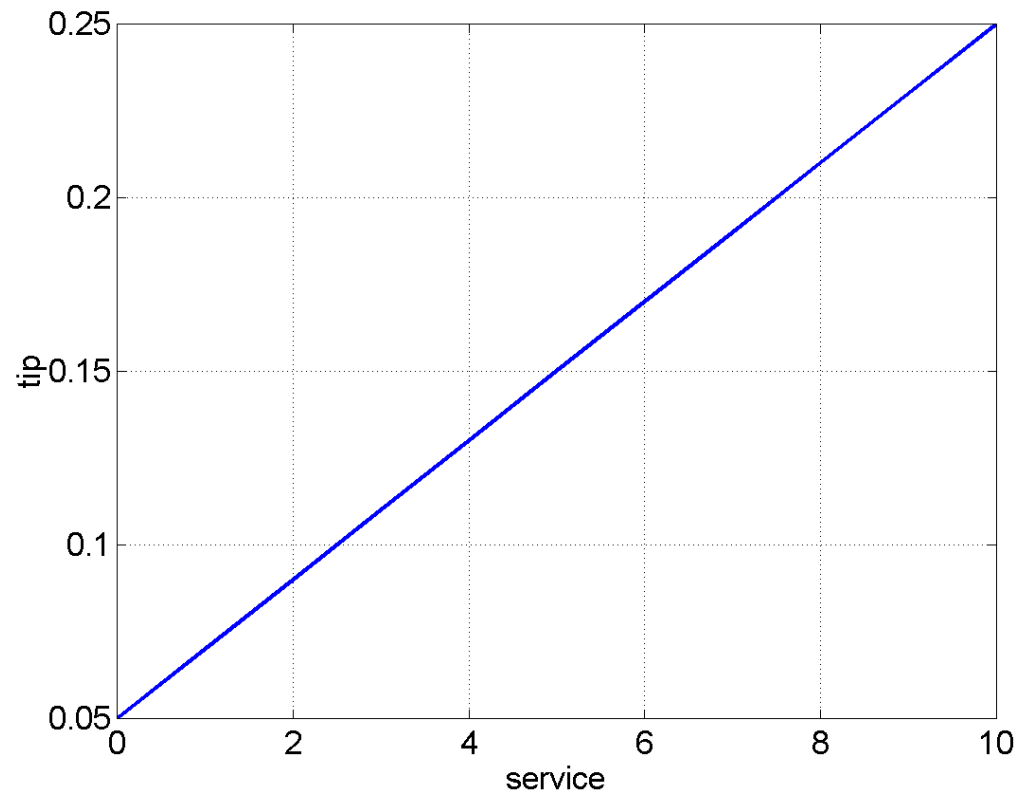
- Tip = 15% of total bill



- What about quality of service?

# Tipping example: The non-fuzzy approach

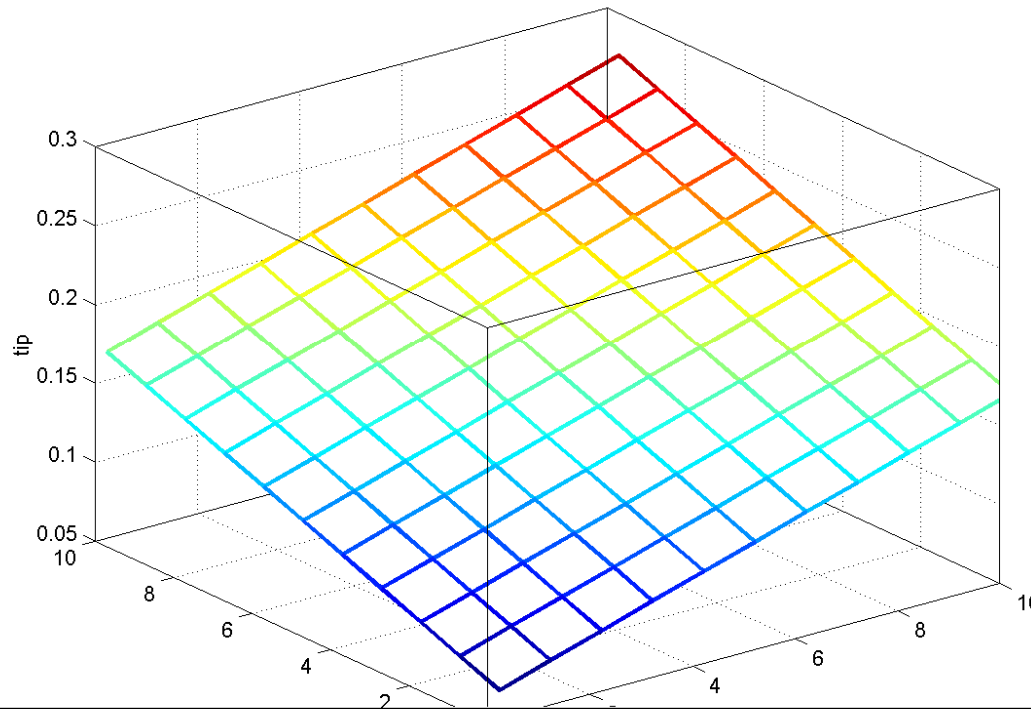
- Tip = linearly proportional to service from 5% to 25%  
 $\text{tip} = 0.20/10 * \text{service} + 0.05$



- What about quality of the food?

# Tipping example: The non-fuzzy approach

- $\text{Tip} = 0.20/20 * (\text{service} + \text{food}) + 0.05$



- We want service to be more important than food quality. E.g., 80% for service and 20% for food.

# Tipping problem: the fuzzy approach



What we want to express is:

1. *If service is poor then tip is cheap*
2. *If service is good the tip is average*
3. *If service is excellent then tip is generous*
4. *If food is rancid then tip is cheap*
5. *If food is delicious then tip is generous*

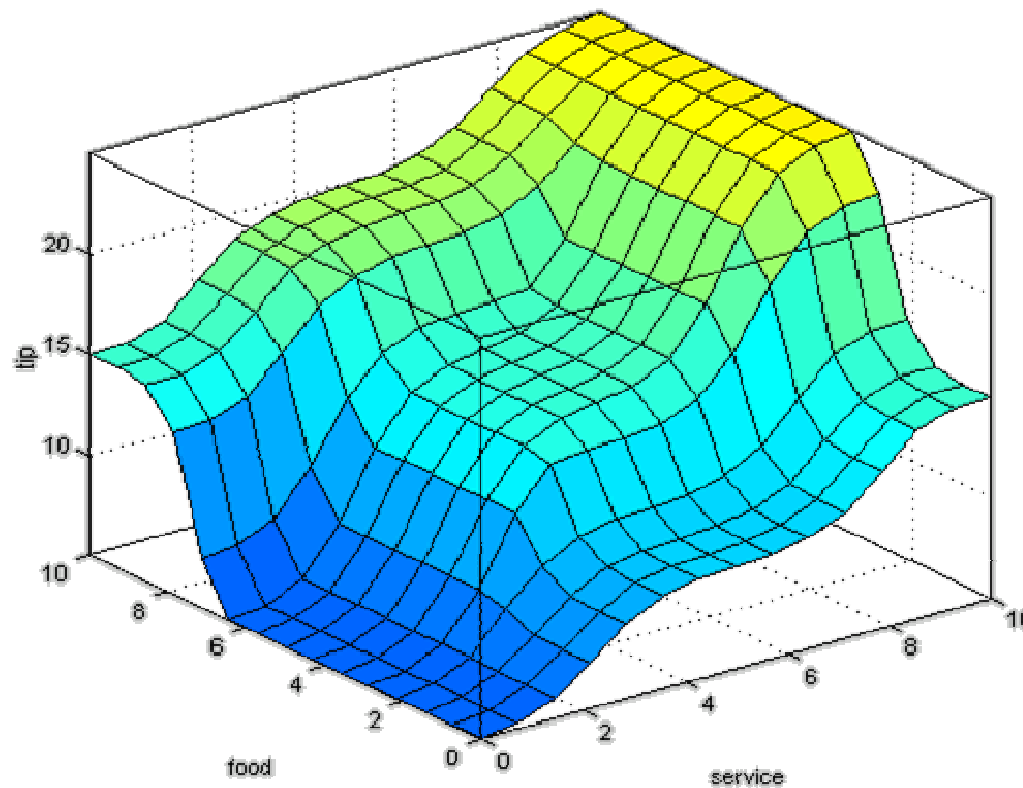
or

1. *If service is poor or the food is rancid then tip is cheap*
2. *If service is good then tip is average*
3. *If service is excellent or food is delicious then tip is generous*

***We have just defined the rules for a fuzzy logic system.***

# Tipping problem: fuzzy solution

Decision function generated using the 3 rules.





## Tipping problem: fuzzy solution



- Before we have a fuzzy solution we need to find out
  - a) how to define terms such as *poor, delicious, cheap, generous etc.*
  - b) how to combine terms using AND, OR and other connectives
  - c) how to combine all the rules into one final output

# Fuzzy sets

- **Boolean/Crisp set A** is a mapping for the elements of S to the set  $\{0, 1\}$ , i.e.,  $A: S \rightarrow \{0, 1\}$
- *Characteristic function:*

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \text{ is an element of set } A \\ 0 & \text{if } x \text{ is not an element of set } A \end{cases}$$

- 
- **Fuzzy set F** is a mapping for the elements of S to the interval  $[0, 1]$ , i.e.,  $F: S \rightarrow [0, 1]$
  - Characteristic function:  $0 \leq \mu_F(x) \leq 1$
  - 1 means full membership, 0 means no membership and anything in between, e.g., 0.5 is called **graded membership**

## Example: Crisp set Tall



- Fuzzy sets and concepts are commonly used in natural language

*John is **tall***

*Dan is **smart***

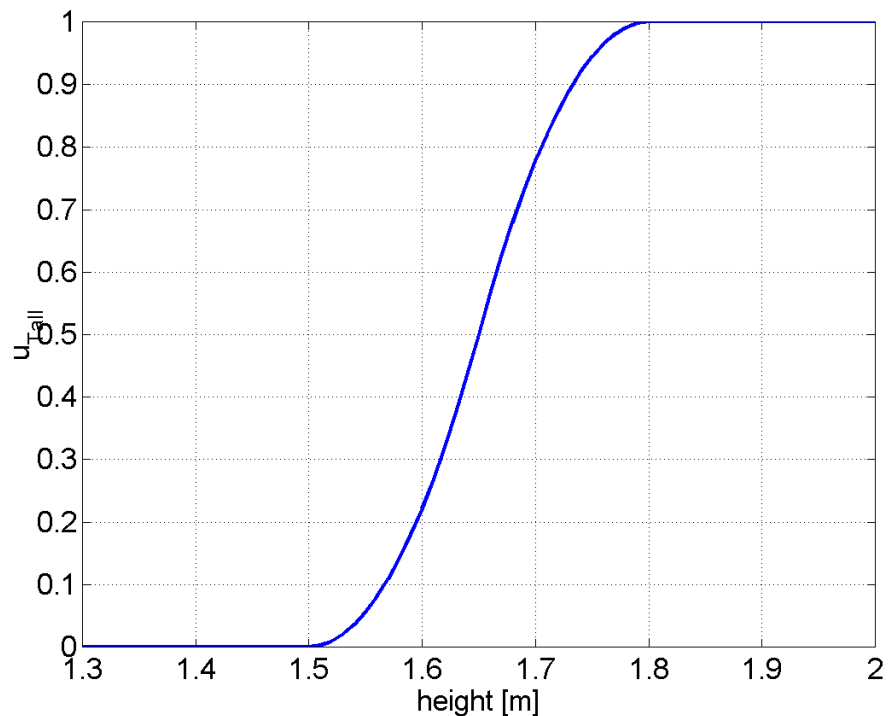
*Alex is **happy***

*The class is **hot***

- E.g., the crisp set **Tall** can be defined as  $\{x \mid \text{height } x > 1.8 \text{ meters}\}$   
But what about a person with a height = 1.79 meters?  
What about 1.78 meters?  
...  
What about 1.52 meters?

