Versatile Mathematics

COMMON MATHEMATICAL APPLICATIONS

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Attributions This book benefited tremendously from others who went before and freely shared their creative work. The following is a short list of those whom we have to thank for their work and their generosity in contributing to the free and open sharing of knowledge.

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  OpenStax College, *Introductory Statistics*. OpenStax College. 19 September 2013. <http://cnx.org/content/col11562/latest/>
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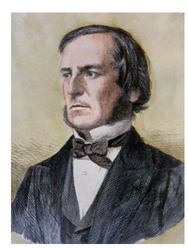
Chapter

1 Logic

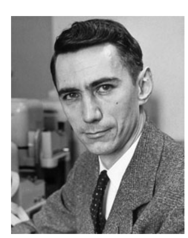
At its heart, every computer circuit—like the memory chip shown above—runs on relatively simple rules of logic. A 19th-century English mathematician named George Boole described a set of rules for abstract logic that seemed to have little practical significance at the time. Nearly a hundred years later, though, a young American student named Claude Shannon, a master's candidate at MIT, noticed that Boole's algebra could be applied to analyzing circuits, leading to tremendous advances in this new field. Today, we rely heavily on computers, and it is intriguing to peer behind the curtain a bit and see how they operate.



George Boole

Computers recognize two states, often written 1 and 0 (or ON and OFF, or TRUE and FALSE). These two states correspond to a high voltage and a low voltage, respectively; early computers used vacuum tubes to represent these states, but the transistor, invented in 1947 at Bell Labs, replaced the vacuum tube as a cheaper, smaller, more reliable alternative.

In this chapter, we will study the fundamentals of logic. We will use values of T and F to represent true and false statements, but everything that we will consider can be applied to computer circuits by simply substituting 1 for T and 0 for F. We will learn how to translate statements in words into symbolic form and how to manipulate that symbolic form. Finally, we will consider complete arguments, which are series of statements that lead to conclusions. We will test whether various arguments are valid or not, and in doing so, we will begin to see the importance of being careful when making an argument.



Claude Shannon

SECTION 1.1 Statements and Logical Operations

To begin our study of logic, we must first define our most basic unit, the **statement**; we'll consider whether specific statements are true or not, and we'll combine statements to form compound statements and ultimately arguments.

The following sentences are statements:

It rained yesterday.
The Packers will win the Super Bowl this year.
Elephants are afraid of mice.

Statements

A **statement** is a claim that is either true or false, but not both.

Notice that it doesn't matter for the definition whether we know if a statement is true or false; it simply has to be one or the other.

What, then, is not a statement? The following are a few examples:

Get some milk at the store.
Why is the sky blue?
This sentence is false.

The first is a command and the second is a question, neither of which have any claim to being true or false. It doesn't make sense to talk about truth values with them.

The third is an interesting one. At first, it looks like a statement, because it makes a claim; however, upon closer inspection, we see that it isn't either true or false. This is known as the *liar's paradox*: if it were true, it would be false, which would mean it is true, which would mean...

In this chapter, we will commonly represent statements with letters to shorten the amount we have to write. For instance, we might define[1] p and q as

p: Today is the longest day of the year.
q: Tomorrow is the shortest day of the year.

That way, if we want to combine these statements, we don't have to write them out over and over again; we can simply write p and q and save a lot of trouble.

In the coming sections, we'll spend a lot of time combining statements in various ways, and it can be easy to get lost in the notation and think of logic as some sort of abstract study of notation. Resist this temptation. Instead, always remember that the letters and symbols we use and manipulate are tied to simple statements like these examples. With every new operation we investigate, our goal is to not just focus on memorizing a list of symbols, but rather to understand why this operation makes sense.

Boolean Logic with Sets

Three basic Boolean operators:

1. And
2. Or
3. Not

Every time you use Google to search the Internet, the search engine turns to Boolean logic to narrow down the search results to give you what you're looking for, using key terms like "and," "or," and "not."

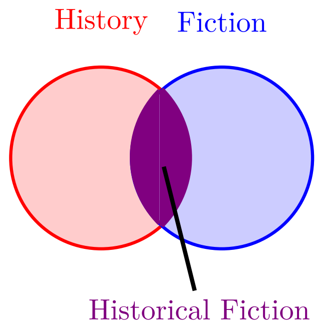
For instance, if you searched for "Frederick," the search engine would pull up results that contain that word (like the city tourism website, the Wikipedia article on the city, maybe the article for Frederick Douglass, etc.). If you searched for "Frederick Community College," the search engine would look for pages that contain those three words; it would search for "Frederick" AND "Community" AND "College."

If instead you search for "Frederick Community −College," the search engine will search for pages that contain "Frederick" AND "Community" but NOT "College," so the FCC website will not be listed.

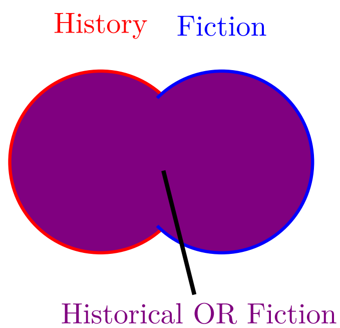
We can think of objects as belonging to sets; in this case, performing a search means looking for objects that belong to a specific list of sets. For instance, if you went to library and searched "History" AND "Fiction," you would find all the books that belong both to the History set and to the Fiction set (i.e. historical fiction).

[1] We typically start with p for *proposition*, another name for a statement

As it turns out, we can deal with logic using either statements or sets. When working with statements, Boolean logic combines multiple statements that are either true or false into a compound statement that is either true or false. When working with sets, Boolean logic combines sets, and a search is considered “true” when it returns an object in that combination of sets.



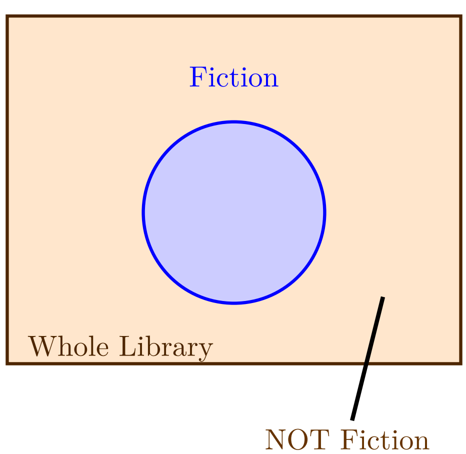
We can draw similar diagrams for the other two basic Boolean operations. For instance, if we searched for “History” OR “Fiction,” the search would return any books that are either historical or fiction, or both.



This is what is called the *inclusive* OR; it includes all values in the first set, plus all values in the second set, plus all values in the overlap. There is also an *exclusive* OR operation (that we won’t consider) that includes all values in the first set and all values in the second set, but *not* the values in the overlap. It is equivalent to saying “Either history or fiction, but not both.”

The inclusive OR is like the following question at a restaurant: “Would you like fries or a drink with that?” It is perfectly acceptable to respond “Both, please.”

Finally, if we searched for NOT “Fiction,” the search[2] would return everything in the database except for those labeled “Fiction.”



Diagrams like these will helpful to visualize combining statements using these three operations.

[2]For obvious reasons, Google gives an error if you only search for NOT something

EXAMPLE 1 SEARCH LOGIC

Suppose we are searching for universities in Mexico. Express a reasonable search using Boolean logic.

Solution

We could search for "Mexico AND university," but we would be likely to get results for universities in New Mexico (in the U.S.). To account for this, we could revise our search terms:

Mexico AND university NOT "New Mexico"

The quotes around New Mexico group those words together, so that the search engine will know that we want to exclude that *string*.

Also, in most search engines, including the word AND is unnecessary, since the search engines that if you provide two keywords, you want results that include both keywords. In Google, a negative sign in front of a terms is used to indicate NOT.

Thus, in Google, this search would be

Mexico university −"New Mexico"

EXAMPLE 2 DESCRIBING A SET

Describe the set of numbers that meet the following conditions:

Even numbers that are greater than 12 and less than 20.

Solution

We could describe this as "numbers that are even" AND "numbers that are greater than 12" AND "numbers that are less than 20," so we're looking for the overlap for these three sets.

The numbers that fit this description are

$$\boxed{\{14, 16, 18\}}$$

Boolean Logic with Statements

We can use these same three operators with statements; instead of thinking of objects belonging to sets, we'll think about whether statements are true or false.

For instance, consider the following two statements.

p: Greece belongs to the European Union.
q: Tokyo is in Iran.

Clearly, p is true and q is false. What about p AND q? In words, this would be the compound statement "Greece belongs to the European Union, and Tokyo is in Iran."

Combining Statements with AND

The compound statement p AND q is only true if p and q are both true; if either one is false, that makes the compound statement false.

In this example, the fact that q is false makes the compound statement break down; just because the first half is true isn't enough.

On the other hand, what if we wrote "Greece belongs to the European Union or Tokyo is in Iran"? In this case, the fact that the first half is true is enough to satisfy the compound statement.

Combining Statements with OR

The compound statement p OR q is true if at least one of p and q are true.

Therefore, p OR q is true, simply because p is true.

Finally, what about NOT, the last of the three basic Boolean operators? What if we wrote "Greece does not belong to the European Union"? This statement would be false, because it is the opposite of a true statement. Conversely, "Tokyo is not in Iran" would be true, because it is the opposite of a false statement.

Negating Statements with NOT

If p is true, then NOT p is false; if p is false, then NOT p is true.

Notation

Fair warning: the notation can be frighteningly unfamiliar at first. Just remember that the ideas are the same as the ones we've already seen; the notation is simply a shorthand that we can use to write compound statements more concisely.

Shorthand Notation for Basic Boolean Operators

AND: $\wedge$

OR: $\vee$

NOT: $\sim$ **or** $\neg$

To remember the difference between $\wedge$ (AND) and $\vee$ (OR), you can think of the $\wedge$ as looking like the "A" in AND. A poor trick, admittedly, but as we do examples you should grow more comfortable with these symbols.

TRANSLATING TO SYMBOLIC FORM — EXAMPLE 3

Let p and q represent the following simple statements:

p : There is life on Mars.
q : There is life on Europa.

Write each of the following statements in symbolic form:

(a) "There is life on both Mars and Europa."

(b) "There is life on neither Mars nor Europa."

(c) "There is life on Mars, but not on Europa."

(d) "There is either life on Mars or no life on Europa."

Solution

(a) This is equivalent to saying "there is life on Mars" AND "there is life on Europa."

$$p \wedge q$$

(b) This is equivalent to saying "there is NOT life on Mars" AND "there is NOT life on Europa."

$$\sim p \wedge \sim q$$

(c) This is equivalent to saying "there is life on Mars" AND "there is NOT life on Europa."

$$p \wedge \sim q$$

(d) This is equivalent to saying "there is life on Mars" OR "there is NOT life on Europa."

$$p \vee \sim q$$

TRY IT

Let p and q represent the following simple statements:

p : We landed a man on the moon.
q : The manned lunar program is dead.

Write each of the following statements in symbolic form:

(a) "We never landed a man on the moon, and in any case the manned lunar program is dead."

(b) "Even though we never landed a man on the moon, the manned lunar program is still alive."

(c) "We landed a man on the moon and the manned lunar program is alive."

(d) "Even though we landed a man on the moon, the manned lunar program is dead."

EXAMPLE 4 TRANSLATING FROM SYMBOLIC FORM

Let p and q represent the following simple statements:

p : Banana bread is delicious.
q : Bacon is delicious.

