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THE EFFECTS OF BINARITY ON PLANET OCCURRENCE RATES MEASURED BY TRANSIT SURVEYS

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ABSTRACT

This note formally derives a general equation for the apparent occurrence rate used in the paper. For some analytic cases, this lets us vet our Monte Carlo code.

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1. PRELIMINARIES

1.1. *Searchable Distance*

We can detect a signal if

$$\frac{\text{signal}}{\text{noise}} \sim \frac{\delta_{\text{obs}}}{(L/d^2)^{-1/2}}, \quad \delta_{\text{obs}} : \text{observed depth} \tag{1}$$

is above some threshold (it would probably make more sense to include the duration information, but that would anyway be a trivial extension and so we omit it here for brevity). Thus, the maximum searchable distance scales as

$$d(\delta_{\text{obs}}, L_{\text{sys}}) \propto \delta_{\text{obs}} \cdot L_{\text{sys}}^{1/2}. \tag{2}$$

We assume that the signal is detected if and only if a given star is searchable.

Assuming that stars are uniformly distributed in space, the number of searchable stars N_{s} is then proportional to

$$N_{\text{s}}(\delta_{\text{obs}}, L_{\text{sys}}) \propto n\delta_{\text{obs}}^3 L_{\text{sys}}^{3/2}, \tag{3}$$

where n is the volume density of *e.g.*, single star, or binary systems. We neglect the dependence of n on stellar type.

1.2. *Relation between Apparent and Actual Stellar Properties*

We assume that the apparent properties of an unresolved binary are the same as those of the primary:

$$M_{\text{a}} = M_1, \quad R_{\text{a}} = R_1, ... \tag{4}$$

We also assume the stellar radius and luminosity is uniquely related to the stellar mass.

Given these assumptions, the total luminosity of a system is

$$L_{\text{sys}} = L_1 + L_2 = L(M_{\text{a}}) + L(qM_{\text{a}}), \tag{5}$$

where $q = M_2/M_1$. Note that L_{sys} is the true value (because M_{a} is the true primary mass), while an observer would estimate a system luminosity of $L(M_{\text{a}})$, based on the apparent stellar parameters. We presume the observer has no a priori knowledge of the distance to a given system – they only measure a flux, assume a stellar mass M_a, and use relations for $L(M_a)$ and $R(M_a)$ to estimate system properties.

1.3. *Apparent Number of Searchable Stars*

Given the apparent signal δ_{obs} and stellar mass M_{a}, the maximum searchable distance for singles and binaries are proportional to $\delta_{\text{obs}} \cdot L(M_{\text{a}})^{1/2}$ and $\delta_{\text{obs}} \cdot [L(M_{\text{a}}) + L(qM_{\text{a}})]^{1/2}$. Thus, the apparent number of searchable stars (i.e. points in the sky), which will be selected by an ignorant observer, is proportional to

$$N_{\text{s,a}}(\delta_{\text{obs}}, M_{\text{a}}) \propto n_{\text{s}}\delta_{\text{obs}}^3 L(M_{\text{a}})^{3/2} \left[1 + \mu(\text{BF}, M_{\text{a}})\right],$$

and can be *defined* as

$$N_{\mathrm{s,a}}(\delta_{\mathrm{obs}}, M_{\mathrm{a}}) \equiv N_{\mathrm{s}}^{0}(\delta_{\mathrm{obs}}, L(M_{\mathrm{a}}))\left[1 + \mu(\mathrm{BF}, M_{\mathrm{a}})\right], \tag{6}$$

where N_{s}^{0} is the number of searchable singles (this agrees with the actual value), $\mathrm{BF} = n_{\mathrm{b}}/(n_{\mathrm{s}} + n_{\mathrm{b}})$ is the binary fraction in a volume-limited sample, and n_s is the number density of singles in a volume-limited sample. Proportionality constants subsumed into the definition of N_s^0 include *e.g.*, the telescope area and the survey duration.

1.3.1. *What is μ?*

In Eq. 6, μ is the ratio of the number of single to double star systems (that are searchable for an observed signal δ_{obs}). If we make various simplifying assumptions, it turns out that this number depends only on the binary fraction and the binary mass ratio distribution in a volume-limited sample. The proof is as follows.

In a shell of thickness $\mathrm{d}r$, there are $n_d(r)4\pi r^2\mathrm{d}r$ searchable double star systems. The number density of searchable double star systems $n_d(r)$ is constant up to the maximum searchable distance for singles, d_0. At greater distances, $n_d(r)$ decreases, because only some binaries have sufficient flux to make the SNR cut. Specifically,

$$n_d(r) = \begin{cases} n_b, & r \leq d_0 \\ n_b \int_{q_{\min}(r)}^{1} \mathrm{d}q\, f_{\mathrm{vl}}(q), & d_0 < r < \sqrt{2}d_0 \end{cases} \tag{7}$$

where $f_{\mathrm{vl}}(q)$ is the mass ratio distribution in a volume-limited sample, $q_{\min}(r)$ is the minimum mass ratio required to provide sufficient flux for searchability. Assuming $L \propto M^{\alpha}$,

$$q_{\min}(r) = \left(\left(\frac{r}{d_0}\right)^2 - 1\right)^{\frac{1}{\alpha}}. \tag{8}$$

Using the common parametrization $f_v(q) \sim q^{\beta}$, and setting $\beta = 0$,

$$n_d(r) = \begin{cases} n_b, & r \leq d_0 \\ n_b(1 - q_{\min}(r)), & d_0 < r < \sqrt{2}d_0. \end{cases} \tag{9}$$

Integrating,

$$N_d = \int \mathrm{d}N_d = \int n_d(r)4\pi r^2\mathrm{d}r \tag{10}$$

$$= 4\pi n_b\left(\frac{d_0^3}{3} + \left[\frac{r^3}{3}\right]_{d_0}^{\sqrt{2}d_0} - \int_{d_0}^{\sqrt{2}d_0}\left(\left(\frac{r}{d_0}\right)^2 - 1\right)^{\frac{1}{\alpha}} r^2\,\mathrm{d}r\right) \tag{11}$$

$$= \frac{4\pi n_b}{3}d_0^3\left(2^{3/2} - 3\int_1^{\sqrt{2}}\mathrm{d}u\, u^2(u^2-1)^{1/\alpha}\right), \tag{12}$$

$$\equiv \frac{4\pi n_b}{3}d_0^3\left(2^{3/2} - 3I\right), \tag{13}$$

where the penultimate line substituted $u = r/d_0$. For the common case $\alpha = 3.5$, the dimensionless integral I evaluates to ≈ 0.4841741.

Thus

$$\mu \equiv \frac{N_d}{N_s^0} = \frac{n_b}{n_s}\left(2^{3/2} - 3I\right) = \frac{\mathrm{BF}}{1-\mathrm{BF}}\left(2^{3/2} - 3I\right). \tag{14}$$

For the twin binary case, $I = 0$, and for the power law volume-limited distribution, I takes the form given in Eq. 12.

