Q1) $f(n) = 2^{n+1} + n^4 - 64$

to be $O(2^n)$

$f(n) < c\,g(n)$ $\quad \exists N, c$ for all $n \geq N$

$2^{n+1} + n^4 - 64 \leq c \cdot 2^n$

$2 \cdot 2^n + n^4 - 64 \leq c \cdot 2^n$

$$2 + \frac{n^4}{2^n} - \frac{64}{2^n} \leq c$$

$c = 3 \quad N = 20$

it is also $\Theta(2^n)$

as

$$2 + \frac{n^4}{2^n} - \frac{64}{2^n} \geq c$$

for
$c = 2$
$n \geq N = 20$

(Q2) i) yes

ii) yes

iii) yes

iv) no

v) no

$n \le c n^{1+\cos(n)}$

since the exponent ranges $[0,2]$

it will always vary betwen greater

and less than n

Q3) i) $F_n \geq 2^{0.5n}$ for $n \geq 6$

if $n = 6$

$F_6 = 8 \geq 2^{0.5 \cdot 6} = 8$

let $k \in \mathbb{N}$

assume $F_k \geq 2^{0.5k}$

if true $k+1$ case must be true

$F_{k+1} \geq 2^{0.5(k+1)}$

$F_k + F_{k-1} \geq \sqrt{2} \cdot 2^{0.5k}$

$2^{0.5k} + 2^{0.5(k-1)} \geq \sqrt{2} \; 2^{0.5k}$

$\cancel{2^{0.5k}}\left(1 + \frac{\sqrt{2}}{2}\right) \geq \sqrt{2} \; \cancel{2^{0.5k}}$

$\left(1 + \frac{\sqrt{2}}{2}\right) \geq \sqrt{2}$

$\therefore F_k \geq 2^{0.5n}$ for $n \geq 6$

Q3 ii) we will use an intermediate step from the previous problem

assume

$F_n \leq 2^{cn}$ for all $n > 0$

$F_{n-1} + F_{n+2} \leq 2^{cn}$

$2^{c(n-1)} + 2^{c(n-2)} \leq 2^{cn}$

$2^{cn} - 2^{c(n-1)} - 2^{c(n-2)} \geq 0$

$2^{cn} - 2^{cn} \cdot 2^{-c} - 2^{cn} \cdot 2^{-2c} \geq 0 \quad \times \dfrac{2^{2c}}{2^{cn}}$ this value is positive

$2^{2c} - 2^{c} - 1 \geq 0$

let $\alpha = 2^{c}$

$\alpha^2 - \alpha - 1 \geq 0$

$\alpha_1 \geq \dfrac{1+\sqrt{5}}{2} \qquad \alpha_2 \leq \dfrac{1-\sqrt{5}}{2}$

$c = \log_2(\alpha_1)$ we reject negative α for real valued log

$c \approx 0.694$

iii) $F_n = \Omega(2^{cn})$

for this to be true

$\lim_{n \to \infty} \left| \frac{F_n}{g 2^{cn}} \right|$ is bounded from below

$F_n \geq g 2^{cn}$

$F_{n-1} + F_{n-2} \geq g 2^{cn}$

similarly

$2^{2c} - 2^{c} - 1 \leq 0$

and with similar substitutions

$c \in \left[0, \log_2\left(\frac{1+\sqrt{5}}{2}\right) \right]$

largest $c = \log_2\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.69$

Q4) i) Show $xy \equiv x' y'$ (mod N)

$x = qN + r_1 \quad y = pN + r_2$

$xy = N(qp + r_1 p + r_2 q) + r_1 r_2 \mod N$

$= r_1 r_2$

we show multiplication only depends on the remainder

similar steps can show the revereso for x', y'

since $x \equiv x' \pmod N = r_1$
$y \equiv y' \pmod N = r_2$

$xy = x'y' \pmod N$

ii) $3^k \mod 2$ by properties of mod

$= \underbrace{3 \cdot 3 \cdots 3}_{k \text{ times}} \mod 2$

$= \underbrace{3 \bmod 2 \times \cdots 3 \bmod 2}_{k \text{ times}}$

$= \quad (3 \bmod 2)^k \quad = \quad 1^k = \underline{1}$

4iii) $\quad 4^{500} \bmod 17$

$4^{4 \times 125} \bmod 17$

$256^{125} \bmod 17$ $\quad$ by previous result

$(256 \bmod 17)^{125} \bmod 17$

$1^{125} \bmod 17$

$= \underline{1}$

Q5) show sum of 3 num is at most 2 digit for base $b \geq 2$

The largest number that can be represented in 2 digits is $b^2 - 1$

the max value that can be represented by 1 digit is $b - 1$

we must show for $a, b, c = b - 1$

$a + b + c \leq b^2 - 1$

$b - 1 + b - 1 + b - 1 \leq b^2 - 1$

$3b - 3 \leq b^2 - 1$

$b^2 - 3b + 2 \geq 0$

$b \in (-\infty, 1] \cup [2, \infty)$

we reject lower boundary and have shown the inequality holds for $b \geq 2$ and so it is true

Q7)

$14x \equiv 7 \pmod{21}$

$x \equiv \dfrac{7 + 21k}{14}$ $k \in \mathbb{N}$

$x \equiv 0.5 + 1.5k$ this

will generate integer solutions for

odd values of k

$q = 2k+1$

$x = 0.5 + 1.5(2k+1)$

$x = 2 + 3k$

$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$x=2$	$x=5$	$x=8$	$x=11$	$x=14$
$k=5$	$k=6$			
$x=17$	$x=20$			

these are the unique solutions mod 21

the equation only has solutions

if for $ax \equiv b \bmod n$

$\gcd(a,n)$ divides b