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THE EFFECTS OF BINARITY ON PLANET OCCURRENCE RATES MEASURED BY TRANSIT SURVEYS

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ABSTRACT

This note formally derives a general equation for the apparent occurrence rate used in the paper.

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1. PRELIMINARIES

1.1. *Searchable Distance*

We can detect a signal if

$$\frac{\text{signal}}{\text{noise}} \sim \frac{\delta_{\rm obs}}{(L/d^2)^{-1/2}}, \quad \delta_{\rm obs} : \text{observed depth} \tag{1}$$

is above some threshold (it would probably make more sense to include the duration information, but that would anyway be a trivial extension and so we omit it here for brevity). Thus, the maximum searchable distance scales as

$$d(\delta_{\rm obs}, L_{\rm sys}) \propto \delta_{\rm obs} \cdot L_{\rm sys}^{1/2}. \tag{2}$$

We assume that the signal is detected if and only if a given star is searchable.

Assuming that stars are uniformly distributed in space, the number of searchable stars $N_{\rm s}$ is then proportional to

$$N_{\rm s}(\delta_{\rm obs}, L_{\rm sys}) \propto n\delta_{\rm obs}^3 L_{\rm sys}^{3/2}, \tag{3}$$

where n is the volume density of *e.g.*, single star, or binary systems. We neglect the dependence of n on stellar type.

1.2. *Relation between Apparent and Actual Stellar Properties*

We assume that the apparent properties of an unresolved binary are the same as those of the primary:

$$M_{\rm a} = M_1, \quad R_{\rm a} = R_1, ... \tag{4}$$

We also assume the stellar radius and luminosity is uniquely related to the stellar mass.

Given these assumptions, the total luminosity of a system is

$$L_{\rm sys} = L_1 + L_2 = L(M_{\rm a}) + L(qM_{\rm a}), \tag{5}$$

where $q = M_2/M_1$. Note that $L_{\rm sys}$ is the true value (because $M_{\rm a}$ is the true primary mass), while an observer would estimate a system luminosity of $L(M_{\rm a})$, based on the apparent stellar parameters. We presume the observer has no a priori knowledge of the distance to a given system – they only measure a flux, assume a stellar mass M_a, and use relations for $L(M_a)$ and $R(M_a)$ to estimate system properties.

1.3. *Apparent Number of Searchable Stars*

Given the apparent signal $\delta_{\rm obs}$ and stellar mass $M_{\rm a}$, the maximum searchable distance for singles and binaries are proportional to $\delta_{\rm obs} \cdot L(M_{\rm a})^{1/2}$ and $\delta_{\rm obs} \cdot [L(M_{\rm a}) + L(qM_{\rm a})]^{1/2}$. Thus, the apparent number of searchable stars (i.e. points in the sky), which will be selected by an ignorant observer, is proportional to

$$N_{\rm s,a}(\delta_{\rm obs}, M_{\rm a}) \propto n_{\rm s}\delta_{\rm obs}^3 L(M_{\rm a})^{3/2} \left[1 + \mu({\rm BF}, M_{\rm a})\right],$$

and we can define

$$N_{\mathrm{s,a}}(\delta_{\mathrm{obs}}, M_{\mathrm{a}}) \equiv N^0_{\mathrm{s}}(\delta_{\mathrm{obs}}, L(M_{\mathrm{a}}))\left[1+\mu(\mathrm{BF}, M_{\mathrm{a}})\right], \tag{6}$$

where N^0_{s} is the number of searchable singles (this agrees with the actual value), $\mathrm{BF} = n_{\mathrm{b}}/(n_{\mathrm{s}} + n_{\mathrm{b}})$ is the binary fraction in a volume-limited sample, and n_s is the number density of singles in a volume-limited sample. Proportionality constants subsumed into the definition of N^0_s include *e.g.*, the telescope area and the survey duration.

1.3.1. *What is μ?*

In Eq. 6, μ is the ratio of the number of single to double star systems (that are searchable for an observed signal δ_{obs}). If we make various simplifying assumptions, it turns out that this number depends only on the binary fraction and the binary mass ratio distribution in a volume-limited sample. The proof is as follows.

