

hw3

1

```
dgp_p1 <- function() {  
  ### Define number of schools, and number of students in the school  
  G <- 50 # number of schools  
  ng <- 10 # number of students in each school  
  n <- G*ng # total n-size  
  
  ### Assign students to schools  
  school <- rep(seq(1,G), ng)  
  school <- sort(school)  
  
  ### Gamma (school-varying intercept for PO's)  
  gamma <- rnorm(G, sd=sqrt(2))[school]  
  
  ### Generate potential outcomes  
  y0 <- gamma + rnorm(n)  
  y1 <- 2 + gamma + rnorm(n)  
  
  ### Put everything into a data-frame  
  data <- data.frame("school" = school, "y0" = y0, "y1" = y1)  
  
  return(data)  
}
```

a

There are 10 students per school and 50 schools total, meaning that there are 500 total students. The ATE here is 2, and the factors that influence the potential outcomes are some random noise from the $\mathcal{N}(0, 1)$ instances added to each one, as well as the school specific random intercept called ‘gamma’.

b

- 1) Complete: Randomly draw 250 students to be treated
- 2) Bernoulli: Probability of treatment for each student is 0.5
- 3) Cluster: Randomly draw 25 schools to be treated
- 4) Stratified: Within each school, randomly draw 5 students to be treated

```
# empty results df to fill
res <- tibble(
  complete = rep(0, 1000),
  bernoulli = rep(0, 1000),
  cluster = rep(0, 1000),
  stratified = rep(0, 1000)
)

for (i in 1:1000) {
  data <- dgp_p1()

  # complete
  row_ids <- sample(1:500, 250)
  complete <- data %>%
    mutate(
      id = row_number(),
      D = if_else(id %in% row_ids, 1, 0)
    ) %>%
    select(-id)

  # bernoulli
  bernoulli <- data %>%
    mutate(D = rbinom(500, 1, 0.5))

  # cluster
  school_ids <- sample(1:50, 25)
  cluster <- data %>%
    group_by(school) %>%
    mutate(
      group_id = cur_group_id(),
      D = if_else(group_id %in% school_ids, 1, 0)
    ) %>%
    ungroup() %>%
    select(-group_id)
```

```

# stratified
within_school_ids <- data %>%
  group_by(school) %>%
  mutate(chosen_id = row_number()) %>%
  slice_sample(n = 5) %>%
  ungroup() %>%
  select(school, chosen_id) %>%
  mutate(D = 1)

stratified <- data %>%
  group_by(school) %>%
  mutate(
    id = row_number()
  ) %>%
  ungroup() %>%
  left_join(
    within_school_ids, by = c("id" = "chosen_id", "school")
  ) %>%
  mutate(D = replace_na(D, 0)) %>%
  select(-id)

data_list <- list(complete, bernoulli, cluster, stratified)

dim <- function(data){

  return(mean(data[data$D == 1, ]$y1) - mean(data[data$D == 0, ]$y0))

}

res[i, ] <- data_list %>%
  map(.f = dim)

}