Write each of the following statements in words:

(a) $p \vee q$

(b) $\sim p \vee q$

(c) $\sim (p \wedge q)$

(d) $\sim p \vee \sim q$

Solution

(a) "Banana bread is delicious or bacon is delicious."

(b) "Either banana bread is not delicious, or bacon is delicious."

(c) "It's not true that both banana bread and bacon are delicious."

(d) "Either banana bread is not delicious or bacon is not delicious."

Notice that the statements in (c) and (d) are logically equivalent; we'll prove this later (it's one of De Morgan's laws).

TRY IT

Let p and q represent the following simple statements:

$$p : 1 + 4 < 5$$
$$q : 1 + 4 = 5$$

Which of the following statements corresponds to $\sim p \wedge \sim q$?

(a) $1 + 4 > 5$

(b) $1 + 4 \geq 5$

(c) $1 + 4 \neq 5$

Truth Tables

Let's summarize what we know about these three operators. We'll begin with the AND operator, and after we've seen how to handle it, we'll use a similar approach with the OR and NOT operators.

AND The combination of two statements using AND is true when both statements are true; otherwise it is false.

1. If p is true and q is true, then $p \wedge q$ is true:

p	q	$p \wedge q$
T	T	T

2. If p is true and q is false, then $p \wedge q$ is false:

p	q	$p \wedge q$
T	F	F

3. If p is false and q is true, then $p \wedge q$ is false:

p	q	$p \wedge q$
F	T	F

4. If p is false and q is false, then $p \wedge q$ is false:

p	q	$p \wedge q$
F	F	F

These are the only four possible combinations of truth values for p and q. We can summarize these four possibilities in a single table; this is called a **truth table**, as shown below.

A computer scientist might write this table using 1 and 0 instead of T and F:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

A truth table handles all the possibilities at once; it lists every combination of whether p and q are true or false, and for each of them, it lists whether the compound statement is true or false. Notice that once we have this table, if we know the truth values of p and q, we can look up the appropriate row in the table to find whether $p \wedge q$ is true or false.

For instance, if p is "All fish are purple" and q is "North America is in the Western Hemisphere," we would look at the row where p is false and q is true–the third row–and note that the compound statement "All fish are purple and North America is in the Western Hemisphere" is therefore false.

Truth tables will be especially useful once we begin to look at more complicated compound statements. If the compound statement is some combination of two statements p and q, the first two columns will be the same as the first two columns shown here, and we'll slowly build up to the final compound statement by systematically adding columns. That's all we'll say for now; it'll make more sense later when we see actual examples.

Truth Tables

A truth table lists all the possible combinations of one or more simple statements (p, q, r, etc.) and calculates the results of applying one or more operation(s) to these simple statements.

When building a truth table with one statement, there will only be two rows:

p	...
T	...
F	...

If we work with two statements, there are four possible combinations of true and false:

p	q	...
T	T	...
T	F	...
F	T	...
F	F	...

In general, if we're working with n statements, we'll need 2^n rows

If we have three statements, we'll have to double the number of rows; we'll need to include the four rows above when r is true, and include them again when r is false:

p	q	r	...
T	T	T	...
T	F	T	...
F	T	T	...
F	F	T	...
T	T	F	...
T	F	F	...
F	T	F	...
F	F	F	...

No matter how we order the rows, we need to account for these eight possibilities.

Now let's look at the truth tables for the other two operations.

OR The combination of two statements using OR is true if at least one of the statements is true; it is only false if both of them are false. The truth table below summarizes this.

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

NOT The negation of a statement is true if the original statement is false; it is false if the original statement is true.

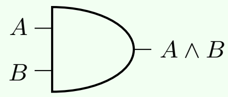
Truth Table for $\sim p$

p	$\sim p$
T	F
F	T

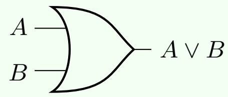
Sidenote: Logic Gates

In circuit design, these three operations are drawn as "gates," or boxes that accept one or two inputs and give the appropriate output in each case. The diagram for each operation is shown below.

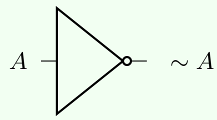
AND



OR

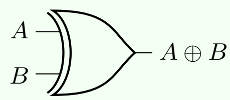


NOT

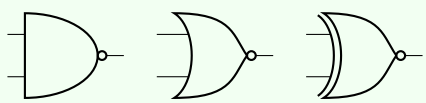


There is a fourth common gate, called the *exclusive OR*, which returns true if one input or the other is true, but not if both are true. This is often called XOR (symbolized by $\oplus$), and represented with the following block.

XOR

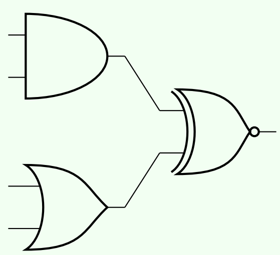


Each of these gates can be negated, which simply inverts all of the truth values. On the diagrams, this is represented with a small "bubble" at the end of the gate.



NAND NOR XNOR

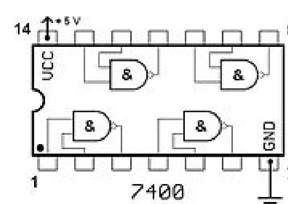
Circuits are created by cascading these gates, using the output of one gate as the input for another.



These circuits form complex logical statements, and scale up to control the logic behind every computer function.

It can be proven that every logical circuit can be written solely in terms of NOR gates, or solely in terms of NAND gates.

For instance, the 7400 chip shown below contains four NAND gates; the remaining two pins supply power and connect the ground.



Combining Operations

As you may imagine, we can describe longer compound statements by using multiple operations. For instance, a university might advertise a scholarship by saying that "a student is eligible if he/she has completed at least two years of college-level work and he/she has at least a 3.0 GPA or he/she has worked full-time for at least five years."

Let's take a look at this statement; first, define p, q, and r to represent the simple pieces that are connected by the operations:

p : "he/she has completed at least two years of college-level work"
q : "he/she has at least a 3.0 GPA"
r : "he/she has worked full-time for at least five years"

Next, try to write the compound statement using the notation we've introduced so far:

$$p \wedge q \vee r$$

There's a problem with this, though; there is some ambiguity in the way the statement is phrased. Does it mean that a student is eligible after completing two years of college with a 3.0 GPA or after working for five years (with no GPA requirement)? Or does it mean that the student is eligible after completing two years of college, as long as they EITHER have a 3.0 GPA OR have worked at least five years?

To handle this ambiguity, we can use parentheses to group statements together. This works like the order of operations in algebra; the parentheses tell us in what order to combine the simple statements.

In this example, if we wanted to make it clear that the first option is the correct one (a student is eligible after completing two years of college with a 3.0 GPA or after working for five years (with no GPA requirement)) we would group the first two statements together:

$$(p \wedge q) \vee r$$

Evaluating Statements with Multiple Operations Suppose we want to take the compound statement above and consider a specific case, whether a specific student meets the minimum requirements for the scholarship. All we have to know is whether p, q, and r are true for this student. Then, if we substitute these values into the compound statement and get a true result, the student meets the requirements; if not, the student does not meet the requirements.

For instance, suppose a student applies who has completed two years of college and has five years of work experience, but only had a 2.5 GPA. For this student, p is true, q is false, and r is true. If we substitute these values into the compound statement, we have

$$(T \wedge F) \vee T$$

Now we can simplify this statement: the parentheses define the order in which we do so:

F $\wedge$ T = F
F $\vee$ T = T

$$\begin{aligned} &(T \wedge F) \vee T \\ &F \vee T \\ &T \end{aligned}$$

For this student, the compound statement simplifies to T, so they meet the requirements.

EVALUATING STATEMENTS

EXAMPLE 5

Evaluate the statement $\sim p \wedge q$ when p is true and q is true.

Solution

Substitute these truth values for p and q, and start by simplifying $\sim p$:

$$\begin{aligned} &\sim T \wedge T \\ &F \wedge T \\ &F \end{aligned}$$

$\sim T = F$

$F \wedge T = F$

Thus, when p and q are both true, $\sim p \wedge q$ is false.

TRY IT

Evaluate the statement $\sim (p \vee q)$ when p and q are both false.

Truth Tables for Statements with Multiple Operations We could repeat the substitution and simplification process shown above every time we encounter a new student applying for the scholarship, but it will be more efficient in the long run to simply account for all the possibilities for the students that could apply; this is precisely what a truth table does for us.

Let's practice with the same example, using the compound statement

$$(p \wedge q) \vee r$$

Since we have three simple statements (p, q, and r) we'll need eight rows to account for all possible combinations of truth values.

p	q	r	...
T	T	T	...
T	F	T	...
F	T	T	...
F	F	T	...
T	T	F	...
T	F	F	...
F	T	F	...
F	F	F	...

Now, to build up to $(p \wedge q) \vee r$, we'll start by including a column for $p \wedge q$, using the rule for AND to create this column. Then we'll add another column that combines this new column with r using the rule for OR. That final column will be the one we're interested in.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	F	T	F	T
F	T	T	F	T
F	F	T	F	T
T	T	F	T	T
T	F	F	F	F
F	T	F	F	F
F	F	F	F	F

This table lists every possible combination of p, q, and r, and the result for $(p \wedge q) \vee r$ in that case. If we were sorting applications for this scholarship, we could look at every student and see which row of this truth table they fit into; if it is one of the first five, they are eligible, but if it is one of the last three, they are ineligible. In reality, we'd probably write a computer program to do the sorting for us, but we'd need to be aware of this logic in order to write that program.

EXAMPLE 6 CONSTRUCTING A TRUTH TABLE

Construct the truth table for $\sim p \wedge \sim q$.

Solution In this case, there are only two simple statements (p and q), so our truth table will only need four rows to account for all the possibilities.

p	q	...
T	T	...
T	F	...
F	T	...
F	F	...

Next, we'll need columns for $\sim p$ and $\sim q$, which will simply invert p and q:

p	q	$\sim p$	$\sim q$	...
T	T	F	F	...
T	F	F	T	...
F	T	T	F	...
F	F	T	T	...

Finally, we'll combine the columns for $\sim p$ and $\sim q$ using the rule for $\wedge$:

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

TRY IT

Fill in the truth table below for the statement $p \vee \sim q$.

p	q	$\sim q$	$p \vee \sim q$
T	T		
T	F		
F	T		
F	F		

Notice that we could have built the truth table in the last example without using the intermediate rows for $\sim p$ and $\sim q$; to do so, we could substitute each combination of p and q into the final statement $\sim p \wedge \sim q$ to jump straight from the first two columns to the final column:

If p is true and q is true:

$$\begin{gathered} \sim T \wedge \sim T \\ F \ \wedge F \\ F \end{gathered}$$

Therefore, in the row where p is T and q is T, the compound statement $\sim p \wedge \sim q$ is F. The other rows can be done similarly.

The reason we tend to build truth tables as shown in the example above–by slowly adding columns and only performing one operation per column–is that this systematic approach tends to be simpler and avoid errors.

CONSTRUCTING A TRUTH TABLE

EXAMPLE 7

Construct the truth table for the statement $\sim (p \vee q)$.

Solution

Again, we are only dealing with two statements p and q, so we begin with the same two columns as before.

p	q	...
T	T	...
T	F	...
F	T	...
F	F	...

Next we add a column for $p \vee q$. Note that we're using the parentheses as a guide for the order; since the parentheses group the $p \vee q$, we'll fill in that column first, then negate it.

p	q	$p \vee q$	...
T	T	T	...
T	F	T	...
F	T	T	...
F	F	F	...

Finally, we negate this last column to get a column for $\sim (p \vee q)$.

