Uncertainty

Chapter 13 of AIMA

Outline

- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule

Uncertainty

Let action A_t = leave for airport "t" minutes before flight
Will A_t get me there on time?

Problems:

1. partial observability (road state, other drivers' plans, etc.)
2. noisy sensors (traffic reports)
3. uncertainty in action outcomes (flat tire, etc.)
4. immense complexity of modeling and predicting traffic

Hence a purely logical approach either

1. risks falsehood: "A_{25} will get me there on time", or
2. leads to conclusions that are too weak for decision making:

"A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

(A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport …)

Methods for handling uncertainty

- **Default** or **nonmonotonic** logic:
  - Assume my car does not have a flat tire
  - Assume A_{25} works unless contradicted by evidence
- Issues: What assumptions are reasonable? How to handle contradiction?

- **Rules with fudge factors**:
  - $A_{25} \mapsto_{0.3}$ get there on time
  - *Sprinkler* $\mapsto_{0.99}$ *WetGrass*
  - *WetGrass* $\mapsto_{0.7}$ *Rain*
- Issues: Problems with combination, e.g., *Sprinkler* causes *Rain*??

- **Probability**
  - Model agent's degree of belief
  - Given the available evidence,
  - A_{25} will get me there on time with probability 0.04

Probability

Probabilistic assertions **summarize** effects of

- **laziness**: failure to enumerate exceptions, qualifications, etc.
- **ignorance**: lack of relevant facts, initial conditions, etc.

Subjective probability:

- Probabilities relate propositions to agent's own state of knowledge

  e.g., $P(A_{25} \mid \text{no reported accidents}) = 0.06$

These are **not** assertions about the world

Probabilities of propositions change with new evidence:

e.g., $P(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15$

Making decisions under uncertainty

Suppose I believe the following:

$P(A_{25}$ gets me there on time | …) = 0.04
$P(A_{90}$ gets me there on time | …) = 0.70
$P(A_{120}$ gets me there on time | …) = 0.95
$P(A_{1440}$ gets me there on time | …) = 0.9999

- Which action to choose?

  Depends on my **preferences** for missing flight vs. time spent waiting, etc.
  - **Utility theory** is used to represent and infer preferences
  - **Decision theory** = probability theory + utility theory

Syntax

- Basic element: **random variable**

- Similar to propositional logic: possible worlds defined by assignment of values to random variables.

- **Boolean** random variables
  e.g., *Cavity* (do I have a cavity?)
- **Discrete** random variables
  e.g., *Weather* is one of <*sunny,rainy,cloudy,snow*>
- Domain values must be exhaustive and mutually exclusive

- Elementary proposition constructed by assignment of a value to a random variable: e.g., *Weather = sunny*, *Cavity = false* (abbreviated as $\neg$*cavity*)

- Complex propositions formed from elementary propositions and standard logical connectives e.g., *Weather = sunny* $\vee$ *Cavity = false*

Syntax

- **Atomic event**: A **complete** specification of the state of the world about which the agent is uncertain

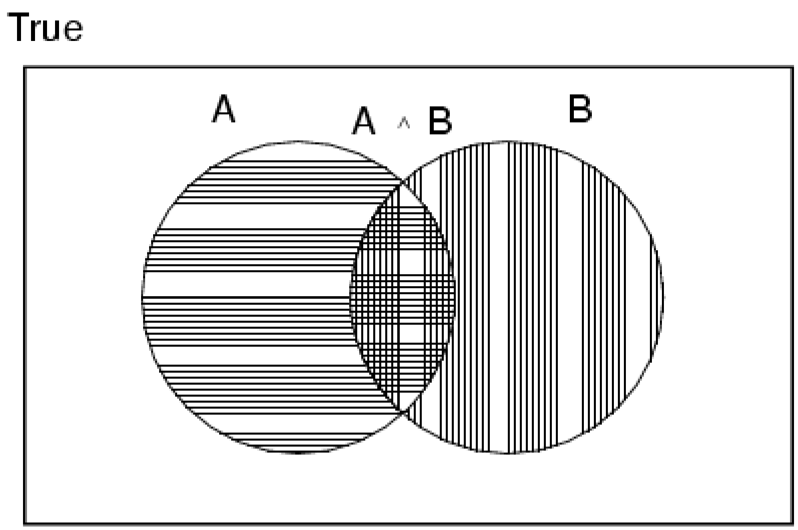
  E.g., if the world consists of only two Boolean variables *Cavity* and *Toothache*, then there are 4 distinct atomic events:

  $Cavity = false \land Toothache = false$
  $Cavity = false \land Toothache = true$
  $Cavity = true \land Toothache = false$
  $Cavity = true \land Toothache = true$

- Atomic events are mutually exclusive and exhaustive

Axioms of probability

- For any propositions *A, B*
  - $0 \leq P(A) \leq 1$
  - $P(true) = 1$ and $P(false) = 0$
  - $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$



Prior probability

- **Prior** or **unconditional probabilities** of propositions
  e.g., P(*Cavity* = true) = 0.1 and P(*Weather* = sunny) = 0.72 correspond
  to belief prior to arrival of any (new) evidence

- **Probability distribution** gives values for all possible assignments:
  P(*Weather*) = <0.72,0.1,0.08,0.1> (**normalized**, i.e., sums to 1)

- **Joint probability distribution** for a set of random variables gives the probability of every atomic event on those random variables
  P(*Weather*,*Cavity*) = a 4 × 2 matrix of values:

Weather =	sunny	rainy	cloudy	snow
Cavity = true	0.144	0.02	0.016	0.02
Cavity = false	0.576	0.08	0.064	0.08

- **Every question about a domain can be answered by the joint distribution**

Conditional probability

- **Conditional** or **posterior probabilities**
  e.g., P(*cavity* | *toothache*) = 0.8
  i.e., given that *toothache* is all I know

- (Notation for conditional distributions:
  P(*Cavity* | *Toothache*) = 2-element vector of 2-element vectors)

- If we know more, e.g., *cavity* is also given, then we have
  P(*cavity* / *toothache,cavity*) = 1

- New evidence may be irrelevant, allowing simplification, e.g.,
  P(*cavity* / *toothache, sunny*) = P(*cavity* | *toothache*) = 0.8
- This kind of inference, sanctioned by domain knowledge, is crucial

Conditional probability

- Definition of conditional probability:
  $P(a \mid b) = P(a \wedge b) / P(b)$ if $P(b) > 0$

- **Product rule** gives an alternative formulation:
  $P(a \wedge b) = P(a \mid b) P(b) = P(b \mid a) P(a)$

- A general version holds for whole distributions, e.g.,
  $\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather \mid Cavity)\, \mathbf{P}(Cavity)$

- (View as a set of 4×2 equations, **not** matrix mult.)

- **Chain rule** is derived by successive application of product rule:

$$\begin{aligned}
\mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1,\ldots,X_{n-1})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\
&= \mathbf{P}(X_1,\ldots,X_{n-2})\, \mathbf{P}(X_{n-1} \mid X_1,\ldots,X_{n-2})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\
&= \ldots \\
&= \prod_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1})
\end{aligned}$$

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$

Inference by enumeration

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	catch	*¬ catch*	*catch*	*¬ catch*
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- For any proposition φ, sum the atomic events where it is true:
  $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$
- P(*toothache*) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$
- P(*toothache* ∨ cavity) = 0.108 + 0.012 + 0.016 + 0.064 + .072 + .008 = .28

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- Can also compute conditional probabilities:

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.016+0.064}{0.108 + 0.012 + 0.016 + 0.064}$$

$$= 0.4$$

Normalization

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- Denominator can be viewed as a **normalization constant** α

P(*Cavity* | *toothache*) = α, **P**(*Cavity,toothache*)

= α, [**P**(*Cavity,toothache,catch*) + **P**(*Cavity,toothache,¬ catch*)]

We can imagine similar equations for the ¬ *Cavity* case. Combining the two sets of equations for *Cavity* and ¬ *Cavity* we can write (more compactly)

= α, [<0.108,0.016> + <0.012,0.064>]

= α, <0.12,0.08> = <0.6,0.4>

General idea: compute distribution on **query variable** (Cavity) by fixing **evidence variables** (toothache) and summing over **hidden variables** (catch)

Inference by enumeration, contd.