1.3.2. *Alternative Derivation of μ*

For a given set of $(\delta_{\mathrm{obs}}, M_{\mathrm{a}})$ and q, the searchable distance for binaries is larger than singles by $\sqrt{1 + L(qM_{\mathrm{a}})/L(M_{\mathrm{a}})}$. Thus,

$$\frac{N_d}{N_s^0} = \int_0^1 \frac{n_b}{n_s}\left[1 + \frac{L(qM_{\mathrm{a}})}{L(M_{\mathrm{a}})}\right]^{3/2} f(q)\mathrm{d}q = \int_0^1 \frac{\mathrm{BF}}{1-\mathrm{BF}}\left[1 + \frac{L(qM_{\mathrm{a}})}{L(M_{\mathrm{a}})}\right]^{3/2} f(q)\mathrm{d}q. \tag{15}$$

If BF is independent of q, $L \sim M^\alpha$, and $f(q) \sim q^\beta$:

$$\mu = \frac{N_d}{N_s^0} = \frac{\mathrm{BF}}{1-\mathrm{BF}} \cdot \frac{1}{1+\beta}\int_0^1 (1+q^\alpha)^{3/2}\, q^\beta \mathrm{d}q, \tag{16}$$

which may be written in a closed form using the hypergeometric function. Integrating by parts and using $u \equiv (1+q^\alpha)^{1/2}$, this reduces to

$$\begin{aligned}\mu = \frac{N_d}{N_s^0} &= \frac{\mathrm{BF}}{1-\mathrm{BF}}\left(\frac{1}{1+\beta}\right)^2 \left\{\left[q^{\beta+1}(1+q^\alpha)^{3/2}\right]_0^1 - \frac{3}{2}\alpha\int_0^1 q^{\alpha+\beta}(1+q^\alpha)^{1/2}\,\mathrm{d}q\right\}, \\ &= \frac{\mathrm{BF}}{1-\mathrm{BF}}\left(\frac{1}{1+\beta}\right)^2\left[2^{3/2} - 3\int_1^{\sqrt{2}} u^2(u^2-1)^{\frac{1+\beta}{\alpha}}\mathrm{d}u\right]. \end{aligned} \tag{17}$$

For $\beta = 0$, the last line is equivalent to Eq.(14).

2. APPARENT OCCURRENCE RATE — GENERAL FORMULA

A group of astronomers wants to measure the mean number of planets of a certain type per star of a certain type. They observe a set points on the sky and detect a set of planets that appear to be of the desired class. They then choose the stars (among those initially selected) around which the planets of interest appeared to be searchable. Given these quantities, they compute the apparent rate density,

$$\Gamma_{\mathrm{a}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}}) = \frac{N_{\mathrm{det}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})}{N_{\mathrm{s,a}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})} \times \frac{1}{p_{\mathrm{tra}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})}. \tag{18}$$

where $\mathcal{P}_{\mathrm{a}}$, $\mathcal{S}_{\mathrm{a}}$ are the apparent planetary/stellar parameters. Here $N_{\mathrm{det}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})$ is the number of detected planets with parameters $\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}}$, *per unit* $\mathcal{P}_{\mathrm{a}}$ *and* $\mathcal{S}_{\mathrm{a}}$. The quantity $N_{\mathrm{s,a}}(\mathcal{P}_{\mathrm{a}}, \mathcal{S}_{\mathrm{a}})$ is the apparent number of searchable stars. The apparent rate for a given class of planets is found by integrating the rate density. For instance, if $\mathcal{P}_{\rm a} = r_a$ and $\mathcal{S}_{\rm a} = M_a$, the apparent rate is

$$\int \mathrm{d}^2\Lambda_{\rm a}(r_{\rm a}, M_{\rm a}) = \int \mathrm{d}r_{\rm a}\mathrm{d}M_{\rm a}\,\Gamma_a(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a}) = \int \mathrm{d}r_{\rm a}\mathrm{d}M_{\rm a}\frac{N_{\rm det}(r_{\rm a}, M_{\rm a})}{N_{\rm s,a}(r_{\rm a}, M_{\rm a})} \times \frac{1}{p_{\rm tra}(r_{\rm a}, M_{\rm a})}. \tag{19}$$

In the presence of dilution, planets with $(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a})$ are associated with systems of many different planetary and stellar properties, so $N_{\rm det}(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a})$ is given by the convolution of the true rate density, $\Gamma(\mathcal{P}, \mathcal{S})$, and $\mathcal{N}(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a}; \mathcal{P}, \mathcal{S})$, the number (per unit $(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a})$) of searchable stars that give $(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a})$ when the true system actually has $(\mathcal{P}, \mathcal{S})$:

$$N_{\rm det}(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a}) = \sum_i N^i_{\rm det}(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a}) = \sum_i \int \mathrm{d}\mathcal{P}\mathrm{d}\mathcal{S}\,\mathcal{N}^i_{\rm s}(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a}; \mathcal{P}, \mathcal{S}) \cdot \Gamma^i(\mathcal{P}, \mathcal{S}) \cdot p_{\rm tra}(\mathcal{P}, \mathcal{S}), \tag{20}$$

where i specifies the type of true host stars (0: single, 1: primary, 2: secondary).

So the problem reduces to the evaluation of

$$\mathcal{N}^i_{\rm s}(\mathcal{P}_{\rm a}, \mathcal{S}_{\rm a}; \mathcal{P}, \mathcal{S}) \tag{21}$$

for planets around single stars, primaries in binaries, and secondaries in binaries.

3. EVALUATION OF $\mathcal{N}^i_{\rm s}$

Let us explicitly write $\mathcal{P} = r$ and $\mathcal{S} = M$; R and L are uniquely determined from the assumed mass–radius–luminosity relation. We neglect the dependence on planetary orbital period. We proceed by evaluating

$$N_{\rm det}(r_a, M_a) = \sum_i N^i_{\rm det}(r_a, M_a) \tag{22}$$

$$N_{\rm det}(r_a, M_a) = \sum_i \int \mathrm{d}r\mathrm{d}M\,\mathcal{N}^i_{\rm s}(r_a, M_a; r, M) \cdot \Gamma^i(r, M) \cdot p_{\rm tra}(r, M), \tag{23}$$

term by term.

3.1. *Single Stars*

For $i = 0$,

$$\mathcal{N}^0_{\rm s}(r_{\rm a}, M_{\rm a}; r, M) = \delta(r_{\rm a} - r)\delta(M_{\rm a} - M)N^0_{\rm s}(r, M), \tag{24}$$

so

$$N^0_{\rm det}(r_{\rm a}, M_{\rm a}) = N^0_{\rm s}(r_{\rm a}, M_{\rm a}) \cdot \Gamma^0(r_{\rm a}, M_{\rm a}) \cdot p_{\rm tra}(r_{\rm a}, M_{\rm a}). \tag{25}$$

If all the stars are singles, this yields

$$\Gamma_{\rm a}(r_{\rm a}, M_{\rm a}) = \Gamma^0(r_{\rm a}, M_{\rm a}), \tag{26}$$

as expected (now $\mu = 0$ and $N_{\rm s,a} = N^0_{\rm s}$) — the true occurrence is recovered.

3.2. *Primaries in Binaries*

The number of primaries with apparent parameters (r_a, M_a) given the true parameters (r, M) is

$$\mathcal{N}_{\rm s}^1(r_a, M_a; r, M) = \int {\rm d}q\, f(q) \mathcal{N}_{{\rm s},q}^1(r_a, M_a, q; r, M), \tag{27}$$

where $f(q)$ is the binary mass ratio distribution.

