hw1

1

Prove each of the following. You may use the results from previous parts of the problem in further proofs. For each proof, I have indicated where you should start. For the purposes of these proofs here, assume that X, Y , and Z are continuous random variables. Assume that a, b and c are constants.

a

$$\begin{aligned}
E[aX+bY+c] &= \int\int (ax+by+c)p(x,y)dxdy \\
&= \int\int axp(x,y)dxdy + \int\int byp(x,y)dxdy + \int\int cp(x,y)dxdy \\
&= a\int x\Big[\int p(x,y)dy\Big]dx + b\int y\Big[\int p(x,y)dx\Big]dy + c\int\int p(x,y)dxdy \\
&= a\int xp(x)dx + b\int yp(y)dy + c\cdot 1 \\
&= aE[X] + bE[Y] + c
\end{aligned}$$

b

$$\begin{aligned}
cov(X,Y) &= E\Big[(X-E[X])(Y-E[Y])\Big] \\
&= E\Big[XY - XE[Y] - YE[X] + E[Y]E[X]\Big] \\
&= E[XY] - E[XE[Y]] - E[YE[X]] + E[E[Y]E[X]] \\
&= E[XY] - E[Y]E[X] - E[X]E[Y] + E[Y]E[X]E[1] \qquad \text{since E[X] and E[Y] are just constants} \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

c

$$
\begin{aligned}
E[Y|X=x] &= \int yp(y|X=x)dy \\
&= \int y\frac{p(y,x)}{p(x)}dy \\
&= \int\int y\frac{p(y,x,z)}{p(x)}dzdy \qquad \text{reverse marginalization} \\
&= \int\int y\frac{p(y|x,z)p(x,z)}{p(x)}dzdy \qquad \text{bayes rule} \\
&= \int \frac{p(x,z)}{p(x)}\Big[\int yp(y|x,z)dy\Big]dz \qquad \text{separating terms} \\
&= \int p(z|x)E[Y|X=x,Z=z]dz \\
&= E\Big[E[Y|X,Z]\;\Big|\;X=x\Big]
\end{aligned}
$$

The transition between the second to last and the last step relies on the fact that since we are conditioning on $X = x$, E[Y|X,Z] is just a function of Z, meaning that the outer expectation is over $p(z|x)$.

d

$$
\begin{aligned}
E[h(X,Y)] &= \int\int h(x,y)p(x,y)dxdy \\
&= \int\int h(x,y)p(y|x)p(x)dxdy \qquad \text{bayes rule} \\
&= \int p(x)\Big[\int h(x,y)p(y|x)dy\Big]dx \qquad \text{rearranging terms} \\
&= \int p(x)E[h(x,y)|X=x] \qquad \text{since we condition on x, E[h(x,y)|x] is over p(y|x)} \\
&= E\Big[E[h(X,Y)|X=x]\Big] \qquad \text{this outer expectation is just over p(x)}
\end{aligned}
$$

e

$$
\begin{aligned}
Var(Y) &= E[Y^2] - E[Y]^2 \\
&= E\Big[E[Y^2|X]\Big] - E[Y]^2 \\
&= E\Big[Var(Y|X) + E[Y|X]^2\Big] - E\Big[E[Y|X]\Big]^2 \\
&= E[Var(Y|X)] + E\Big[E[Y|X]^2\Big] - E\Big[E[Y|X]\Big]^2 \\
&= E[Var(Y|X)] + Var(E[Y|X]) \qquad \text{defintion of variance}
\end{aligned}
$$

2

Prove each of the following statements. In each, let $X_{(n)}$ and $Y_{(n)}$ be sequences of random variables, and let c and d be constants.

a

Choose some $\varepsilon > 0$. We can write.

$$
P\bigg(\Big|(X_{(n)} + Y_{(n)}) - (c+d)\Big| \geq \varepsilon\bigg) = P\bigg(\Big|(X_{(n)} - c) + (Y_{(n)} - d)\Big| \geq \varepsilon\bigg)
$$

But if the event $\Big|(X_{(n)} - c) + (Y_{(n)} - d)\Big| \geq \varepsilon$ happened then the event

- "$|X_{(n)} - c| \geq \varepsilon/2$ or $|Y_{(n)} - d| \geq \varepsilon/2$" necessarily happened as well