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Notice that the column for $\sim (p \vee q)$ here is identical to the column for $\sim p \wedge \sim q$ in the previous example. Again, this is no accident; this is another representation of one of De Morgan's laws.

TRY IT

Fill in the truth table below for the statement $\sim p \wedge (p \vee \sim q)$.

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim p \wedge (p \vee \sim q)$
T	T				
T	F				
F	T				
F	F				

EXAMPLE 8 CONSTRUCTING A TRUTH TABLE

Construct the truth table for the statement $(p \wedge \sim q) \vee (r \wedge \sim p)$.

Solution Now we have three pieces (p, q, and r), so we'll need three starting columns and eight rows to account for all the possibilities.

p	q	r	...
T	T	T	...
T	F	T	...
F	T	T	...
F	F	T	...
T	T	F	...
T	F	F	...
F	T	F	...
F	F	F	...

Next, we'll need $\sim p$ and $\sim q$ in the final statement, so we'll add a column for each of those.

p	q	r	$\sim p$	$\sim q$	...
T	T	T	F	F	...
T	F	T	F	T	...
F	T	T	T	F	...
F	F	T	T	T	...
T	T	F	F	F	...
T	F	F	F	T	...
F	T	F	T	F	...
F	F	F	T	T	...

Next, add two more columns: one for $p \wedge \sim q$ and one for $r \wedge \sim p$. The last step will be to combine these two columns with $\vee$.

p	q	r	$\sim p$	$\sim q$	$p \wedge \sim q$	$r \wedge \sim p$	...
T	T	T	F	F	F	F	...
T	F	T	F	T	T	F	...
F	T	T	T	F	F	T	...
F	F	T	T	T	F	T	...
T	T	F	F	F	F	F	...
T	F	F	F	T	T	F	...
F	T	F	T	F	F	F	...
F	F	F	T	T	F	F	...

Finally, combine these last two columns with the OR rule.

p	q	r	$\sim p$	$\sim q$	$p \wedge \sim q$	$r \wedge \sim p$	$(p \wedge \sim q) \vee (r \wedge \sim p)$
T	T	T	F	F	F	F	F
T	F	T	F	T	T	F	T
F	T	T	T	F	F	T	T
F	F	T	T	T	F	T	T
T	T	F	F	F	F	F	F
T	F	F	F	T	T	F	T
F	T	F	T	F	F	F	F
F	F	F	T	T	F	F	F

Quantified Statements

Some statements only make a claim about a specific category. For instance, one might say that "all politicians are dishonest." In this case, only those who fall into the category of politician are considered. This is an example of a **quantified statement**.

The words "all" or "some" (similarly "none" or "not all") are called **quantifiers**, since they refer to quantities, limiting the scope of a statement to a given category.

Quantifiers

The Universal Quantifier
The universal quantifier refers to **all** of a given category. The symbol $\forall$ is used to mean "for all."

($\forall$ is an upside-down A)

Ex: To say that all human beings are mortal, we could write

"$\forall$ human beings x, x is mortal" or
"$\forall x$ such that x is a human being, x is mortal"

Note: "none" is a variation of "for all." For instance, "no pig is beautiful" could be written "$\forall p$ such that p is a pig, p is not beautiful."

The Existential Quantifier
The existential quantifier refers to **some** of a given category. The symbol $\exists$ is used to mean "there exists." This is equivalent to saying that "there is at least one."

($\exists$ is a backwards E)

Ex: To say that some dogs are retrievers, we could write

"$\exists x$ such that x is a dog and x is a retriever"

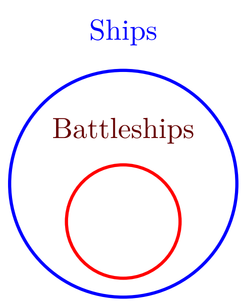
We can draw diagrams to represent quantified statements, like the following examples. We'll use circles to represent categories, but the size of the circles is not relevant; we're only interested in the location and interaction of the circles.

DIAGRAM FOR THE UNIVERSAL QUANTIFIER — EXAMPLE 9

Draw a diagram to represent the statement that "all battleships are ships."

Solution

The diagram below looks like a Venn diagram. The outer circle represents all ships, and the inner circle represents battleships.

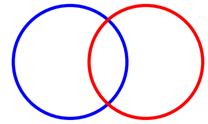


Notice that anything in the battleship category is also automatically in the ship category; hence, this represents the quantified statement that "all battleships are ships."

EXAMPLE 10 DIAGRAM FOR THE EXISTENTIAL QUANTIFIER

Draw a diagram to represent the statement that "some clouds are white."

Solution Now, the left circle represents all white things, and the right circle represents all clouds.



Notice that there is some overlap, which is the clouds that are white. However, there are things that are white that are not clouds (Paper! Snow! A ghost!) and some clouds that are not white. This diagram therefore accurately represents the statement that "some clouds are white."

Negating Quantified Statements

Negating a quantified statement is not initially an intuitive process. For instance, suppose we have a false statement like "all clouds are white." If we negate it, we should get a true statement.

You may be tempted to write the negation as "all clouds are not white." However, this too is a false statement, so it cannot be the negation of the first false statement.

The correct negation would be "NOT all clouds are white," which can also be written "some clouds are not white" or "there is at least one cloud that is not white."

Negating a Quantified Statement

The negation of the statement "all A are B" is "some A are not B."
The negation of the statement "some A are B" is "all A are not B."

EXAMPLE 11 NEGATING QUANTIFIED STATEMENTS

Write the negation of each of the following quantified statements:

(a) All vegetarians eat carrots

(b) Some birds are flightless

(c) Every car salesman is dishonest

(d) There are some foods that cause cancer

Solution

(a) *Some vegetarians do not eat carrots*

(b) *All birds are not flightless* OR *No birds are flightless* OR *All birds can fly*

(c) *At least one car salesman is not dishonest*

(d) *No foods cause cancer*

Notice that there can be several equivalent ways to phrase something in English that all correspond to the same logical statement.

TRY IT Write the negation of the statement "All dogs go to heaven."

Exercises 1.1

In exercises 1–10, determine whether or not each sentence is a statement.

1. 1024 is the smallest 4-digit number that is a perfect square.

2. Doc Brown's time machine requires 1.21 jigowatts of power to operate.

3. All elephants are purple.

4. Don't do drugs.

5. $64 = 2^5$

6. Lizard people run the government.

7. What are you holding?

8. Don't believe everything you read.

9. More soldiers died from sickness than combat in the Revolutionary War.

10. This statement is a lie.

In exercises 11–13, let p and q represent the following statements:

p : It will rain this weekend.
q : We can go to the mall on Saturday.

Write each of the following statements in symbolic form.

11. "It won't rain this weekend."

12. "Either it will rain this weekend, or we can go to the mall on Saturday."

13. "We can't go to the mall on Saturday, but at least it won't rain this weekend."

In exercises 14–16, let p and q represent the following statements:

p : Interest rates are low.
q : It is not time to buy a house.

Write each of the following statements in symbolic form.

14. "It is time to buy a house; interest rates are low."

15. "Interest rates are not low, and it is not time to buy a house."

16. "Either it is time to buy a house, or interest rates are not low."

In exercises 17–22, let p and q represent the following statements:

p : You study hard.
q : You graduate with honors.

Write each of the following statements in symbolic form.

17. "You study hard and you graduate with honors."

18. "You don't study hard, but you graduate with honors anyway."

19. "You study hard, but you still don't graduate with honors."

20. "You study hard or you don't graduate with honors."

21. "You don't study hard or you graduate with honors anyway."

22. "You don't study hard, or you don't graduate with honors."

In exercises 23–26, let p and q represent the following statements:

p : Kevin Durant will win the MVP award.
q : The Lakers will win the NBA championship.

Write each of the following statements in words.

23. $p \vee q$

24. $p \wedge q$

25. $\sim p \vee q$

26. $p \wedge \sim q$

In exercises 27–30, let p and q represent the following statements:

p : This is a dog.
q : This is a mammal.

Write each of the following statements in words.

27. $\sim p \wedge q$ **28.** $p \wedge \sim q$ **29.** $\sim p \vee \sim q$ **30.** $\sim q$

In exercises 31–34, let p and q represent the following statements:

p : Eggs make a good breakfast.
q : Bacon is not healthy.

Write each of the following statements in words.

31. $\sim p$ **32.** $p \vee \sim q$ **33.** $\sim p \wedge \sim q$ **34.** $\sim p \vee q$

In exercises 35–40, evaluate the truth value of the given statement for the given truth values of p, q, and r.

35. $p \vee \sim q$, where p is T and q is F.

36. $\sim p \wedge q$, where p is F and q is F.

37. $p \wedge (q \vee \sim r)$, where p is T, q is F, and r is F.

38. $\sim (p \wedge \sim p)$, where p is F.

39. $(p \wedge q) \vee r$, where p is F, q is T, and r is T.

40. $(p \vee (\sim p \vee q)) \wedge \sim (q \wedge r)$, where p is F, q is F, and r is T.

In exercises 41–46, fill in the blanks in each truth table.

41.

p	q	$\sim p$	$\sim p \wedge q$
T	T		
T	F		
F	T		
F	F		

42.

p	q	$\sim p$	$p \wedge q$	$(p \wedge q) \vee \sim p$
T	T			
T	F			
F	T			
F	F			

43.

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim p \wedge (p \vee \sim q)$
T	T				
T	F				
F	T				
F	F				

44.

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$p \wedge \sim q$	$(p \wedge \sim q) \vee (\sim p \vee q)$
T	T					
T	F					
F	T					
F	F					

45.

p	q	r	$\sim p$	$\sim r$	$q \vee \sim r$	$\sim p \wedge (q \vee \sim r)$
T	T	T				
T	F	T				
F	T	T				
F	F	T				
T	T	F				
T	F	F				
F	T	F				
F	F	F				

46.

p	q	r	$\sim p$	$\sim q$	$\sim r$	$\sim p \ \vee q$	$p \vee \sim r$	$\sim p \vee \sim q$	$(\sim p \ \vee q) \wedge (p \vee \sim r)$	$[(\sim p \ \vee q) \wedge (p \vee \sim r)] \wedge (\sim p \vee \sim q)$
T	T	T								
T	F	T								
F	T	T								
F	F	T								
T	T	F								
T	F	F								
F	T	F								
F	F	F								

In exercises 47–50, write the negation of each quantified statement.

47. All mammals are land-based.

48. Some horses are not domesticated.

49. Some trains are late.

50. All basketball players are not tall.

SECTION 1.2 Conditionals and Equivalence

In this section, we'll introduce two more ways to combine statements.

If-Then

Conditional A presidential candidate might say something like "If I am elected, I will reduce taxes by 20%." This is an example of a *conditional statement*, or an *if-then statement.* A conditional statement consists of a condition and an implication.

Condition	Implication
I am elected	I will reduce taxes by 20%

If-and-Only-If

Biconditional In the example of a conditional statement above, the candidate made a claim about what would happen if they WERE elected, and didn't make any mention of what would happen if they WEREN'T elected.

For another example, suppose a mother tells her son, "If you clean your room, you can have a popsicle before dinner." She made no claim about what would happen if he didn't clean his room; based on that statement, it's possible that she'd let him have one even if he failed to clean his room.

On the other hand, if she said, "You can have a popsicle before dinner IF AND ONLY IF you clean your room," she would account for all possibilities: if he cleans his room, he will get a popsicle; if he doesn't clean, he won't get a popsicle.

This is an example of a *biconditional statement*, or an *if-and-only-if statement.* This is stronger than a conditional statement, because it accounts for all possibilities.

