SYDE 556/750

Simulating Neurobiological Systems
Lecture 6: Recurrent Dynamics

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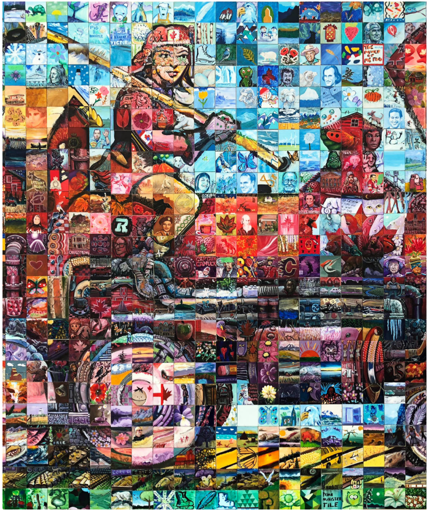
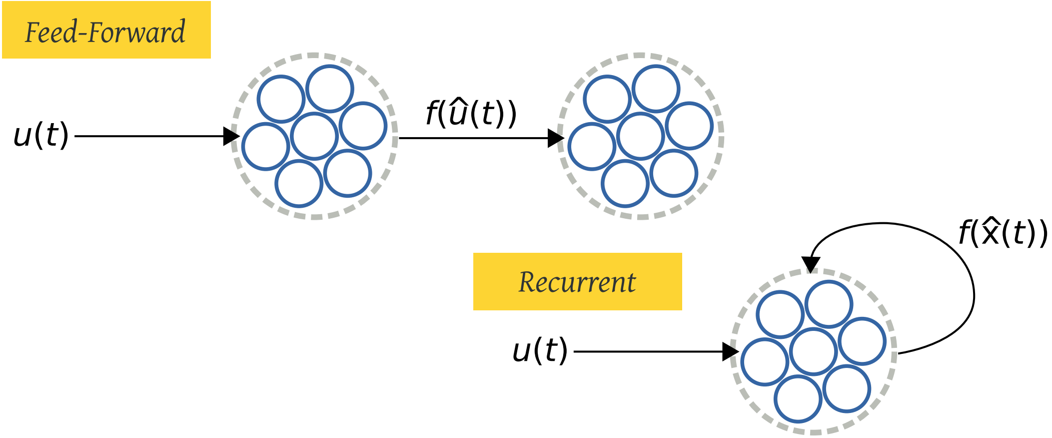
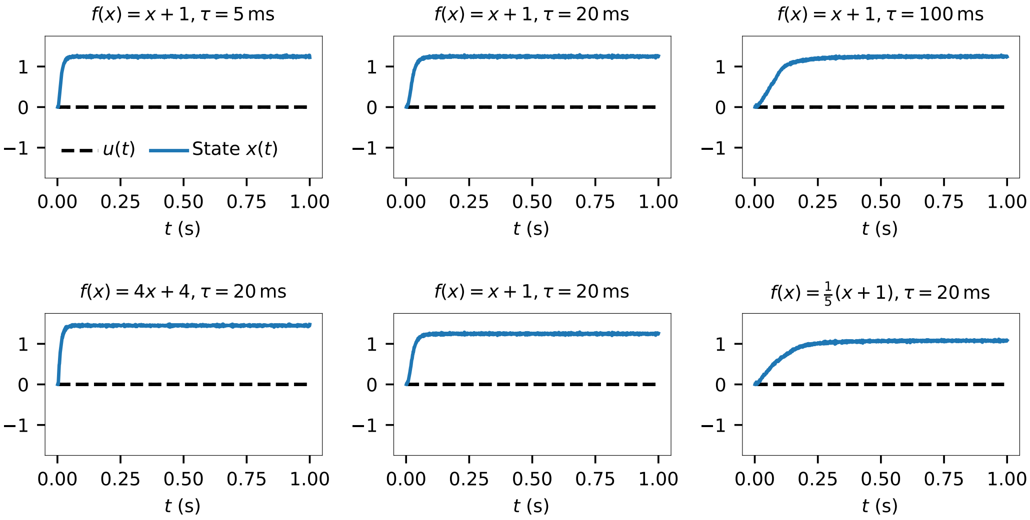