Since we assume $\mathcal{S}_{\rm a} = \mathcal{S}_1$,

$$\mathcal{N}_{{\rm s},q}^1(r_a, M_a, q; r, M) \propto \delta(M_a - M). \tag{28}$$

In this case, $\mathcal{N}_{{\rm s},q}^1$ is non-zero only at $r_a = R_a\sqrt{\delta_{\rm obs}}$, and the observed depth is

$$\delta_{\rm obs} = \left[\frac{r}{R(M_a)}\right]^2 \times \frac{L(M_a)}{L_{\rm sys}(M_a, q)} \equiv \left[\frac{r}{R(M_a)}\right]^2 \times \mathcal{A}(q, M_{\rm a})^2 \tag{29}$$

The normalization of $\mathcal{N}_{{\rm s},q}^1$ is given by the number of binaries that are searchable for a signal $\delta_{\rm obs}$ and have the mass ratio q (cf. 1.3.2):

$$N_{\rm s}^0(\delta_{\rm obs}, L(M_a)) \cdot \frac{n_b}{n_s}\left[\frac{L_{\rm sys}(M_a, q)}{L(M_{\rm a})}\right]^{3/2} = N_{\rm s}^0(\delta_{\rm obs}, L(M_a)) \cdot \frac{\rm BF}{1 - {\rm BF}} \cdot \frac{1}{\mathcal{A}(q, M_{\rm a})^3}. \tag{30}$$

Thus,

$$\mathcal{N}_{{\rm s},q}^1(r_a, M_a, q; r, M) = N_{\rm s}^0(\delta_{\rm obs}, L(M_a)) \cdot \frac{\rm BF}{1 - {\rm BF}} \cdot \frac{1}{\mathcal{A}(q, M_{\rm a})^3} \times \delta\left[r_a - r\mathcal{A}(q, M_{\rm a})\right] \delta(M_a - M). \tag{31}$$

Note that while the observed depth is a function of the true parameters (Eq. 29), since we are counting detected planets of a given *apparent* size, we will not need to worry about this dependence. In Eq. 27, we will have $f(q) \sim q^\beta$, the volume-limited mass ratio distribution (Eq. 30 accounts for the Malmquist bias).

3.3. *Secondaries in Binaries*

In this case, $M = qM_1 = qM_a$, so

$$\mathcal{N}_{{\rm s},q}^2(r_a, M_a, q; r, M) \propto \delta\left(M_a - \frac{M}{q}\right). \tag{32}$$

Again $\mathcal{N}_{\rm s}^2$ is non-zero only at $r_a = R_a\sqrt{\delta_{\rm obs}}$, but this time

$$\delta_{\rm obs} = \left[\frac{r}{R(qM_a)}\right]^2 \times \frac{L(qM_a)}{L_{\rm sys}(M_a, q)}. \tag{33}$$

The normalization remains the same as the previous case (we are counting the searchable stars at a given observed depth $\delta_{\rm obs}$, total luminosity of the binary is the same). Thus,

$$\mathcal{N}_{\rm s}^2(r_a, M_a; r, M) = \int {\rm d}q\, f(q) \mathcal{N}_{{\rm s},q}^2(r_a, M_a, q; r, M), \tag{34}$$

where

$$\mathcal{N}_{\rm s}^2(r_a, M_a; r, M; q) = N_{\rm s}^0(\delta_{\rm obs}, L(M_a)) \cdot \frac{\rm BF}{1-{\rm BF}} \cdot \frac{1}{\mathcal{A}(q, M_{\rm a})^3} \times \delta\left(r_a - r\sqrt{\left[\frac{R(M_a)}{R(qM_a)}\right]^2 \frac{L(qM_a)}{L_{\rm sys}(M_a, q)}}\right) \delta\left(M_a - \frac{M}{q}\right). \quad (35)$$

One might worry in Eq. 32 that we opt to write $\mathcal{N}_{\rm s}^2 \propto \delta(M_a - M/q)$, rather than $\propto \delta(M_a q - M)$ or some other seemingly function with the same functional dependence, but a different normalization once integrated. We do this because the delta function in Eq. 32 is defined with respect to the measure $\mathrm{d}M_{\rm a}$, not $\mathrm{d}M$. This means that it should be unity when integrated over $\mathrm{d}M_{\rm a}$. Ultimately, this is based on how we defined $\mathcal{N}_{\rm s}^2$ as a number per $r_{\rm a}$, per $M_{\rm a}$.

One might also worry because in Eq. 35, it is hard to see any explicit indication that secondaries are harder to search for planets of a given true size than primaries (or singles). This piece of intuition is accounted in the delta functions. If one were to instead go through the exercise of saying "how many searchable secondaries are there in terms of the *true* parameters?" you would get exactly the same answer[1].

4. RESULT

4.1. *Marginalization over the True Properties*

Let's integrate out $\mathcal{P}$ and $\mathcal{S}$.

$$N_{\rm det}^0(r_{\rm a}, M_{\rm a}) = \int {\rm d}r{\rm d}M\, \mathcal{N}_{\rm s}^0(r_{\rm a}, M_{\rm a}; r, M) \cdot \Gamma^0(r, M) \cdot p_{\rm tra}(M) \quad (36)$$

$$= \int {\rm d}r{\rm d}M\, N_{\rm s}^0(\delta_{\rm obs}, L(M_{\rm a}))\delta(r_{\rm a} - r)\delta(M_{\rm a} - M) \cdot \Gamma^0(r, M) \cdot p_{\rm tra}(M) \quad (37)$$

$$= N_{\rm s}^0(\delta_{\rm obs}, L(M_{\rm a})) \cdot \Gamma^0(r_{\rm a}, M_{\rm a}) \cdot p_{\rm tra}(M_{\rm a}). \quad (38)$$

$$N_{\rm det}^1(r_{\rm a}, M_{\rm a}) = \int {\rm d}r{\rm d}M\, \mathcal{N}_{\rm s}^1(r_{\rm a}, M_{\rm a}; r, M) \cdot \Gamma^1(r, M) \cdot p_{\rm tra}(M) \quad (39)$$

$$= \int {\rm d}q\, f(q) \int {\rm d}r{\rm d}M\, \mathcal{N}_{{\rm s},q}^1(r_{\rm a}, M_{\rm a}, q; r, M) \cdot \Gamma^1(r, M) \cdot p_{\rm tra}(M) \quad (40)$$

$$= N_{\rm s}^0(\delta_{\rm obs}, L(M_{\rm a})) \cdot p_{\rm tra}(M_{\rm a}) \cdot \frac{\rm BF}{1-{\rm BF}} \int \frac{{\rm d}q}{\mathcal{A}^4}\, f(q)\, \Gamma^1\left(\frac{r_a}{\mathcal{A}}, M_{\rm a}\right), \quad (41)$$

where

$$\mathcal{A}(q, M_{\rm a}) = \sqrt{\frac{L(M_{\rm a})}{L_{\rm sys}(M_{\rm a}, q)}}. \quad (42)$$