Conditional: If-Then

The following are all examples of conditional statements:

If you get an average of 90% or higher in this course, you'll receive an A.
If you don't pay your taxes, then the IRS will fine you.
It will rain tomorrow if this tropical storm stays in the area.

Notice that in the last example, the statement is written in reverse order; the implication is the first half of the sentence, and the condition is the second half. Pay attention to where the word IF appears.

A conditional statement can also provide an alternate result for what will occur if the condition is not met. For instance, one could say, "If the weather is clear tomorrow, we'll go hiking. Otherwise, we'll go to the mall."

What occurs depends on the truth value of the condition.

If the weather is clear tomorrow...	Result
T	We'll go hiking.
F	We'll go to the mall.

Conditional

When the truth of one statement depends on the truth of another statement, this forms a conditional. A conditional statement has the form

If p, then q. or If p, then q. Otherwise, r.

The conditional "If p then q" is written

Also written as $p \implies q$

$$p \to q$$

Let's practice with the notation by translating from word statements to their symbolic form.

USING CONDITIONAL NOTATION — EXAMPLE 1

Let p represent the statement "You give me \$10,000" and let q represent the statement "I will give you my car." Write the following statements symbolically.

(a) If you give me \$10,000, I will give you my car.

$$p \to q$$

(b) I won't give you my car if you give me \$10,000.

$$p \to \sim q$$

(c) If you don't give me \$10,000, I won't give you my car.

$$\sim p \to \sim q$$

(d) If I don't give you my car, you don't give me \$10,000.

$$\sim q \to \sim p$$

TRY IT

Let p represent the statement "You give me \$10,000" and let q represent the statement "I will give you my car." Which of the following represents the statement "If I give you my car, you will give me \$10,000"?

(a) $\sim p \to q$

(b) $q \to \sim p$

(c) $q \to p$

(d) $\sim q \to p$

We can also reverse this by taking a statement written in symbols, and finding a way to express this in words. As with other statements, there are several different ways of expressing statements like these in words. Let's look at a few examples.

USING CONDITIONAL NOTATION — EXAMPLE 2

Let p represent the statement "You are hurt" and let q represent the statement "You use a bandage." Write the following statements in words.

(a) $p \to q$: "If you are hurt, you use a bandage."

(b) $p \to \sim q$: "When you are hurt, you don't use a bandage."

(c) $\sim p \to q$: "You use a bandage if you aren't hurt."

(d) $q \to p$: "If you use a bandage, you are hurt."

(e) $\sim q \to \sim p$: "If you don't use a bandage, you aren't hurt."

TRY IT

Let p represent the statement "You are hurt" and let q represent the statement "You use a bandage." Which of the following statements corresponds to $\sim q \to p$?

(a) "Don't use a bandage if you are hurt."

(b) "Use a bandage if you aren't hurt."

(c) "Use a bandage if you are hurt."

(d) "If you don't use a bandage, you are hurt."

Sidenote: Programming With If Statements

Conditional statements are a basic and important piece of computer programming; they tell the program what to do, depending on the value of some variable.

For instance, suppose we want to write a simple program that takes a student's grade and determines whether or not the student is passing. In C++, the snippet of code that makes this decision would look like the following:

```
if (grade > 70) {
    cout << "You are passing this course."; //This prints to the screen
}
else {
    cout << "You are failing this course.";
}
```

CONDITIONAL STATEMENTS WITH EXCEL

Spreadsheet programs also use conditional statements extensively. For example, suppose a student's grade is stored in cell `A1` of a spreadsheet, and we wanted to calculate whether the student is passing or failing. In Excel, we would write in another cell (where we want `P` or `F` to appear):

```
=IF(A1>70,"P","F")
```

This expression will check whether the condition (`A1>70`) is true. If it is, the cell will be filled in with `P`; if not, it will be filled in with `F`.

The format is `=IF(condition, value-if-true, value-if-false)`.

EXAMPLE 3 CONDITIONAL STATEMENT IN EXCEL

An accountant needs to withhold 15% of her client's income for taxes if the client's income is below \$30,000. If the income is above \$30,000, she needs to withhold 20%. Write a statement in Excel that would calculate the amount to withhold, if the income is stored in cell `A1`.

Solution In words, we would write: "If income < \$30,000, multiply by 0.15; otherwise, multiply by 0.2." In Excel, that would look like

```
=IF(A1<30000, 0.15*A1, 0.2*A1)
```

We can also combine multiple statements in the condition, as shown in the following example.

EXAMPLE 4 CONDITIONAL STATEMENT WITH MULTIPLE CONDITIONS

Suppose that in a spreadsheet, cell `A1` contains annual income, and cell `A2` contains the number of dependents. A certain tax credit applies to someone with no dependents who earns less than \$10,000, or to someone with dependents who earns less than \$20,000. Write a rule that describes this.

Solution There are two ways to get this tax credit:
Income < \$10,000 AND dependents = 0 OR
Income < \$20,000 AND dependents > 0

In Excel, an `AND` operation is written `AND(first-statement, second-statement)`, and an `OR` operation is written similarly.

Thus, to write this conditional statement, we would enter

```
=IF(OR(AND(A1<10000, A2=0),AND(A2<20000, A2>0)),"Credit","No credit")
```

Truth Table for a Conditional Statement

Consider a conditional statement like "If the Cardinals win the next game, they'll win the World Series." To fill out a truth table for $p \to q$, where p is "the Cardinals win the next game" and q is "they win the World Series," we'll evaluate the truth of $p \to q$ for each possible combination of p and q.

If p is true and q is true: then the Cardinals win the next game and win the World Series, so the statement $p \to q$ wasn't disproven. Thus, in this case $p \to q$ is true.

p	q	$p \to q$
T	T	T

If p is true and q is false: then the Cardinals win the next game but they *don't* win the World Series, so the statement $p \to q$ *was* disproven. Thus, in this case $p \to q$ is false.

p	q	$p \to q$
T	F	F

If p is false and q is true: then the Cardinals *don't* win the next game but they *do* win the World Series. In this case, the statement $p \to q$ *wasn't* disproven, because the statement only made a claim of what would happen if p occurred. Since p didn't occur, $p \to q$ never broke down. Under traditional logic, in this case $p \to q$ is true, since it was never disproven.

p	q	$p \to q$
F	T	T

If p is false and q is false: then the Cardinals don't win the next game and they don't win the World Series. Using the same logic as the last case, in this case $p \to q$ is defined as true, since it makes no claim on what should happen if p is false.

p	q	$p \to q$
F	F	T

If the Cardinals don't win the next game, it doesn't matter whether they win the World Series or not; the given statement is presumed to be true.

Truth Table for a Conditional Statement

The conditional statement $p \to q$ is only false when the condition (p) is true and the implication (q) is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

With regard to the mother who promised the popsicle to her son if he cleaned his room, if he doesn't clean his room, she is free to give him the popsicle or not; no matter what she does, she's keeping her word, because her promise only applied to what would happen if he did clean his room. If he does clean his room, though, she is forced to give him the popsicle, or else her promise was a lie.

Remember this key to a conditional statement: it is only false when the condition is met, but the promised result doesn't occur.

EXAMPLE 5 TRUTH TABLES WITH CONDITIONALS

Construct a truth table for $\sim (q \to p)$.

Solution First, after placing the columns for p and q, we'll need a column for $q \to p$, and then finally we'll negate this column. Notice that the implication direction is reversed from what we've seen before; this just means that q is now the condition and p is the implication. The only false value in this column will occur when q is true and p is false.

p	q	$q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

Finally, negating this column:

p	q	$q \to p$	$\sim (q \to p)$
T	T	T	F
T	F	T	F
F	T	F	T
F	F	T	F

TRY IT

Fill in the truth table below for $\sim p \to q$.

p	q	$\sim p$	$\sim p \to q$
T	T		
T	F		
F	T		
F	F		

Just for practice, let's try another example.

EXAMPLE 6 TRUTH TABLES WITH CONDITIONALS

Construct a truth table for $\sim r \ \wedge (q \to\sim p)$.

Solution The table is shown below; it is left to the reader to verify.

p	q	r	$\sim p$	$q \to\sim p$	$\sim r$	$\sim r \ \wedge (q \to\sim p)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	T	F	F
F	F	T	T	T	F	F
T	T	F	F	F	T	F
T	F	F	F	T	T	T
F	T	F	T	T	T	T
F	F	F	T	T	T	T

The next example leads to a new definition.

TAUTOLOGY **EXAMPLE 7**

Construct a truth table for $[(p \to q) \land p] \to q$.

Solution

At first, this looks daunting, but if we break it down, we notice that we'll need a column for $p \to q$ (which we've done before), we'll need to combine that with the p column using $\land$ (which we practiced in the last section), and finally we'll need to use that column as the condition side of a conditional statement with q as the implication.

The filled-in table looks like the one below.

p	q	$p \to q$	$(p \to q) \land p$	$[(p \to q) \land p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This is actually an example of an *argument*, which we'll cover in more detail later in this chapter. This is what we call a *valid* argument, which we prove by noting that the last column is all T's.

Having the last column full of T's is something that we haven't seen before; this is an example of what is called a *tautology*.

Tautologies and Self-Contradictions

A **tautology** is a statement that is always true. For our purposes, it is a statement involving p and q, for instance, that is true no matter what combination of p and q are true or false. This is the same as saying that it is a statement whose column in a truth table is all T's.

The opposite of a tautology is a **self-contradiction**, which is a statement that is always false (a column in a truth table that is full of F's).

A simple example of a tautology is the statement $p \lor \sim p$:

p	$\sim p$	$p \lor \sim p$
T	F	T
F	T	T

For any statement p, either it will be true, or its negation will be true, so $p \lor \sim p$ is always true; it is a tautology.

TRY IT

Construct a truth table to determine whether $(p \lor q) \land (\sim p \land \sim q)$ is a tautology, a self-contradiction, or neither.

Biconditional: If-and-Only-If

At the beginning of this section, we introduced the idea of the *biconditional*, which is a stronger statement than the conditional; the biconditional states not only the result of some condition being met, but also that that result will not occur if that condition is not met.

For instance, consider a statement like "You will pass this course if and only if your average score is over 70%." This is equivalent to saying "If your average score is over 70%, you will pass this course, and if not, you will not pass this course."

Biconditional

The biconditional "p if and only if q" is written

$$p \leftrightarrow q$$

Let's build the truth table for the statement above.

If p is true and q is true: then your score is over 70% and you pass this course. In this case, the biconditional is true, because this situation fits the claim.

p	q	$p \leftrightarrow q$
T	T	T

If p is true and q is false: then your score is over 70% and you don't pass this course. In this case, the biconditional is false, because the claim didn't match what happened.

p	q	$p \leftrightarrow q$
T	F	F

If p is false and q is true: then your score is not over 70% and you pass this course. This is a case where the one-way conditional was true, because it didn't make any claim about what would occur when the condition wasn't met. However, now the claim is that if your score is not over 70%, you do not pass the course, so this situation makes a lie of the claim.

p	q	$p \leftrightarrow q$
F	T	F

If p is false and q is false: then your score is not over 70% and you do not pass this course. In this case, the biconditional is true, because that's exactly what the second part of the claim said.

p	q	$p \leftrightarrow q$
F	F	T

Truth Table for a Biconditional Statement

The biconditional statement $p \leftrightarrow q$ is true whenever p and q have identical truth values; it is false when their truth values are different.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

This is another way of saying that $p \leftrightarrow q$ claims that p and q are equivalent; $p \leftrightarrow q$ is true precisely when they *are* equivalent.