But because the first event happening means necessarily that the second happened, then we can say that the first event is a subset of the second event. Of course if $A \subseteq B$ then $P(A) \leq P(B)$ so we can say that

$$
P\bigg(\Big|(X_{(n)} + Y_{(n)}) - (c+d)\Big| \geq \varepsilon\bigg) \leq P\bigg(\Big|X_{(n)} - c\Big| \geq \varepsilon/2 \bigcup \Big|Y_{(n)} - d\Big| \geq \varepsilon/2\bigg)
$$

We also know that $P(A \cup B) \leq P(A) + P(B)$ so

$$
\begin{aligned}
P\bigg(\Big|(X_{(n)} + Y_{(n)}) - (c+d)\Big| \geq \varepsilon\bigg) &\leq P\bigg(\Big|X_{(n)} - c\Big| \geq \varepsilon/2 \bigcup \Big|Y_{(n)} - d\Big| \geq \varepsilon/2\bigg) \\
&\leq P\bigg(\Big|X_{(n)} - c\Big| \geq \varepsilon/2\bigg) + P\bigg(\Big|Y_{(n)} - d\Big| \geq \varepsilon/2\bigg)
\end{aligned}
$$

But since we know that $X_{(n)} \xrightarrow{p} c$ and $Y_{(n)} \xrightarrow{p} d$ then we know that

$$P\left(\left|X_{(n)} - c\right| \geq \varepsilon/2\right) + P\left(\left|Y_{(n)} - d\right| \geq \varepsilon/2\right) \to 0 \qquad \text{as } n \to \infty$$

Importantly these are sequences of numbers and not random variables so we can indeed say that since each term individually converges to zero, then so does their sum. We are not using what we were trying to prove.

But now what we have is that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|(X_{(n)} + Y_{(n)}) - (c+d)\right| \geq \varepsilon\right) = 0$$

which is exactly what it means for $X_{(n)} + Y_{(n)} \xrightarrow{p} c + d$

b

to do

c

If $X_{(n)} \xrightarrow{p} c$ then we also know that $\frac{X_{(n)}}{c} \xrightarrow{p} 1$

Next let $\varepsilon > 0$ be given and take

$$\begin{aligned}
\left|\frac{c}{X_{(n)}} - 1\right| \geq \varepsilon \iff & -\varepsilon \leq \frac{c}{X_{(n)}} - 1 \leq \varepsilon \\
&= 1 - \varepsilon \leq \frac{c}{X_{(n)}} \leq 1 + \varepsilon \\
&= \frac{c}{1+\varepsilon} \leq X_{(n)} \leq \frac{c}{1-\varepsilon} \\
&= \frac{c}{1+\varepsilon} - c \leq X_{(n)} - c \leq \frac{c}{1-\varepsilon} - c \\
&= \frac{c - c(1+\varepsilon)}{1+\varepsilon} \leq X_{(n)} - c \leq \frac{c - c(1-\varepsilon)}{1-\varepsilon} \\
&= -\frac{c\varepsilon}{1+\varepsilon} \leq X_{(n)} - c \leq \frac{c\varepsilon}{1-\varepsilon}
\end{aligned}$$

we're almost there, but from here we can not that since $\varepsilon > 0$ then we have that $c\varepsilon/(1+\varepsilon) < c\varepsilon/(1-\varepsilon)$ which we can leverage to say that

$$-\frac{c\varepsilon}{1+\varepsilon} \leq X_{(n)} - c \leq \frac{c\varepsilon}{1-\varepsilon} \implies -\frac{c\varepsilon}{1-\varepsilon} \leq X_{(n)} - c \leq \frac{c\varepsilon}{1-\varepsilon} \implies |X_{(n)} - c| \leq \frac{c\varepsilon}{1-\varepsilon}$$

Essentially the logic is that if we know that $-a \leq X_{(n)} - c$ and $a < b$ then $-b \leq X_{(n)} - c$ as well. In any event, we have made our event **larger** and so we can say that this event has a larger probability of occuring. To reiterate, in the first chunk of steps above we had shown a direct bi-conditional, and then in this last step we made our event slightly larger so we can write:

$$P\left(\left|\frac{c}{X_{(n)}} - 1\right| \geq \varepsilon\right) \leq P\left(|X_{(n)} - c| \leq \frac{c\varepsilon}{1-\varepsilon}\right)$$

We are given that $X_{(n)} \xrightarrow{p} c$ and so we certainly know that the right hand term goes to zero as $n \to \infty$. This then tells us that

$$\frac{c}{X_{(n)}} \xrightarrow{p} 1 \iff \frac{1}{X_{(n)}} \xrightarrow{p} \frac{1}{c}$$

as desired.