In a shell of thickness $\mathrm{d}r$, there are $n_d(r)4\pi r^2\mathrm{d}r$ searchable double star systems. The number density of searchable double star systems $n_d(r)$ is constant up to the maximum searchable distance for singles, d_0. At greater distances, $n_d(r)$ decreases, because only some binaries have sufficient flux to make the SNR cut. Specifically,

$$n_d(r) = \begin{cases} n_b, & r \le d_0 \\ n_b \int_{q_{\min}(r)}^{1} \mathrm{d}q\, f(q), & d_0 < r < \sqrt{2}d_0 \end{cases} \tag{7}$$

where $f(q)$ is the mass ratio distribution in a volume-limited sample, $q_{\min}(r)$ is the minimum mass ratio required to provide sufficient flux for searchability. Assuming $L \propto M^\alpha$,

$$q_{\min}(r) = \left(\left(\frac{r}{d_0}\right)^2 - 1\right)^{\frac{1}{\alpha}}. \tag{8}$$

Using the common parametrization $f(q) \sim q^\beta$, and setting $\beta = 0$,

$$n_d(r) = \begin{cases} n_b, & r \le d_0 \\ n_b(1 - q_{\min}(r)), & d_0 < r < \sqrt{2}d_0. \end{cases} \tag{9}$$

Integrating,

$$N_d = \int \mathrm{d}N_d = \int n_d(r)4\pi r^2\mathrm{d}r \tag{10}$$

$$= 4\pi n_b \left(\frac{d_0^3}{3} + \left[\frac{r^3}{3}\right]_{d_0}^{\sqrt{2}d_0} - \int_{d_0}^{\sqrt{2}d_0} \left(\left(\frac{r}{d_0}\right)^2 - 1\right)^{\frac{1}{\alpha}} r^2\,\mathrm{d}r\right) \tag{11}$$

$$= \frac{4\pi n_b}{3} d_0^3 \left(2^{3/2} - 3\int_1^{\sqrt{2}} \mathrm{d}u\, u^2(u^2-1)^{1/\alpha}\right), \tag{12}$$

$$\equiv \frac{4\pi n_b}{3} d_0^3 \left(2^{3/2} - 3I\right), \tag{13}$$

where the penultimate line substituted $u = r/d_0$. For the common case $\alpha = 3.5$, the dimensionless integral I evaluates to ≈ 0.484174.

Thus

$$\mu \equiv \frac{N_d}{N_s^0} = \frac{n_b}{n_s}\left(2^{3/2} - 3I\right) = \frac{\mathrm{BF}}{1-\mathrm{BF}}\left(2^{3/2} - 3I\right). \tag{14}$$

For the twin binary case, $I = 0$, and for the power law volume-limited distribution, I takes the form given in Eq. 12.

2. APPARENT OCCURRENCE RATE — GENERAL FORMULA

A group of astronomers wants to measure the mean number of planets of a certain type per star of a certain type. They observe a set points on the sky and detect N_{det} planets that appear to be of the desired class. They then choose the stars (among those initially selected) around which the planets of interest appeared to be searchable. Finally they compute the apparent occurrence,

$$\Lambda_{\mathrm{a}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}}) = \frac{N_{\mathrm{det}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})}{N_{\mathrm{s,a}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})} \times \frac{1}{p_{\mathrm{tra}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})}. \tag{15}$$

where $\mathcal{P}_{\mathrm{a}}$, $\mathcal{S}_{\mathrm{a}}$ are the apparent planetary/stellar parameters.

In the presence of dilution, planets with $(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})$ are associated with systems of many different planetary and stellar properties, so $N_{\mathrm{det}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})$ is given by the convolution of the true occurrence, $\Lambda(\mathcal{P}, \mathcal{S})$, and number of searchable stars that give $(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})$ when the true system actually has $(\mathcal{P}, \mathcal{S})$, $\mathcal{N}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}}; \mathcal{P}, \mathcal{S})$:

$$N_{\mathrm{det}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}}) = \sum_i N_{\mathrm{det}}^i(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}}) = \sum_i \int \mathrm{d}\mathcal{P}\mathrm{d}\mathcal{S}\, \mathcal{N}_{\mathrm{s}}^i(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}}; \mathcal{P}, \mathcal{S}) \cdot \Lambda^i(\mathcal{P}, \mathcal{S}) \cdot p_{\mathrm{tra}}(\mathcal{P}, \mathcal{S}), \tag{16}$$

where i specifies the type of true host stars (0: single, 1: primary, 2: secondary).

