SYDE 556/750

**Simulating Neurobiological Systems
Lecture 6: Recurrent Dynamics**

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- ▶ Slide design: Andreas Stöckel
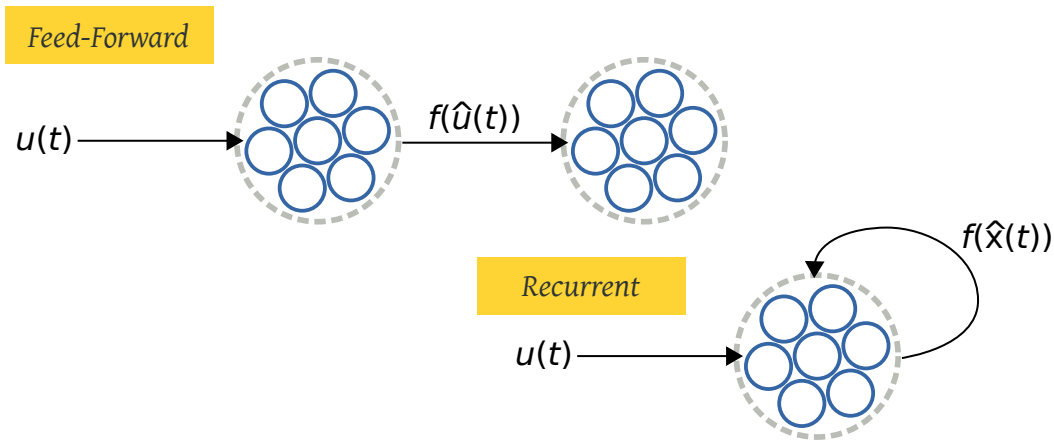- ▶ Content: Terry Stewart, Andreas Stöckel, Chris Eliasmith
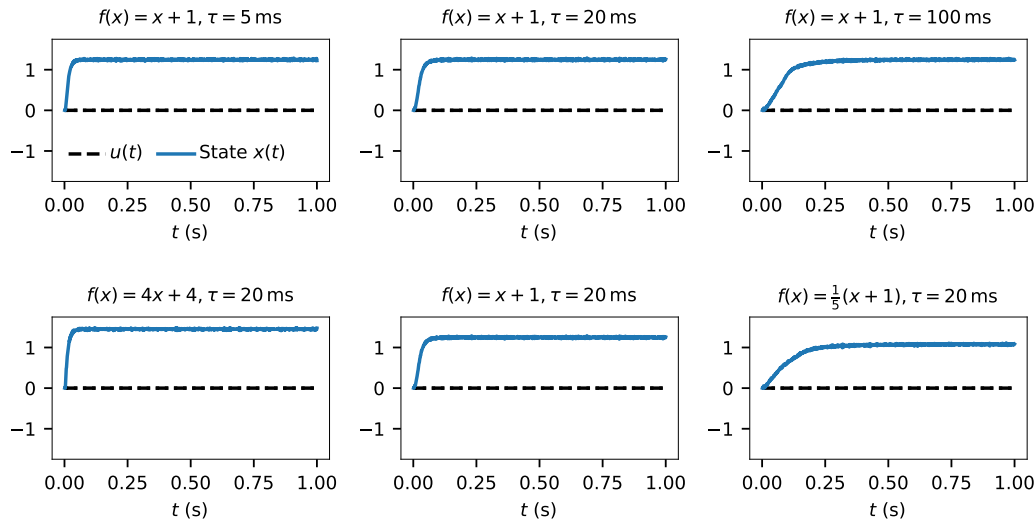
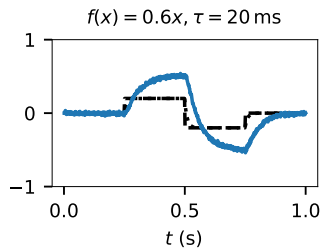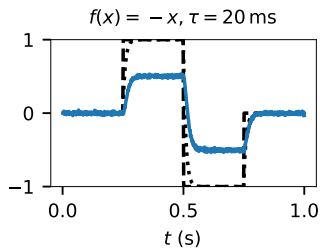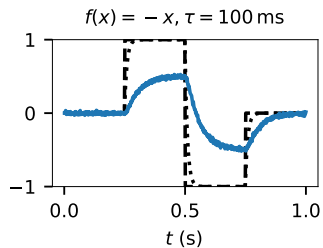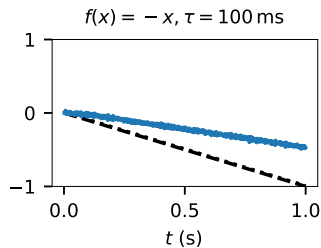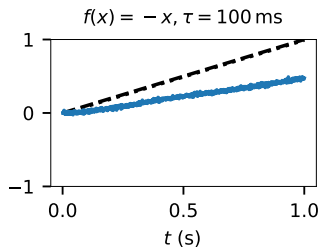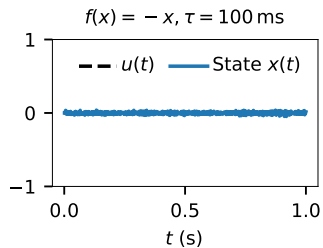UNIVERSITY OF **WATERLOO** | **FACULTY OF ENGINEERING**