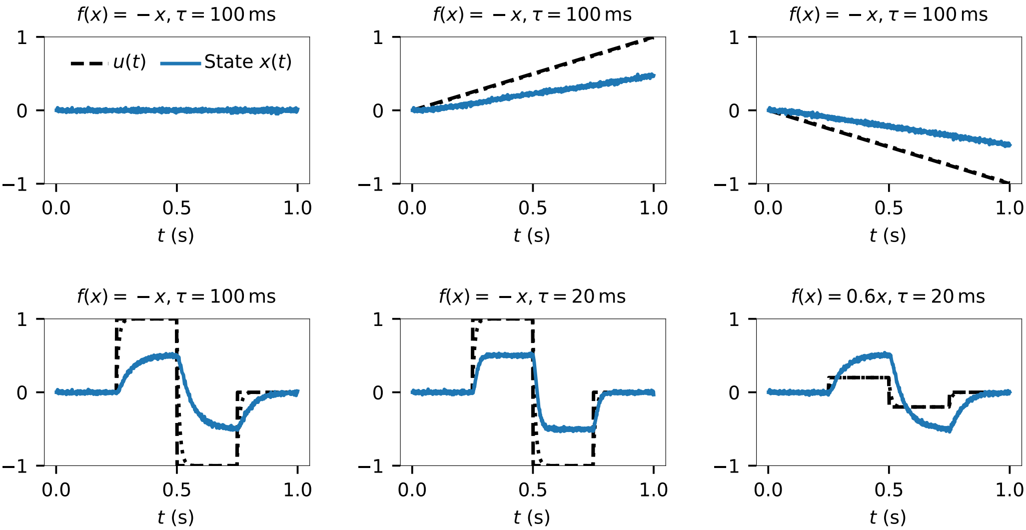
Feed Forward vs. Recurrent Connections



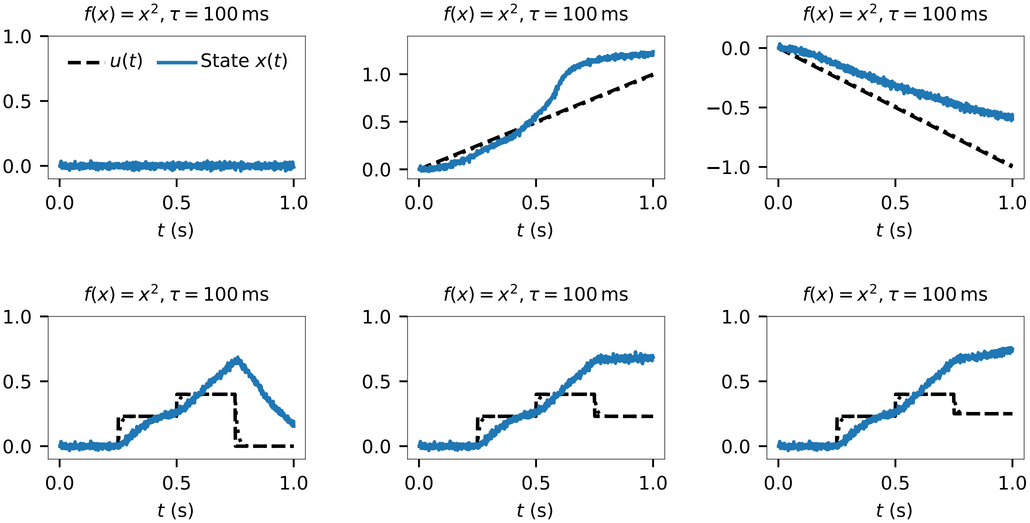
Recurrence Experiments (I)