So the problem reduces to the evaluation of

$$\mathcal{N}_{\mathrm{s}}^i(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}}; \mathcal{P}, \mathcal{S}) \tag{17}$$

for planets around single stars, primaries in binaries, and secondaries in binaries.

3. EVALUATION OF $\mathcal{N}_{\mathrm{s}}^i$

Let us explicitly write $\mathcal{P} = r$ and $\mathcal{S} = M$; R and L are uniquely determined from the assumed mass–radius–luminosity relation. We neglect the dependence on planetary orbital period. We proceed by evaluating

$$N_{\mathrm{det}}(r_a, M_a) = \sum_i N_{\mathrm{det}}^i(r_a, M_a) \tag{18}$$

$$N_{\mathrm{det}}(r_a, M_a) = \sum_i \int \mathrm{d}r\mathrm{d}M\, \mathcal{N}_{\mathrm{s}}^i(r_a, M_a; r, M) \cdot \Lambda^i(r, M) \cdot p_{\mathrm{tra}}(r, M), \tag{19}$$

term by term.

3.1. *Single Stars*

For $i = 0$,

$$\mathcal{N}_{\rm s}^0(r_{\rm a}, M_{\rm a}; r, M) = \delta(r_{\rm a} - r)\delta(M_{\rm a} - M)N_{\rm s}^0(r, M), \tag{20}$$

so

$$N_{\rm det}^0(r_{\rm a}, M_{\rm a}) = N_{\rm s}^0(r_{\rm a}, M_{\rm a}) \cdot \Lambda^0(r_{\rm a}, M_{\rm a}) \cdot p_{\rm tra}(r_{\rm a}, M_{\rm a}). \tag{21}$$

If all the stars are singles, this yields

$$\Lambda_{\rm a}(r_{\rm a}, M_{\rm a}) = \Lambda^0(r_{\rm a}, M_{\rm a}), \tag{22}$$

as expected (now $\mu = 0$ and $N_{\rm s,a} = N_{\rm s}^0$) — the true occurrence is recovered.

3.2. *Primaries in Binaries*

Since we assume $\mathcal{S}_{\rm a} = \mathcal{S}_1$,

$$\mathcal{N}_{\rm s}^1(r_a, M_a; r, M) \propto \delta(M_a - M). \tag{23}$$

In this case, $\mathcal{N}_{\rm s}^1$ is non-zero only at $r_a = R_a\sqrt{\delta_{\rm obs}}$, and the observed depth is

$$\delta_{\rm obs} = \left[\frac{r}{R(M_a)}\right]^2 \times \frac{L(M_a)}{L_{\rm sys}(M_a, q)}. \tag{24}$$

The normalization of $\mathcal{N}_{\rm s}^1$ is given by the number of binaries that are searchable for a signal $\delta_{\rm obs}$:

$$N_{\rm s}^0(\delta_{\rm obs}, L(M_a)) \cdot \mu({\rm BF}, M_a). \tag{25}$$

Thus, the number of primaries with apparent parameters (r_a, M_a) given the true parameters (r, M) is

$$\mathcal{N}_{\rm s}^1(r_a, M_a; r, M) = \int {\rm d}q\, f(q)\mathcal{N}_{{\rm s},q}^1(r_a, M_a; r, M; q), \tag{26}$$

where $f(q)$ is the binary mass ratio distribution in the SNR-limited sample and

$$\begin{aligned}\mathcal{N}_{{\rm s},q}^1(r_a, M_a; r, M; q) = N_{\rm s}^0(\delta_{\rm obs}, L(M_a)) \cdot \mu({\rm BF}, M_a) \\ \times \delta\left(r_a - r\sqrt{\frac{L(M_a)}{L_{\rm sys}(M_a, q)}}\right)\delta(M_a - M).\end{aligned} \tag{27}$$

Note that while the observed depth is a function of the true parameters (Eq. 24), since we are counting detected planets of a given *apparent* size, we will not need to worry about this dependence. As discussed in the main text, we have $f(q) \sim q^\beta(1 + q^\alpha)^{3/2}$.

