hw1

1

Prove each of the following. You may use the results from previous parts of the problem in further proofs. For each proof, I have indicated where you should start. For the purposes of these proofs here, assume that X, Y , and Z are continuous random variables. Assume that a, b and c are constants.

a

$$E[aX + bY + c] = \int \int (ax + by + c)p(x, y)dxdy$$
$$= \int \int axp(x,y)dxdy + \int \int byp(x,y)dxdy + \int \int cp(x,y)dxdy$$
$$= a \int x \left[\int p(x,y)dy \right] dx + b \int y \left[\int p(x,y)dx \right] dy + c \int \int p(x,y)dxdy$$
$$= a \int xp(x)dx + b \int yp(y)dy + c \cdot 1$$
$$= aE[X] + bE[Y] + c$$

b

$$cov(X, Y) = E\left[(X - E[X])(Y - E[Y]) \right]$$
$$= E\left[XY - XE[Y] - YE[X] + E[Y]E[X] \right]$$
$$= E[XY] - E[XE[Y]] - E[YE[X]] + E[E[Y]E[X]]$$
$$= E[XY] - E[Y]E[X] - E[X]E[Y] + E[Y]E[X]E[1] \quad \text{since E[X] and E[Y] are just constants}$$
$$= E[XY] - E[X]E[Y]$$

c

$$E[Y|X = x] = \int yp(y|X = x)dy$$

$$= \int y\frac{p(y, x)}{p(x)}dy$$

$$= \int \int y\frac{p(y, x, z)}{p(x)}dzdy \quad \text{reverse marginalization}$$

$$= \int \int y\frac{p(y|x, z)p(x, z)}{p(x)}dzdy \quad \text{bayes rule}$$

$$= \int \frac{p(x, z)}{p(x)}\left[\int yp(y|x, z)dy\right]dz \quad \text{separating terms}$$

$$= \int p(z|x)E[Y|X = x, Z = z]dz$$

$$= E\left[E[Y|X, Z] \,\middle|\, X = x\right]$$

The transition between the second to last and the last step relies on the fact that since we are conditioning on $X = x$, E[Y|X,Z] is just a function of Z, meaning that the outer expectation is over $p(z|x)$.

d

$$E[h(X, Y)] = \int \int h(x, y)p(x, y)dxdy$$

$$= \int \int h(x, y)p(y|x)p(x)dxdy \quad \text{bayes rule}$$

$$= \int p(x)\left[\int h(x, y)p(y|x)dy\right]dx \quad \text{rearranging terms}$$

$$= \int p(x)E[h(x, y)|X = x] \quad \text{since we condition on x, E[h(x,y)|x] is over p(y|x)}$$

$$= E\left[E[h(X, Y)|X = x]\right] \quad \text{this outer expectation is just over p(x)}$$

e

$$
\begin{aligned}
Var(Y) &= E[Y^2] - E[Y]^2 \\
&= E\Big[E[Y^2|X]\Big] - E[Y]^2 \\
&= E\Big[Var(Y|X) + E[Y|X]^2\Big] - E\Big[E[Y|X]\Big]^2 \\
&= E[Var(Y|X)] + E\Big[E[Y|X]^2\Big] - E\Big[E[Y|X]\Big]^2 \\
&= E[Var(Y|X)] + Var(E[Y|X]) \qquad \text{defintion of variance}
\end{aligned}
$$