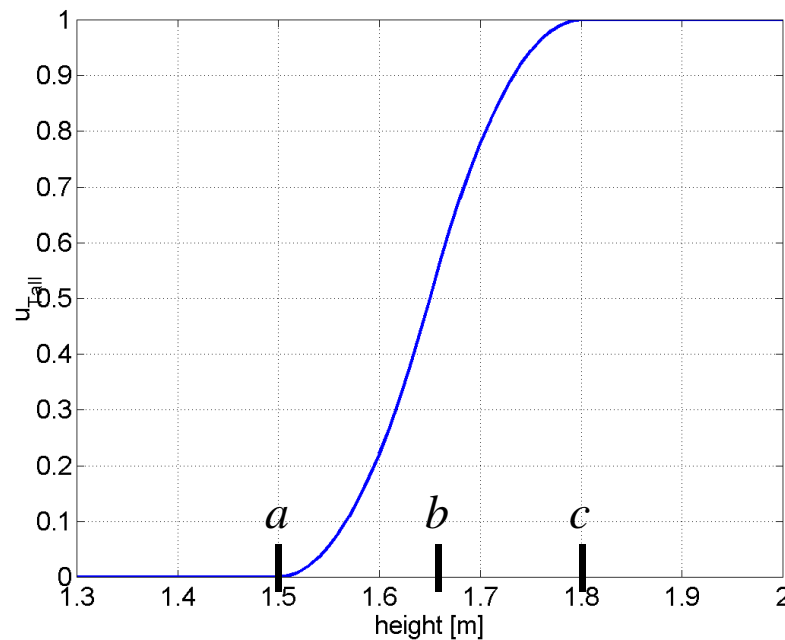
## Example: Fuzzy set Tall

- In a fuzzy set a person with a height of 1.8 meters would be considered tall to a **high degree**  
A person with a height of 1.7 meters would be considered tall to a lesser degree etc.
- The function can change for basketball players, Danes, women, children etc.



# Membership functions: S-function

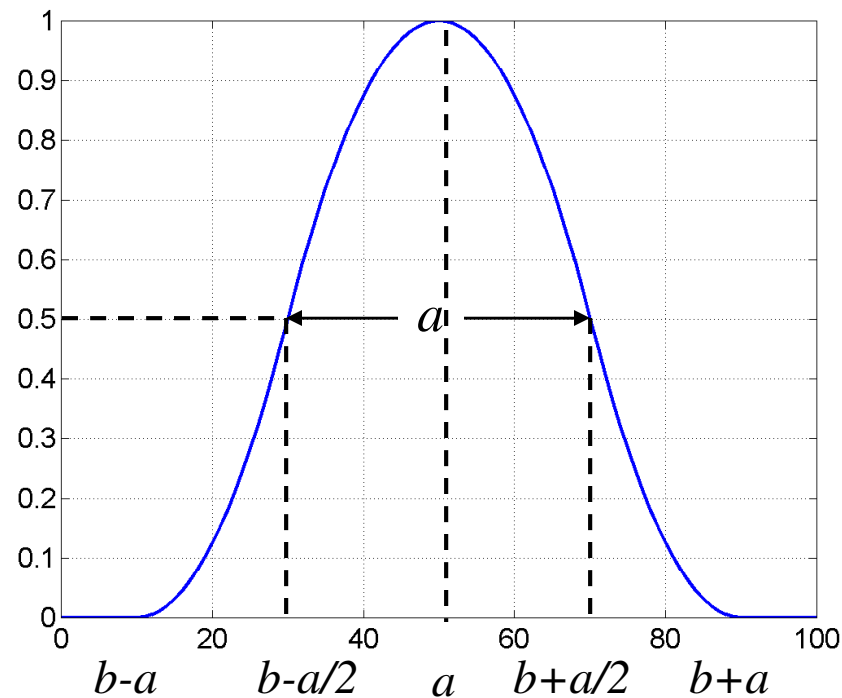
- The S-function can be used to define fuzzy sets
- $S(x, a, b, c) =$ 
  - 0 for  $x \leq a$
  - $2(x-a/c-a)^2$  for  $a \leq x \leq b$
  - $1 - 2(x-c/c-a)^2$  for  $b \leq x \leq c$
  - 1 for  $x \geq c$



# Membership functions: $\Pi$ -Function

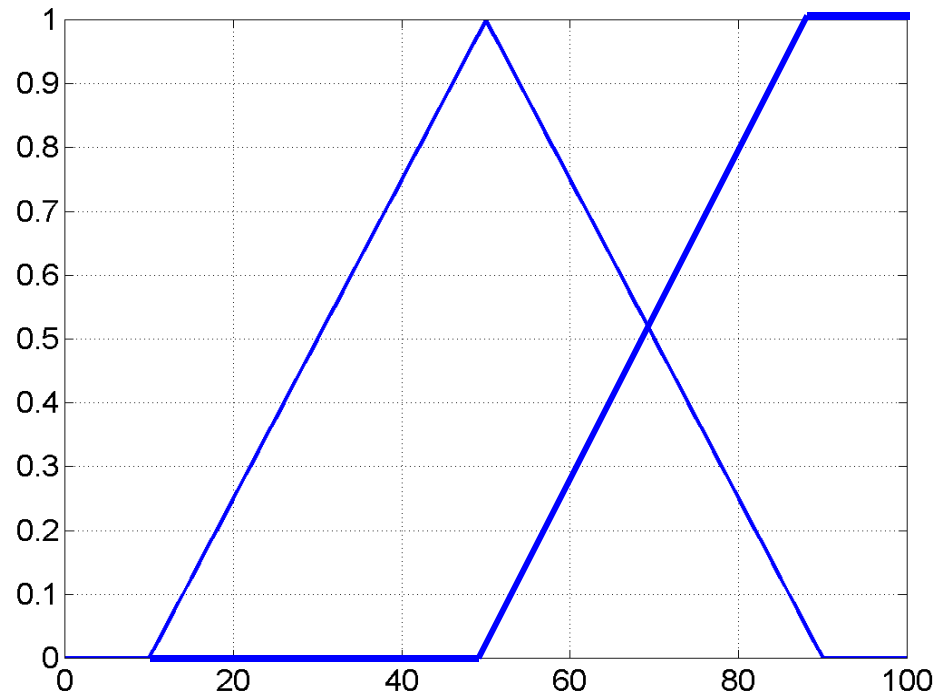
- $\Pi(x, a, b) =$ 
  - $S(x, b-a, b-a/2, b)$  for  $x \leq b$
  - $1 - S(x, b, b+a/2, a+b)$  for  $x \geq b$

E.g., *close* (to  $a$ )

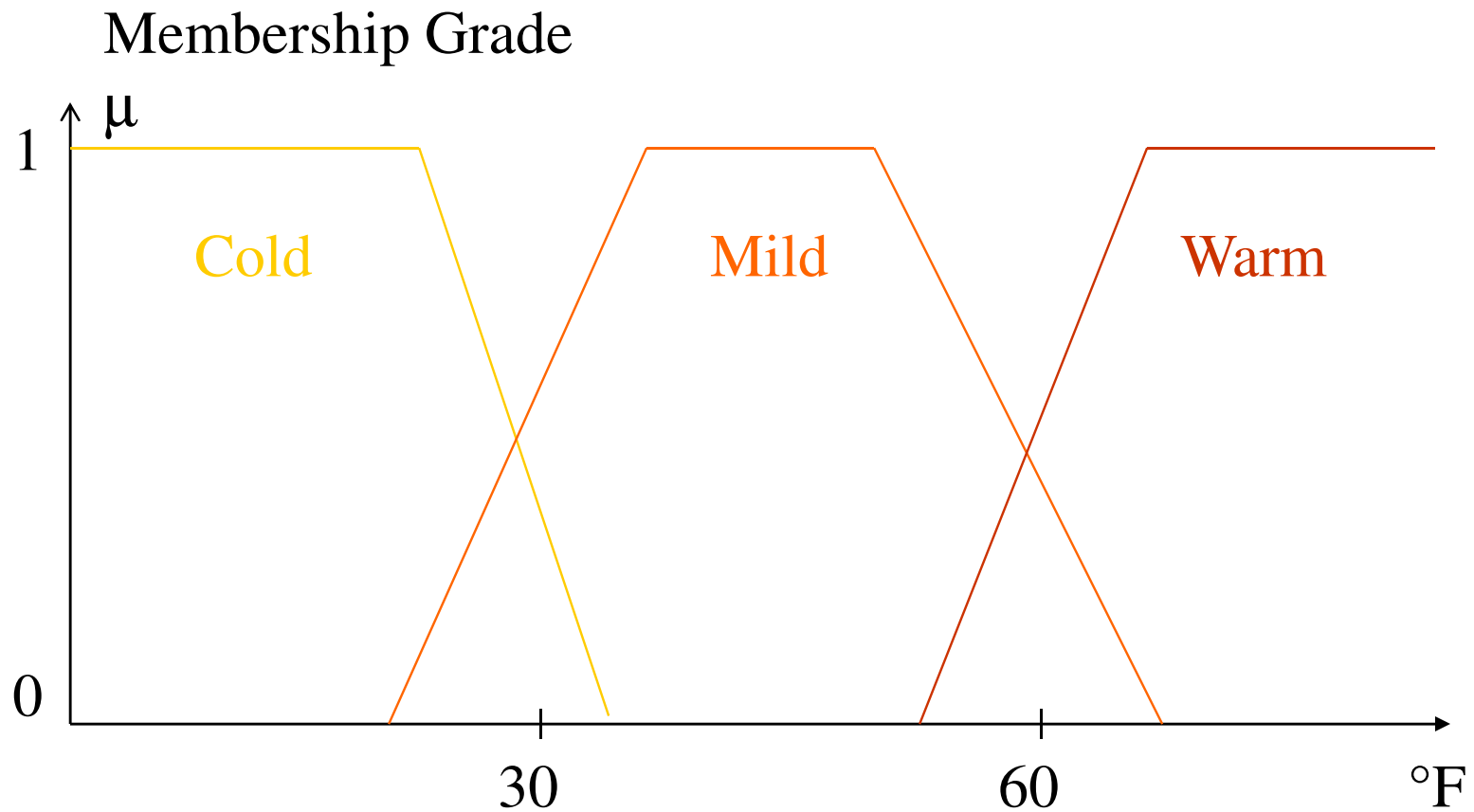


# Simple membership functions

- Piecewise linear: triangular etc.
- Easier to represent and calculate  $\Rightarrow$  saves computation