```

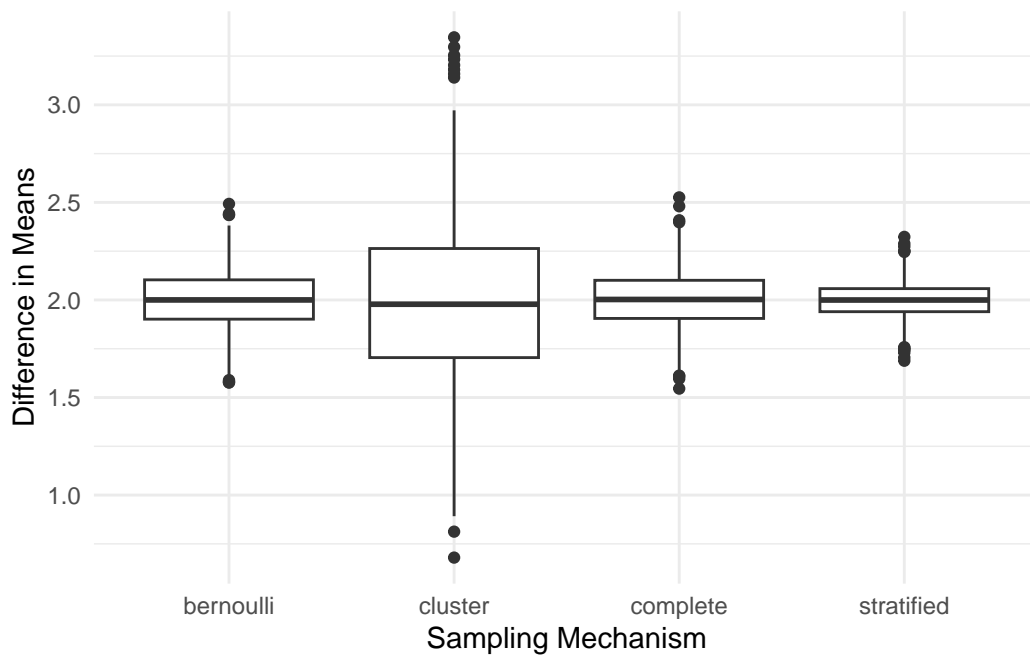
c

Each design is indeed unbiased for the ATE. The variances for the bernoulli and complete randomizations are pretty similar, while the cluster randomization has the largest variance and the stratified randomization has the smallest variance.

```

res %>%
  pivot_longer(
    cols = everything(),
    names_to = "sampling_mechanism",
    values_to = "dim"
  ) %>%
  ggplot(aes(x = sampling_mechanism, y = dim)) +
  geom_boxplot() +
  theme_minimal() +
  labs(
    x = "Sampling Mechanism",
    y = "Difference in Means"
  )

```



```

res %>%
  summarise(across(everything(), var))

```

```

# A tibble: 1 x 4
  complete bernoulli cluster stratified
  <dbl>     <dbl>    <dbl>    <dbl>
1  0.0204   0.0228   0.169    0.00789

```

d

I think it makes sense that cluster randomization would have the highest variance because there are sort of two sources of variability being incorporated: variability within each chosen cluster and the variability between all of the clusters. The data is generated such that there is already variability between the clusters and by re-randomizing in such a way that entire clusters are randomly selected to be treated we essentially enhance that source of variability.

One reason that we would expect the stratified randomization to have the lowest variance is because, unlike in the cluster randomization case, we don't have extra between-cluster variance. I think this is also reason that it has lower variance than the bernoulli and complete randomization cases, as in each re-randomization we ensure that observations from each cluster are included in the treatment. This gives us better balance across clusters and thus avoids more of that unwanted between-cluster variance.

2

a

Making sure to only perform the analysis on units that are eligible to receive the subsidy, we simply regress our response variable on the treatment category column.

```
data_raw <- read.dta13("/Users/joshuayamamoto/Downloads/BD-SAN-FINAL.dta") %>%
  mutate(r4_recode = case_when(
    r4_any_od_adults == "Yes" ~ 1L,
    r4_any_od_adults == "No" ~ 0L
  ))

data <- data_raw %>%
  filter(eligible == "Eligible") %>%
  select(r4_recode, treat_cat_3) %>%
  drop_na()

coef(summary(lm(r4_recode ~ treat_cat_3 , data)))[ -1,1] %>%
  as.data.frame() %>%
  rename('Estimate' = '.')
```

	Estimate
treat_cat_3LPP Only	-0.07935631
treat_cat_3Supply Only	-0.09581019
treat_cat_3Loser (Low)	-0.09035256
treat_cat_3Loser (Med)	-0.15611116

```
treat_cat_3Loser (High) -0.13388554
treat_cat_3Winner (Low) -0.10971356
treat_cat_3Winner (Med) -0.19489560
treat_cat_3Winner (High) -0.16496828
```

b

I think that the main reason has to do with the fact that it would spark controversy and issues within clusters if some individuals were afforded better treatments than others. Furthermore there would likely be a lot of spillover if the treatment was individually assigned since friends and family would probably let each other benefit from their treatment.