Typically, we are interested in
the posterior joint distribution of the **query variables** **Y**
given specific values **e** for the **evidence variables** **E**

Let the **hidden variables** be **H** = **X** - **Y** - **E**

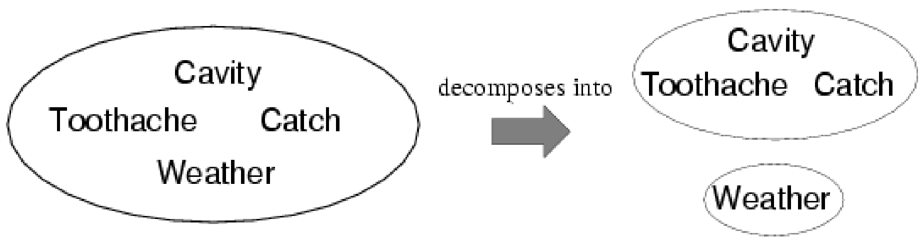
Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbf{P}(\mathbf{Y} \mid \mathbf{E} = \mathbf{e}) = \alpha\mathbf{P}(\mathbf{Y},\mathbf{E} = \mathbf{e}) = \alpha\Sigma_h\mathbf{P}(\mathbf{Y},\mathbf{E}= \mathbf{e}, \mathbf{H} = \mathbf{h})$$

- The terms in the summation are joint entries because **Y**, **E** and **H** together exhaust the set of random variables
- Obvious problems:
  1. Worst-case time complexity $O(d^n)$ where d is the largest arity
  2. Space complexity $O(d^n)$ to store the joint distribution
  3. How to find the numbers for $O(d^n)$ entries?

Independence

- *A* and *B* are independent iff
  P(*A*|*B*) = **P**(*A*) or **P**(*B*|*A*) = **P**(*B*) or **P**(A, B) = **P**(*A*) **P**(*B*)



P(*Toothache, Catch, Cavity, Weather*) = **P**(*Toothache, Catch, Cavity*) **P**(*Weather*)

- 32 entries reduced to 12; for *n* independent biased coins, $O(2^n) \rightarrow O(n)$

- **Absolute independence** powerful but unfortunately rare

- Dentistry is a large field with hundreds of variables, none of which are independent. What to do? We need a more refined idea of independence…

Conditional independence

- $\mathbf{P}(Toothache, Cavity, Catch)$ has $2^3 = 8$ independent entries

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  (1) $\mathbf{P}(catch \mid toothache, cavity) = \mathbf{P}(catch \mid cavity)$

- The same independence holds if I haven't got a cavity:
  (2) $\mathbf{P}(catch \mid toothache, \neg cavity) = \mathbf{P}(catch \mid \neg cavity)$

- *Catch* is **conditionally independent** of *Toothache* given *Cavity*:
  $\mathbf{P}(Catch \mid Toothache, Cavity) = \mathbf{P}(Catch \mid Cavity)$

- Equivalent statements:
  $\mathbf{P}(Toothache \mid Catch, Cavity) = \mathbf{P}(Toothache \mid Cavity)$
  $\mathbf{P}(Toothache, Catch \mid Cavity) = \mathbf{P}(Toothache \mid Cavity)\, \mathbf{P}(Catch \mid Cavity)$

Conditional independence contd.