3.3. *Secondaries in Binaries*

In this case, $M = qM_1 = qM_a$, so

$$\mathcal{N}_\mathrm{s}^2(r_a, M_a; r, M) \propto \delta\left(M_a - \frac{M}{q}\right). \tag{28}$$

Again $\mathcal{N}_\mathrm{s}^2$ is non-zero only at $r_a = R_a\sqrt{\delta_\mathrm{obs}}$, but this time

$$\delta_\mathrm{obs} = \left[\frac{r}{R(qM_a)}\right]^2 \times \frac{L(qM_a)}{L_\mathrm{sys}(M_a, q)}. \tag{29}$$

The normalization remains the same as the previous case (we are counting the searchable stars at a given observed depth δ_obs, total luminosity of the binary is the same). Thus,

$$\mathcal{N}_\mathrm{s}^2(r_a, M_a; r, M) = \int \mathrm{d}q\, f(q) \mathcal{N}_{\mathrm{s},q}^2(r_a, M_a; r, M; q), \tag{30}$$

where

$$\begin{aligned}\mathcal{N}_\mathrm{s}^2(r_a, M_a; r, M; q) = N_\mathrm{s}^0(\delta_\mathrm{obs}, L(M_a)) \cdot \mu(\mathrm{BF}, M_a)& \\ \times\, \delta\left(r_a - r\sqrt{\left[\frac{R(M_a)}{R(qM_a)}\right]^2 \frac{L(qM_a)}{L_\mathrm{sys}(M_a, q)}}\right) \delta\left(M_a - \frac{M}{q}\right).&\end{aligned} \tag{31}$$

4. RESULT

4.1. *Marginalization over the True Properties*

Let's integrate out $\mathcal{P}$ and $\mathcal{S}$.

$$N_\mathrm{det}^0(r_\mathrm{a}, M_\mathrm{a}) = \int \mathrm{d}r\mathrm{d}M\, \mathcal{N}_\mathrm{s}^0(r_\mathrm{a}, M_\mathrm{a}; r, M) \cdot \Lambda^0(r, M) \cdot p_\mathrm{tra}(M) \tag{32}$$

$$= \int \mathrm{d}r\mathrm{d}M\, N_\mathrm{s}^0(\delta_\mathrm{obs}, L(M_\mathrm{a}))\delta(r_\mathrm{a} - r)\delta(M_\mathrm{a} - M) \cdot \Lambda^0(r, M) \cdot p_\mathrm{tra}(M) \tag{33}$$

$$= N_\mathrm{s}^0(\delta_\mathrm{obs}, L(M_\mathrm{a})) \cdot \Lambda^0(r_\mathrm{a}, M_\mathrm{a}) \cdot p_\mathrm{tra}(M_\mathrm{a}). \tag{34}$$

$$N_\mathrm{det}^1(r_\mathrm{a}, M_\mathrm{a}) = \int \mathrm{d}r\mathrm{d}M\, \mathcal{N}_\mathrm{s}^1(r_\mathrm{a}, M_\mathrm{a}; r, M) \cdot \Lambda^1(r, M) \cdot p_\mathrm{tra}(M) \tag{35}$$

$$= \int \mathrm{d}q\, f(q) \int \mathrm{d}r\mathrm{d}M\, \mathcal{N}_{\mathrm{s},q}^1(r_\mathrm{a}, M_\mathrm{a}; r, M; q) \cdot \Lambda^1(r, M) \cdot p_\mathrm{tra}(M) \tag{36}$$

$$= N_\mathrm{s}^0(\delta_\mathrm{obs}, L(M_\mathrm{a})) \cdot p_\mathrm{tra}(M_\mathrm{a}) \cdot \mu(\mathrm{BF}, M_\mathrm{a}) \int \frac{\mathrm{d}q}{\mathcal{A}}\, f(q)\, \Lambda^1(r_a/\mathcal{A}, M_\mathrm{a}), \tag{37}$$

where

$$\mathcal{A}(q, M_{\rm a}) = \sqrt{\frac{L(M_{\rm a})}{L_{\rm sys}(M_{\rm a}, q)}}. \tag{38}$$