3

We simply apply Chebyshev's inequality to $X_{(n)}$, choosing any ε, to get

$$P\left(\left|X_{(n)} - E[X_{(n)}]\right| \geq \varepsilon\right) \leq \frac{var(X_{(n)})}{\varepsilon^2}$$

And since we know that $var(X_{(n)}) \to 0$, then we know that

$$\frac{1}{\varepsilon^2} \cdot var(X_{(n)}) \to \frac{1}{\varepsilon^2} \cdot 0 = 0$$

Of course probabilities are bounded below by zero, so we get that the probability expression is squeezed towards zero as n increases. And since we also know that $E[X_{(n)}] \to c$, then when we take the limit of the above expression we get

$$\lim_{n\to\infty} P\left(\left|X_{(n)} - c\right| \geq \varepsilon\right) = 0$$

which is exactly what it means for $X_{(n)} \xrightarrow{p} c$

4

We know that $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0,\sigma^2)$ and $\hat{\sigma} \xrightarrow{p} \sigma$ and we can write

$$\frac{\sqrt{n}(\hat{\theta}-\theta)}{\hat{\sigma}} = \frac{\sigma}{\hat{\sigma}}\frac{\sqrt{n}(\hat{\theta}-\theta)}{\sigma}$$

Next we apply the continuous mapping theorem we can say that

$$\frac{\sigma}{\hat{\sigma}} \xrightarrow{p} \frac{\sigma}{\sigma} \qquad \text{and} \qquad \frac{\sqrt{n}(\hat{\theta}-\theta)}{\sigma} \xrightarrow{d} N(0,1)$$

Next Slutsky's theorem tells us that

$$\frac{\sqrt{n}(\hat{\theta}-\theta)}{\hat{\sigma}} = \frac{\sigma}{\hat{\sigma}}\frac{\sqrt{n}(\hat{\theta}-\theta)}{\sigma} \xrightarrow{d} 1 \cdot N(0,1)$$

And by definition of convergence in distribution we get that

$$\lim_{n\to\infty} P\left(-z_{1-\alpha/2} \le \frac{\sqrt{n}(\hat{\theta}-\theta)}{\hat{\sigma}} \le z_{1-\alpha/2}\right) = 1-\alpha$$

We can then do some rearranging within the parentheses

$$\begin{aligned}
-z_{1-\alpha/2} \le \frac{\sqrt{n}(\hat{\theta}-\theta)}{\hat{\sigma}} \le z_{1-\alpha/2} &= -z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \le \hat{\theta}-\theta \le z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \\
&= \hat{\theta} - z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \le -\theta \le \hat{\theta} + z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \\
&= \hat{\theta} - z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \le \theta \le \hat{\theta} + z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \qquad \text{symmetry of normal dist}
\end{aligned}$$

so we have that

$$\lim_{n\to\infty} P\left(\hat{\theta} - z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \le \theta \le \hat{\theta} + z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}}\right) = 1-\alpha$$

as desired.

5

a

Well in OLS we choose β such that $\mathbf{X}^T(\mathbf{Y} - \mathbf{X}\beta) = 0$ ($\hat{\beta}_{OLS}$ solves this equation).

We can write this out as follows

$$\begin{pmatrix} 1 & \dots & 1 \\ X_{1,1} & \dots & X_{n,1} \\ \vdots & \ddots & \vdots \\ X_{1,k-1} & \dots & X_{n,k-1} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

but immediately we can see that an immediate consequence of this is that

$$1 \cdot e_1 + 1 \cdot e_2 + \dots + 1 \cdot e_n = \sum_{i=1}^{n} e_i = 0$$

so of course $\frac{1}{n}\sum e_i = 0$ as well.

b

No, in order for the CEF of Y to be linear we need the expected value of the errors conditional on X to be zero. The OLS solution gives us a way to estimate $E[Y|X]$ but doesn't tell us anything about the true underlying data generating process.