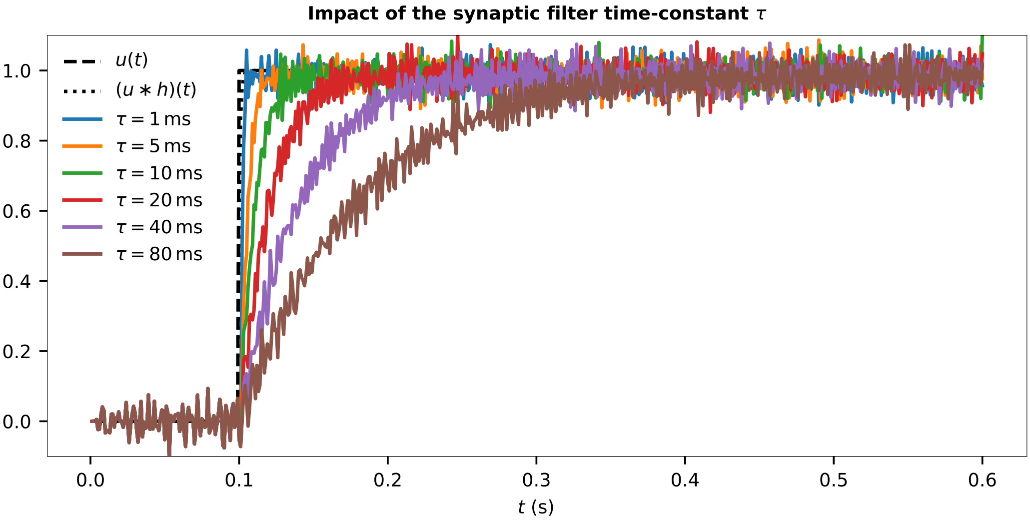
Recurrence Experiments (II)



Recurrence Experiments (III)



Making Sense of Dynamics



Behaviour of a Recurrent Connection

Feed-forward

$$x(t) = g(u(t)) * h(t)$$

Recurrent

$$\begin{aligned}
x(t) &= (g(u(t)) + f(x(t))) * h(t) \\
X(s) &= (G(s) + F(s))H(s) \\
X(s) &= (G(s) + F(s))\frac{1}{1 + s\tau} \\
X(s) + s\tau X(s) &= G(s) + F(s) \\
s\tau X(s) &= G(s) + F(s) - X(s) \\
sX(s) &= \frac{F(s) - X(s)}{\tau} + \frac{G(s)}{\tau} \\
\frac{dx}{dt} &= \frac{f(x(t)) - x(t)}{\tau} + \frac{g(u(t))}{\tau}
\end{aligned}$$

Behaviour of a Recurrent Connection

$$\frac{dx}{dt} = \frac{f(x(t)) - x(t)}{\tau} + \frac{g(u(t))}{\tau}$$

If we want this

$$\frac{dx}{dt} = a(x) + b(u)$$

Then we set

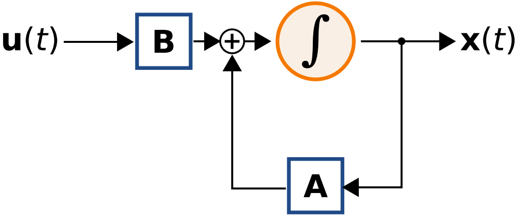
$$f(x) = \tau a(x) + x$$
$$g(x) = \tau b(u)$$

And we get

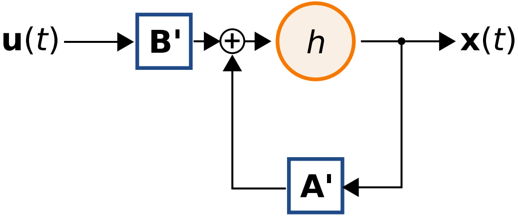
$$\frac{dx}{dt} = \frac{(\tau a(x) + x) - x}{\tau} + \frac{(\tau b(u))}{\tau}$$
$$\frac{dx}{dt} = a(x) + b(u)$$