Feed Forward vs. Recurrent Connections

Feed-Forward

$u(t)$ ⟶ $f(\hat{u}(t))$ ⟶

Recurrent

$u(t)$ ⟶ $f(\hat{x}(t))$

Recurrence Experiments (I)

Recurrence Experiments (II)

Recurrence Experiments (III)
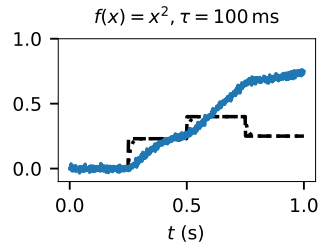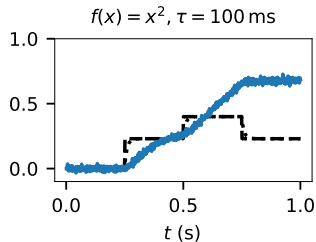
Making Sense of Dynamics

Impact of the synaptic filter time-constant τ

Legend:
- $u(t)$
- $(u * h)(t)$
- $\tau = 1\,\mathrm{ms}$
- $\tau = 5\,\mathrm{ms}$
- $\tau = 10\,\mathrm{ms}$
- $\tau = 20\,\mathrm{ms}$
- $\tau = 40\,\mathrm{ms}$
- $\tau = 80\,\mathrm{ms}$

t (s)
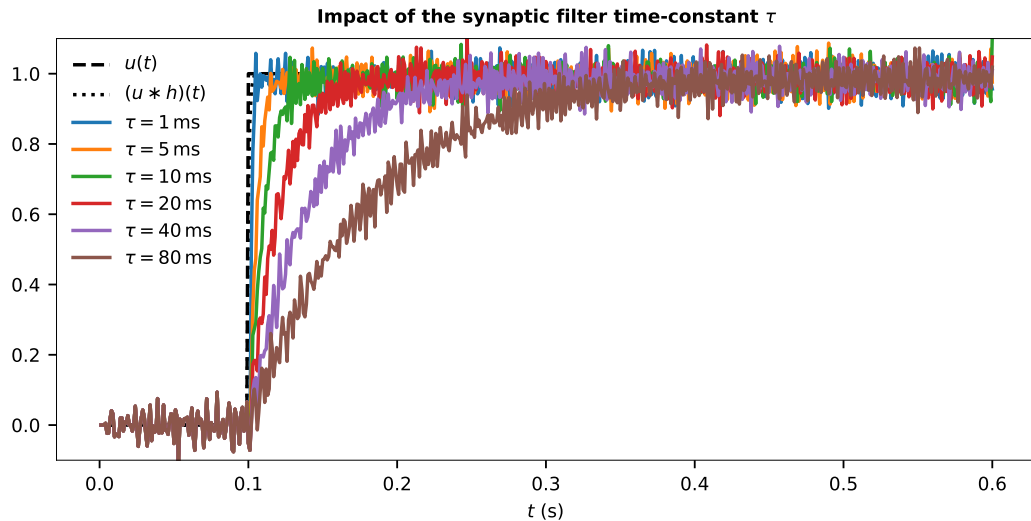
Behaviour of a Recurrent Connection
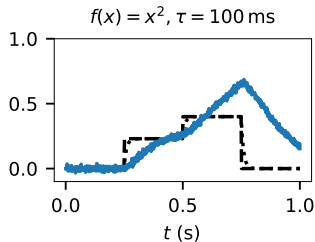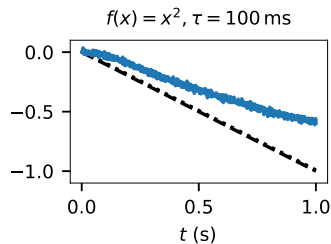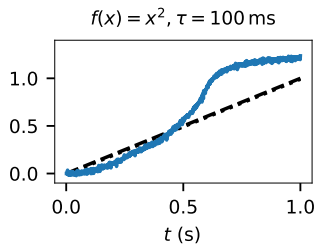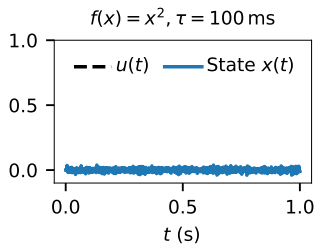
Feed-forward