6

```r
library(tidyverse)
library(patchwork)
library(moderndive)

dgp <- function(n) {
  x <- rnorm(n)
  epsilon <- (x^2 + 1) * (rchisq(n, df=1)-1)
  y <- 1 + x + epsilon

  df <- tibble(
    y = y,
    x = x
```

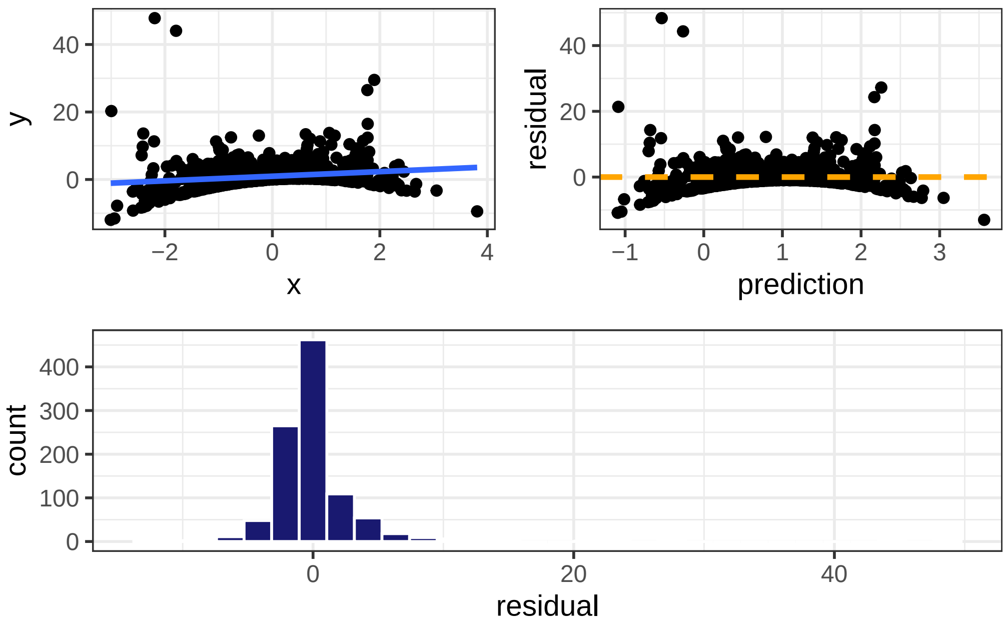
```
  )
}
```

a

From the plots we can see that linearity and equal variability do not hold as the residuals do not follow a uniform band across the range of predicted values. We might be inclined to think that normality holds from looking at the histogram of the residuals, but the scatterplot of y and x with the linear model shows us that the residuals are certainly not normally distributed.

```r
set.seed(1)
df <- dgp(1000)
mod <- lm(y ~ x, df)

df_preds <- df %>%
  mutate(
    prediction = predict(mod, newx = x),
    residual = y - prediction
    )
```

```r
## linearity
p1 <- df_preds %>%
  ggplot(aes(x, y)) +
  geom_point() +
  geom_smooth(method = "lm", se = F) +
  theme_bw()

p2 <- df_preds %>%
  ggplot(aes(x = prediction, y = residual)) +
  geom_point() +
  geom_hline(yintercept = 0, color = "orange", linetype = 2, linewidth = 1) +
  theme_bw()

p3 <- df_preds %>%
  ggplot(aes(x = residual)) +
  geom_histogram(fill = "midnightblue", color = "white") +
  theme_bw()


(p1 + p2) / p3
```



b

Linearity holds as shown below

$$
\begin{aligned}
E[\varepsilon|X] &= E[(X^2+1)\cdot(\chi_1^2-1) \mid X] \\
&= E[(X^2\cdot\chi_1^2+\chi_1^2-X^2-1 \mid X] \\
&= E[X^2\cdot\chi_1^2 \mid X]+E[\chi_1^2 \mid X]-E[X^2 \mid X]-E[1 \mid X] \\
&= X^2E[\chi_1^2 \mid X]+E[\chi_1^2 \mid X]-X^2E[1 \mid X]-E[1 \mid X] \\
&= X^2\cdot 1+1-X^2\cdot 1-1 \\
&= 0
\end{aligned}
$$

Normality certainly doesn't hold as the errors were generated by a process where we added one to the squared values of the covariates and multiplied these values by a chi-squared distribution, meaning that the errors cannot be normally distributed.