c

```
randomization_scheme <- function() {

  ## 1: Cluster randomization of villages to treatment conditions.
  ## note: stratify by number of neighborhoods per village (1-2 and 3+)

  ## Control, LPP Only, LPP + Subsidy, Supply only.

  # first figure out the strata that each village belongs to
  strata_labels <- data_raw %>%
    distinct(vid, cid) %>%
    count(vid) %>%
    mutate(strata = case_when(
      n < 3 ~ "s1",
      n >= 3 ~ "s2"
    )) %>%
    select(-n)

  # extract strata labels for each village
  s1_villages <- strata_labels[strata_labels$strata == "s1", ]$vid
  s2_villages <- strata_labels[strata_labels$strata == "s2", ]$vid

  ## s1 has 50 villages
  ## s2 has 57 villages

  ## 22 in control, 12 LPP only, 63 in LPP + Subsidy, 10 in Supply Only
  ## (following SM from paper)
```

```

## they overweight the LPP + Subsidy group since it will be further
## partitioned in the next stage

# randomly assign the first level of treatment conditions to entire villages by strata
# did this as complete randomization
s1_clusters <- sample(
  c(
    rep("Control", 11), rep("LPP_Only", 6),
    rep("LPP+Subsidy", 29), rep("Supply_Only", 4)
  ),
  size = 50, replace = F
)

s2_clusters <- sample(
  c(
    rep("Control", 11), rep("LPP_Only", 6),
    rep("LPP+Subsidy", 34), rep("Supply_Only", 6))
  ,
  size = 57, replace = F
)

# create new tibble with the treatment that each village belongs to
cluster_labels <- tibble(
  vid = c(s1_villages, s2_villages),
  treat1 = c(s1_clusters, s2_clusters)
)

# rejoin strata and treatment labels to original data
data_s1 <- data_raw %>%
  left_join(strata_labels, by = "vid") %>%
  left_join(cluster_labels, by = "vid")

## 2. Cluster randomization of neighborhoods in the "LPP + Subsidy" condition
## into a 2x3 set of subtreatments indicating if they get just the subsidy,
## or the subsidy + supply, and the level of the intensity of the vouchers.

## first cluster randomize into LPP + Subsidy and LPP + Subsidy + Supply

## this is done at the neighborhood level so first get a list of the
## distinct neighborhoods who received "LPP + Subsidy"
n_cid <- data_s1 %>%

```

```

filter(treat1 == "LPP+Subsidy") %>%
distinct(cid) %>%
nrow()

# recover how many cid should be given LPP + Subsidy and LPP + Subsidy + Supply
# I chose to just balance this as best as possible

# randomly assign neighborhoods within LPP + Subsidy to either LPP + Subsidy
# or LPP + Subsidy + Supply using bernoulli randomization
stage2_labels <- data_s1 %>%
  filter(treat1 == "LPP+Subsidy") %>%
  distinct(cid) %>%
  mutate(stage2id = rbinom(n(), 1, 0.5)) %>%
  mutate(stage2grp = case_when(
    stage2id == 1 ~ "LPP+Subsidy",
    T ~ "LPP+Subsidy+Supply"
  ))

# within LPP + Subsidy neighborhoods do a bernoulli random assignment of intensity
p1 <- stage2_labels %>%
  filter(stage2grp == "LPP+Subsidy") %>%
  mutate(intensity = sample(c("High", "Medium", "Low"), n(), replace = T))

# within LPP + Subsidy + Supply neighborhoods do bernoulli random
# assignment of intensity
p2 <- stage2_labels %>%
  filter(stage2grp == "LPP+Subsidy+Supply") %>%
  mutate(intensity = sample(c("High", "Medium", "Low"), n(), replace = T))

stage2_labels_final <- rbind(p1, p2)

# join the intensity labels to the original data by neighborhood
data_s2 <- data_s1 %>%
  left_join(stage2_labels_final, by = "cid") %>%
  mutate(treat1 = case_when(
    is.na(stage2grp) ~ treat1,
    T ~ stage2grp
  )) %>%