$$x(t) = g(u(t)) * h(t)$$

Recurrent

$$x(t) = (g(u(t)) + f(x(t))) * h(t)$$
$$X(s) = (G(s) + F(s))H(s)$$
$$X(s) = (G(s) + F(s))\frac{1}{1 + s\tau}$$
$$X(s) + s\tau X(s) = G(s) + F(s)$$
$$s\tau X(s) = G(s) + F(s) - X(s)$$
$$sX(s) = \frac{F(s) - X(s)}{\tau} + \frac{G(s)}{\tau}$$
$$\frac{dx}{dt} = \frac{f(x(t)) - x(t)}{\tau} + \frac{g(u(t))}{\tau}$$

Behaviour of a Recurrent Connection

$$\frac{dx}{dt} = \frac{f(x(t)) - x(t)}{\tau} + \frac{g(u(t))}{\tau}$$

If we want this

$$\frac{dx}{dt} = a(x) + b(u)$$

Then we set

$$f(x) = \tau a(x) + x$$
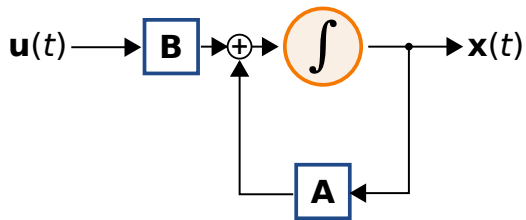$$g(x) = \tau b(u)$$

And we get

$$\frac{dx}{dt} = \frac{(\tau a(x) + x) - x}{\tau} + \frac{(\tau b(u))}{\tau}$$
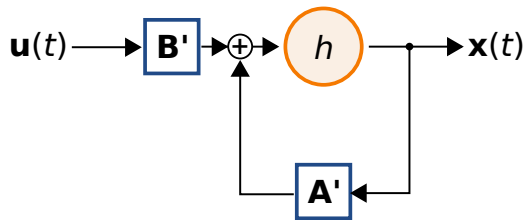$$\frac{dx}{dt} = a(x) + b(u)$$

Implementing Dynamics using a Neural Ensemble

Evaluating an LTI
using an Integrator

Evaluating an LTI
using a Synaptic Filter

Implementing Dynamical Systems as a Neural Ensemble

LTI System

$$\varphi(\mathbf{u}, \mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\varphi'(\mathbf{u}, \mathbf{x}) = \mathbf{A}'\mathbf{x} + \mathbf{B}'\mathbf{u}$$
$$\mathbf{A}' = \tau\mathbf{A} + \mathbf{I}$$
$$\mathbf{B}' = \tau\mathbf{B}\,.$$

Additive Time-Invariant System

$$\varphi(\mathbf{u}, \mathbf{x}) = f(\mathbf{x}) + g(\mathbf{u})$$
$$\varphi'(\mathbf{u}, \mathbf{x}) = f'(\mathbf{x}) + g'(\mathbf{u})$$
$$f'(\mathbf{x}) = \tau f(\mathbf{x}) + \mathbf{x}$$
$$g'(\mathbf{u}) = \tau g(\mathbf{u})$$

"General" Recipe

Scale the original dynamics by τ, add feedback $\mathbf{x}$

NEF Principle 3: Dynamics

Time-Invariant Dynamical System

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = f(\mathbf{x}(t), \mathbf{u}(t))$$

Linear Time-Invariant (LTI) Dynamical System

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

NEF Principle 3 – Dynamics
Neural dynamics are characterized by considering neural representations as control theoretic state variables. We can use control theory (and dynamical systems theory) to analyse and construct these systems.