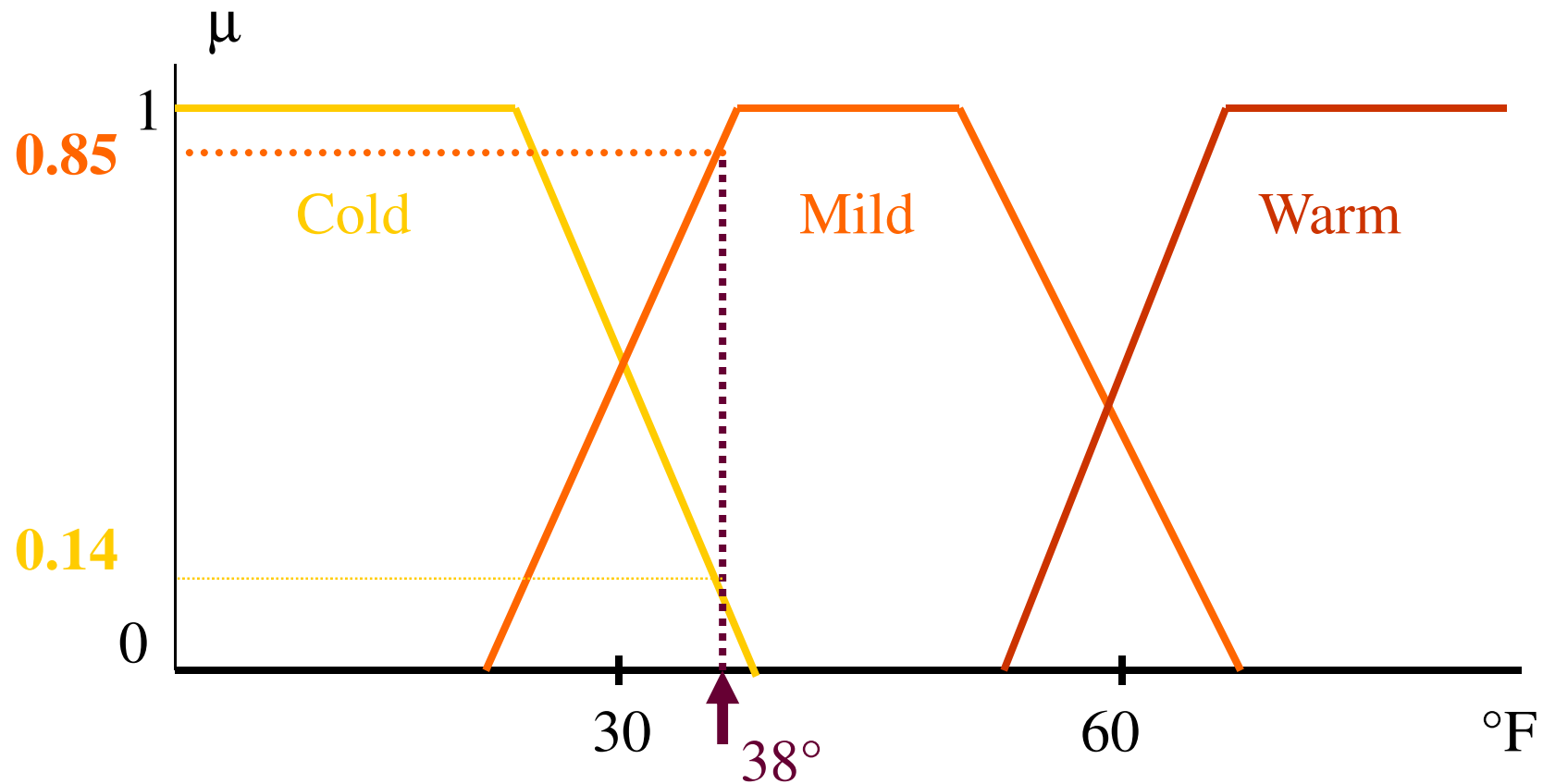
# Fuzzy Sets





# Observation

An observed temperature of 38 is cold with a belief of 0.14, Mild with a belief of 0.85 and warm with a belief of 0



## Other representations of fuzzy sets

- A finite set of elements:

$$F = \mu_1/x_1 + \mu_2/x_2 + \dots \mu_n/x_n$$

+ means (Boolean) set union

- For example:

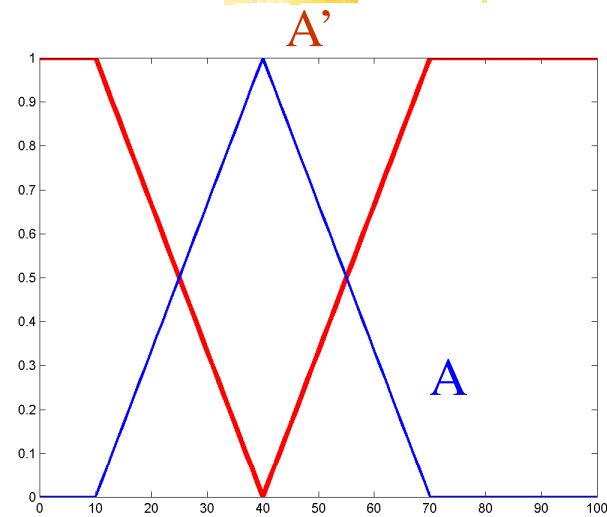
$$\text{TALL} = \{0/1.0, 0/1.2, 0/1.4, 0.2/1.6, 0.8/1.7, 1.0/1.8\}$$

# Fuzzy set operators

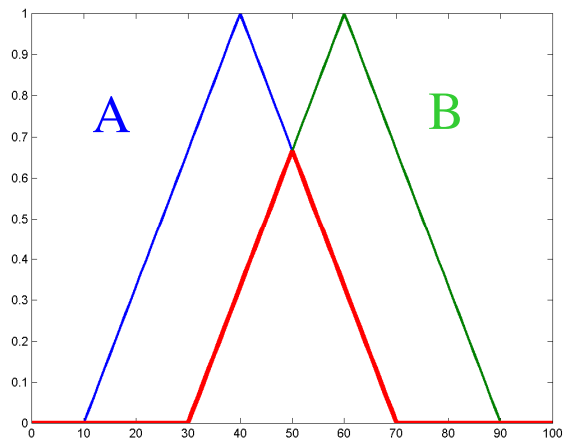


- Equality  
 $A = B$   
 $\mu_A(x) = \mu_B(x)$  for all  $x \in X$
- Complement  
 $A'$   
 $\mu_{A'}(x) = 1 - \mu_A(x)$  for all  $x \in X$
- Containment  
 $A \subseteq B$   
 $\mu_A(x) \leq \mu_B(x)$  for all  $x \in X$
- Union  
 $A \cup B$   
 $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$  for all  $x \in X$
- Intersection  
 $A \cap B$   
 $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$  for all  $x \in X$

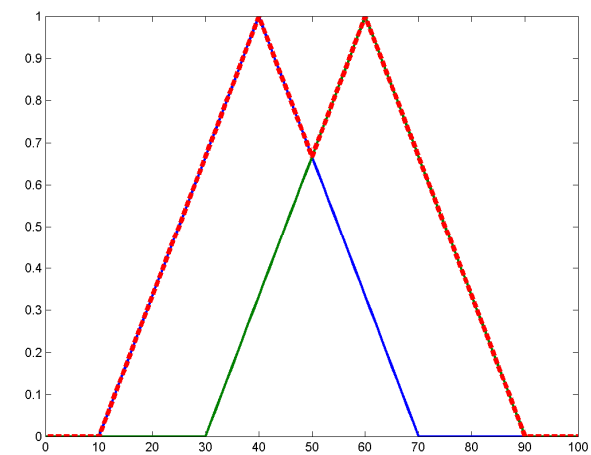
# Example fuzzy set operations



$A \cap B$



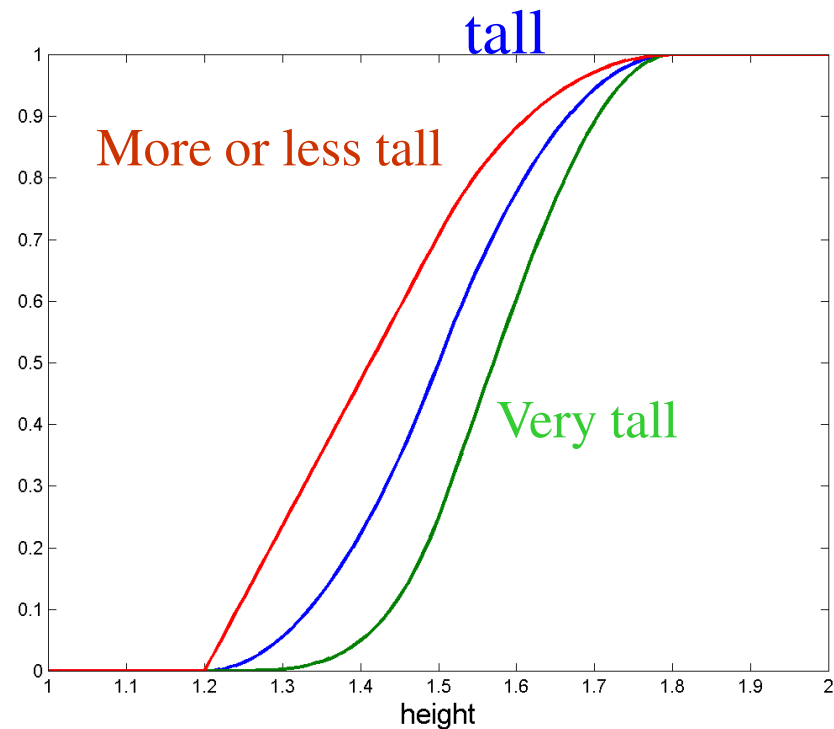
$A \cup B$



# Linguistic Hedges

- Modifying the meaning of a fuzzy set using hedges such as *very*, *more or less*, *slightly*, *etc.*

- "*Very F*" =  $F^2$
- "*More or less F*" =  $F^{1/2}$
- *etc.*



# Fuzzy relations

- A fuzzy relation for N sets is defined as an extension of the crisp relation to include the membership grade.

$$R = \{\mu_R(x_1, x_2, \dots, x_N)/(x_1, x_2, \dots, x_N) \mid x_i \in X, i=1, \dots, N\}$$