```



```

select(-stage2grp) %>%
mutate(treat2 = case_when(
  is.na(intensity) ~ treat1,
  T ~ intensity
)) %>%
select(-intensity)

## 3: Randomization of the vouchers for households in the "LPP + Subsidy"
## conditions, given their intensity assignment.
## note: by household

# high- 75%
# medium- 50%
# low- 25%
# per the paper

# use bernoulli randomization to assign winners of the lottery by
# household to low intensity neighborhoods using the varying
# success probabilities from above

s2_1 <- data_s2 %>%
  filter(treat2 == "Low") %>%
  mutate(lottery = sample(
    c("Winner", "Loser"), n(),
    replace = T, prob = c(0.25, 0.75)
  ))

s2_2 <- data_s2 %>%
  filter(treat2 == "Medium") %>%
  mutate(lottery = sample(
    c("Winner", "Loser"), n(),
    replace = T, prob = c(0.5, 0.5)
  ))

s2_3 <- data_s2 %>%
  filter(treat2 == "High") %>%
  mutate(lottery = sample(
    c("Winner", "Loser"), n(),
    replace = T, prob = c(0.75, 0.25)
  ))

```

```

# formatting
lottery_labels <- rbind(s2_1, s2_2, s2_3) %>%
  mutate(lottery = paste0(lottery, " (", treat2, ")")) %>%
  select(hhid, lottery)

# rejoin everything together
data_s3 <- data_s2 %>%
  left_join(lottery_labels, by = "hhid") %>%
  mutate(treat3 = case_when(
    is.na(lottery) ~ treat2,
    T ~ lottery
  )) %>%
  select(-lottery) %>%
  select(hhid, cid, vid, r4_recode, eligible, treat3, treat_cat_3)

return(data_s3)
}

```

d

The results from the randomization inference are saved and reloaded in for the analysis portion so that the code doesn't have to be re-run every single time

```

null_dist <- data.frame(dim = rep(0, 5000))

for(i in 1:5000) {
  iter <- randomization_scheme()
  iter <- iter %>%
    filter(eligible == "Eligible") %>%
    filter(treat3 == "Winner (Low)" | treat3 == "Control") %>%
    drop_na()

  null_dist[i, ] <- coef(lm(r4_recode ~ treat3, iter))[2]
}

null_dist <- read_csv(here("data", "hw3_p2_null_dist.csv"))

data_sub <- data_raw %>%

```

```

filter(eligible == "Eligible") %>%
filter(treat_cat_3 == "Winner (Low)" | treat_cat_3 == "Control") %>%
  mutate(r4_recode = case_when(
    r4_any_od_adults == "Yes" ~ 1L,
    r4_any_od_adults == "No" ~ 0L
  )) %>%
drop_na()

true_dim <- coef(lm(r4_recode ~ treat_cat_3, data_sub))[2]

p_val_one_sided <- mean(null_dist$dim < true_dim)
p_val_two_sided <- mean(null_dist$dim < -abs(true_dim)) +
  mean(null_dist$dim > abs(true_dim))

one_sided <- null_dist %>%
  mutate(filler = ifelse(dim > true_dim, "more", "less")) %>%
  ggplot(aes(x = dim, fill = filler)) +
  geom_histogram(binwidth = 0.0058, show.legend = F) +
  scale_fill_manual(values = c("#4c5c4d", "#a9ccaa")) +
  geom_vline(
    xintercept = true_dim, color = "red",
    linetype = "dashed", linewidth = 1.2
  ) +
  theme_minimal() +
  annotate(
    geom = "text", x = -0.17, y = 150,
    label = "p-value = 0.007", color = "#d11611"
  )

two_sided <- null_dist %>%
  mutate(filler = ifelse(
    dim < -abs(true_dim) | dim > abs(true_dim), "shade", "no-shade"
  )) %>%
  ggplot(aes(x = dim, fill = filler)) +
  geom_histogram(binwidth = 0.0055, show.legend = F) +
  geom_vline(
    xintercept = -abs(true_dim), color = "red",
    linetype = "dashed", linewidth = 1.2
  ) +