- Write out full joint distribution using chain rule:
  P(*Toothache, Catch, Cavity*)
  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch, Cavity*)
  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch* | *Cavity*) **P**(*Cavity*)
  = **P**(*Toothache* | *Cavity*) **P**(*Catch* | *Cavity*) **P**(Cavity)
  I.e., 2 + 2 + 1 = 5 independent numbers

- **In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*.**

- **Conditional independence is our most basic and robust form of knowledge about uncertain environments.**

Bayes' Rule

- Product rule $P(a \wedge b) = P(a \mid b)\, P(b) = P(b \mid a)\, P(a)$
  $\Rightarrow$ **Bayes' rule:** $P(a \mid b) = P(b \mid a)\, P(a) / P(b)$

- or in distribution form
  $$\mathbf{P}(Y|X) = \mathbf{P}(X|Y)\, \mathbf{P}(Y) / \mathbf{P}(X) = \alpha \mathbf{P}(X|Y)\, \mathbf{P}(Y)$$

- Useful for assessing **diagnostic** probability from **causal** probability:

  - $P(Cause|Effect) = P(Effect|Cause)\, P(Cause) / P(Effect)$

  - E.g., let *M* be meningitis, *S* be stiff neck:
    $P(m|s) = P(s|m)\, P(m) / P(s) = 0.8 \times 0.0001 / 0.1 = 0.0008$

  - Note: posterior probability of meningitis still very small!

Bayes' Rule and conditional independence

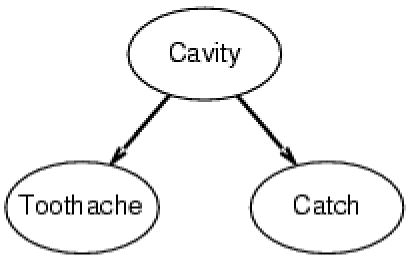
$\mathbf{P}(Cavity \mid toothache \wedge catch)$

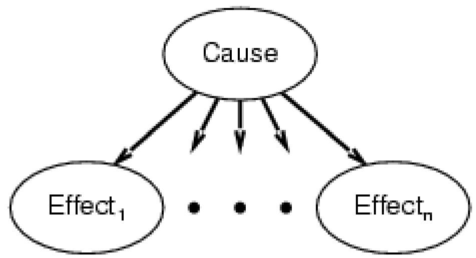
$= \alpha\mathbf{P}(toothache \wedge catch \mid Cavity)\ \mathbf{P}(Cavity)$

$= \alpha\mathbf{P}(toothache \mid Cavity)\ \mathbf{P}(catch \mid Cavity)\ \mathbf{P}(Cavity)$

- This is an example of a **naïve Bayes** model:

  $\mathbf{P}(\text{Cause},\text{Effect}_1, \ldots ,\text{Effect}_n) = \mathbf{P}(\text{Cause})\ \pi_i\mathbf{P}(\text{Effect}_i|\text{Cause})$





- Total number of parameters is **linear** in *n*

Summary

- Probability is a rigorous formalism for uncertain knowledge
- **Joint probability distribution** specifies probability of every **atomic event**
- Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
- **Independence** and **conditional independence** provide additional tools to reduce the size of the analysis to be performed