Low-pass Filter Example

Desired behaviour

$$\frac{dx}{dt} = \frac{u - x}{\tau_{desired}}$$

Feed-forward input

$$g(u) = \tau_{synapse}(\frac{u}{\tau_{desired}})$$

$$g(u) = \frac{\tau_{synapse}}{\tau_{desired}} u$$

Recurrent
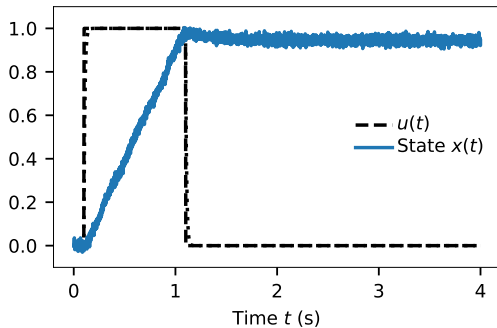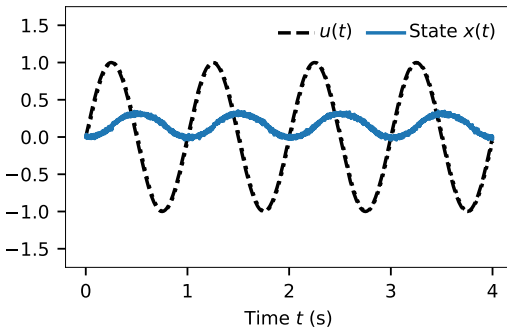
$$f(x) = \tau_{synapse}(\frac{-x}{\tau_{desired}}) + x$$

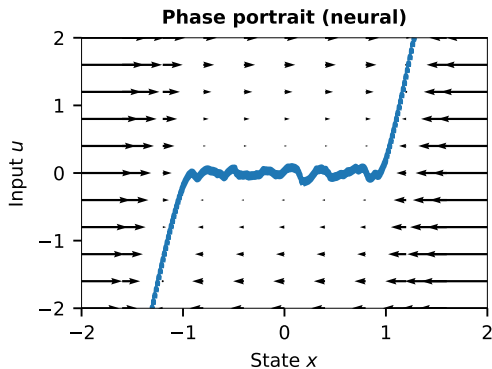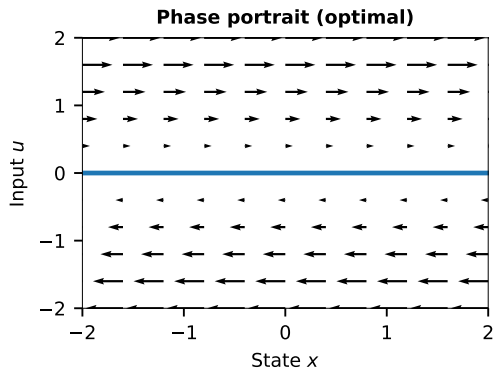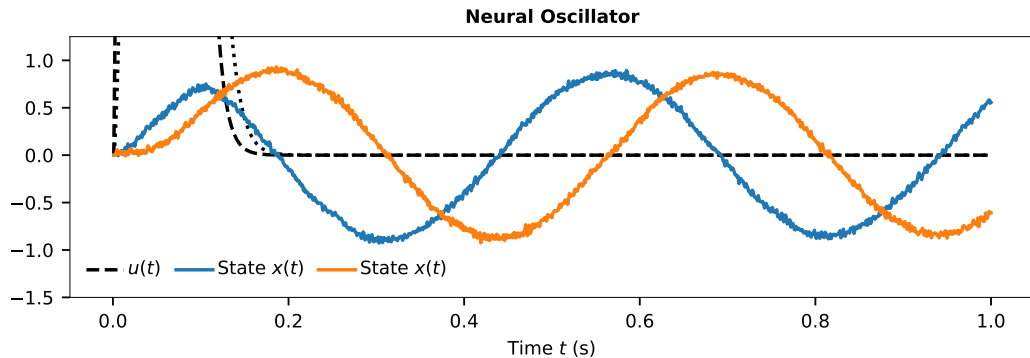$$f(x) = (1 - \frac{\tau_{synapse}}{\tau_{desired}})x$$

Integrator Example (I)
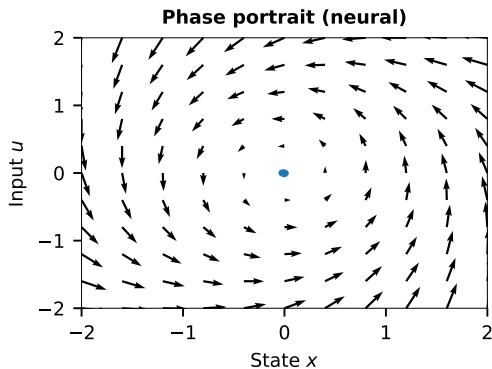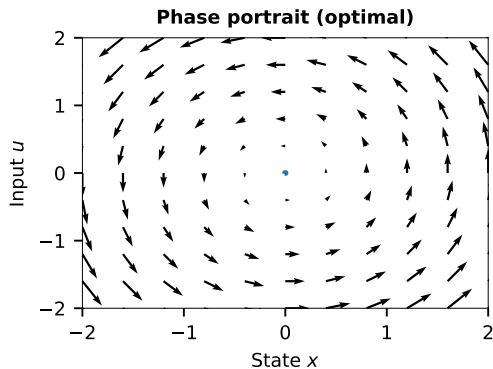
Integrator Example (II)
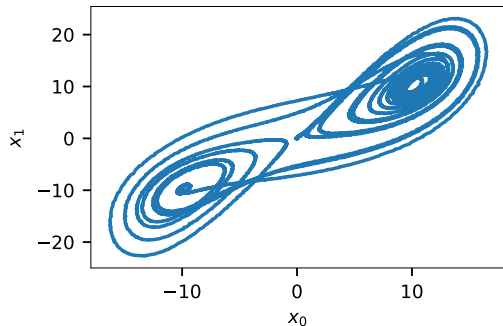
Oscillator Example (I)