EXAMPLE 8 TRUTH TABLES WITH BICONDITIONALS

Construct the truth table for $\sim q \leftrightarrow \sim p$.

Solution For this example, we'll need columns for $\sim q$ and $\sim p$. Then, to construct the final column, just look for where $\sim q$ and $\sim p$ are identical; these are where $\sim q \leftrightarrow \sim p$ is true, and it is false everywhere else.

p	q	$\sim q$	$\sim p$	$\sim q \leftrightarrow \sim p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

Equivalence

We used the word *equivalent* in discussing the biconditional statement above, saying that the biconditional is true when the truth values of two statements are identical.

We can go further than this; if two compound statements have identical truth values no matter what combination of p and q are true, these two statements are said to be equivalent.

Note: another way to say this is to say that a and b are equivalent if $a \leftrightarrow b$ is a tautology.

Equivalent Statements

Two statements are **equivalent**, symbolized $\equiv$, if their columns in a truth table are identical.

EQUIVALENT STATEMENTS — EXAMPLE 9

Show that $\sim (\sim p) \equiv p$.

Solution

To show this, we'll construct a truth table that contains columns for both $\sim (\sim p)$ and p, and we'll show that these columns are identical.

p	$\sim p$	$\sim (\sim p)$
T	F	T
F	T	F

Since the columns for p and $\sim (\sim p)$ are identical, we conclude that saying "NOT NOT p" is the same as saying "p."

EQUIVALENT STATEMENTS — EXAMPLE 10

Show that $(p \vee q) \vee r \equiv p \vee (q \vee r)$.

Solution

To show this, we'll build a truth table that contains a column for $(p \vee q) \vee r$ and a column for $p \vee (q \vee r)$, and show that these columns are identical.

The truth table is

p	q	r	$p \vee q$	$q \vee r$	$(p \vee q) \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	F	T	T	T	T	T
F	T	T	T	T	T	T
F	F	T	F	T	T	T
T	T	F	T	T	T	T
T	F	F	T	F	T	T
F	T	F	T	T	T	T
F	F	F	F	F	F	F

Since the columns for $(p \vee q) \vee r$ and $p \vee (q \vee r)$ are identical, we have proven that

$$(p \vee q) \vee r \equiv p \vee (q \vee r).$$

This is called an associative law.

In the next example, we'll look at alternate ways of stating a conditional statement, and show their equivalence. We will also see one example of a statement that sometimes looks equivalent but isn't.

EXAMPLE 11 EQUIVALENT STATEMENTS TO A CONDITIONAL

Select the statement that is not equivalent to the following statement:

If it is raining, I need a jacket.

(a) It's not raining or I need a jacket.

(b) I need a jacket or it's not raining.

(c) If I need a jacket, it's raining.

(d) If I don't need a jacket, it's not raining.

Solution To find which of these are equivalent to the original, we'll need to define simple statements and construct a truth table to compare all the alternatives. Whichever statements have identical truth columns will be the ones that are equivalent.

If p is "It is raining" and q is "I need a jacket," then the original statement is $p \to q$. The other statements are

(a) $\sim p \vee q$

(b) $q \vee \sim p$

(c) $q \to p$

(d) $\sim q \to \sim p$

Now all that remains is to build the truth table.

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \vee q$	$q \vee \sim p$	$q \to p$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	F	F	T	F
F	T	T	F	T	T	T	F	T
F	F	T	T	T	T	T	T	T

Note that the only alternative that isn't equivalent to $p \to q$ is $q \to p$.

The only statement that isn't equivalent is

(c) If I need a jacket, it's raining.

This is called the *converse* of $p \to q$.

TRY IT Pick the converse of the statement "If it is raining, then there are clouds in the sky."

(a) There aren't clouds in the sky or it is raining.

(b) If there are clouds in the sky, it is raining.

(c) If there aren't clouds in the sky, it isn't raining.

(d) It is raining and there are clouds in the sky.

Notice that in that example, we found that $\sim p \vee q$ is equivalent to $p \to q$. This is why we consider the three operations from the previous section to be the *basic* operations, because others like the conditional and biconditional can be re-phrased in terms of AND, OR, and NOT.

Therefore, we can say something like "If you eat a pound of cotton candy, you'll feel sick" or "Either you didn't eat a pound of cotton candy, or you feel sick"; these are equivalent. To see why, think about how we evaluate the truth of a conditional statement—a conditional statement $p \to q$ is only false when p is true ($\sim p$ is false) and q is false. Similarly, $\sim p \vee q$ is only false when $\sim p$ is false and q is false.

Conditional and Biconditional Using Basic Operations

We can write both the conditional and the biconditional in terms of the basic operations AND, OR, and NOT.

Specifically,

$$p \to q \;\equiv\; \sim p \vee q \qquad \text{and} \qquad p \leftrightarrow q \;\equiv\; (p \wedge q) \vee (\sim p \wedge \sim q)$$

Think about why the equivalence shown for the biconditional makes sense. This states that claiming that two statements are equivalent is the same as claiming that they're either both true $(p \wedge q)$ or both false $(\sim p \wedge \sim q)$. Of course, this is precisely what equivalence means.

Statements Related to the Conditional

For every conditional statement $p \to q$, we can rearrange the terms in three common ways by reversing the arrow and negating one or the other or both of p and q.

Converse, Inverse, and Contrapositive

Name	In words	In symbols
	If p, then q.	$p \to q$
Converse	If q, then p.	$q \to p$
Inverse	If not p, then not q.	$\sim p \to \sim q$
Contrapositive	If not q, then not p.	$\sim q \to \sim p$

EXAMPLE 12

CONVERSE, INVERSE, AND CONTRAPOSITIVE

Consider the valid statement, "If you live in Frederick, you live in Maryland." Write the converse, inverse, and contrapositive of this statement.

Solution

(a) Converse: "If you live in Maryland, you live in Frederick."

(b) Inverse: "If you don't live in Frederick, you don't live in Maryland."

(c) Contrapositive: "If you don't live in Maryland, you don't live in Frederick."

TRY IT

Write the converse, inverse, and contrapositive of the statement "If it is summer, the sun is shining."

This example already gives us an idea of which of these is equivalent to the original conditional, because only one of the three is also true—the contrapositive (assuming that when we refer to Frederick, we mean Frederick, MD).

The contrapositive is logically equivalent to the original statement. To show this, we can build a truth table with the four related statements.

If we have a conditional statement (like $p \to q$ or $q \to p$), we can obtain an equivalent conditional statement by switching the direction and negating both p and q.

p	q	Conditional $p \to q$	Converse $q \to p$	Inverse $\sim p \to \sim q$	Contrapositive $\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

Exercises 1.2

In exercises 1–8, let p and q represent the following statements:

p : You studied.
q : You passed this course.

Write each of the following statements in symbolic form.

1. If you passed this course, you studied.

2. You passed this course if and only if you studied.

3. You didn't study if and only if you passed this course.

4. You didn't study if you didn't pass this course.

Write each of the following statements in words.

5. $p \to q$

6. $\sim p \leftrightarrow \sim q$

7. $\sim q \to p$

8. $\sim p \to q$

In exercises 9–16, let p, q, and r represent the following statements:

p : The ice cream truck is here.
q : The pool is open.
r : It is summertime.

Write each of the following statements in symbolic form.

9. If the ice cream truck is here, and the pool is open, then it is summertime.

10. It isn't summertime if the ice cream truck isn't here and the pool isn't open.

11. It's summertime if and only if the pool is open.

12. If it isn't summertime, the pool isn't open or the ice cream truck isn't here.

Write each of the following statements in words.

13. $(p \vee q) \to r$

14. $r \to (p \wedge q)$

15. $p \leftrightarrow q$

16. $p \leftrightarrow (q \wedge r)$

In exercises 17–24, fill in the blanks in each truth table.

17.

p	q	$q \to p$	$\sim (q \to p)$
T	T		
T	F		
F	T		
F	F		

18.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$
T	T			
T	F			
F	T			
F	F			

19.

p	q	$p \vee q$	$q \to (p \vee q)$
T	T		
T	F		
F	T		
F	F		

20.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(p \wedge q) \to (\sim p \wedge \sim q)$
T	T					
T	F					
F	T					
F	F					

21.

p	q	$q \to p$	$p \to q$	$(q \to p) \wedge (p \to q)$
T	T			
T	F			
F	T			
F	F			

22.

p	q	$p \to q$	$\sim q$	$(p \to q) \wedge \sim q$
T	T			
T	F			
F	T			
F	F			

23.

p	q	r	$\sim p$	$\sim p \vee r$	$(\sim p \vee r) \to q$
T	T	T			
T	F	T			
F	T	T			
F	F	T			
T	T	F			
T	F	F			
F	T	F			
F	F	F			

24.

p	q	r	$\sim p$	$\sim r$	$q \to \sim p$	$\sim r \wedge (q \to \sim p)$
T	T	T				
T	F	T				
F	T	T				
F	F	T				
T	T	F				
T	F	F				
F	T	F				
F	F	F				

In exercises 25–28, determine if each statement is a tautology or not.

25. $p \wedge \sim p$

26. $p \vee \sim p$

27. $(p \wedge q) \vee (\sim p \wedge q) \leftrightarrow q$

28. $(q \to p) \vee (\sim q \to \sim p)$

In exercises 29–30, select the statement that is equivalent to the one given.

29. "Either the Cardinals or the Yankees will win the World Series."

(a) If the Cardinals don't win the World Series, the Yankees will win it.

(b) The Cardinals and the Yankees will win the World Series.

(c) If the Cardinals win the World Series, the Yankees will not win it.

(d) If the Yankees win the World Series, the Cardinals will not.

30. "If the light is on, someone is home."

(a) The light is on and someone is home.

(b) If someone is home, the light is on.

(c) Either someone is not home, or the light is on.

(d) If someone is not home, the light is not on.

In exercises 31–32, write the converse, inverse, and contrapositive of the given statement.

31. If you drive through the field, you see fireflies.

32. If you go to office hours, you pass the test.

SECTION 1.3 Logic Rules

In this section, we'll review some of the logical equivalencies that we've already seen, and we'll encounter some new ones. We'll show how each equivalency can be proven with a truth table by finding columns that are identical, but we'll also want to have an intuitive grasp of them as well. After all, since we're studying logic, the results we find should be sensible.

Each of the equivalencies in this section essentially give an alternate way of phrasing a logical statement. Since we use many alternate constructions when we build an argument, it can be a very rewarding and enlightening study to break different forms down and find which are logically equivalent.

Reviewing Equivalent Conditional Statements

We'll start with some review, pointing out again the four common conditional structures.

Name	In words	In symbols
Conditional	If p, then q.	$p \to q$
Converse	If q, then p.	$q \to p$
Inverse	If not p, then not q.	$\sim p \to \sim q$
Contrapositive	If not q, then not p.	$\sim q \to \sim p$

Equivalent Conditional Statements

A conditional statement and its contrapositive are equivalent:

$$p \to q \equiv \sim q \to \sim p$$
$$q \to p \equiv \sim p \to \sim q$$

EXAMPLE 1 EQUIVALENT CONDITIONAL STATEMENTS

Write a statement that is equivalent to the following:

If you don't have the new security update,
you are vulnerable to viruses and other attacks.

Solution The contrapositive of this statement is equivalent to it. Remember that to construct the contrapositive of a conditional statement, we reverse the direction and negate both pieces. Thus, the contrapositive of this statement is

If you are not vulnerable to viruses and other attacks,
you have the new security update.