```

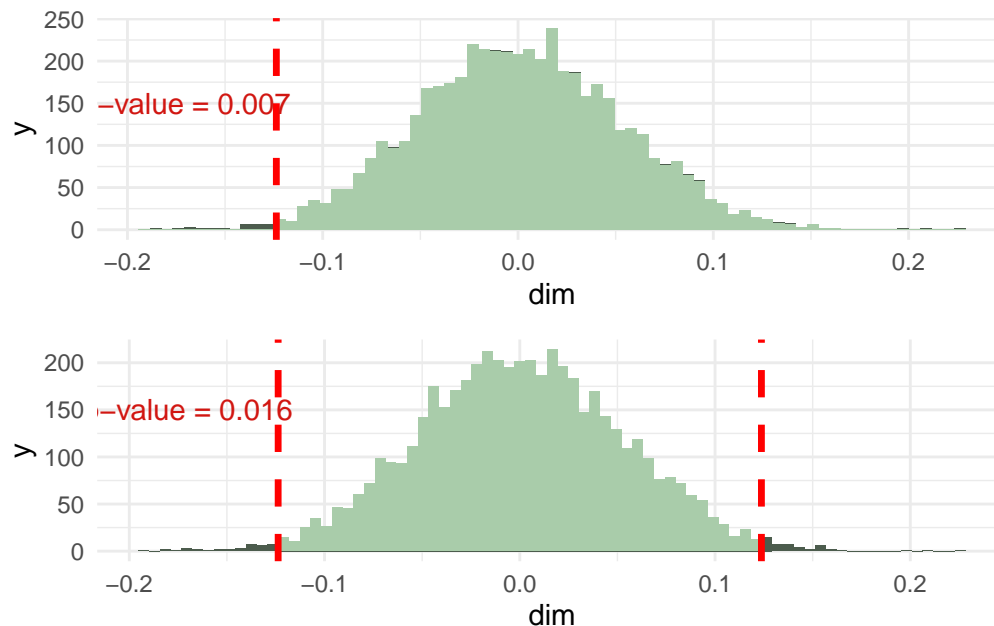
```

geom_vline(
  xintercept = abs(true_dim), color = "red",
  linetype = "dashed", linewidth = 1.2
) +
scale_fill_manual(values = c("#a9ccaa", "#4c5c4d")) +
theme_minimal() +
annotate(
  geom = "text", x = -0.17, y = 150,
  label = "p-value = 0.016", color = "#d11611"
)

```

We can interpret the p-value as follows: the probability of observing a similar treatment effect under the null hypothesis (which says that the treatment effect is zero for all units) is 0.007. And the probability of observing a similar absolute treatment effect under the sharp null is 0.016.

one_sided / two_sided



Because these p-values are significant we can reject the sharp null. Essentially we have shown that across many hypothetical treatment randomizations under the sharp null, we would very rarely expect to see an observed treatment effect as extreme as the observed one.

e

I think my findings align with the authors original findings because I followed their randomization scheme as closely as possible. Besides small potential differences in complete vs bernoulli randomization during the different phases of the process, everything else followed what the authors did. I guess this reassures me that the effect found in the paper was a significant one, but it still feels a bit like the intricacies of the randomization scheme could have played a role in getting the result.