Implementing Dynamics using a Neural Ensemble

Evaluating an LTI
using an Integrator



Evaluating an LTI
using a Synaptic Filter



Implementing Dynamical Systems as a Neural Ensemble

LTI System

$$
\begin{aligned}
\varphi(\mathbf{u}, \mathbf{x}) &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\varphi'(\mathbf{u}, \mathbf{x}) &= \mathbf{A}'\mathbf{x} + \mathbf{B}'\mathbf{u} \\
\mathbf{A}' &= \tau\mathbf{A} + \mathbf{I} \\
\mathbf{B}' &= \tau\mathbf{B}\,.
\end{aligned}
$$

Additive Time-Invariant System

$$
\begin{aligned}
\varphi(\mathbf{u}, \mathbf{x}) &= f(\mathbf{x}) + g(\mathbf{u}) \\
\varphi'(\mathbf{u}, \mathbf{x}) &= f'(\mathbf{x}) + g'(\mathbf{u}) \\
f'(\mathbf{x}) &= \tau f(\mathbf{x}) + \mathbf{x} \\
g'(\mathbf{u}) &= \tau g(\mathbf{u})
\end{aligned}
$$

“General” Recipe

Scale the original dynamics by τ, add feedback $\mathbf{x}$

NEF Principle 3: Dynamics

Time-Invariant Dynamical System

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = f(\mathbf{x}(t), \mathbf{u}(t))$$

Linear Time-Invariant (LTI) Dynamical System

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

NEF Principle 3 – Dynamics
Neural dynamics are characterized by considering neural representations as control theoretic state variables. We can use control theory (and dynamical systems theory) to analyse and construct these systems.

Low-pass Filter Example

Desired behaviour

$$\frac{dx}{dt} = \frac{u - x}{\tau_{desired}}$$

Feed-forward input

$$g(u) = \tau_{synapse}(\frac{u}{\tau_{desired}})$$

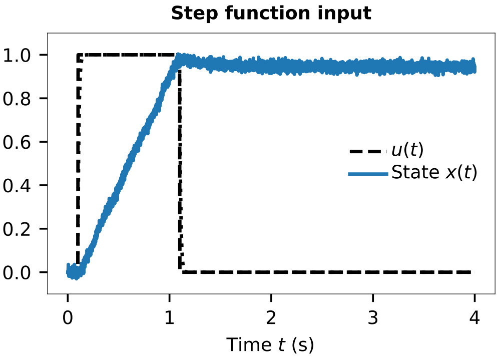
$$g(u) = \frac{\tau_{synapse}}{\tau_{desired}} u$$

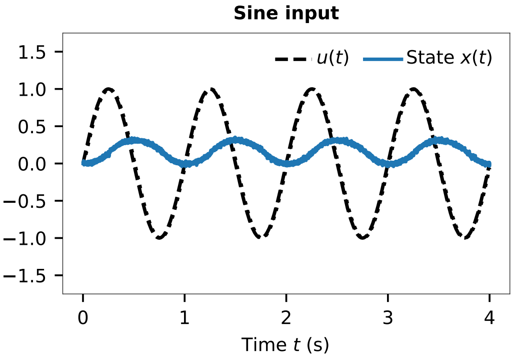
Recurrent

$$f(x) = \tau_{synapse}(\frac{-x}{\tau_{desired}}) + x$$

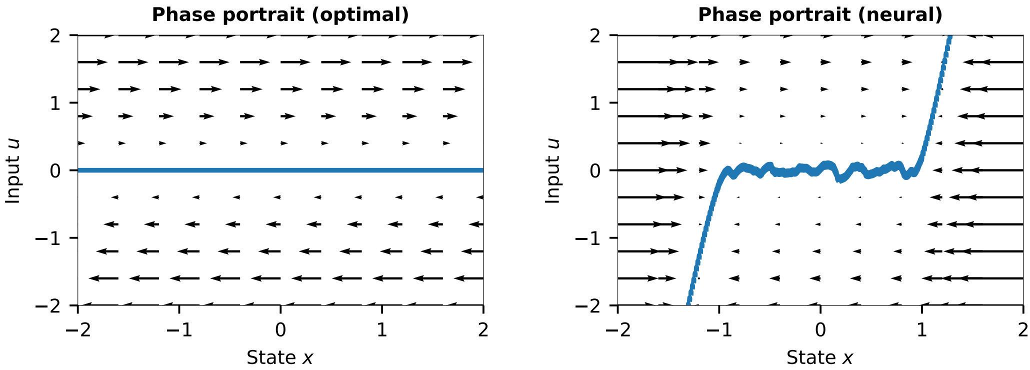
$$f(x) = (1 - \frac{\tau_{synapse}}{\tau_{desired}})x$$