[1] LB did this, in his handwritten notes `2017/12/09.0-2017/12/11/.0`

Finally,

$$N_{\rm det}^2(r_{\rm a}, M_{\rm a}) = \int {\rm d}r{\rm d}M\, \mathcal{N}_{\rm s}^2(r_{\rm a}, M_{\rm a}; r, M) \cdot \Gamma^2(r, M) \cdot p_{\rm tra}(M) \tag{43}$$

$$= \int {\rm d}q\, f(q) \int {\rm d}r{\rm d}M\, \mathcal{N}_{{\rm s},q}^2(r_{\rm a}, M_{\rm a}, q; r, M) \cdot \Gamma^2(r, M) \cdot p_{\rm tra}(M) \tag{44}$$

$$= N_{\rm s}^0(\delta_{\rm obs}, L(M_{\rm a})) \cdot \frac{\rm BF}{1-{\rm BF}} \int \frac{{\rm d}r{\rm d}q}{\mathcal{A}^3}\, q f(q) \delta(r_a - r\mathcal{B}) \Gamma^2(r, qM_{\rm a}) p_{\rm tra}(qM_{\rm a}) \tag{45}$$

$$= N_{\rm s}^0(\delta_{\rm obs}, L(M_{\rm a})) \cdot \frac{\rm BF}{1-{\rm BF}} \int \frac{q{\rm d}q}{\mathcal{A}^3\mathcal{B}}\, f(q)\, \Gamma^2(r_a/\mathcal{B}, qM_{\rm a})\, p_{\rm tra}(qM_{\rm a}), \tag{46}$$

where

$$\mathcal{B}(q, M_{\rm a}) = \frac{R(M_{\rm a})}{R(qM_{\rm a})} \sqrt{\frac{L(qM_{\rm a})}{L_{\rm sys}(M_{\rm a}, q)}}. \tag{47}$$

4.2. *Final Formula*

Note again that the denominators of $\Gamma_{\rm a}$ are the same for singles, primaries, and secondaries: $N_{\rm s,a}(\delta_{\rm obs}, M_{\rm a})$ and $p_{\rm tra}(M_{\rm a})$. This is because we are calculating the occurrence at the same apparent planet/star properties, and the observer can never distinguish binaries from singles (and adopt the primary properties for binaries). Using the results above, the apparent rate density,

$$\Gamma_{\rm a}(r_{\rm a}, M_{\rm a}) = \frac{1}{N_{\rm s,a}(r_{\rm a}, M_{\rm a}) p_{\rm tra}(r_{\rm a}, M_{\rm a})} \times \sum_i N_{\rm det}^i(r_{\rm a}, M_{\rm a}), \tag{48}$$

evaluates to

$$\Gamma_{\rm a}(r_{\rm a}, M_{\rm a}) = \frac{1}{1+\mu({\rm BF}, M_{\rm a})} \times \left\{ \Gamma^0(r_{\rm a}, M_{\rm a}) + \frac{\rm BF}{1-{\rm BF}} \left[\int \frac{{\rm d}q}{\mathcal{A}^4}\, f(q)\, \Gamma^1\left(\frac{r_{\rm a}}{\mathcal{A}}, M_{\rm a}\right) + \int \frac{q{\rm d}q}{\mathcal{A}^3\mathcal{B}}\, f(q)\, \Gamma^2\left(\frac{r_{\rm a}}{\mathcal{B}}, qM_{\rm a}\right) \frac{R(qM_{\rm a})}{R(M_{\rm a})} q^{-1/3} \right] \right\}. \tag{49}$$

4.2.1. *Simplifying above general result*

To simplify, drop the explicit BF dependences, and assume $R \propto M$, so $R(qM_a)/R(M_a) = q$. Also, move all superscript numbers to subscripts. Then the apparent rate density is

$$\Gamma_{\rm a}(r_{\rm a}, M_{\rm a}) = \frac{1}{1+\mu} \times \left\{ \Gamma^0(r_{\rm a}, M_{\rm a}) + \frac{\rm BF}{1-{\rm BF}} \left[\int \frac{{\rm d}q}{\mathcal{A}^4}\, f(q)\, \Gamma^1\left(\frac{r_{\rm a}}{\mathcal{A}}, M_{\rm a}\right) + \int \frac{q{\rm d}q}{\mathcal{A}^3\mathcal{B}}\, f(q)\, \Gamma^2\left(\frac{r_{\rm a}}{\mathcal{B}}, qM_{\rm a}\right) q^{2/3} \right] \right\}. \tag{50}$$

For the power law case when $L \sim M^\alpha$,

$$\mathcal{A}(q) = (1+q^\alpha)^{-1/2}, \quad \mathcal{B}(q) = q^{-1}(1+q^{-\alpha})^{-1/2} \tag{51}$$

5. EXAMPLES

5.1. *Twin Binary*

We have $f(q) = \delta(q-1)$. Since all stars have the exact same mass (both true mass and apparent mass), we will drop the M_a dependence. The apparent rate density is then

$$\Gamma_\mathrm{a}(r_\mathrm{a}) = \frac{1}{1+\mu(\mathrm{BF})}\left\{\Gamma_0(r_\mathrm{a}) + \mu(\mathrm{BF})\cdot\sqrt{2}\cdot\left[\Gamma_1(\sqrt{2}\,r_\mathrm{a}) + \Gamma_2(\sqrt{2}\,r_\mathrm{a})\right]\right\}, \tag{52}$$

where

$$\mu(\mathrm{BF}) = \int_0^1 \frac{\mathrm{BF}}{1-\mathrm{BF}}\left[1+\frac{L(qM_\mathrm{a})}{L(M_\mathrm{a})}\right]^{3/2} f(q)\,\mathrm{d}q = 2^{3/2}\cdot\frac{\mathrm{BF}}{1-\mathrm{BF}}. \tag{53}$$

5.1.1. *Same Planets (Model # 1)*

If $\Gamma_i(r) = Z_i\cdot\delta(r-r_0)$ (all the planets have the same radius), by using the identity $\delta(ax+b) = \delta(x+b/a)/|a|$,

$$\Gamma_\mathrm{a}(r_\mathrm{a}) = \frac{1}{1+\mu(\mathrm{BF})}\left[Z_0\cdot\delta(r_\mathrm{a}-r_0) + (Z_1+Z_2)\cdot\mu(\mathrm{BF})\cdot\delta\left(r_\mathrm{a}-\frac{r_0}{\sqrt{2}}\right)\right]. \tag{54}$$

This reproduces Eq.(9) of the draft (after removing the p_det terms)[2]. In the limit of $\mathrm{BF}\to 1$ ($\mu\to\infty$), the above formula yields $\Gamma_\mathrm{a} = (Z^1+Z^2)\,\delta(r_\mathrm{a}-r_0/\sqrt{2})$.