Oscillator Example (II)
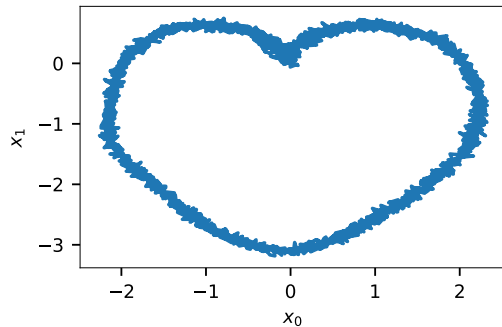
Lorentz Attractor
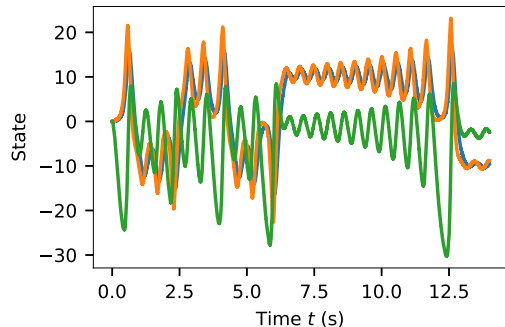
$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} 10x_2(t) - 10x_1(t) \\ -x_1(t)x_3(t) - x_2(t) \\ x_1(t)x_2(t) - \frac{8}{3}(x_3(t) + 28) - 28 \end{pmatrix}$$
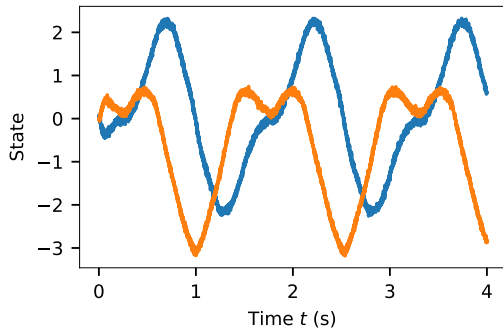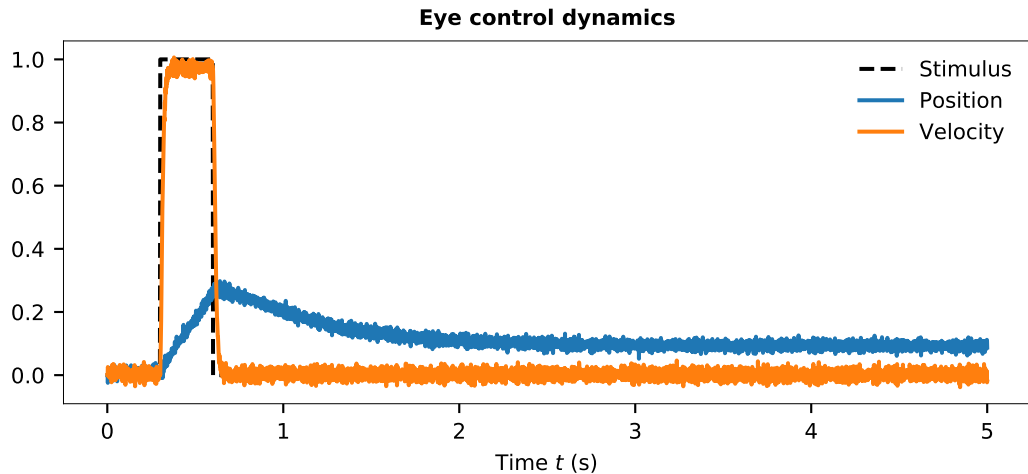
Heart Shape

Horizontal Eye Control

Eye control dynamics

Image sources

Title slide
"The Canada 150 Mosaic Mural"
Author: Mosaic Canada Murals.
From Wikimedia.