f

```
grid <- true_dim + seq(from = -0.25, to = 0.25, length.out = 300)

Null_Dist_df <- matrix(data = rep(0, 300000), ncol = 300)

for(i in 1:1000){
  d_new <- randomization_scheme()$treat3
  New_DIM <- c(rep(0, 300))
  for(v in 1:length(grid)){
    new_po <- data_raw %>%
      mutate(D_old = case_when(
        treat_cat_3 == "Control" ~ 0,
        T ~ 1
      )) %>%
      mutate(D_new_raw = d_new) %>%
      mutate(D_new = case_when(
        D_new_raw == "Control" ~ 0,
        T ~ 1
      )) %>%
      select(r4_recode, D_old, D_new) %>%
      drop_na() %>%
      mutate(
        Y_0 = r4_recode - D_old*grid[v],
        Y_1 = r4_recode + (1-D_old)*grid[v],
        Y_new = D_new*Y_1 + (1 - D_new)*Y_0
      )

    fit <- lm(Y_new ~ D_new, new_po)
    dim_new <- fit$coefficients["D_new"]
    New_DIM[v] <- dim_new
  }
}
```

```

Null_Dist_df[i, ] <- New_DIM

}

grid <- true_dim + seq(from = -0.25, to = 0.25, length.out = 300)
Null_Dist_df <- read_csv(here("data", "hw3_p2_fisher_ci.csv"))

p_value_vec <- rep(0, 300)

for(i in 1:length(grid)){
  # Get corresponding null distribution and null value
  temp_null_dist <- Null_Dist_df[,i]
  v <- grid[i]
  # absolute difference between observed and assumed
  diff <- abs(true_dim-v)
  # Get adjusted p-value -- probability of as far or further away from assumed
  temp_p <-
    mean(temp_null_dist >= v + diff) +
    mean(temp_null_dist <= v - diff)
  # Add to p-value vector
  p_value_vec[i] <- temp_p
}

over <- which(p_value_vec>0.05)
index_lwr <- over[1]-1
index_upr <- over[length(over)]+1
ci <- c(grid[index_lwr], grid[index_upr])

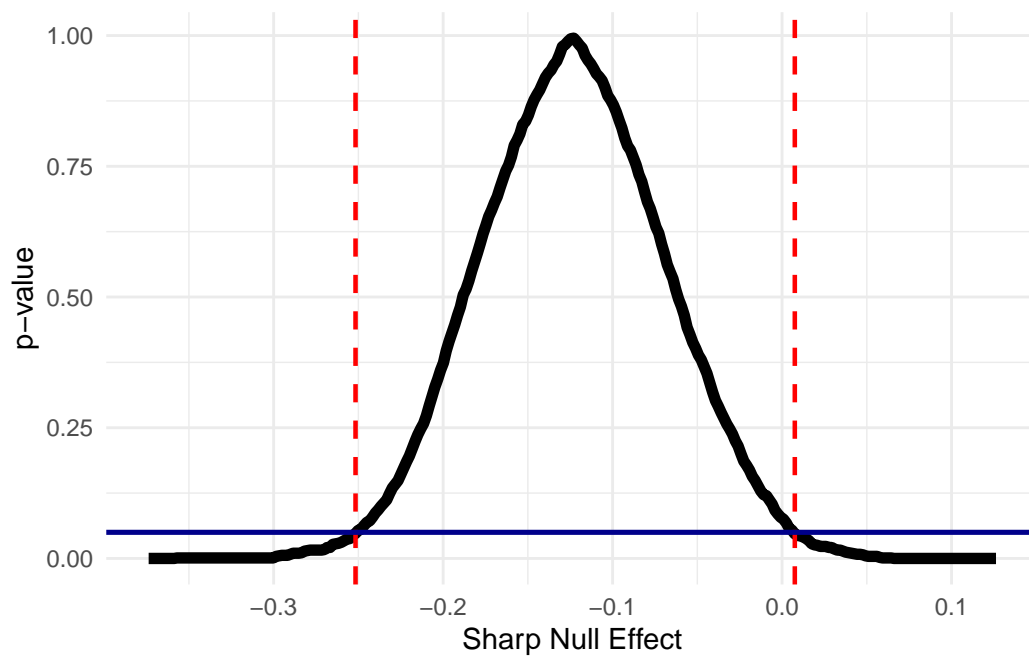
tibble(
  p_value = p_value_vec,
  sharp_null_effect = grid
) %>%
  ggplot(aes(x = sharp_null_effect, y = p_value)) +
  geom_line(linewidth = 2) +
  geom_vline(
    xintercept = ci[1], linetype = "dashed",
    linewidth = 0.8, color = "red"
  ) +
  geom_vline(

```

```

    xintercept = ci[2], linetype = "dashed",
    linewidth = 0.8, color = "red"
  ) +
  geom_hline(
    yintercept = 0.05, color = "darkblue",
    linewidth = 0.9
  ) +
  theme_minimal() +
  labs(
    x = "Sharp Null Effect",
    y = "p-value"
  )
)

```



This confidence interval $(-0.2517, 0.0075)$ tells us the range of observed treatment effect within which we would not reject the sharp null hypothesis. I