5.1.2. *Power Law Distribution of Planets*

Imagine instead that we took $\Gamma_i(r) = Z_i f(r) = Z_i r^\delta/\mathcal{N}_r$, for $f(r)$ the radius shape function, $\mathcal{N}_r$ the shape function's normalization, and Z_i the number of planets per star. From Eq. 52, we get an apparent rate density

$$\Gamma_a(r_a) = \frac{r_a^\delta}{\mathcal{N}_r}\left[\frac{Z_0}{1+\mu(\mathrm{BF})} + 2^{\frac{\delta+1}{2}}\frac{\mu(\mathrm{BF})}{1+\mu(\mathrm{BF})}(Z_1+Z_2)\right]. \tag{55}$$

Consider the case when the Z_i's are identical. If we compare $\Gamma_a(r_a)$ with $\Gamma(r)$, we must impose that r_a and r are indistinguishable (i.e.: the observer cannot tell them apart). Note that the true rate density for stars selected in the sample is

$$\Gamma_\mathrm{sel}(r) = \frac{r^\delta}{\mathcal{N}_r}\left[\frac{Z_0}{1+2\mu} + \frac{\mu}{1+2\mu}(Z_1+Z_2)\right], \tag{56}$$

while the true rate density for a volume-limited sample of stars is

$$\Gamma_\mathrm{vl}(r) = \frac{r^\delta}{\mathcal{N}_r}\left[\frac{Z_0}{1+2(n_b/n_s)} + \frac{(n_b/n_s)}{1+2(n_b/n_s)}(Z_1+Z_2)\right], \tag{57}$$

for n_b and n_s the number density of binaries and singles in a volume-limited sample.

[2] The above equation also follows by just thinking about the problem and explicitly writing out the answer (cf. LB's handwritten notes `2017/11/26.9`), though the formal derivation is a good check.

The stand-alone term "true rate density" is ill defined, and should be avoided. In different contexts, it might refer to either a) $\Gamma_{\rm sel}(r)$, b) $\Gamma_{\rm vl}(r)$, or even c) $\Gamma_0(r)$, the rate around singles. The meaning that most people probably *want* true rate density to have is $\Gamma_0(r)$ — simply "I have a star that is single; how many planets are there?". This is also a sensible definition because it is *independent* of any complications of planet formation that would make Z_1 or Z_2 different from Z_0. One way to deal with this, rather than making the arbitrary choice, would be to just plot out all the possible options. In general though, we will give highest preference to $\Gamma_0(r)$, since it can be compared with the results of other surveys.

We could define a "correction factor" relative to Γ_0, or to $\Gamma_{\rm sel}$, or to $\Gamma_{\rm vl}$. If the Z_i's are equal, these are all identical:

$$\begin{aligned} X_\Gamma &\equiv \left.\frac{\Gamma_a(r_a)}{\Gamma_0(r)}\right|_{r_a\to r,\ Z_i\text{'s equal}} \\ &= \left.\frac{\Gamma_a(r_a)}{\Gamma_{\rm sel}(r)}\right|_{r_a\to r,\ Z_i\text{'s equal}} \\ &= \left.\frac{\Gamma_a(r_a)}{\Gamma_{\rm vl}(r)}\right|_{r_a\to r,\ Z_i\text{'s equal}} \\ &= \frac{1+2^{\frac{\delta+3}{2}}\mu}{1+\mu}. \end{aligned} \tag{58}$$

For the case of BF = 0.1, $\mu \approx 0.153$. Taking $\delta = -2.92$ from Howard et al. (2012), we get a correction factor $X_\Gamma = \Gamma_a/\Gamma = 1.004$. In other words, the apparent rate density is an *over*estimate of the rate density of selected stars, with a relative error $\delta\Gamma = |\Gamma - \Gamma_{\rm a}|/\Gamma$ of 0.4%.

As a side-note, this particular model (twin binaries, power law planet distribution) is a good unit test for the numerics. More realistically, we'll take a power law above some cutoff (to avoid the divergence for $\delta < 0$), but Eq. 55 will tell us the apparent rate density within the region of parameter space where only the power law matters.

5.2. *Power Law World*

Now assume $f(q) = q^\beta/\mathcal{N}_q$, for $\mathcal{N}_q$ the normalization. As previously noted, if we assume

$$L(M) \sim M^\alpha \sim R^\alpha, \tag{59}$$

we have

$$\mathcal{A}(q) = (1+q^\alpha)^{-1/2}, \quad \mathcal{B}(q) = q^{-1}(1+q^{-\alpha})^{-1/2}. \tag{60}$$

This gives a value of $\mu(\mathrm{BF})$ stated in Eq. 14. Now recall Eq. 50 for the apparent rate density. We restate it here:

$$\begin{aligned} \Gamma_{\rm a}(r_{\rm a}, M_{\rm a}) = \frac{1}{1+\mu} \times \Bigg\{ \Gamma^0(r_{\rm a}, M_{\rm a}) + \frac{\mathrm{BF}}{1-\mathrm{BF}} \Bigg[\int \frac{\mathrm{d}q}{\mathcal{A}^4} f(q)\, \Gamma^1\left(\frac{r_{\rm a}}{\mathcal{A}}, M_{\rm a}\right) \\ + \int \frac{q\mathrm{d}q}{\mathcal{A}^3\mathcal{B}} f(q)\, \Gamma^2\left(\frac{r_{\rm a}}{\mathcal{B}}, qM_{\rm a}\right) q^{2/3} \Bigg] \Bigg\}. \end{aligned} \tag{61}$$

Since in this model we will have stars with different masses, it makes more sense to keep the explicit "M_a" dependence in our equations.

5.2.1. *Same Planets (Model # 2)*

For our "Model #2", we assume

$$\Gamma_i(r, M) = Z_i \cdot \delta(r - r_0) M^\gamma / \mathcal{N}_M, \tag{62}$$

where $\mathcal{N}_M$ has the correct units to make Z_i dimensionless. In most cases we'll take $\gamma = 0$, but we'll carry it through the calculations here.

For $i = 1$, call the integral on the left of the $[\ldots]$ "I_1". Then

$$I_1(r_\mathrm{a}, M_\mathrm{a}) \equiv \int \frac{\mathrm{d}q}{\mathcal{A}^4} f(q) \Gamma_1\left(\frac{r_\mathrm{a}}{\mathcal{A}}, M_\mathrm{a}\right) \tag{63}$$

$$= \frac{1}{\mathcal{N}_q} \int \mathrm{d}q\, q^\beta (1 + q^\alpha)^2 \Gamma_1(r_a (1 + q^\alpha)^{1/2}, M_\mathrm{a}) \tag{64}$$

$$= \frac{Z_1 M_\mathrm{a}^\gamma}{\mathcal{N}_q \mathcal{N}_M} \int \mathrm{d}q\, q^\beta (1 + q^\alpha)^2 \delta(r_a (1 + q^\alpha)^{1/2} - r_0) \tag{65}$$

$$= \frac{Z_1 M_\mathrm{a}^\gamma}{\mathcal{N}_q \mathcal{N}_M} \frac{2}{\alpha} \frac{r_0}{r_a^2} \left(\left(\frac{r_0}{r_a}\right)^2 - 1\right)^{\frac{1-\alpha}{\alpha}} \int \mathrm{d}q\, q^\beta (1 + q^\alpha)^2 \delta\left(q - \left(\left(\frac{r_0}{r_a}\right)^2 - 1\right)^{\frac{1}{\alpha}}\right) \tag{66}$$

$$= \frac{Z_1 M_\mathrm{a}^\gamma}{\mathcal{N}_q \mathcal{N}_M} \frac{2}{\alpha} \frac{r_0}{r_a^2} \left(\left(\frac{r_0}{r_a}\right)^2 - 1\right)^{\frac{\beta-\alpha+1}{\alpha}} \left(\frac{r_0}{r_a}\right)^4. \tag{67}$$

The penultimate line used the fact that $r_a(1 + q^\alpha)^{1/2} = r_0$ has only one root:

$$q(r_a) = \left(\left(\frac{r_0}{r_a}\right)^2 - 1\right)^{1/\alpha}, \tag{68}$$

and for any function $f(q)$ with a single root q_i,

$$\delta(f(q)) = \frac{\delta(q - q_i)}{|f'(q_i)|}. \tag{69}$$

Eq. 67 is a good unit test for numerics.