Bayesian networks

AIMA Chapter 14

Outline

- Syntax
- Semantics

Bayesian networks

- A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions

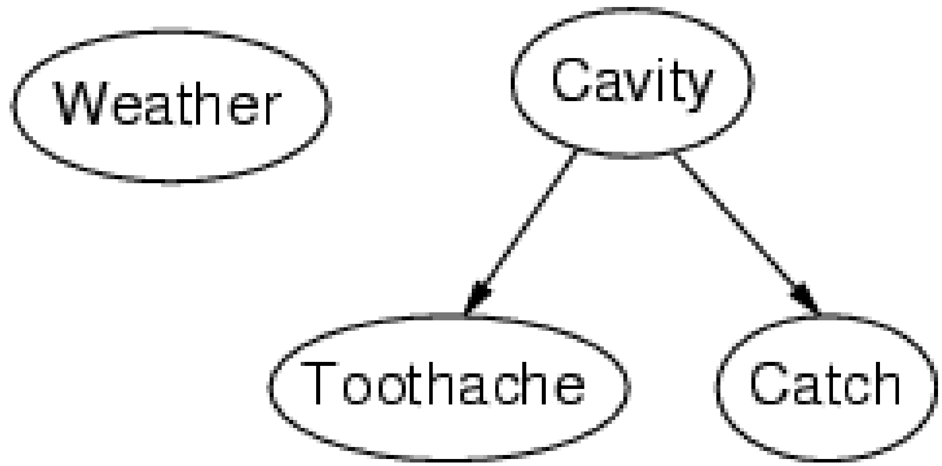
- Syntax:
  - a set of nodes, one per variable
  - a directed, acyclic graph (link ≈ "directly influences")
  - a conditional distribution for each node given its parents:

$$\mathbf{P}(X_i \mid \text{Parents}(X_i))$$

- In the simplest case, conditional distribution represented as a **conditional probability table** (CPT) giving the distribution over X_i for each combination of parent values

Example

- Topology of network encodes conditional independence assertions:



- *Weather* is independent of the other variables
- *Toothache* and *Catch* are conditionally independent given *Cavity*