3

a

$$\begin{aligned}
-B_n - E_n &= -\left(\frac{1}{n_1} \sum_{D_i=1} f_1(X_i) - \frac{1}{n} \sum_{i=1}^n f_1(X_i)\right) + \left(\frac{1}{n_0} \sum_{D_i=0} f_0(X_i) - \frac{1}{n} \sum_{i=1}^n f_0(X_i)\right) - \\
&\quad \left(\frac{1}{n_1} \sum_{D_i=1} \varepsilon_i(1) - \frac{1}{n} \sum_{i=1}^n \varepsilon_i(1)\right) + \left(\frac{1}{n_0} \sum_{D_i=0} \varepsilon_i(0) - \frac{1}{n} \sum_{i=1}^n \varepsilon_i(0)\right) \\
&= -\frac{1}{n_1} \sum_{D_i=1} f_1(X_i) + \frac{1}{n_0} \sum_{D_i=0} f_0(X_i) - \frac{1}{n_1} \sum_{D_i=1} \varepsilon_i(1) + \frac{1}{n_0} \sum_{D_i=0} \varepsilon_i(0) + \text{SATE} \\
&= -\frac{1}{n_1} \sum_{D_i=1} [f_1(X_i) + \varepsilon_i(1)] + \frac{1}{n_0} \sum_{D_i=0} [f_0(X_i) + \varepsilon_i(0)] + \text{SATE} \\
&= -\frac{1}{n_1} \sum_{D_i=1} Y_i(1) + \frac{1}{n_0} \sum_{D_i=0} Y_i(0) + \text{SATE}
\end{aligned}$$

so now we can say that

$$\begin{aligned}
\hat{\tau}_{dim} - B_n - E_n &= \frac{1}{n_1} \sum_{D_i=1} Y_i(1) - \frac{1}{n_0} \sum_{D_i=0} Y_i(0) - \frac{1}{n_1} \sum_{D_i=1} Y_i(1) + \frac{1}{n_0} \sum_{D_i=0} Y_i(0) + \text{SATE} \\
&= \text{SATE}
\end{aligned}$$

as desired.

b

First recall that we can express $\hat{\tau}_{ols}$ as follows by regressing $(Y - X^T \hat{\beta}_{ols})$ on D . If we simplify the expression down we get the following:

$$\begin{aligned}
\hat{\tau}_{ols} &= \left(\frac{1}{n_1} \sum_{D_i=1} Y_i - \frac{1}{n_0} \sum_{D_i=1} Y_i\right) - \left(\frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\beta}_{ols} - \frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{ols}\right) \\
&= \hat{\tau}_{dim} - \left(\frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\beta}_{ols} - \frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{ols}\right)
\end{aligned}$$

Next we can simplify the term $-\hat{\beta}_{n,ols}$ as follows:

$$\begin{aligned}
-\hat{\beta}_{n,ols} &= -\left(\frac{n_1\hat{\tau}_{ols}}{n_1} + \frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\beta}_{ols} - \frac{n\hat{\tau}_{ols}}{n} - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\beta}_{ols}\right) + \left(\frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{ols} - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\beta}_{ols}\right) \\
&= -\left(\frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\beta}_{ols} - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\beta}_{ols}\right) + \left(\frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{ols} - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\beta}_{ols}\right) \\
&= -\frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\beta}_{ols} + \frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{ols} \\
&= -\left(\frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\beta}_{ols} - \frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{ols}\right)
\end{aligned}$$

Finally if we take $\hat{\tau}_{dim} - \hat{\beta}_{n,ols}$ along with the first equality that we recovered, we get

$$\begin{aligned}
\hat{\tau}_{dim} - \hat{\beta}_{n,ols} &= \hat{\tau}_{dim} - \left(\frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\beta}_{ols} - \frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{ols}\right) \\
&= \hat{\tau}_{ols}
\end{aligned}$$