Equal Variance does not hold as $\text{var}(\varepsilon \mid X=x)$ varies with x as is shown below

$$
\begin{aligned}
\text{var}(\varepsilon \mid X = x) &= E[\varepsilon^2 \mid X = x] \\
&= E\Big[(X^2+1)^2(\chi_1^2 - 1)^2 \mid X = x\Big] \\
&= (x^2+1)^2 E\Big[(\chi_1^2)^2 - 2\chi_1^2 + 1 \mid X = x\Big] \\
&= (x^2+1)^2\Big(E\Big[(\chi_1^2)^2 \mid X = x\Big] - 2E[\chi_1^2 \mid X = x] + 1\Big) \\
&= (x^2+1)^2\Big(E\Big[(\chi_1^2)^2 \mid X = x\Big] - 1\Big) \\
&= (x^2+1)^2\Big(\text{var}(\chi_1^2 \mid X = x) + \Big(E[\chi_1^2 \mid X = x]\Big)^2 - 1\Big) \\
&= (x^2+1)^2(2 + 1^2 - 1) \\
&= 2 \cdot (x^2+1)^2
\end{aligned}
$$

c

i

t confidence interval, using traditional linear regression inference

```
get_regression_table(mod) %>%
  filter(term == "x") %>%
  select(lower_ci, upper_ci) %>%
  as.numeric()
```

```
[1] 0.462 0.904
```

ii

percentile bootstrap confidence interval (1000 bootstrap replicates)

```
set.seed(100)
df_list <- list()
for(i in 1:1000){
  data <- df %>%
    slice_sample(n = 1000, replace = T)

  df_list[[i]] <- data
}
```

```r
get_coef <- function(data) {
  mod <- lm(y ~ x, data)
  return(coef(summary(mod))[2,1])
}

boot_coefs <- df_list %>%
  map_dbl(.f = get_coef)

quantile(boot_coefs, c(0.025, 0.975))
```

```
     2.5%     97.5% 
0.2405628 1.0597162 
```

iii

normal approximation confidence interval using the bootstrapped standard deviation (1000 bootstrap replicates)

```r
coef(summary(mod))[2, 1] + c(-1, 1)*qnorm(0.975)*sd(boot_coefs)
```

```
[1] 0.2872003 1.0795604
```

iv

normal approximation confidence interval using robust standard errors

```r
library(sandwich)

varBeta <- vcovHC(mod, type="HC")
se <- sqrt(diag(varBeta))[2]

coef(summary(mod))[2, 1] + c(-1, 1)*qnorm(0.975)*se
```

```
[1] 0.2900729 1.0766878
```

d

```
set.seed(4)

df_list <- list()

for(i in 1:1000) {
  df_list[[i]] <- list(
    df_10 = dgp(10),
    df_100 = dgp(100),
    df_1000 = dgp(1000)
  )
}

getter <- function(df){
  model <- lm(y ~ x, df)
  beta <- coef(summary(model))[2]
  trad_ci <- get_regression_table(model) %>%
    filter(term == "x") %>%
    select(lower_ci, upper_ci) %>%
    mutate(method = "trad")

  varBeta <- vcovHC(model, type="HC")
  se <- sqrt(diag(varBeta))[2]
  endpts <- coef(summary(model))[2, 1] + c(-1, 1)*qnorm(0.975)*se
  robust_ci <- tibble(
    lower_ci = endpts[1],
    upper_ci = endpts[2],
    method = "robust"
  )

  all_ci <- rbind(trad_ci, robust_ci)
  return(list(beta, all_ci))
}

res_list <- df_list %>%
  map(.f = ~map(.x, .f = getter))

coef_df <- data.frame()
ci_df <- data.frame()
```