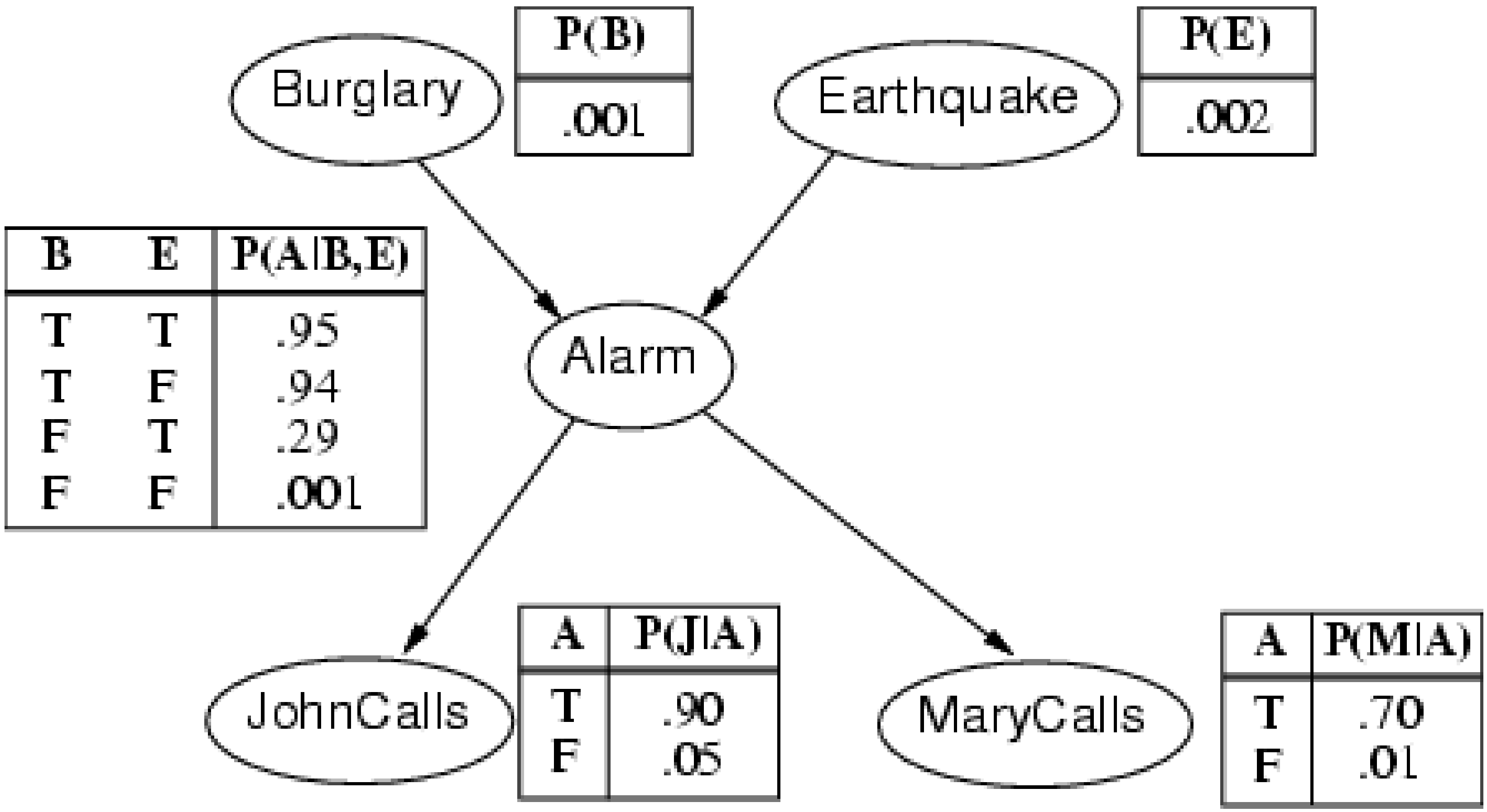
Example

- I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

- Variables: *Burglary*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*

- Network topology reflects "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call, but not every time because she listens to loud music
  - The alarm can cause John to call too, but not every time because he is sometimes confused by the phone ringing
  - John and Mary do not have any way to communicate or coordinate their calling

Example contd.



P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

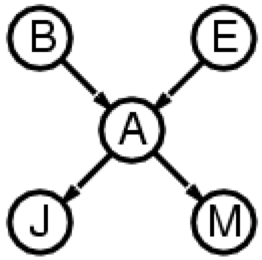
A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

Conditional Probability Tables (CPTs)

Compactness

- A CPT for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values
- Each row requires one number p for $X_i = true$ (the number for $X_i = false$ is just $1-p$)

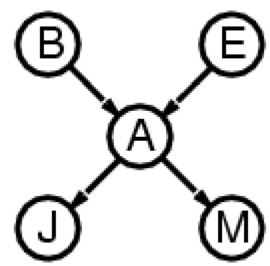


- If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers
- I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution
- For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5-1 = 31$)

Semantics

The full joint distribution is defined as the product of the local conditional distributions:

$$\mathbf{P}(X_1, \ldots, X_n) = \prod_{i=1}^{n} \mathbf{P}(X_i \mid Parents(X_i))$$



e.g., $\mathbf{P}(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$$= \mathbf{P}(j \mid a)\, \mathbf{P}(m \mid a)\, \mathbf{P}(a \mid \neg b, \neg e)\, \mathbf{P}(\neg b)\, \mathbf{P}(\neg e)$$

Constructing Bayesian networks

- 1. Choose an ordering of variables $X_1, \ldots, X_n$
- 2. For i = 1 to n
  - add X_i to the network
  - select parents from $X_1, \ldots, X_{i-1}$ such that

$$\mathbf{P}(X_i \mid Parents(X_i)) = \mathbf{P}(X_i \mid X_1, \ldots X_{i-1})$$

This choice of parents guarantees:

$$\mathbf{P}(X_1, \ldots, X_n) = \prod_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1}) \quad \text{(chain rule)}$$

$$= \prod_{i=1}^{n} \mathbf{P}(X_i \mid Parents(X_i)) \quad \text{(by construction)}$$

Intuitively: connect the variables that DIRECTLY influence each other using arrows. Identify absolutely independent variables. Then, focus on conditional independence

Summary

- Bayesian networks provide a natural representation for (causally induced) conditional independence
- Topology + CPTs = compact representation of joint distribution
- Generally easy for domain experts to construct