Attempting the same thing for $i = 2$,

$$I_2(r_\mathrm{a}, M_\mathrm{a}) \equiv \int \frac{q\mathrm{d}q}{\mathcal{A}^3 \mathcal{B}} f(q) \Gamma_2\left(\frac{r_\mathrm{a}}{\mathcal{B}}, q M_\mathrm{a}\right) q^{2/3} \tag{70}$$

$$= Z_2 \int \frac{q\mathrm{d}q}{\mathcal{A}^3 \mathcal{B}} f(q) \delta\left(\frac{r_\mathrm{a}}{\mathcal{B}} - r_0\right) \frac{(q M_\mathrm{a})^\gamma}{\mathcal{N}_M} q^{2/3} \tag{71}$$

$$= \frac{Z_2 M_\mathrm{a}^\gamma}{\mathcal{N}_q \mathcal{N}_M} \int \mathrm{d}q\, q^{\beta+\gamma+\frac{8}{3}} (1 + q^\alpha)^{3/2} (1 + q^{-\alpha})^{1/2} \delta(r_a q (1 + q^{-\alpha})^{1/2} - r_0). \tag{72}$$

Unfortunately, analytic roots of $r_a q(1+q^{-\alpha})^{1/2} = r_0$ do not exist. If the roots are found numerically then

$$I_2(r_\mathrm{a}, M_\mathrm{a}) = \frac{Z_2}{\mathcal{N}_q} \sum_{q_i \in \{\mathrm{roots}\}} q_i^{\beta+\gamma+\frac{8}{3}} (1+q_i^\alpha)^{3/2} (1+q_i^{-\alpha})^{1/2}, \tag{73}$$

but this doesn't really give much insight.

5.2.2. *Power law planets (Model # 6)*

Power law planets are a slightly more realistic mass ratio distribution. If we now assume

$$\Gamma_i(r, M) = Z_i \times \frac{r^\delta}{\mathcal{N}_r} \times \frac{M^\gamma}{\mathcal{N}_M}, \tag{74}$$

where Z_i is dimensionless, and the normalization constants carry the units (each side has units $[r^{-1}M^{-1}]$). From Eq. 50, we can write

$$\Gamma_\mathrm{a}(r_\mathrm{a}, M_\mathrm{a}) = \frac{1}{1+\mu} \left(\Gamma_0(r_\mathrm{a}, M_\mathrm{a}) + \frac{\mathrm{BF}}{1-\mathrm{BF}} \left[I_1(r_a, M_\mathrm{a}) + I_2(r_a, M_\mathrm{a}) \right] \right), \tag{75}$$

where $I_1(r_\mathrm{a}, M_\mathrm{a})$ and $I_2(r_\mathrm{a}, M_\mathrm{a})$ are defined in the same way as in Eqs. 63 and 70. Doing the calculation, we find

$$\begin{aligned} I_1(r_\mathrm{a}, M_\mathrm{a}) &= \int \frac{\mathrm{d}q}{\mathcal{A}^4} f(q) \Gamma^1 \left(\frac{r_\mathrm{a}}{\mathcal{A}}, M_\mathrm{a} \right) && (76) \\ &= \frac{Z_1}{\mathcal{N}_q \mathcal{N}_r \mathcal{N}_M} r_\mathrm{a}^\delta M_\mathrm{a}^\gamma \int \mathrm{d}q\, q^\beta (1+q^\alpha)^{\frac{\delta+4}{2}} && (77) \end{aligned}$$

and

$$\begin{aligned} I_2(r_\mathrm{a}, M_\mathrm{a}) &= \int \frac{q\,\mathrm{d}q}{\mathcal{A}^3 \mathcal{B}} f(q) \Gamma^2 \left(\frac{r_\mathrm{a}}{\mathcal{B}}, q M_\mathrm{a} \right) q^{2/3} && (78) \\ &= \frac{Z_2}{\mathcal{N}_q \mathcal{N}_r \mathcal{N}_M} r_\mathrm{a}^\delta M_\mathrm{a}^\gamma \int \mathrm{d}q\, q^{\beta+\gamma+\delta+\frac{8}{3}} (1+q^\alpha)^{\frac{3}{2}} (1+q^{-\alpha})^{\frac{\delta+1}{2}}. && (79) \end{aligned}$$

The net effect of binarity is to just *scale* the overall distribution!

Note that the true rate density for a volume-limited sample is

$$\begin{aligned} \Gamma_\mathrm{vl}(r, M) &= \frac{r^\delta M^\gamma}{\mathcal{N}_r \mathcal{N}_M} \left[\frac{Z_0}{1+2(n_b/n_s)} + \frac{(n_b/n_s)}{1+2(n_b/n_s)} (Z_1 + Z_2) \right] && (80) \\ &= \frac{r^\delta M^\gamma Z_0}{\mathcal{N}_r \mathcal{N}_M} \qquad \text{(assuming } Z_0 = Z_1 = Z_2\text{)}, && (81) \end{aligned}$$

which when $Z_0 = Z_1 = Z_2$ is equal to Γ_0, and Γ_sel. The correction factor with respect to single stars, $X_\Gamma = \Gamma_\mathrm{a}/\Gamma_0$, is then

$$\begin{aligned} X_\Gamma &\equiv \left. \frac{\Gamma_\mathrm{a}(r_\mathrm{a}, M_\mathrm{a})}{\Gamma_0(r, M)} \right|_{r_\mathrm{a} \to r,\ M_\mathrm{a} \to M,\ Z_i\text{'s equal}} && (82) \\ &= \frac{1}{1+\mu} \left[1 + \frac{1}{\mathcal{N}_q} \frac{\mathrm{BF}}{1-\mathrm{BF}} \left(\int \mathrm{d}q\, q^\beta (1+q^\alpha)^{\frac{\delta+4}{2}} + \int \mathrm{d}q\, q^{\beta+\gamma+\delta+\frac{8}{3}} (1+q^\alpha)^{\frac{3}{2}} (1+q^{-\alpha})^{\frac{\delta+1}{2}} \right) \right]. && (83) \end{aligned}$$

For $\alpha = 3.5$, $\beta = 0$, $\gamma = 0$, $\delta = -2.92$, the sum of the integrals in $(\ldots) \approx 1.41425$. For BF $= 0.44$, this yields $\Gamma_\mathrm{a}/\Gamma_0 = 1.01448$, an apparent rate which is *higher* than the true rate by 1.4%! Note that this result is the same as $\Gamma_\mathrm{a}/\Gamma_\mathrm{sel}$ and $\Gamma_\mathrm{a}/\Gamma_\mathrm{vl}$ in the limit where all the Z_i's are equal.