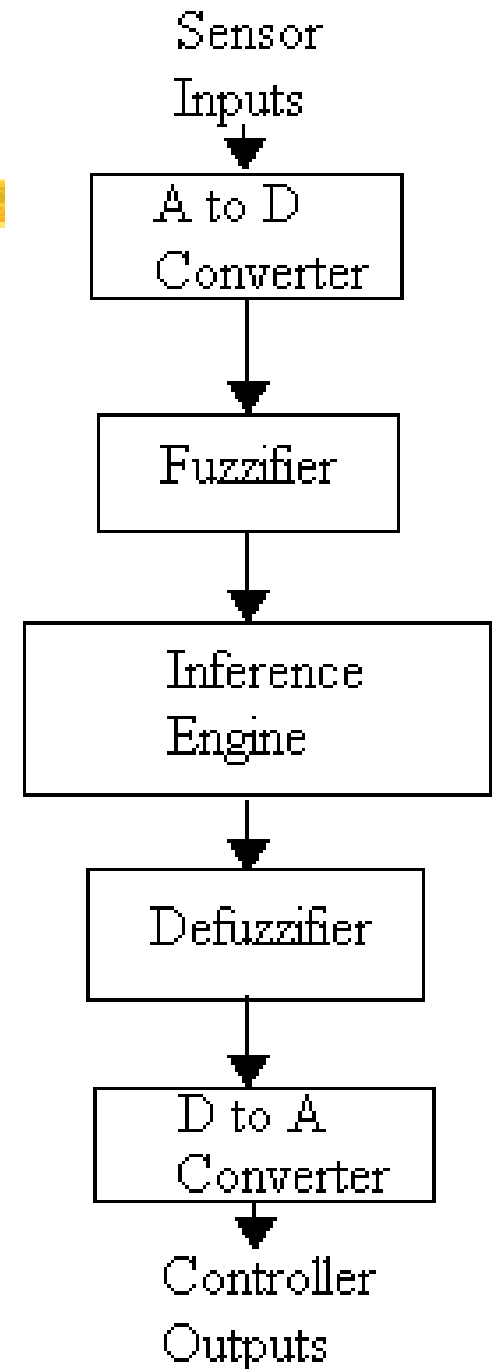
which associates the membership grade,  $\mu_R$ , of each tuple.

- E.g.

$$\text{Friend} = \{0.9/(\text{Manos}, \text{Nacho}), 0.1/(\text{Manos}, \text{Dan}), \\ 0.8/(\text{Alex}, \text{Mike}), 0.3/(\text{Alex}, \text{John})\}$$

# Fuzzy inference

- Fuzzy logical operations
- Fuzzy rules
- Fuzzification
- Implication
- Aggregation
- Defuzzification



# Fuzzy logical operations

- AND, OR, NOT, etc.
- **NOT**  $A = A' = 1 - \mu_A(x)$
- $A$  **AND**  $B = A \cap B = \min(\mu_A(x), \mu_B(x))$
- $A$  **OR**  $B = A \cup B = \max(\mu_A(x), \mu_B(x))$

From the following truth tables it is seen that fuzzy logic is a **superset** of Boolean logic.

min(A,B)

A	B	A and B
0	0	0
0	1	0
1	0	0
1	1	1

max(A,B)

A	B	A or B
0	0	0
0	1	1
1	0	1
1	1	1

1-A

A	not A
0	1
1	0



## If-Then Rules



- Use fuzzy sets and fuzzy operators as the **subjects** and **verbs** of fuzzy logic to form rules.

**if x is A then y is B**

where A and B are linguistic terms defined by fuzzy sets on the sets X and Y respectively.

This reads

**if x == A then y = B**

## Evaluation of fuzzy rules



- In Boolean logic:  $p \Rightarrow q$   
if p is true then q is true
- In fuzzy logic:  $p \Rightarrow q$   
if p is true to some degree then q is true to some degree.

$0.5p \Rightarrow 0.5q$  (partial premise implies partially)

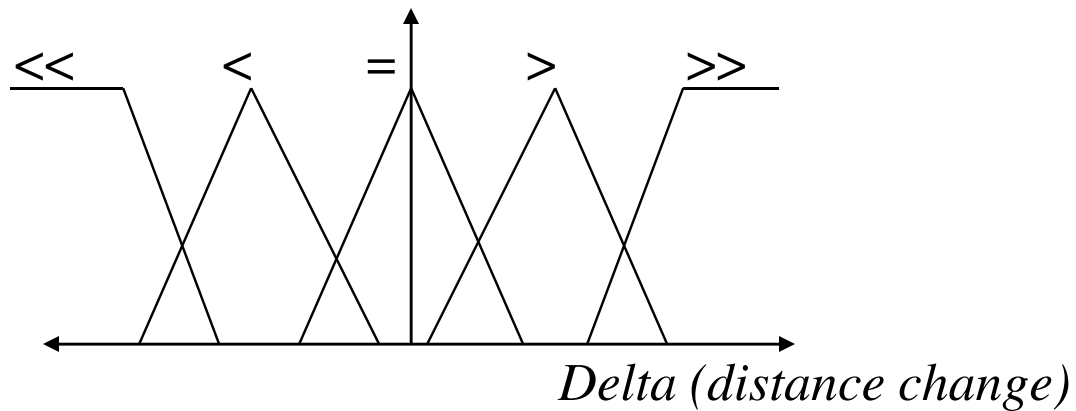
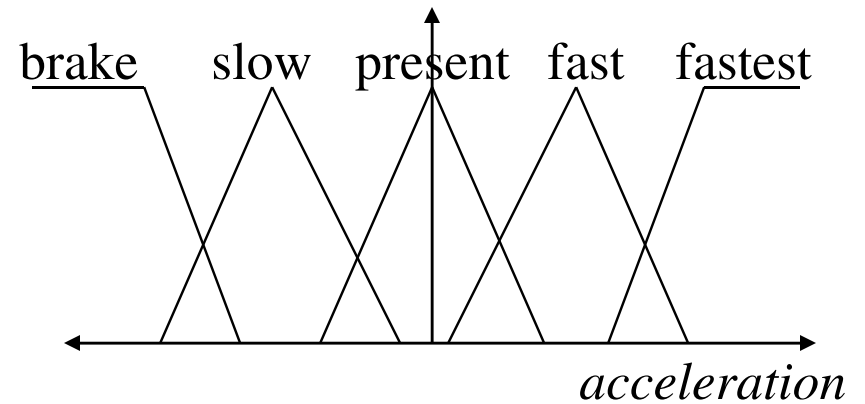
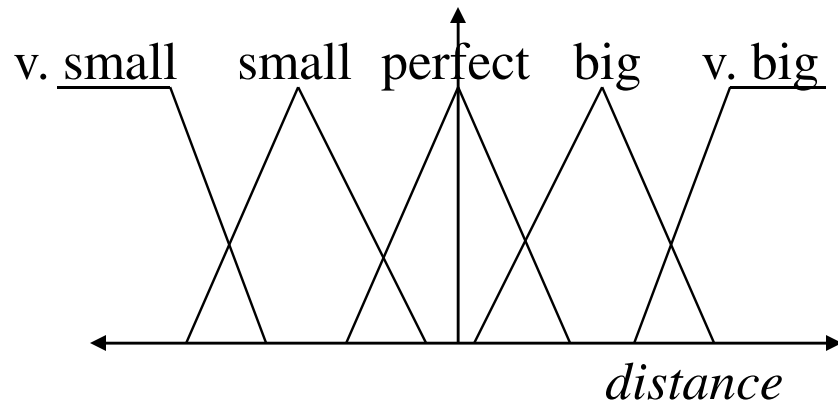
- How?

# Fuzzy Rules

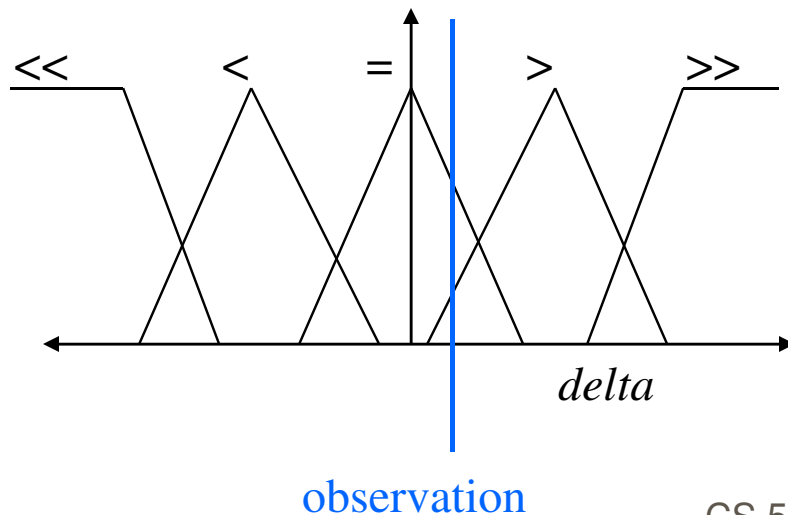
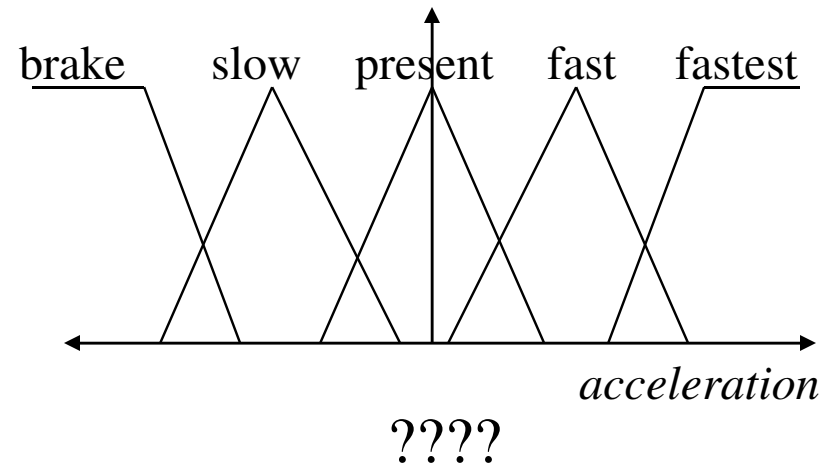
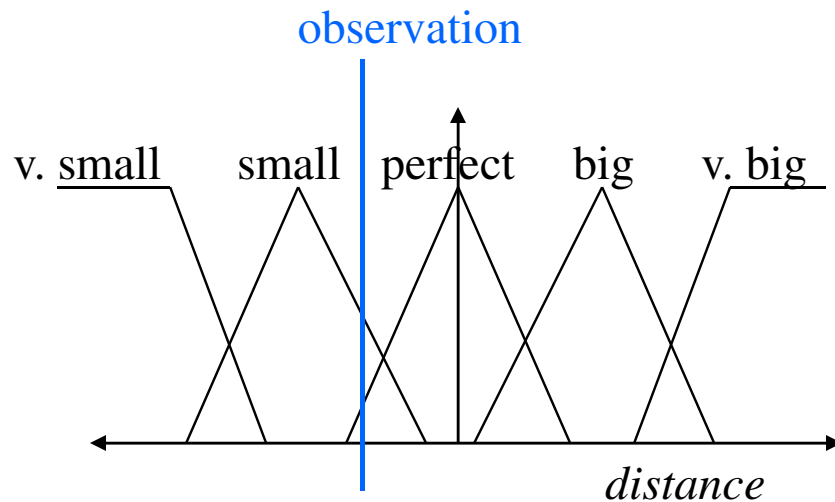


- Example: “If our distance to the car in front is small, and the distance is decreasing slowly, then decelerate quite hard”
  - Fuzzy variables in blue
  - Fuzzy sets in red
- *QUESTION:* Given the distance and the change in the distance, what acceleration should we select?