Finally,

$$\begin{aligned}
N^2_{\rm det}(r_{\rm a}, M_{\rm a}) &= \int {\rm d}r{\rm d}M\, \mathcal{N}^2_{\rm s}(r_{\rm a}, M_{\rm a}; r, M) \cdot \Lambda^2(r, M) \cdot p_{\rm tra}(M) &(39)\\
&= \int {\rm d}q\, f(q) \int {\rm d}r{\rm d}M\, \mathcal{N}^2_{{\rm s},q}(r_{\rm a}, M_{\rm a}; r, M; q) \cdot \Lambda^2(r, M) \cdot p_{\rm tra}(M) &(40)\\
&= N^0_{\rm s}(\delta_{\rm obs}, L(M_{\rm a}))\mu({\rm BF}, M_{\rm a}) \int {\rm d}r{\rm d}q\, q f(q)\delta(r_a - r\mathcal{B})\Lambda^2(r, qM_{\rm a})p_{\rm tra}(qM_{\rm a}) &(41)\\
&= N^0_{\rm s}(\delta_{\rm obs}, L(M_{\rm a})) \cdot \mu({\rm BF}, M_{\rm a}) \int \frac{q{\rm d}q}{\mathcal{B}}\, f(q)\, \Lambda^2(r_a/\mathcal{B}, qM_{\rm a})\, p_{\rm tra}(qM_{\rm a}), &(42)
\end{aligned}$$

where

$$\mathcal{B}(q, M_{\rm a}) = \frac{R(M_{\rm a})}{R(qM_{\rm a})}\sqrt{\frac{L(qM_{\rm a})}{L_{\rm sys}(M_{\rm a}, q)}}. \tag{43}$$

4.2. *Final Formula*

Note again that the denominators of $\Lambda_{\rm a}$ are the same for singles, primaries, and secondaries: $N_{\rm s,a}(\delta_{\rm obs}, M_{\rm a})$ and $p_{\rm tra}(M_{\rm a})$. This is because we are calculating the occurrence at the same apparent planet/star properties, and the observer can never distinguish binaries from singles (and adopt the primary properties for binaries). Using the results above, the apparent occurrence rate,

$$\Lambda_{\rm a}(r_{\rm a}, M_{\rm a}) = \frac{1}{N_{\rm s,a}(r_{\rm a}, M_{\rm a})p_{\rm tra}(r_{\rm a}, M_{\rm a})} \times \sum_i N^i_{\rm det}(r_{\rm a}, M_{\rm a}), \tag{44}$$

evaluates to

$$\begin{aligned}
\Lambda_{\rm a}(r_{\rm a}, M_{\rm a}) = \frac{1}{1+\mu({\rm BF}, M_{\rm a})} \times \Bigg\{ \Lambda^0(r_{\rm a}, M_{\rm a}) + \mu({\rm BF}, M_{\rm a}) \Bigg[\int \frac{{\rm d}q}{\mathcal{A}}\, f(q)\, \Lambda^1\left(\frac{r_{\rm a}}{\mathcal{A}}, M_{\rm a}\right)\\
+ \int \frac{q{\rm d}q}{\mathcal{B}}\, f(q)\, \Lambda^2\left(\frac{r_{\rm a}}{\mathcal{B}}, qM_{\rm a}\right)\, \frac{R(qM_{\rm a})}{R(M_{\rm a})} q^{-1/3} \Bigg] \Bigg\}. \qquad (45)
\end{aligned}$$

4.2.1. *Simplifying above general result*

To simplify, drop the explicit M_a and BF dependences, and assume $R \propto M$, so $R(qM_a)/R(M_a) = q$. Also, move all superscript numbers to subscripts. Then the apparent occurrence rate is

$$\Lambda_{\rm a}(r_{\rm a}) = \frac{1}{1+\mu}\left(\Lambda_0(r_{\rm a}) + \mu\left[\int \frac{{\rm d}q}{\mathcal{A}(q)} f(q)\Lambda_1\left(\frac{r_{\rm a}}{\mathcal{A}(q)}\right) + \int \frac{q{\rm d}q}{\mathcal{B}(q)} f(q)\Lambda_2\left(\frac{r_{\rm a}}{\mathcal{B}(q)}\right) q^{2/3}\right]\right). \tag{46}$$