as desired

c

First we'll simplify the expression for $-\hat{\beta}_{n, Lin}$

$$\begin{aligned}
-\hat{\beta}_{n, Lin} &= -\left(\frac{1}{n_1} \sum_{D_i=1} (X_i^T(\hat{\beta}_{Lin} + \hat{\gamma}_{Lin}) + \hat{\tau}_{Lin}) - \frac{1}{n} \sum_{i=1}^n (X_i^T(\hat{\beta}_{Lin} + \hat{\gamma}_{Lin}) + \hat{\tau}_{Lin})\right) + \\
&\quad \left(\frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{Lin} - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\beta}_{Lin}\right) \\
&= -\left(\frac{1}{n_1} \sum_{D_i=1} X_i^T(\hat{\beta}_{Lin} + \hat{\gamma}_{Lin}) - \frac{1}{n} \sum_{i=1}^n X_i^T(\hat{\beta}_{Lin} + \hat{\gamma}_{Lin})\right) + \left(\frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{Lin} - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\beta}_{Lin}\right) \\
&= -\left(\frac{1}{n_1} \sum_{D_i=1} X_i^T(\hat{\beta}_{Lin} + \hat{\gamma}_{Lin}) - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\beta}_{Lin} - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\gamma}_{Lin}\right) + \\
&\quad \left(\frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{Lin} - \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\beta}_{Lin}\right) \\
&= -\frac{1}{n_1} \sum_{D_i=1} X_i^T(\hat{\beta}_{Lin} + \hat{\gamma}_{Lin}) + \frac{1}{n} \sum_{i=1}^n X_i^T \hat{\gamma}_{Lin} + \left(\frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{Lin}\right)
\end{aligned}$$

next we can write out some of the terms of $\hat{\tau}_{Lin}$

$$\begin{aligned}
\hat{\tau}_{Lin} &= \underset{\tau}{\operatorname{argmin}} \sum_{i=1}^n \left[Y_i - \left(\tau D_i + (X_i - \bar{X})^T \hat{\beta}_{Lin} + D_i (X_i - \bar{X})^T \hat{\gamma}_{Lin} \right) \right]^2 \\
&= \underset{\tau}{\operatorname{argmin}} \sum_{i=1}^n \left[\left(Y_i - (X_i - \bar{X})^T \hat{\beta}_{Lin} - D_i (X_i - \bar{X})^T \hat{\gamma}_{Lin} \right) - \tau D_i \right]^2 \\
&= \frac{1}{n_1} \sum_{D_i=1} \left(Y_i - (X_i - \bar{X})^T \hat{\beta}_{Lin} - (X_i - \bar{X})^T \hat{\gamma}_{Lin} \right) - \frac{1}{n_0} \sum_{D_i=0} \left(Y_i - (X_i - \bar{X})^T \hat{\beta}_{Lin} \right) \\
&= \underbrace{\left[\frac{1}{n_1} \sum_{D_i=1} Y_i - \frac{1}{n_0} \sum_{D_i=0} Y_i \right]}_{\hat{\tau}_{dim}} - \frac{1}{n_1} \sum_{D_i=1} (X_i - \bar{X})^T \hat{\beta}_{Lin} - \frac{1}{n_1} \sum_{D_i=1} (X_i - \bar{X})^T \hat{\gamma}_{Lin} + \\
&\quad \frac{1}{n_0} \sum_{D_i=0} (X_i - \bar{X})^T \hat{\beta}_{Lin} \\
&= \hat{\tau}_{dim} - \frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\beta}_{Lin} + \bar{X}^T \hat{\beta}_{Lin} - \frac{1}{n_1} \sum_{D_i=1} X_i^T \hat{\gamma}_{Lin} + \bar{X}^T \hat{\gamma}_{Lin} + \frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{Lin} - \bar{X}^T \hat{\beta}_{Lin} \\
&= \hat{\tau}_{dim} - \underbrace{\frac{1}{n_1} \sum_{D_i=1} X_i^T (\hat{\beta}_{Lin} + \hat{\gamma}_{Lin}) + \frac{1}{n} \sum_{D_i=1} X_i^T \hat{\gamma}_{Lin} + \frac{1}{n_0} \sum_{D_i=0} X_i^T \hat{\beta}_{Lin}}_{\hat{\beta}_{n, Lin}} \\
&= \hat{\tau}_{dim} - \hat{\beta}_{n, Lin}
\end{aligned}$$

as desired.

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