Integrator Example (I)

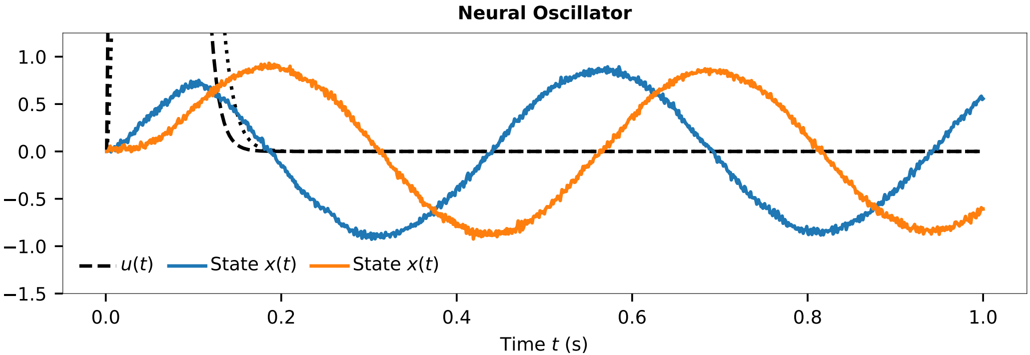




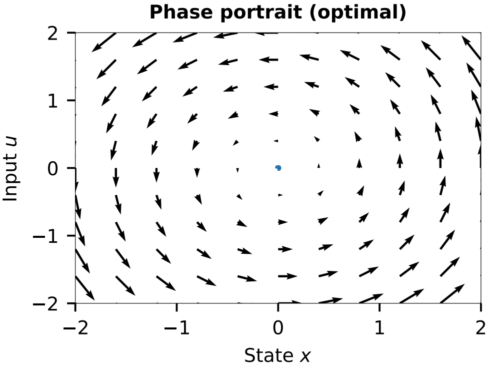
Integrator Example (II)

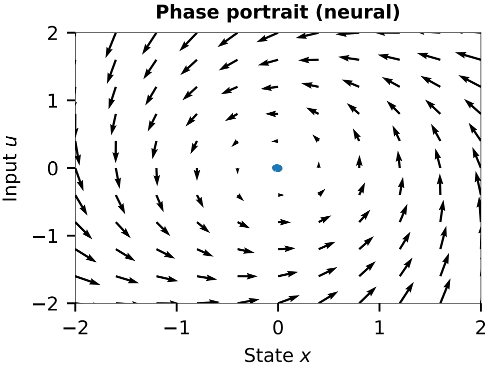


Oscillator Example (I)

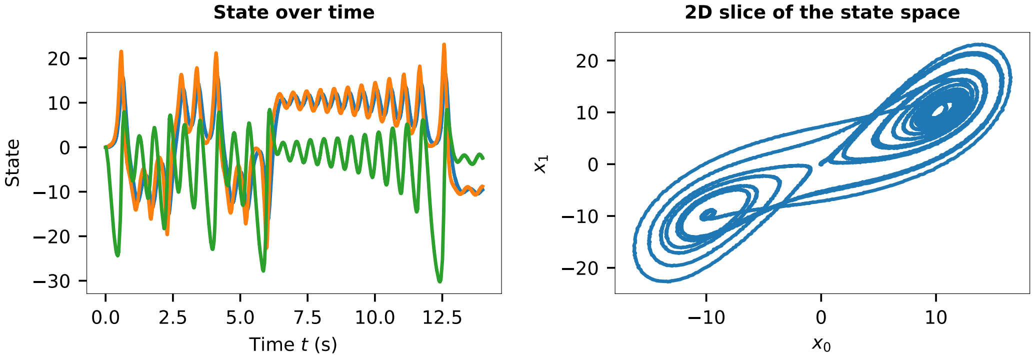


Oscillator Example (II)