# Fuzzification: Set Definitions

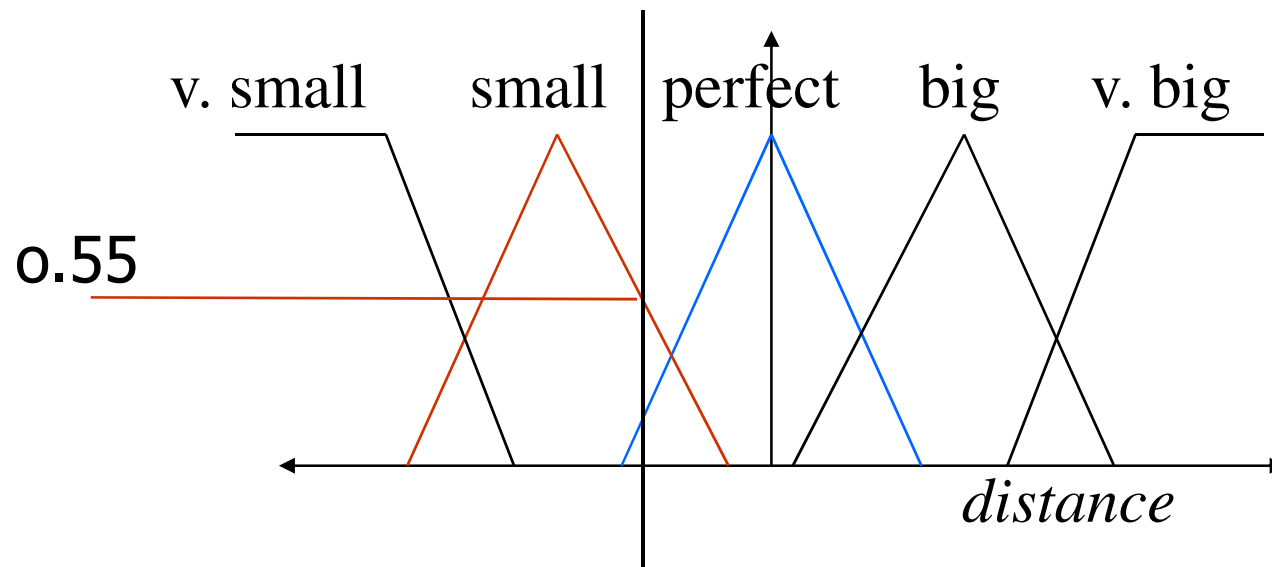


# Fuzzification: Instance



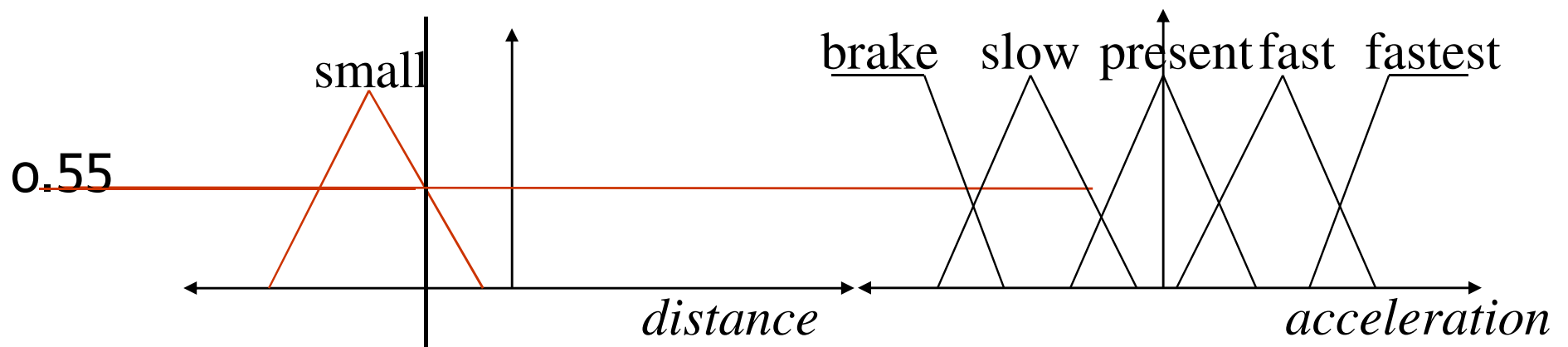
- Distance could be considered small or perfect
- Delta could be stable or growing
- What acceleration?

## Fuzzification: Instance



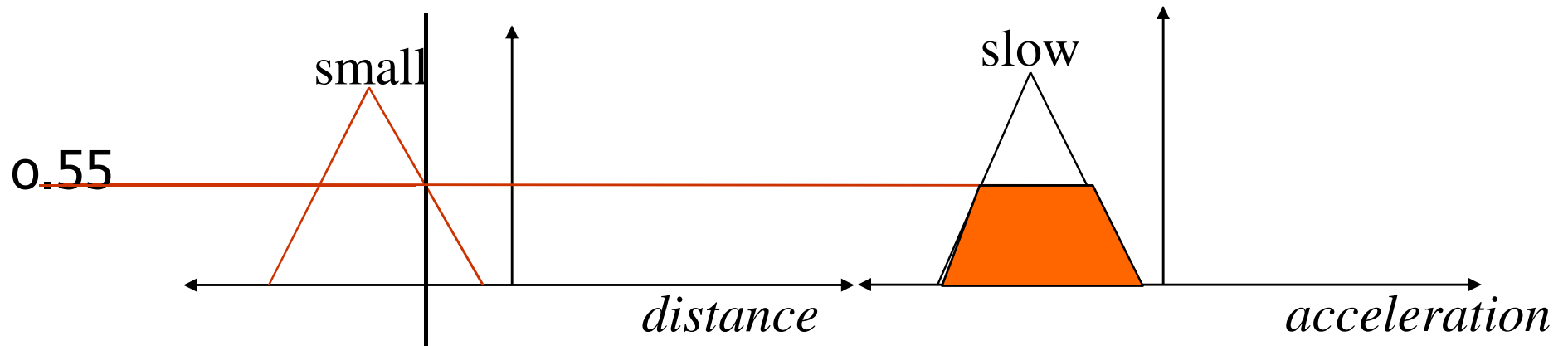
IF distance is Small THEN Slow Down

# Rule Evaluation



Distance is small, then you slow down.  
Question: What is the weight to slow down?

# Rule Evaluation



## Clipping approach (others are possible):

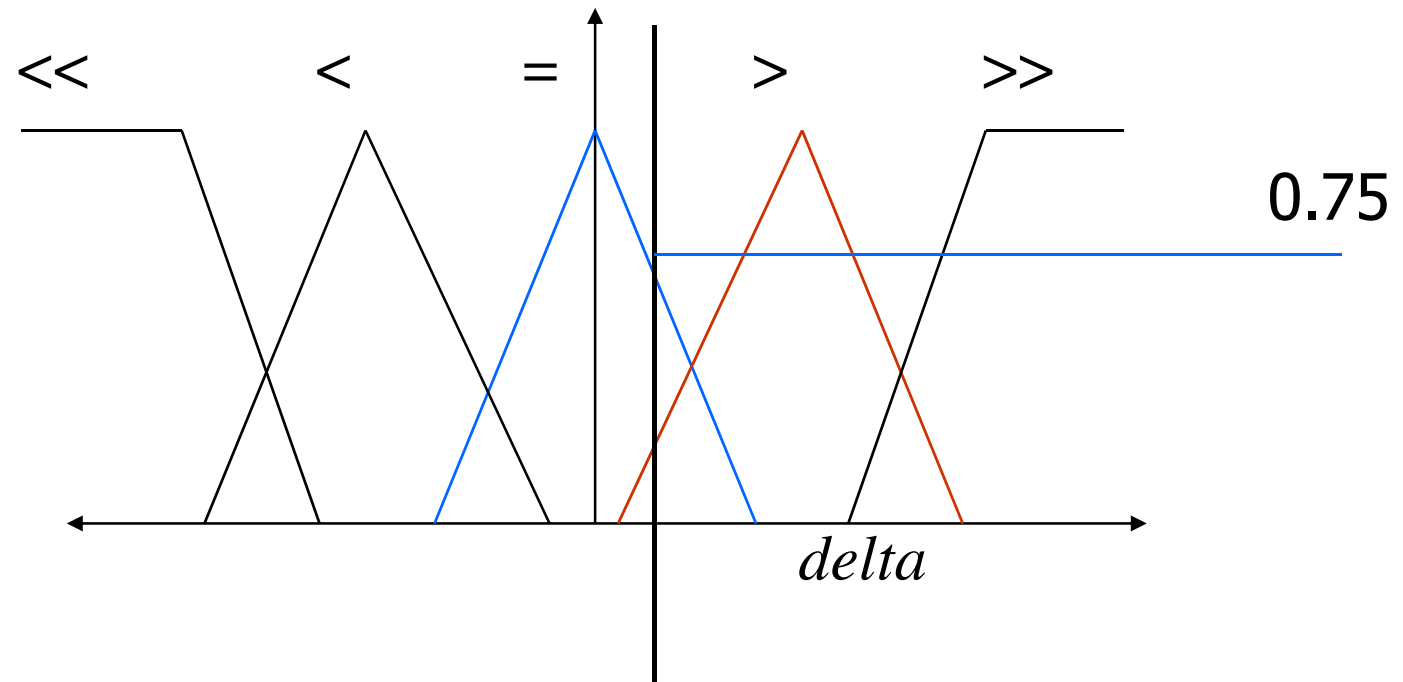
Clip the fuzzy set for "slow" (the consequent) at the height given by our belief in the premises (0.55)

We will then consider the clipped AREA (orange) when making our final decision

Rationale: if belief in premises is low, clipped area will be very small  
But if belief is high it will be close to the whole unclipped area