For the power law case when $L \sim M^\alpha$,

$$\mathcal{A}(q) = (1+q^\alpha)^{-1/2}, \quad \mathcal{B}(q) = q^{-1}(1+q^{-\alpha})^{-1/2} \tag{47}$$

5. EXAMPLES

5.1. *Twin Binary*

We have $f(q) = \delta(q-1)$. The apparent rate is

$$\Lambda_{\rm a}(r_{\rm a}) = \frac{1}{1+\mu({\rm BF})}\left\{\Lambda_0(r_{\rm a}) + \mu({\rm BF})\cdot\sqrt{2}\cdot\left[\Lambda_1(\sqrt{2}\,r_{\rm a}) + \Lambda_2(\sqrt{2}\,r_{\rm a})\right]\right\}, \tag{48}$$

where

$$\mu({\rm BF}) = \int {\rm d}q\, f(q)\frac{\rm BF}{1-{\rm BF}}\left[1+\frac{L(qM_{\rm a})}{L(M_{\rm a})}\right]^{3/2} = 2^{3/2}\cdot\frac{\rm BF}{1-{\rm BF}}. \tag{49}$$

5.1.1. *Same Planets (Model # 1)*

If $\Lambda_i(r, M) = Z_i\cdot\delta(r-r_0)$ (all the planets have the same radius), by using the identity $\delta(ax+b) = \delta(x+b/a)/|a|$,

$$\Lambda_{\rm a}(r_{\rm a}) = \frac{1}{1+\mu({\rm BF})}\left[Z_0\cdot\delta(r_{\rm a}-r_0) + (Z_1+Z_2)\cdot\mu({\rm BF})\cdot\delta\left(r_{\rm a}-\frac{r_0}{\sqrt{2}}\right)\right]. \tag{50}$$

This reproduces Eq.(9) of the draft (after removing the $p_{\rm det}$ terms). In the limit of BF $\to 1$ ($\mu\to\infty$), the above formula yields $\Lambda_{\rm a} = (Z^1+Z^2)\,\delta(r_{\rm a}-r_0/\sqrt{2})$.

Eq. 50 is technically abusing notation, because Λ must be dimensionless, and $\delta(r-r_0)$ has units of inverse length. Since the equation is meant to convey the idea: "at a given apparent radius, the apparent number of planets per star is x", and the delta function is more succinct than listing out discrete cases, we leave it as-is.

5.1.2. *Power Law Distribution of Planets*

Imagine instead that we took $\Lambda_i(r, M) = Z_i r^\delta$. From Eq. 48, we get an apparent rate

$$\Lambda_a(r_a) = r_a^\delta\left[\frac{Z_0}{1+\mu({\rm BF})} + 2^{\frac{\delta}{2}+1}\frac{\mu({\rm BF})}{1+\mu({\rm BF})}(Z_1+Z_2)\right]. \tag{51}$$

Consider the case when the Z_i's are identical. If we compare $\Lambda_a(r_a)$ with $\Lambda(r)$, we must impose that r_a and r are indistinguishable (i.e.: the observer cannot tell them apart). Note that the true rate is

$$\Lambda(r) = r^\delta\left[\frac{Z_0}{1+2\mu} + \frac{\mu}{1+2\mu}(Z_1+Z_2)\right] \tag{52}$$

We then get a correction factor

$$X_\Lambda \equiv \left.\frac{\Lambda_a(r_a)}{\Lambda(r)}\right|_{r_a\to r} = \frac{2^{\frac{\delta}{2}+2}\mu+1}{\mu+1}. \tag{53}$$

For the case of BF $= 0.1$, $\mu\approx 0.153$. Taking $\delta = -2.92$ from Howard et al. (2012), we get a correction factor $X_\Lambda = \Lambda_a/\Lambda = 1.06$. In other words, the apparent rate is an *over*estimate of the true rate, with a relative error of 6%.