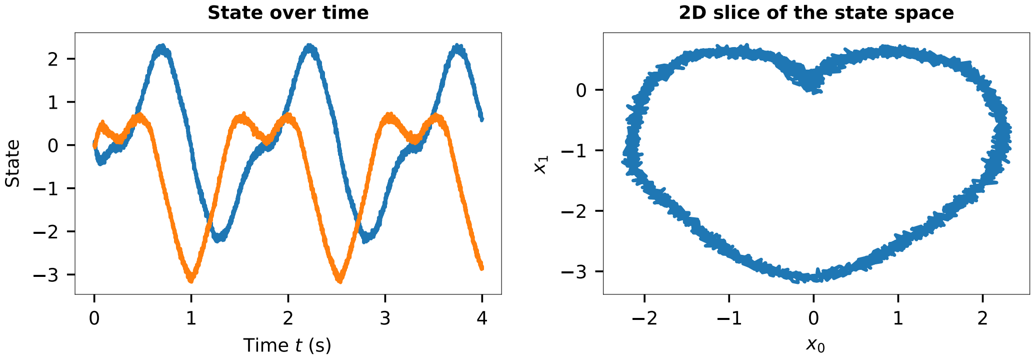


Lorentz Attractor



$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} 10x_2(t) - 10x_1(t) \\ -x_1(t)x_3(t) - x_2(t) \\ x_1(t)x_2(t) - \frac{8}{3}(x_3(t) + 28) - 28 \end{pmatrix}$$

Heart Shape



State over time
State
2
1
0
−1
−2
−3
0
1
2
3
4
Time t (s)
2D slice of the state space
x_1
0
−1
−2
−3
−2
−1
0
1
2
x_0

Horizontal Eye Control

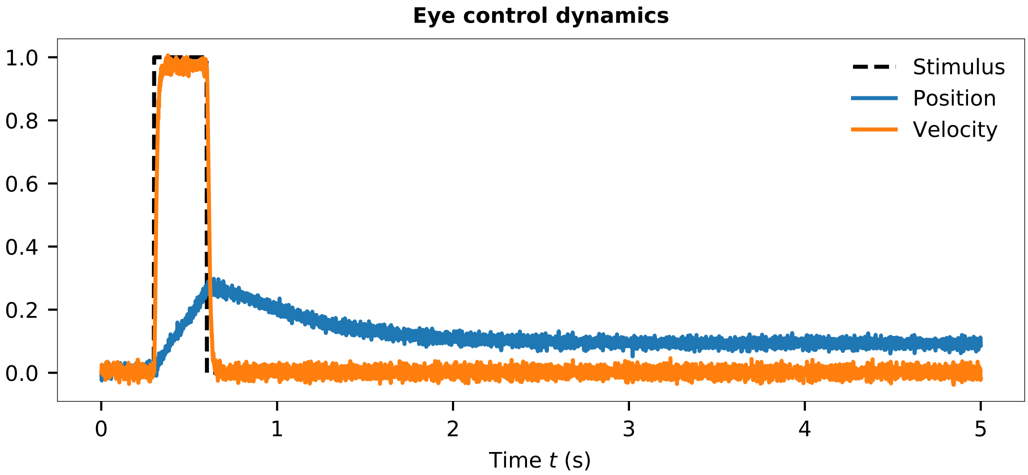


Image sources

Title slide
"The Canada 150 Mosaic Mural"
Author: Mosaic Canada Murals.
From Wikimedia.