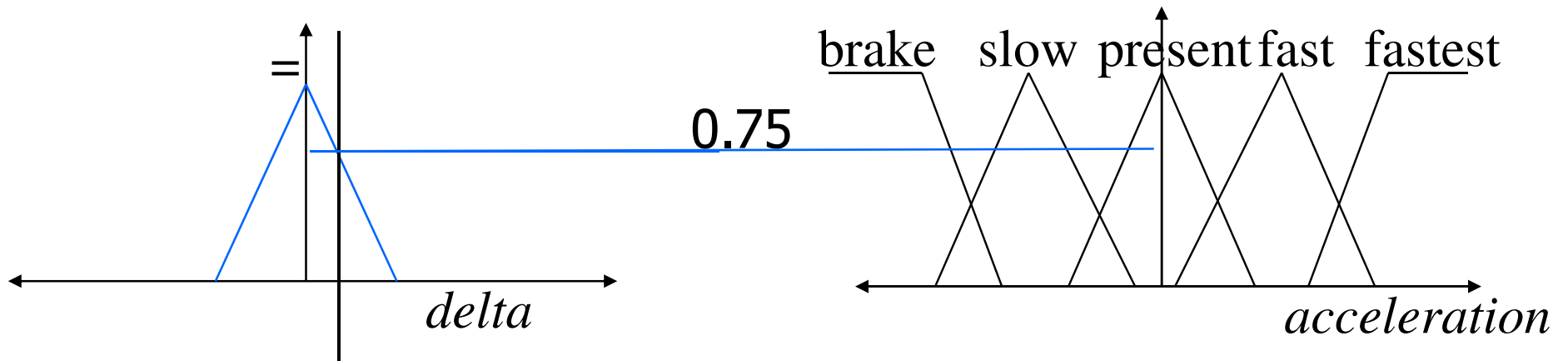


## Fuzzification: Instance



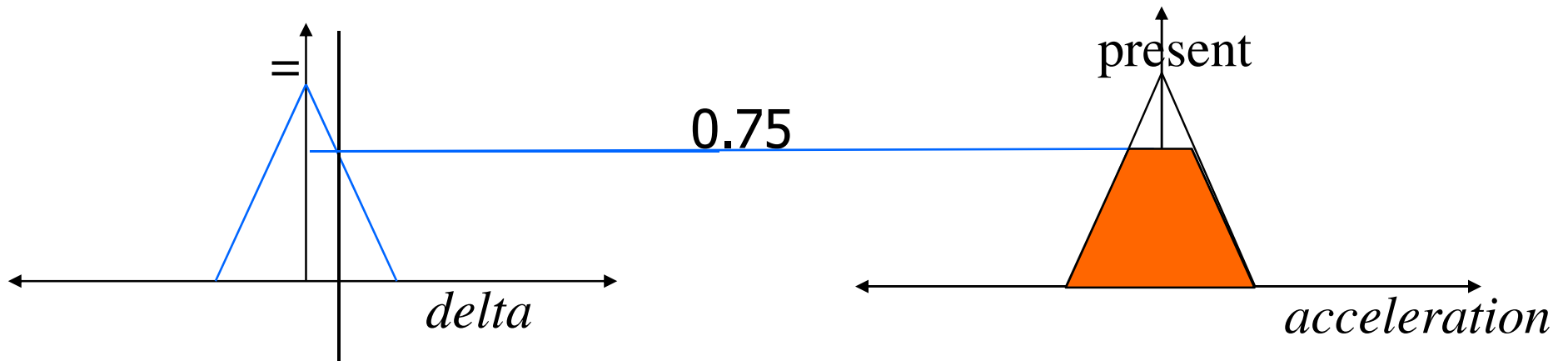
IF change in distance is = THEN Keep the speed

# Rule Evaluation



Distance is not growing, then keep present acceleration

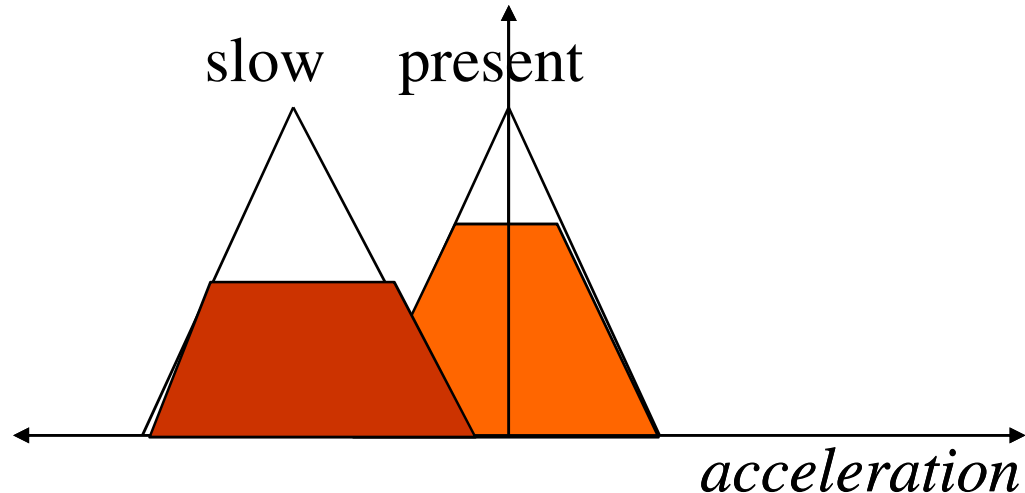
# Rule Evaluation



Distance is not growing, then keep present acceleration

# Rule Aggregation

How do we make a final decision? From each rule we have  
Obtained a clipped area. But in the end we want a single  
Number output: our desired acceleration

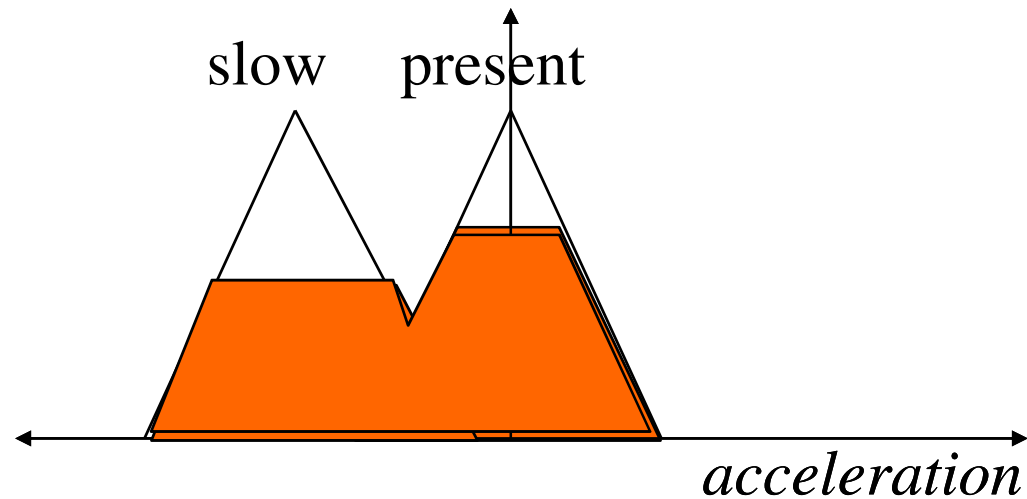


From distance



From delta (distance change)

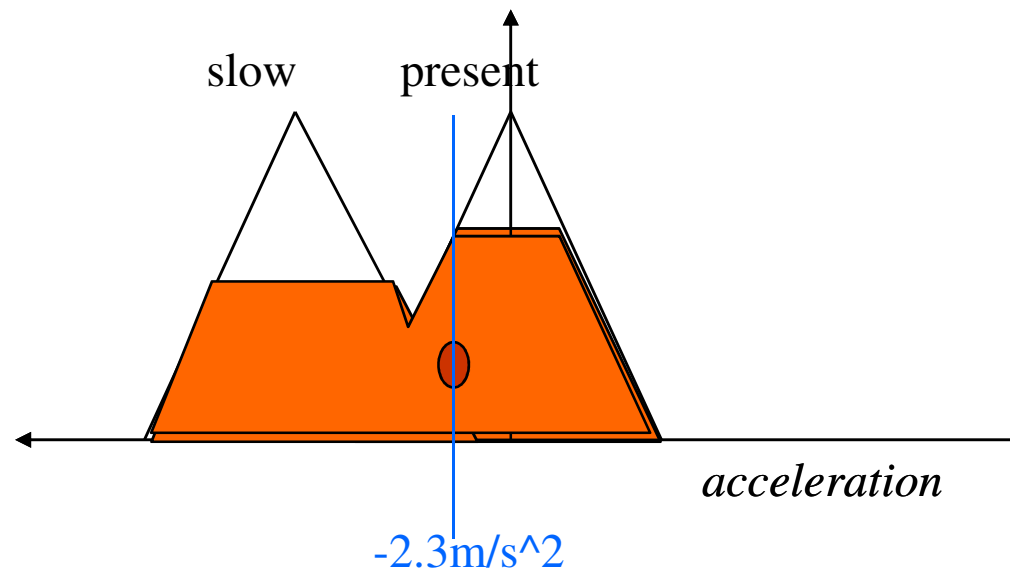
# Rule Aggregation



In the rule aggregation step, we merge all clipped areas into One (taking the union).

Intuition: rules for which we had a strong belief that their premises were satisfied  
Will tend to “pull” that merged area towards their own central value, since their  
Clipped areas will be large

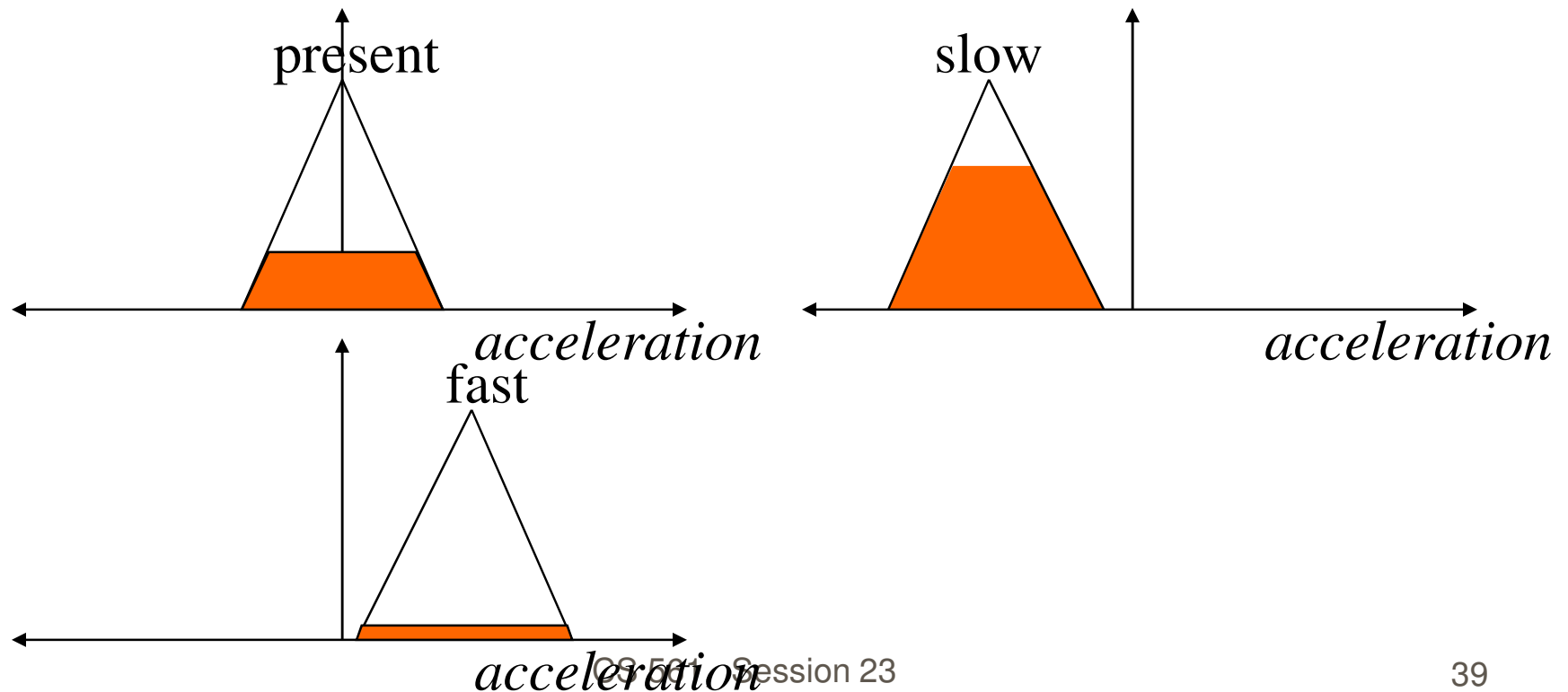
# Defuzzification



In the last step, defuzzification, we return as our acceleration Value the x coordinate of the center of mass of the merged area