TRY IT Write a statement that is equivalent to the following:

If you take violin lessons, you can't take guitar lessons.

Again, note that we have many different ways of saying the same thing in words. In that example, the original statement could also have been written

If you don't have the new security update,
you are not safe from viruses and other attacks.

In that case, the contrapositive would be

If you are safe from viruses and other attacks,
you have the new security update.

Notice that this result is the same as the result in the example, but these are just two of the many ways that someone could phrase this conditional statement. Being able to break a statement like this down to its logical structure is therefore a powerful analytic tool.

Negating a Conditional Statement

Consider the following conditional statement:

If Aaron Rodgers has a good game, the Packers will win.

What if we wanted to negate this statement, that is, to write its opposite? Think back to how we defined when a conditional statement is true: this is true if whenever the condition occurs, the result occurs as well. Thus, the only case in which it is false is when the condition occurs but the result does NOT occur:

Aaron Rodgers has a good game, but the Packers do NOT win.

This gives us an idea what the negation of a conditional statement should be; we can verify this with a truth table.

EXAMPLE 2 NEGATING A CONDITIONAL STATEMENT

Show that

$$\sim (p \to q) \;\equiv\; p \wedge \sim q.$$

Solution

We can set up a truth table to prove this; we'll include a column for $p \to q$ and one for $p \wedge \sim q$ and note that they are opposites to show that the negation of each is the other.

p	q	$\sim q$	$p \wedge \sim q$	$p \to q$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Since the columns for $p \to q$ and $p \wedge \sim q$ are exactly opposite,

$$\sim (p \to q) \;\equiv\; p \wedge \sim q.$$

Negating a Conditional Statement

The negation of $p \to q$ is when p occurs and q does not occur:

$$\sim (p \to q) \;\equiv\; p \wedge \sim q$$

Note that this is not the same as the *inverse* of the conditional statement: $\sim (p \to q) \not\equiv \;\sim p \to \sim q$

EXAMPLE 3 NEGATING A CONDITIONAL STATEMENT

Write the negation of each of the following conditional statements.

(a) If the door is unlocked, the alarm sounds.

The door is unlocked and the alarm doesn't sound.

(b) If you don't drive carefully, you won't get better gas mileage.

You don't drive carefully and you get better gas mileage.

(c) If you buy the Juicer 5000, you won't regret it!

You bought the Juicer 5000 and regretted it.

TRY IT

Write the negation of the following statement.

If you fail this test, you'll fail the course.

Distributive Rules

Recall from algebra that *distributing* means taking something like

$$2(x+4)$$

and writing it as

$$2 \cdot x + 2 \cdot 4 = 2x + 8,$$

applying the multiplication to each of the terms in parentheses. There are similar distribution laws when it comes to logical operations; we'll state them first, then prove them.

Distributive Rules

$$\begin{aligned} p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r) \\ p \vee (q \wedge r) &\equiv (p \vee q) \wedge (p \vee r) \end{aligned}$$

EXAMPLE 4 DISTRIBUTIVE RULES

Prove that

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$

Solution To prove this, we'll set up a truth table with a column for each side and note that these columns are identical.

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	F	T	T	F	T	T	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F
T	T	F	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	F	F	F	F	F	F

Based on the fact that the last two columns are identical, we have proven that

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$

TRY IT

Fill in the missing values of the truth table below to prove that

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T					
T	F	T					
F	T	T					
F	F	T					
T	T	F					
T	F	F					
F	T	F					
F	F	F					

USING DISTRIBUTIVE RULES

EXAMPLE 5

Write statements that are logically equivalent to the ones below.

(a) The suspect has blue eyes, and either he has a visible scar on his cheek or he has a beard.

(b) Either we invest in basic research and train engineers, or our space program will fail.

Solution

We will begin each example by defining the components of the statement, then looking for the appropriate distributive law above.

(a) Let p represent "the suspect has blue eyes," q represent "the suspect has a visible scar on his cheek," and r represent "the suspect has a beard."

Then the full statement can be written symbolically as

$$p \wedge (q \vee r)$$

This is equivalent to

$$(p \wedge q) \vee (p \wedge r).$$

In words, this equivalent statement is

Either the suspect has blue eyes and a scar on his cheek,
or the suspect has blue eyes and a beard.

(b) Let p represent "we invest in basic research," q represent "we train engineers," and r represent "our space program will fail."

Then the full statement can be written symbolically as

$$(p \wedge q) \vee r$$

Now we know that this is equivalent to

$$(p \vee r) \wedge (q \vee r).$$

In words, this equivalent statement is

We will either invest in basic research or our space program will fail,
and we will either train engineers or our space program will fail.

TRY IT

Write a statement that is logically equivalent to the following:

You will either pass or fail this course, and your grade is based on your work.

Depending on the situation, these distributive rules may or may not be intuitive. Again, the power of studying logic like this is that we can find absolute rules that hold even when they refer to statements that are hard to understand in words.

De Morgan's Laws

De Morgan's Laws are used in computer science to rewrite Boolean expressions so that a circuit can be built using only one kind of logic gate (a NAND gate or a NOR gate). This makes the circuit cheaper to build, since there are fewer types of hardware needed.

Augustus De Morgan, a 19th-century British mathematician and logician, formalized a pair of laws that describe how to negate an AND or an OR. Although these principles were known before De Morgan, his name is attached to them because he introduced them to logic in their current form.

Let's think about how to negate $p \wedge q$ first. For example, consider the statement

Julie watched *Braveheart* and *A Beautiful Mind*.

This statement claims that she watched both movies; if she didn't watch either of them, the statement is false. If she EITHER didn't watch *Braveheart* OR didn't watch *A Beautiful Mind*, then this claim is disproven.

This gives us an idea of how to negate $p \wedge q$. Since p AND q is true when p and q are both true, and false otherwise, to negate $p \wedge q$, we just need one of them to be negated.

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

This is one of De Morgan's laws, which states that the negation of p AND q is the negation of p OR the negation of q.

The other law is very similar. If we want to negate $p \vee q$, we need to negate both of them, so we negate p AND negate q:

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

De Morgan's Laws

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

EXAMPLE 6 PROVING DE MORGAN'S LAWS

Prove that $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

Solution

Recall: to prove that two statements are equivalent, we build a truth table that includes a column for each, then show that those two columns are identical.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Since the last two columns are identical, this truth table provides proof that

$$\sim (p \wedge q) \equiv \sim p \vee \sim q.$$

TRY IT

Fill in the truth table below to prove that $\sim (p \vee q) \equiv \sim p \wedge \sim q$.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T					
T	F					
F	T					
F	F					

USING DE MORGAN'S LAWS — EXAMPLE 7

Use De Morgan's Laws to write a statement that is equivalent to

It is not true that jeans and tuxedo jackets fit the dress code for a wedding.

Solution

We'll begin by defining simple statements p and q as
p: Jeans fit the dress code for a wedding.
q: Tuxedo jackets fit the dress code for a wedding.

Then the original statement is

$$\sim (p \wedge q).$$

Using De Morgan's Laws, we know that

$$\sim (p \wedge q) \;\equiv\; \sim p \vee \sim q.$$

Finally, we can rewrite $\sim p \vee \sim q$ in words as

Either jeans do not fit the dress code for a wedding,
or tuxedo jackets do not fit the dress code for a wedding.

TRY IT

Use De Morgan's Laws to write a statement that is equivalent to

It is not true that the Bears or the Falcons won on Sunday.

USING DE MORGAN'S LAWS — EXAMPLE 8

Write the negation of the following statement.

I will buy either this sweater or these pants.

Solution

Define p and q:
p: I will buy this sweater.
q: I will buy these pants.

The original statement is

$$p \vee q$$

so its negation is

$$\sim (p \vee q) \;\equiv\; \sim p \wedge \sim q.$$

In words, the negation is

I won't buy this sweater and I won't buy these pants.

TRY IT

Write the negation of the following statement.

It is Tuesday and it's raining.

EXAMPLE 9 USING DE MORGAN'S LAWS

Write the negation of the following statement.

Volvo makes trucks, and doesn't make train engines.

Solution

Define p and q:
p: Volvo makes trucks.
q: Volvo makes train engines.

The original statement is

$$p \wedge \sim q$$

so its negation is

$$\sim (p \wedge \sim q) \ \equiv \sim p \vee q.$$

In words, the negation is

Either Volvo doesn't make trucks, or they make train engines.

Notice that we could have also defined q to be "Volvo doesn't make train engines," but either way, when we negated it, the second half becomes "they make train engines."

TRY IT

Write the negation of the following statement.

Either you don't do your homework or you fail this course.

EXAMPLE 10 USING DE MORGAN'S LAWS

Write the contrapositive of the following statement.

If it is not windy, we can swim and we cannot sail.

Solution

Again, we begin by defining simple statements. This time there are three pieces, so we define p, q, and r.
p: It is windy.
q: We can swim.
r: We can sail.
Notice that we defined each simple statement as a positive statement (without the word NOT in it).

The original statement can then be written as

$$\sim p \to (q \wedge \sim r).$$

Recall that we form the contrapositive by reversing the arrow and negating both sides. The contrapositive is thus

$$\sim (q \wedge \sim r) \to \sim (\sim p).$$

Now we can use De Morgan's Laws to rewrite the left-hand side.

$$(\sim q \vee r) \to p$$

In words, this is equivalent to the statement

If we cannot swim or we can sail, then it is windy.

TRY IT

Write the contrapositive of the following statement.

If you call this number or go to the website, you will get a discount on your next visit.

Exercises 1.3

In exercises 1–8, write the negation of each conditional statement.

1. If a fruit is blue, then it is not a banana.

2. If the storm comes through, that awning will blow away.

3. You'll catch a cold if you don't take Vitamin C.

4. You'll get a three percent return on your investment if you invest with us.

5. If you get an engineering degree, you'll be offered a job as soon as you graduate.

6. If you pass this course, you will graduate this semester.

7. If your score is between 12 and 17, you will place into the first course.

8. If your GPA is over 3.7 and you live on campus, you are eligible for this scholarship.

In exercises 9–12, use the distributive laws to write a statement that is logically equivalent to each given statement.

9. Either the bridge will hold, or those cables will snap and the roadway will crack.

10. You either meet the job requirements or you don't, but you will not get the job.

11. Either get your grades up and get a job, or you won't get a car.

12. This band is from Texas, and they have either three or four members.

In exercises 13–16, use De Morgan's Laws to write a statement that is logically equivalent to each given statement.

13. It is not true that North Dakota and East Dakota are both states.

14. It is not true that this chapter covers logic and finance.

15. It is not true that this book is entertaining or educational.

16. It is not true that today is Wednesday or Thursday.

In exercises 17–20, use De Morgan's Laws to write the negation of each given statement.

17. I pay taxes and I vote.

18. Either the Packers or the Broncos will win the Super Bowl.

19. Either that smoothie contains green vegetables, or it isn't as healthy as it looks.

20. Class isn't over, and that clock is fast.

In exercises 21–24, write the contrapositive of each conditional statement.

21. If he is guilty, he won't testify at his trial.

22. If the cat is running, he either spotted a mouse or he spotted a squirrel.

23. If you do not report for jury duty, or you falsify your information, you will be prosecuted.

24. If you give your plants water and sunlight, they will survive.

SECTION 1.4 Arguments

Premise: the basis of an argument or proof

A logical argument builds a conclusion from a set of *premises*. Our job in this section will be to sift through various arguments and find which are valid and which are invalid.