More realistically, we'll take a power law above some cutoff (to avoid the divergence for $\delta < 0$), but nonetheless the above indicates we're going to need to write a different argument about whether binarity might explain the HJ rate discrepancy (apparently, it goes the wrong way!)

5.2.3. *Constant rate density of planets*

Combining a power law distribution with a constant below some minimum radius would be reasonable (Model #3). Here, we'll have

$$\begin{aligned}\Gamma_i(r, M) &= Z_i \times \frac{r^\delta}{\mathcal{N}_r} \times \frac{M^\gamma}{\mathcal{N}_M} \qquad (r > r_\ell), \\ &= Z_i \times \frac{r_\ell^\delta}{\mathcal{N}_r} \times \frac{M^\gamma}{\mathcal{N}_M} \equiv c_\ell \qquad (r < r_\ell). \end{aligned} \tag{84}$$

Once again,

$$\Gamma_\mathrm{a}(r_\mathrm{a}, M_\mathrm{a}) = \frac{1}{1+\mu}\left(\Gamma_0(r_\mathrm{a}, M_\mathrm{a}) + \frac{\mathrm{BF}}{1-\mathrm{BF}}\left[I_1(r_a, M_\mathrm{a}) + I_2(r_a, M_\mathrm{a})\right]\right). \tag{85}$$

At $r_\mathrm{a} > r_\ell$, the integrals $I_1(r_\mathrm{a}, M_\mathrm{a})$ and $I_2(r_\mathrm{a}, M_\mathrm{a})$ are given by Eqs. 77 and 79 respectively.

Case $i = 1$:—At $r_\mathrm{a} < r_\ell$, they can be evaluated as follows:

$$\begin{aligned} I_1(r_\mathrm{a}, M_\mathrm{a}) &\equiv \int \frac{\mathrm{d}q}{\mathcal{A}^4} f(q)\Gamma_1\left(\frac{r_\mathrm{a}}{\mathcal{A}}, M_\mathrm{a}\right) &(86)\\ &= \int_{r_\mathrm{a}/\mathcal{A} > r_\ell} \frac{\mathrm{d}q}{\mathcal{A}^4} f(q)\Gamma_1\left(\frac{r_\mathrm{a}}{\mathcal{A}}, M_\mathrm{a}\right) + \int_{r_\mathrm{a}/\mathcal{A} < r_\ell} \frac{\mathrm{d}q}{\mathcal{A}^4} f(q)\Gamma_1\left(\frac{r_\mathrm{a}}{\mathcal{A}}, M_\mathrm{a}\right) &(87)\\ &= \int_{q_\ell(r_\mathrm{a})}^1 \frac{\mathrm{d}q}{\mathcal{A}^4} f(q)\Gamma_1\left(\frac{r_\mathrm{a}}{\mathcal{A}}, M_\mathrm{a}\right) + \int_0^{q_\ell(r_\mathrm{a})} \frac{\mathrm{d}q}{\mathcal{A}^4} f(q)\Gamma_1\left(\frac{r_\mathrm{a}}{\mathcal{A}}, M_\mathrm{a}\right) &(88)\\ &= \frac{Z_1 r_\mathrm{a}^\delta M_\mathrm{a}^\gamma}{\mathcal{N}_q\mathcal{N}_r\mathcal{N}_M}\int_{q_\ell(r_\mathrm{a})}^1 \mathrm{d}q\, q^\beta(1+q^\alpha)^{\frac{\delta+4}{2}} + \frac{Z_1 r_\ell^\delta M_\mathrm{a}^\gamma}{\mathcal{N}_q\mathcal{N}_r\mathcal{N}_M}\int_0^{q_\ell(r_\mathrm{a})} \mathrm{d}q\,(1+q^\alpha)^2 q^\beta, &(89) \end{aligned}$$

where $q_\ell(r_\mathrm{a})$ is found by inverting $r_\mathrm{a}/\mathcal{A} = r_\ell$:

$$q_\ell(r_\mathrm{a}) = \left(\left(\frac{r_\ell}{r_\mathrm{a}}\right)^2 - 1\right)^{\frac{1}{\alpha}}. \tag{90}$$

Case $i = 2$:—For $i = 2$, the situation is slightly more complicated. I will specify that $\alpha = 3.5$, or else the existence of different classes of minima in $r_\mathrm{a}/\mathcal{B} = r_\ell$ might confuse the matter. If so, then there are two subcases. *For subcase #1,* $\left(\frac{r_\ell}{r_\mathrm{a}}\right)^2 > 2$. Then

$$I_2(r_\mathrm{a}, M_\mathrm{a}) = \int \frac{q^{5/3}\,\mathrm{d}q}{\mathcal{A}^3\mathcal{B}} f(q)\Gamma^2\left(\frac{r_\mathrm{a}}{\mathcal{B}}, qM_\mathrm{a}\right) \tag{91}$$

$$= \int_0^{q_{\ell,1}(r_\mathrm{a})}(\ldots) + \int_{q_{\ell,1}(r_\mathrm{a})}^{q_t}(\ldots) + \int_{q_t}^{1}(\ldots) \tag{92}$$

$$= \int_0^{q_{\ell,1}(r_\mathrm{a})}(\ldots) + \int_{q_{\ell,1}(r_\mathrm{a})}^{1}(\ldots), \tag{93}$$

where $(\ldots)$ indicates the same integrand as in the first line. Here, q_t is the turning point of $q^2(1+q^{-\alpha})$ (see Figs. 1 and 2), and $q_{\ell,1}(r_\mathrm{a})$ is the sole root of

$$q^2(1+q^{-\alpha}) = \left(\frac{r_\ell}{r_\mathrm{a}}\right)^2, \tag{94}$$

which must be found numerically. In Eq 92, the left-most term has $r_\mathrm{a}/\mathcal{B} > r_\ell$, while the right two terms have $r_\mathrm{a}/\mathcal{B} < r_\ell$ (and so they can be safely merged). Simplifying,

$$\begin{aligned} I_2(r_\mathrm{a}, M_\mathrm{a}) = {}& \frac{1}{\mathcal{N}_q}\int_0^{q_{\ell,1}(r_\mathrm{a})} \mathrm{d}q\, q^{8/3}(1+q^\alpha)^{3/2} q^\beta \Gamma^2\left(\frac{r_\mathrm{a}}{\mathcal{B}}, qM_\mathrm{a}\right) \\ &+ \frac{1}{\mathcal{N}_q}\int_{q_{\ell,1}(r_\mathrm{a})}^{1} \mathrm{d}q\, q^{8/3}(1+q^\alpha)^{3/2} q^\beta \Gamma^2\left(\frac{r_\mathrm{a}}{\mathcal{B}}, qM_\mathrm{a}\right), \end{aligned} \tag{95}$$

and plugging in the rate densities, we get

$$\begin{aligned} I_2(r_\mathrm{a}, M_\mathrm{a}) = {}& \frac{Z_2 r_\mathrm{a}^\delta M_\mathrm{a}^\gamma}{\mathcal{N}_q\mathcal{N}_r\mathcal{N}_M}\int_0^{q_{\ell,1}(r_\mathrm{a})} \mathrm{d}q\, q^{\beta+\gamma+\delta+\frac{8}{3}}(1+q^\alpha)^{\frac{3}{2}}(1+q^{-\alpha})^{\frac{\delta+1}{2}} \\ &+ \frac{Z_2 r_\ell^\delta M_\mathrm{a}^\gamma}{\mathcal{N}_q\mathcal{N}_r\mathcal{N}_M}\int_{q_{\ell,1}(r_\mathrm{a})}^{1} \mathrm{d}q\, q^{\beta+\gamma+\frac{8}{3}}(1+q^\alpha)^{3/2}(1+q^{-\alpha})^{\frac{1}{2}}. \qquad \left(\text{for } \left(\frac{r_\ell}{r_\mathrm{a}}\right)^2 > 2\right) \end{aligned} \tag{96}$$