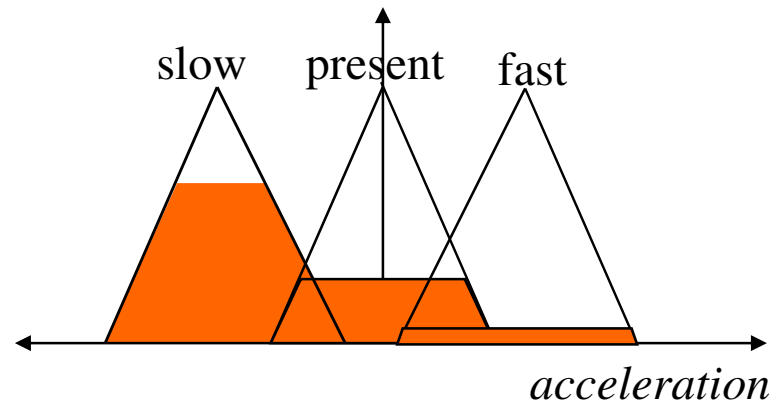
## Rule Aggregation: Another case

- Convert our belief into action
  - For each rule, clip action fuzzy set by belief in rule



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- Convert our belief into action
  - For each rule, clip action fuzzy set by belief in rule





## Matching for Example



- Relevant rules are:
  - If distance is small and delta is growing, maintain speed
  - If distance is small and delta is stable, slow down
  - If distance is perfect and delta is growing, speed up
  - If distance is perfect and delta is stable, maintain speed

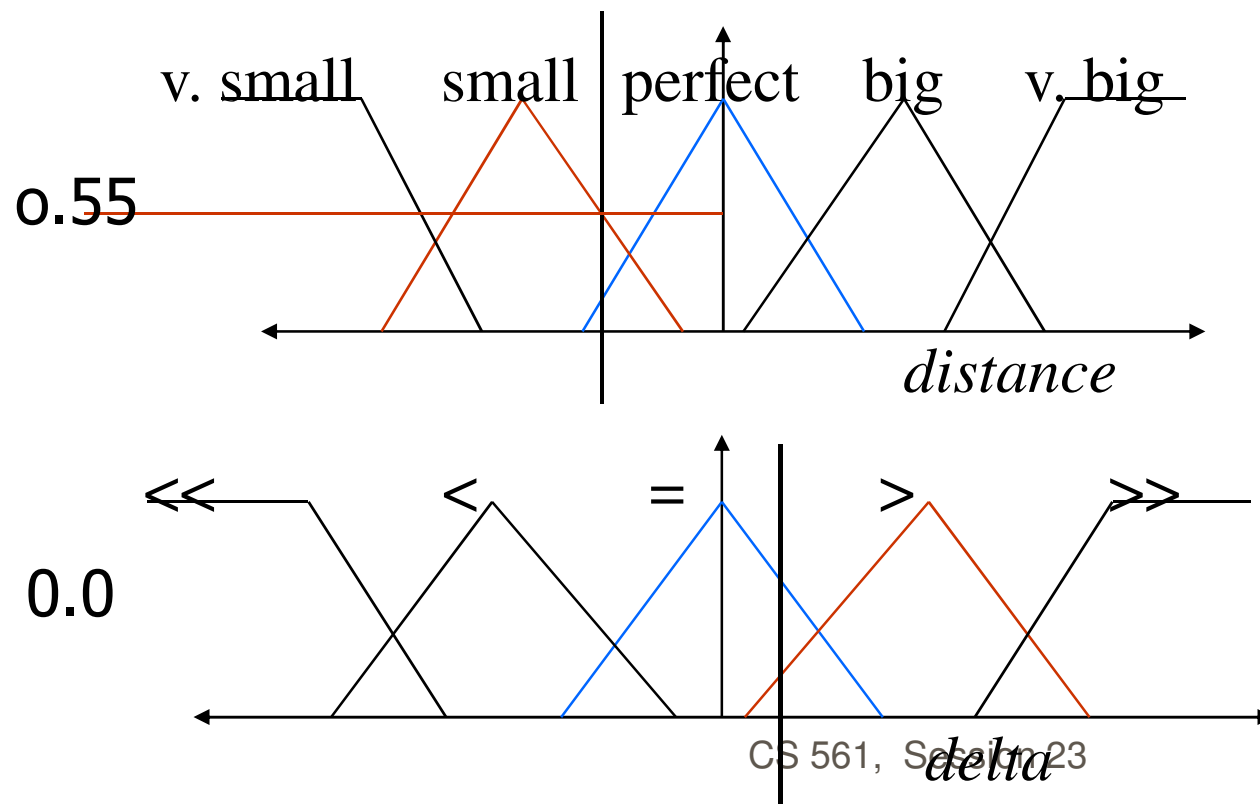
## Matching for Example



- For first rule, distance is small has 0.75 truth, and delta is growing has 0.3 truth
  - So the truth of the **and** is 0.3
- Other rule strengths are 0.6, 0.1 and 0.1