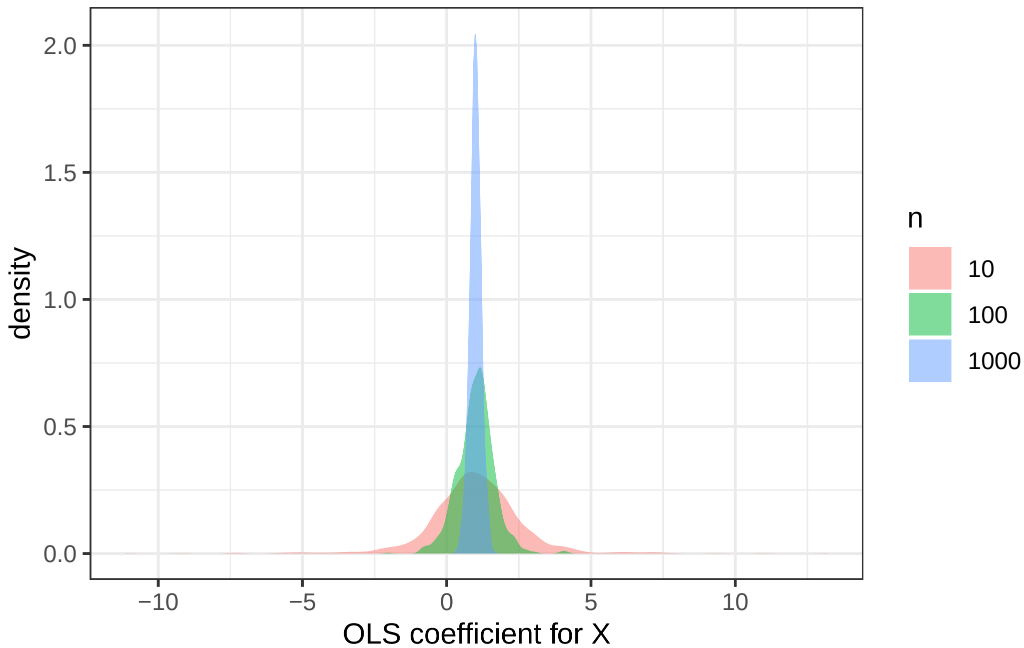
```r
for (i in 1:1000){

  coef_10 <- res_list[[i]]$df_10[[1]]
  coef_100 <- res_list[[i]]$df_100[[1]]
  coef_1000 <- res_list[[i]]$df_1000[[1]]

  ci_10 <- res_list[[i]]$df_10[[2]] %>%
    mutate(n = 10, iter = i)


  ci_100 <- res_list[[i]]$df_100[[2]] %>%
    mutate(n = 100, iter = i)

  ci_1000 <- res_list[[i]]$df_1000[[2]] %>%
    mutate(n = 1000, iter = i)

  res_ci <- rbind(ci_10, ci_100, ci_1000)

  res_coef <- tibble(
    n = c(10, 100, 1000),
    beta = c(coef_10, coef_100, coef_1000),
    iter = rep(i, 3)
  )

  ci_df <- rbind(ci_df, res_ci)
  coef_df <- rbind(coef_df, res_coef)

}
```

e

As we'd expect, the distributions have lower variance as n increases.

```r
library(here)
coef_df <- read_csv(here("data", "hw1_coef_df.csv"))
ci_df <- read_csv(here("data", "hw1_ci_df.csv"))

coef_df %>%
  ggplot(aes(x = beta, fill = factor(n))) +
  geom_density(color = NA, alpha = 0.5) +
```

```
  theme_bw() +
  labs(
    fill = "n",
    x = "OLS coefficient for X"
  )
```



Additionally, the OLS estimate was indeed unbiased. We would indeed expect this as we mathematically proved that linearity holds ($E[\varepsilon \mid X] = 0$)

```
coef_df %>%
  group_by(n) %>%
  summarise(mean = mean(beta))
```

```
# A tibble: 3 x 2
      n  mean
  <dbl> <dbl>
1    10 1.01
2   100 1.01
3  1000 0.996
```

f

The traditional confidence interal struggle mightily for all n sizes and this makes sense as the normality and equal variance assumptions are not met. The robust SE confidence intervals also had significant undercoverage for small sample sizes which makes sense because the squared residuals only "figure out" the variance structure of the errors in asymptopia and so we'd expect it to perform poorly with small samples. In contrast, it performs very well with the larger sample sizes, achieving essentially perfect coverage for samples of size 1000.

```r
ci_df %>%
  left_join(coef_df, by = c("n", "iter")) %>%
  rowwise() %>%
  mutate(contains = between(1, lower_ci, upper_ci)) %>%
  ungroup() %>%
  group_by(n, method) %>%
  summarise(coverage = mean(contains))
```

```
# A tibble: 6 x 3
# Groups:   n [3]
      n method coverage
  <dbl> <chr>     <dbl>
1    10 robust    0.745
2    10 trad      0.821
3   100 robust    0.928
4   100 trad      0.723
5  1000 robust    0.954
6  1000 trad      0.715
```