There are two basic types of arguments: inductive and deductive. We'll focus in this section on deductive arguments, but here we'll give a few examples of inductive arguments for the sake of interest. The difference between an inductive argument and a deductive argument is that an inductive argument builds from specific examples to draw a general principle, and a deductive argument is the reverse, using a general principle to imply specific examples.

Argument Types

An **inductive argument** uses a collection of specific examples as its premises and proposes a general principle as its conclusion.

$$\text{Specific} \longrightarrow \text{General}$$

A **deductive argument** uses one or more general principles as its premises and proposes a specific situation as its conclusion.

$$\text{General} \longrightarrow \text{Specific}$$

EXAMPLE 1 INDUCTIVE ARGUMENT

The following is an example of an inductive argument:
"When I went to the store last week I forgot my wallet, and when I went today I forgot my wallet. I always forget my wallet when I go to the store."

Specific examples

Premises:
I forgot my wallet last week.
I forgot my wallet today.

General principle

Conclusion:
I always forget my wallet.

This is a fairly weak argument, since it is based on only two instances.

An example of a stronger inductive argument is one like the following:

Every day for the past year, a plane has flown over my house at 2 pm.
A plane will fly over my house every day at 2 pm.

This is a stronger argument because it has a larger set of specific examples to support the conclusion.

Inductive arguments can never prove the conclusion true, but it can provide evidence to suggest that it may be true. This evidence tends to be stronger if there are more examples.

Every medical study is an example of an inductive argument. For instance, to test a new blood pressure medication, a pharmaceutical company might design a trial with 100 patients with high blood pressure. If they give half of them the new medication, and that half experiences a dramatic improvement in blood pressure, they can conclude that the medication accomplishes their goal. While that doesn't technically *prove* that claim by a logical standard, it provides strong enough evidence that the company can confidently market the new drug.

Unlike inductive arguments, deductive arguments can be definitively analyzed for validity.

DEDUCTIVE ARGUMENT

EXAMPLE 2

The following is an example of a deductive argument:
"All cats are mammals. Since a tiger is a cat, a tiger is a mammal."

Premises:
All cats are mammals.
A tiger is a cat.

General principles

Conclusion:
A tiger is a mammal.

Specific example

A valid argument is one that has a solid chain of implication leading from the premises to the conclusion.

For the rest of this section, the word *argument* will refer to *deductive argument*.

Valid Deductive Argument

A deductive argument is considered **valid** if the conclusion follows logically from the premises. In other words, an argument is valid if the conclusion is true whenever the premises are true.

We will evaluate the validity of deductive arguments in two ways:

1. Using diagrams
2. Building truth tables

Notice that when we say that an argument is **valid**, we don't focus on whether or not the premises are true; we just want to prove that IF they are true, the conclusion must logically follow. We can also talk about **true** arguments, which are valid arguments with premises that are provably true.

We need to start with a valid argument, though, and then we can start to investigate whether the premises are true or not.

Evaluating Arguments With Diagrams

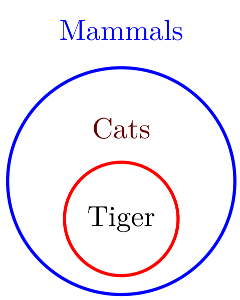
EVALUATING AN ARGUMENT WITH A DIAGRAM

EXAMPLE 3

Consider the argument from example 2:
"All cats are mammals. Since a tiger is a cat, a tiger is a mammal."

The first premise, that all cats are mammals, is a quantified statement using the universal quantifier. We drew a diagram like the one below to illustrate statements like this. Saying that all cats are mammals is the same as saying that the set of cats is a subset of the set of mammals.

This is called an *Euler diagram*, named for Leonhard Euler.



The second premise, that a tiger is a cat, places the tiger within the circle that represents cats. Clearly, the tiger must also lie in the circle that represents mammals, so this argument is valid.

Typically, arguments with premises that involve the universal quantifier like this one are handled naturally with diagrams.

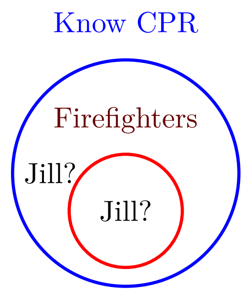
EXAMPLE 4 EVALUATING AN ARGUMENT WITH A DIAGRAM

Argument:
"All firefighters know CPR. Jill knows CPR. Therefore, Jill is a firefighter."

Premises:
All firefighters know CPR.
Jill knows CPR.

Conclusion:
Jill is a firefighter.

Based on the first premise, the set of firefighters must lie within the set of those who know CPR. Based on the second premise, Jill is somewhere in the circle of those who know CPR, but it isn't clear whether she is inside the firefighter set.



Therefore, this argument is not valid, because just being in the set of those who know CPR is not enough to guarantee being in the set of firefighters.

TRY IT

Is the following argument valid or not?

"No cows are purple. Fido is not a cow, so Fido is purple."

Evaluating Arguments With Truth Tables

As we've seen, arguments that involve claims about quantified statements are often easy to visualize with diagrams, as in the examples above.

If, on the other hand, an argument involves conditional statements, it is more natural to analyze the argument with a truth table. Remember, an argument is valid if the conclusion is true every time the premises are true. We'll build a truth table to see if the premises lead to the conclusion every time they're true.

Evaluating Arguments With Truth Tables

An argument is valid if the premises imply the conclusion. Thus, to analyze an argument, we can construct a truth table for the statement

$$[(\text{premise 1}) \wedge (\text{premise 2}) \wedge \ldots (\text{premise } n)] \rightarrow \text{ conclusion.}$$

If this column is all true, that means that the premises DO imply the conclusion for every possibility, so the argument is valid. If there is at least one false value in this column, the argument is invalid.

Fallacies

An invalid argument is called a **fallacy**.

EVALUATING AN ARGUMENT WITH A TRUTH TABLE

EXAMPLE 5

Consider the following argument:
"If you take a lot of math classes, you will lose sleep. You are taking a lot of math classes. Therefore, you will lose sleep."

Solution

This is a simple example, and it should be obvious that this is a valid argument, but we'll use this to illustrate the process of testing an argument by building a truth table. First, define the premises and conclusion:

In words

Premises:
If you take a lot of math classes, you will lose sleep.
You are taking a lot of math classes.

Conclusion:
You will lose sleep.

Let's define the simple statements that appear in the premises and conclusion:
p: "You take a lot of math classes."
q: "You lose sleep."

We can now write the argument more concisely using these letters to represent the statements:

Symbolically

Premises:
$p \to q$
p

Conclusion:
q

The full argument is $[(p \to q) \wedge p] \to q$. We will now build a truth table for this argument.

p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the last column is all true values, we've proven that this argument is valid.

TRY IT

Determine if the following argument is valid.

Premises:
If I have a shovel I can dig a hole.
I dug a hole.

Conclusion:
Therefore, I had a shovel.

EXAMPLE 6 EVALUATING AN ARGUMENT WITH A TRUTH TABLE

Is the following argument valid, or is it a fallacy?

"If the defendant's DNA is found at the crime scene, then we can have him stand trial. He is standing trial. Consequently, we have found evidence of his DNA at the crime scene."

Solution

Premises:
$p \to q$: If his DNA is found at the crime scene, we can have him stand trial.
q: He is standing trial.

Conclusion:
q: We found evidence of his DNA at the crime scene.
The structure of the argument is

$$[(p \to q) \wedge q] \to p$$

which can be analyzed with the following truth table.

p	q	$p \to q$	$(p \to q) \wedge q$	$[(p \to q) \wedge q] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Since a false value appears in the last column, this argument is a fallacy.

Notice the fallacy in the last example. When we make a statement like "If the defendant's DNA is found at the crime scene, we can have him stand trial," we are not excluding the possibility that something else could lead to him standing trial. Thus, just knowing that he's standing trial is not enough to conclude that his DNA was found. In fact, looking at the truth table, we can see that the one case where the argument breaks down is the case where p is false, and q is true, meaning the case where his DNA was not found at the crime scene and yet he is standing trial.

Common Valid and Invalid Arguments

There are several common valid arguments and common fallacies, and we'd like to be able to recognize them. If we can strip an argument down to its basic form and match it to one of these common forms, we won't have to build a truth table every time. This way, we'll be able to analyze and build arguments much more efficiently.

Notice that we've already seen an example of a valid argument and a fallacy. These two are common forms called, respectively, direct reasoning and the fallacy of the converse.

This argument form is sometimes called *modus ponens*, which is Latin for "method of affirming."

The symbol $\therefore$ means "therefore"

Direct Reasoning and the Fallacy of the Converse

A valid argument using **direct reasoning** follows the pattern

$$\begin{array}{ll} p \to q & \text{Premise} \\ p & \text{Premise} \\ \hline \therefore q & \text{Conclusion} \end{array}$$

The **fallacy of the converse** follows the pattern

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

The fallacy of the converse tries to turn a conditional statement around and make the implication imply the condition.

We'll look at other pairs like this, valid arguments and the fallacies that look similar to them.

VALID AND INVALID ARGUMENTS

EXAMPLE 7

Is the following argument valid, or is it a fallacy?

"If my computer crashes, I'll lose all my photos. I haven't lost all my photos. Therefore, my computer hasn't crashed."

Solution

Premises:
$p \to q$: If my computer crashes, I'll lose all my photos.
$\sim q$: I haven't lost all my photos.

Conclusion:
$\sim p$: My computer hasn't crashed.
The structure of the argument is

$$[(p \to q) \wedge \sim q] \to \sim p$$

which can be analyzed with the following truth table.

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim q$	$[(p \to q) \wedge \sim q] \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since the last column is all T's, the argument is valid. This is an example of **contrapositive reasoning**.

VALID AND INVALID ARGUMENTS

EXAMPLE 8

Is the following argument valid, or is it a fallacy?

"If I am at the beach, then I get sunburned. I'm not at the beach, so I won't get sunburned."

Solution

Premises:
$p \to q$: If I'm at the beach, I get sunburned
$\sim p$: I'm not at the beach.

Conclusion:
$\sim q$: I won't get sunburned.
The structure of the argument is

$$[(p \to q) \wedge \sim p] \to \sim q$$

which can be analyzed with the following truth table.

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim p$	$[(p \to q) \wedge \sim p] \to \sim q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Since the last column contains an F, this argument is a fallacy. This is an example of the **fallacy of the inverse**.

TRY IT

What form does the following argument match?
"If you jump off that cliff, you'll break your leg."
"I won't jump off the cliff, so I won't break my leg."

This argument form is sometimes called *modus tollens*, which is Latin for "method of denying."

Contrapositive Reasoning and the Fallacy of the Inverse

A valid argument using **contrapositive reasoning**[a] follows the pattern

$$\begin{array}{l} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

The **fallacy of the inverse** follows the pattern

$$\begin{array}{l} p \to q \\ \underline{\sim p} \\ \therefore \sim q \end{array}$$

[a]This argument form is much less intuitive than direct reasoning, but very powerful.

The next pair of arguments don't involve conditional statements; instead, they involve an OR statement, sometimes called a *disjunction*.

EXAMPLE 9 VALID AND INVALID ARGUMENTS

Is the following argument valid, or is it a fallacy?

"I'm either cold or hot. I'm not cold, so I am hot."

Solution

Premises:
$p \vee q$: I'm either cold or hot.
$\sim p$: I'm not cold.

Conclusion:
q: I am hot.
The structure of the argument is

$$[(p \vee q) \wedge \sim p] \to q$$

which can be analyzed with the following truth table.

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$[(p \vee q) \wedge \sim p] \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

This is a valid argument, called **disjunctive reasoning**. Essentially, this argument says that there are only two alternatives, p and q. If one of them is not true, the other must be.