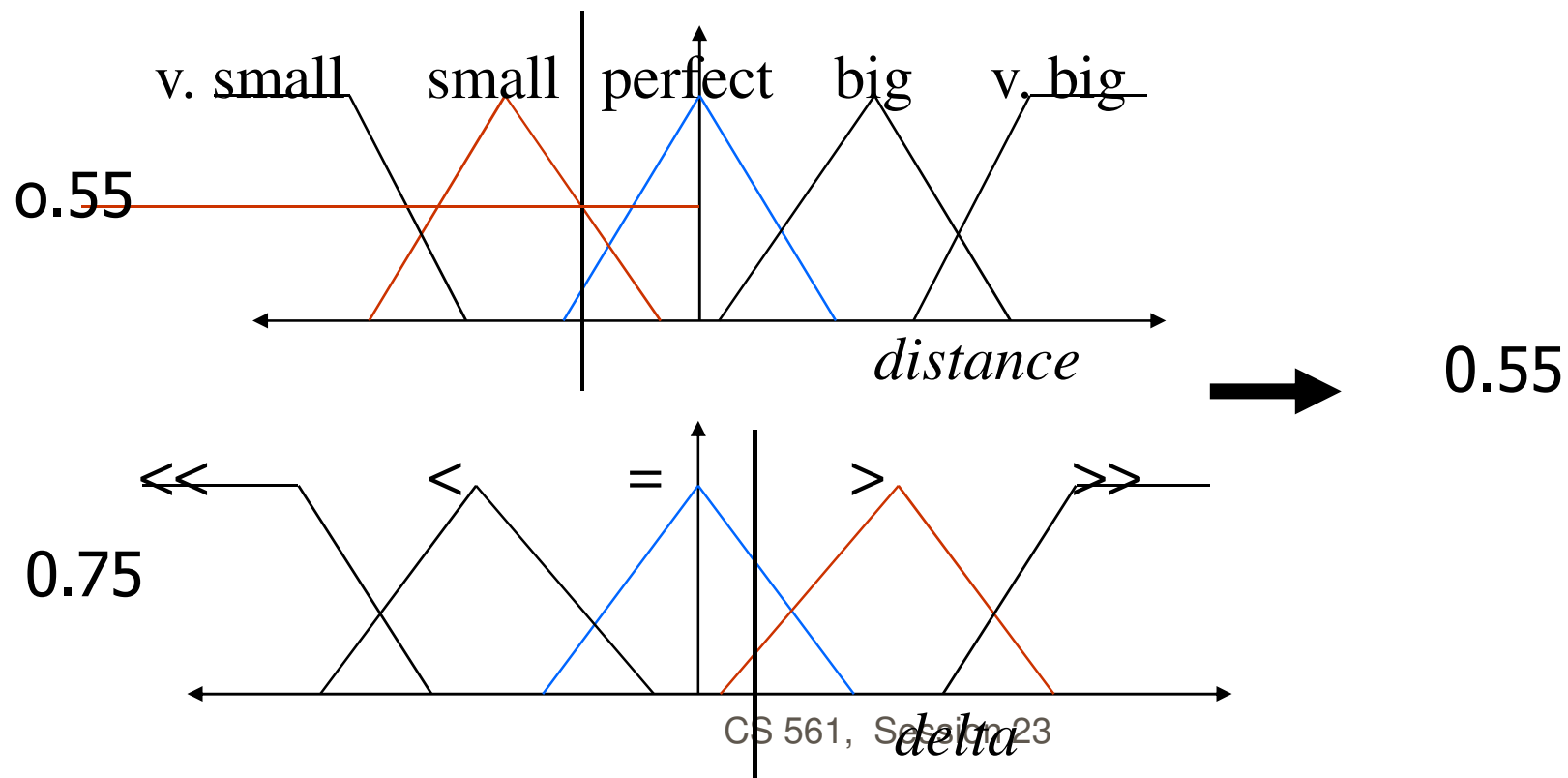
# AND/OR Example

- IF Distance Small AND change in distance negative THEN high deceleration



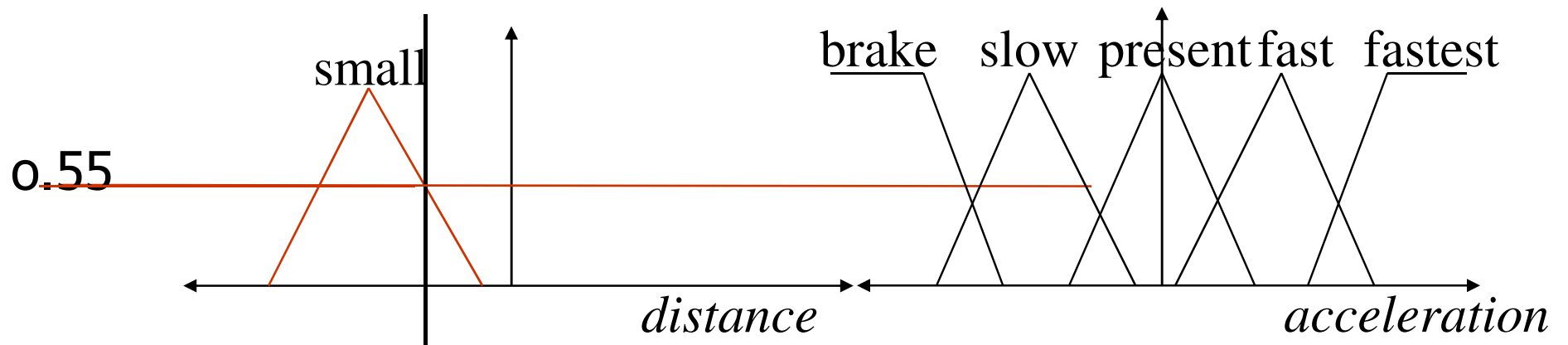
# AND/OR Example

- IF Distance Small AND change in distance = THEN slow deceleration



# AND/OR Example

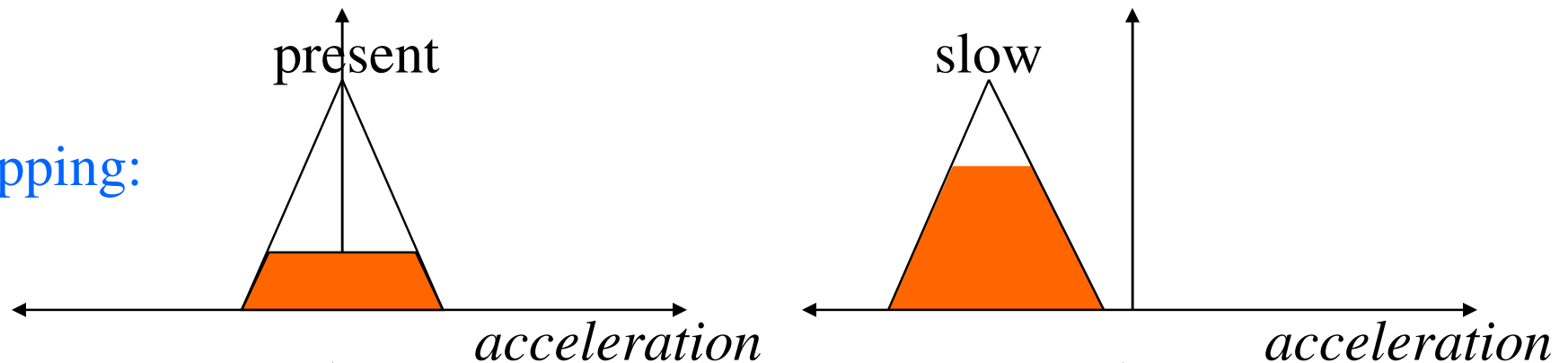
- IF Distance Small AND change in distance = THEN slow deceleration



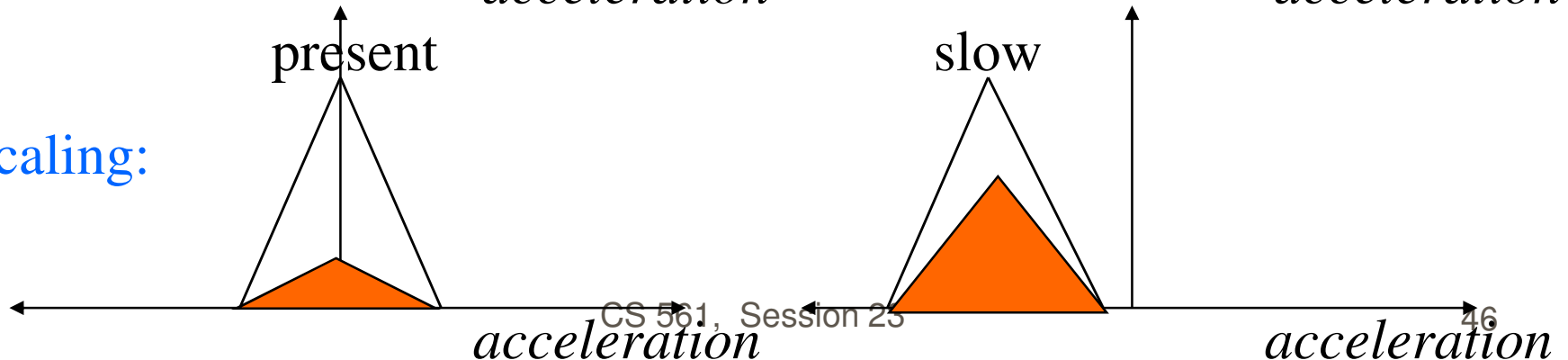
# Scaling vs. Clipping

Instead of clipping, another approach is to scale the fuzzy set  
By the belief in the premises

Clipping:



Scaling:



## Summary: If-Then rules



1. Fuzzify inputs:

Determine the degree of membership for all terms in the premise.

If there is one term then this is the degree of support for the consequence.

2. Apply fuzzy operator:

If there are multiple parts, apply logical operators to determine the degree of support for the rule.

## Summary: If-Then rules



3. Apply implication method:

Use degree of support for rule to shape output fuzzy set of the consequence.

How do we then combine several rules?



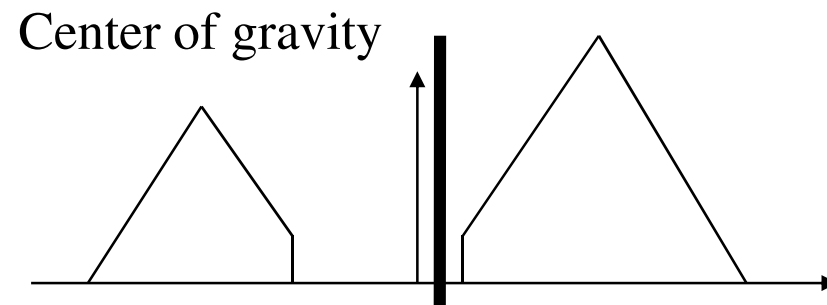
## Multiple rules



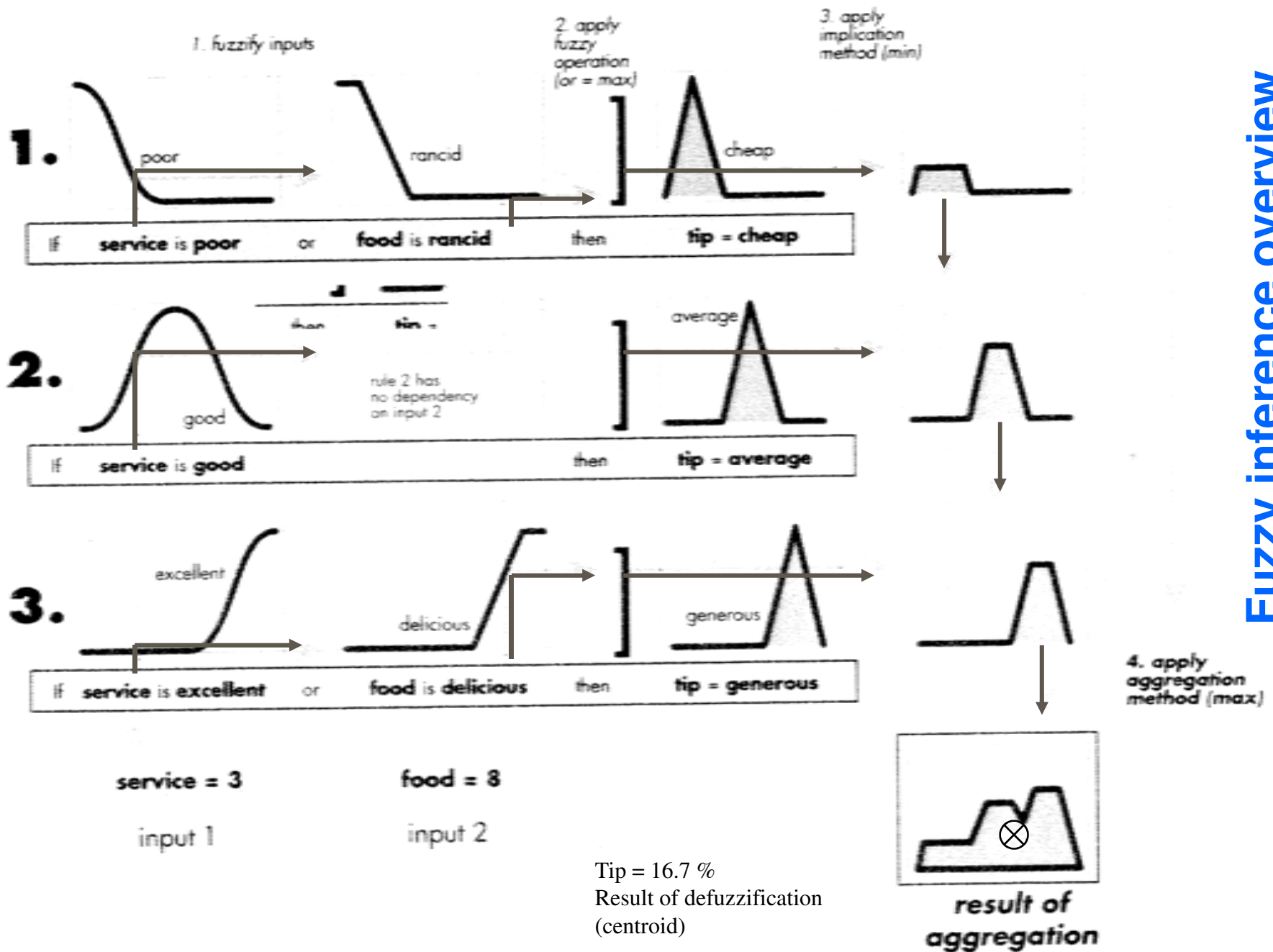
- We aggregate the outputs into a single fuzzy set which combines their decisions.
- The input to aggregation is the list of truncated fuzzy sets and the output is a single fuzzy set for each variable.
- **Aggregation rules:** max, sum, etc.
- As long as it is commutative then the order of rule exec is irrelevant.

## Defuzzify the output

- Take a fuzzy set and produce a single crisp number that represents the set.
- Practical when making a decision, taking an action etc.



# Fuzzy inference overview



## Limitations of fuzzy logic



- How to determine the membership functions? Usually requires fine-tuning of parameters
- Defuzzification can produce undesired results

# Fuzzy tools and shells



- Matlab's Fuzzy Toolbox
- FuzzyClips
- Etc.