For subcase #2, if $q_t^2(1+q_t^{-\alpha}) < \left(\frac{r_\ell}{r_\mathrm{a}}\right)^2 < 2$, then there are two roots, $q_{\ell,1}$ (which is $< q_t$) and $q_{\ell,2}$ (which is $> q_t$). Each must be found numerically. Their existence makes the integrals silly:

$$I_2(r_\mathrm{a}, M_\mathrm{a}) = \int \frac{q^{5/3}\,\mathrm{d}q}{\mathcal{A}^3\mathcal{B}} f(q)\Gamma^2\left(\frac{r_\mathrm{a}}{\mathcal{B}}, qM_\mathrm{a}\right) \tag{97}$$

$$= \int_0^{q_{\ell,1}(r_\mathrm{a})}(\ldots) + \int_{q_{\ell,1}(r_\mathrm{a})}^{q_t}(\ldots) + \int_{q_t}^{q_{\ell,2}(r_\mathrm{a})}(\ldots) + \int_{q_{\ell,2}(r_\mathrm{a})}^{1}(\ldots) \tag{98}$$

$$= \int_0^{q_{\ell,1}(r_\mathrm{a})}(\ldots) + \int_{q_{\ell,1}(r_\mathrm{a})}^{q_{\ell,2}(r_\mathrm{a})}(\ldots) + \int_{q_{\ell,2}(r_\mathrm{a})}^{1}(\ldots), \tag{99}$$

where the left and right-most terms have $r_\mathrm{a}/\mathcal{B} > r_\ell$, and the center term has $r_\mathrm{a}/\mathcal{B} < r_\ell$. Numerically, $q_t = 0.9210924370$ and $q_t^2(1+q^{-\alpha}) = 1.9796263301$.

Plugging in the rate densities, we get

$$\begin{aligned} I_2(r_\mathrm{a}, M_\mathrm{a}) = \frac{Z_2 r_\mathrm{a}^\delta M_\mathrm{a}^\gamma}{\mathcal{N}_q \mathcal{N}_r \mathcal{N}_M} \int_0^{q_{\ell,1}(r_\mathrm{a})} \mathrm{d}q\, q^{\beta+\gamma+\delta+\frac{8}{3}} (1+q^\alpha)^{\frac{3}{2}} (1+q^{-\alpha})^{\frac{\delta+1}{2}} \\ + \frac{Z_2 r_\ell^\delta M_\mathrm{a}^\gamma}{\mathcal{N}_q \mathcal{N}_r \mathcal{N}_M} \int_{q_{\ell,1}(r_\mathrm{a})}^{q_{\ell,2}(r_\mathrm{a})} \mathrm{d}q\, q^{\beta+\gamma+\frac{8}{3}} (1+q^\alpha)^{3/2} (1+q^{-\alpha})^{\frac{1}{2}} \\ + \frac{Z_2 r_\mathrm{a}^\delta M_\mathrm{a}^\gamma}{\mathcal{N}_q \mathcal{N}_r \mathcal{N}_M} \int_{q_{\ell,2}(r_\mathrm{a})}^{1} \mathrm{d}q\, q^{\beta+\gamma+\delta+\frac{8}{3}} (1+q^\alpha)^{\frac{3}{2}} (1+q^{-\alpha})^{\frac{\delta+1}{2}} \\ \ldots \text{ for } q_t^2(1+q_t^{-\alpha}) < \left(\frac{r_\ell}{r_\mathrm{a}}\right)^2 < 2. \end{aligned} \tag{100}$$

This should actually be enough to integrate semi-analytically with a computer.

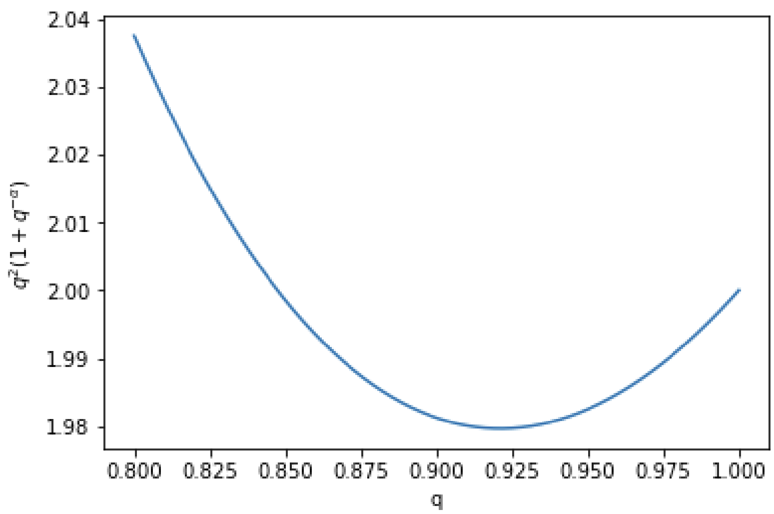

Figure 1. For $\alpha = 3.5$, there is a single minimum of $q^2(1+q^{-\alpha})$.

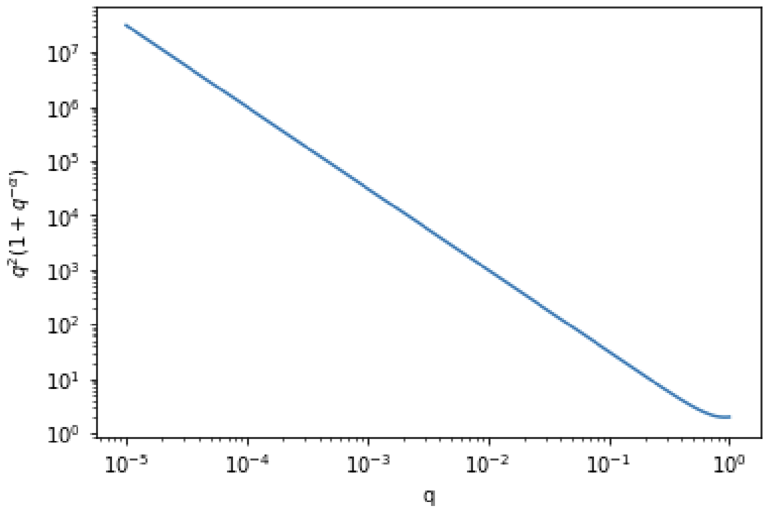

Figure 2. For $\alpha = 3.5$, there is a single minimum of $q^2(1 + q^{-\alpha})$.