5.2. *Power Law World*

If we assume

$$L(M) \sim M^\alpha \sim R^\alpha, \tag{54}$$

we find

$$\mathcal{A}(q) = (1+q^\alpha)^{-1/2}, \quad \mathcal{B}(q) = q^{-1}(1+q^{-\alpha})^{-1/2}. \tag{55}$$

This gives a value of $\mu(\mathrm{BF})$ stated in Eq. 14. Now recall Eq. 46 for the apparent occurrence rate. We restate it here:

$$\Lambda_\mathrm{a}(r_\mathrm{a}) = \frac{1}{1+\mu}\left(\Lambda_0(r_\mathrm{a}) + \mu\left[\int \frac{\mathrm{d}q}{\mathcal{A}(q)} f(q)\Lambda_1\left(\frac{r_\mathrm{a}}{\mathcal{A}(q)}\right) + \int \frac{q\mathrm{d}q}{\mathcal{B}(q)} f(q)\Lambda_2\left(\frac{r_\mathrm{a}}{\mathcal{B}(q)}\right) q^{2/3}\right]\right). \tag{56}$$

5.2.1. *Same Planets (Model # 2)*

For our "Model #2", we again want $\Lambda_i(r) = Z_i \cdot \delta(r - r_0)$. To achieve this, write

$$f(q) = \frac{q^\beta(1+q^\alpha)^{3/2}}{\mathcal{N}_q}, \tag{57}$$

for $\mathcal{N}_q$ the normalization. For $i=1$, call the integral on the left the $[\ldots]$ "I_1". Then

$$\begin{aligned} I_1(r_a) &\equiv \int \frac{\mathrm{d}q}{\mathcal{A}(q)} f(q)\Lambda_1\left(\frac{r_\mathrm{a}}{\mathcal{A}(q)}\right) && (58)\\ &= \frac{1}{\mathcal{N}_q}\int \mathrm{d}q\, q^\beta(1+q^\alpha)^2\Lambda_1(r_a(1+q^\alpha)^{1/2}) && (59)\\ &= \frac{Z_1}{\mathcal{N}_q}\int \mathrm{d}q\, q^\beta(1+q^\alpha)^2\delta(r_a(1+q^\alpha)^{1/2} - r_0) && (60)\\ &= \frac{Z_1}{\mathcal{N}_q}\left(\left(\frac{r_0}{r_a}\right)^2 - 1\right)^{\beta/\alpha}\left(\frac{r_0}{r_a}\right)^4. && (61) \end{aligned}$$

The penultimate line used the fact that $r_a(1+q^\alpha)^{1/2} = r_0$ has only one root:

$$q(r_a) = \left(\left(\frac{r_0}{r_a}\right)^2 - 1\right)^{1/\alpha}. \tag{62}$$

Eq. 61 should be used as a sanity check on any Monte Carlo numerics.

Attempting the same thing for $i=2$,

$$\begin{aligned} I_2(r_a) &\equiv \int \frac{q\mathrm{d}q}{\mathcal{B}(q)} f(q)\Lambda_2\left(\frac{r_\mathrm{a}}{\mathcal{B}(q)}\right) q^{2/3} && (63)\\ &= \frac{Z_2}{\mathcal{N}_q}\int \mathrm{d}q\, q^{\beta+\frac{8}{3}}(1+q^\alpha)^{3/2}(1+q^{-\alpha})^{1/2}\delta(r_a q(1+q^{-\alpha})^{1/2} - r_0). && (64) \end{aligned}$$

Unfortunately, analytic roots of $r_a q(1+q^{-\alpha})^{1/2} = r_0$ do not exist. Schematically, since for any function $g(x)$,

$$\delta(g(x)) = \sum_{x_i \in \{\mathrm{roots}\}} \frac{\delta(x - x_i)}{|g'(x_i)|}, \tag{65}$$

if the roots are found numerically then

$$I_2(r_a) = \frac{Z_2}{\mathcal{N}_q} \sum_{q_i \in \{\text{roots}\}} q_i^{\beta+\frac{8}{3}} (1+q_i^{\alpha})^{3/2} (1+q_i^{-\alpha})^{1/2}. \tag{66}$$

But this doesn't really give much insight.

5.2.2. *Power law planets (Model # 3)*

Assume now that

$$\Lambda(r) \sim r^{\delta} \tag{67}$$

and keep calculating...