EXAMPLE 10 VALID AND INVALID ARGUMENTS

Is the following argument valid, or is it a fallacy?

"A musician can play the guitar or the piano. She can play the guitar, so she can't play the piano."

Solution

Premises:
$p \vee q$: She can play the guitar or she can play the piano.
p: She can play the guitar.

Conclusion:
$\sim q$: She can't play the piano.
The structure of the argument is

$$[(p \vee q) \wedge p] \to \sim q.$$

The following truth table analyzes this argument.

p	q	$\sim q$	$p \vee q$	$(p \vee q) \wedge p$	$[(p \vee q) \wedge p] \rightarrow \sim q$
T	T	F	T	T	F
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	T

This is a fallacy, called **misuse of disjunctive reasoning**.

Disjunctive Reasoning and Its Misuse

A valid argument using **disjunctive reasoning** follows the pattern

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array} \quad \text{OR} \quad \begin{array}{l} p \vee q \\ \sim q \\ \hline \therefore p \end{array}$$

Either p or q is true. If we know that one isn't true, the other must be.

A fallacy called the **misuse of disjunctive reasoning** follows the pattern

$$\begin{array}{l} p \vee q \\ p \\ \hline \therefore \sim q \end{array} \quad \text{OR} \quad \begin{array}{l} p \vee q \\ q \\ \hline \therefore \sim p \end{array}$$

This ignores the fact that the OR is inclusive; p can be true or q can be true or both can be true.

VALID AND INVALID ARGUMENTS

EXAMPLE 11

Is the following argument valid, or is it a fallacy?

"If you wear an enormous cowboy hat, people will stare. If people stare at you, you get embarrassed. Therefore, if you wear an enormous cowboy hat, you will be embarrassed."

Solution

Premises:
$p \rightarrow q$: If you wear an enormous cowboy hat, people will stare.
$q \rightarrow r$: If people stare at you, you get embarrassed.

Conclusion:
$p \rightarrow r$: If you wear an enormous cowboy hat, you will be embarrassed.
The structure of the argument is

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

which can be analyzed with the following truth table.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	T	F	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	F	T	F	T	F	T
F	F	F	T	T	T	T	T

This is a valid argument, called **transitive reasoning**. Transitive reasoning means stringing conditional statements together, where the conclusion of one statement forms the condition of the next. As long as we can form a chain like these, we can string many conditional statements together and have a valid conclusion that leads from the beginning to the end.

EXAMPLE 12 VALID AND INVALID ARGUMENTS

Is the following argument valid, or is it a fallacy?

"If you watch *Old Yeller*, you will cry. If you cry, your shirt will be stained. Your shirt is stained, so you must have watched *Old Yeller*."

Solution

Premises:
$p \to q$: If you watch *Old Yeller*, you will cry.
$q \to r$: If you cry, your shirt will be stained.

Conclusion:
$r \to p$: Your shirt is stained, so you must have watched *Old Yeller*.
The structure of the argument is

$$[(p \to q) \wedge (q \to r)] \to (r \to p)$$

which can be analyzed with the following truth table.

p	q	r	$p \to q$	$q \to r$	$r \to p$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (r \to p)$
T	T	T	T	T	T	T	T
T	F	T	F	T	T	F	T
F	T	T	T	T	F	T	F
F	F	T	T	T	F	T	F
T	T	F	T	F	T	F	T
T	F	F	F	T	T	F	T
F	T	F	T	F	T	F	T
F	F	F	T	T	T	T	T

This is a fallacy, called **misuse of transitive reasoning**. The correct conclusion to this argument is $p \to r$, so concluding that $r \to p$ means thinking that the conditional statement and its converse are equivalent, which we have shown before that they are not.

Transitive Reasoning and Its Misuse

A valid argument using **transitive reasoning** follows the pattern

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array} \quad \text{OR} \quad \begin{array}{l} p \to q \\ q \to r \\ \hline \therefore \sim r \to \sim p \end{array}$$

Note: the two forms shown are equivalent, because a conditional statement $p \to r$ and its contrapositive $\sim r \to \sim p$ are equivalent.

A fallacy called the **misuse of transitive reasoning** tries to build a transitive chain, but fails. One possibility form for this fallacy is the following.

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore r \to p \end{array} \quad \text{OR} \quad \begin{array}{l} p \to q \\ q \to r \\ \hline \therefore \sim p \to \sim r \end{array}$$

The difference between transitive reasoning and its misuse can often be subtle in words, but it is clearer when we break down the argument. Notice that there are multiple ways to misuse transitive reasoning; only the chain of implication shown above is valid. Other arguments that look like transitive reasoning can be analyzed the same way that we analyzed the previous example.

TRY IT

Is the following argument valid?

Premises:
If I go to the party, I'll be really tired tomorrow.
If I go to the party, I'll get to see friends.

Conclusion: If I don't see friends, I won't be tired tomorrow.

VALID AND INVALID ARGUMENTS — EXAMPLE 13

Is the following argument valid, or is it a fallacy?

"If I work hard, I'll get a raise. If I get a raise, I'll buy a boat. If I don't buy a boat, I must not have worked hard."

Solution

Premises:
$p \to q$: If I work hard, I'll get a raise.
$q \to r$: If I get a raise, I'll buy a boat.

Conclusion:
$\sim r \to \sim p$: If I don't buy a boat, I must not have worked hard.

We could construct a truth table for this argument, but we notice that this is a form of transitive reasoning that is valid, using the contrapositive. Therefore, we conclude that this is a valid argument.

Lewis Carroll, the author of *Alice in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected in a transitive chain.

For example, find a logical conclusion from the following premises.
All babies are illogical.
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

If we let the following letters represent each piece:
b: "is a baby"
i: "is illogical"
d: "is despised"
m: "can manage a crocodile"

then we can write the premises as
$b \to i$
$m \to \sim d$
$i \to d$

From the first and third premises, we can conclude that $b \to d$; babies, therefore are despised. Using the contrapositive of the second statement, $d \to \sim m$, we can conclude that $b \to \sim m$; therefore, babies cannot manage crocodiles.

Examples of Valid Arguments and Fallacies

The following tables summarize the forms of valid arguments and fallacies that we've seen illustrated.

Common Valid Arguments

Direct Reasoning	Contrapositive Reasoning	Disjunctive Reasoning	Transitive Reasoning
$\begin{array}{l} p \to q \\ \underline{p \quad\quad} \\ \therefore q \end{array}$	$\begin{array}{l} p \to q \\ \underline{\sim q \quad\quad} \\ \therefore \sim p \end{array}$	$\begin{array}{l} p \vee q \\ \underline{\sim p \quad} \\ \therefore q \end{array}$ $\quad$ $\begin{array}{l} p \vee q \\ \underline{\sim q \quad} \\ \therefore p \end{array}$	$\begin{array}{l} p \to q \\ \underline{q \to r \quad} \\ \therefore p \to r \end{array}$ $\quad$ $\begin{array}{l} p \to q \\ \underline{q \to r \quad\quad\quad} \\ \therefore \sim r \to \sim p \end{array}$

Common Fallacies

Fallacy of the Converse	Fallacy of the Inverse	Disjunctive Misuse	Transitive Misuse
$\begin{array}{l} p \to q \\ \underline{q \quad\quad} \\ \therefore p \end{array}$	$\begin{array}{l} p \to q \\ \underline{\sim p \quad\quad} \\ \therefore \sim q \end{array}$	$\begin{array}{l} p \vee q \\ \underline{p \quad\quad} \\ \therefore \sim q \end{array}$ $\quad$ $\begin{array}{l} p \vee q \\ \underline{q \quad\quad} \\ \therefore \sim p \end{array}$	$\begin{array}{l} p \to q \\ \underline{q \to r \quad} \\ \therefore r \to p \end{array}$ $\quad$ $\begin{array}{l} p \to q \\ \underline{q \to r \quad\quad\quad} \\ \therefore \sim p \to \sim r \end{array}$

Fallacies in Common Language

We'll conclude this chapter with a list of a few common logical fallacies, most of which are not related to the logical structures we've seen so far. However, it is useful to be aware of these.

Ad Hominem

An *ad hominem* (Latin "to the person") argument attacks the person making the argument, ignoring the argument itself.

Example of Ad Hominem "Jane says that whales aren't fish, but she's only in second grade, so she can't be right."

Appeal to Ignorance

This type of argument assumes something is true because it hasn't been proven false.

Example of Appeal to Ignorance "Nobody has proven that photo isn't of Bigfoot, so it must be a photo of Bigfoot."

Appeal to Authority

These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, that alone is not enough for real proof. This is especially true when the authority is speaking on something outside their area of expertise.

Example of Appeal to Authority "Jennifer Hudson lost weight with Weight Watchers, so their program must work."

Appeal to Consequence

This concludes that a premise is true or false based on whether the consequences are desirable or not.

Example of Appeal to Consequences "Humans will travel faster than light; faster-than-light travel would be beneficial for space colonization."

False Dilemma

A false dilemma falsely frames an argument as an "either or" choice, without allowing for additional options.

Example of False Dilemma "Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens."

Circular Reasoning

Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

Example of Circular Reasoning "I shouldn't have gotten a C in that class; I'm an A student!"

Straw Man

A straw man argument involves misrepresenting an opponent's argument in a less favorable way to make it easier to attack.

Example of Straw Man Argument "Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against terrorist attacks."

Post Hoc, Ergo Propter Hoc (often abbreviated Post Hoc)

A post hoc argument assumes that because two things happened sequentially, the first must have caused the second.
Post hoc, ergo propter hoc means "after this, therefore because of this."

Example of Post Hoc "Today I wore a red shirt and my football team won! I need to wear a red shirt every time they play to make sure they keep winning."

Correlation Implies Causation

Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related, one must have caused the other.

Example of Correlation Error "When ice cream sales are high, drowning deaths are high as well. Therefore, ice cream must cause people to drown." This argues a causal relation between these two, when in fact both are dependent on the weather; during hot summer months, people are more likely to buy ice cream and also more likely to go swimming.

TRY IT

Identify the logical fallacy in each of the following arguments.

(a) Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.

(b) Since the 1950's, both the atmospheric carbon dioxide level and obesity level have increased. Therefore, higher atmospheric carbon dioxide causes obesity.

(c) The oven was working fine until you started using it, so you must have broken it.

(d) You can't give me a D in the class – I can't afford to retake it.

(e) The senator wants to increase support for food stamps. He wants to take the taxpayers' hard-earned money and give it away to lazy people. This isn't fair so we shouldn't do it.

This is certainly not an exhaustive list of all logical fallacies, but it accounts for many of the most common ones.

Exercises 1.4

In exercises 1–4, determine whether each argument is inductive or deductive.

1. The last mayor was honest. The current mayor is honest. All mayors are honest.

2. Every word has the letter *e* in it. Your name has the letter *e* in it.

3. All deserts have some plant life. Some plants live in the Gobi Desert.

4. The sun rose yesterday, and it has risen every other day of my life. Therefore, the sun rises every day.

In exercises 5–6, use a diagram (an Euler diagram) to evaluate whether each argument is valid.

5. Every concert pianist can play *Chopsticks*. Ellen can play *Chopsticks*, so she must be a concert pianist.

6. Every computer program has some bugs in it. Microsoft Excel is a computer program, so it has some bugs.

In exercises 7–14, use a truth table to determine whether the given argument is valid.

7.
$$\begin{array}{l} p \to \sim q \\ \underline{q \qquad\quad} \\ \therefore \sim p \end{array}$$

8.
$$\begin{array}{l} p \to q \\ \underline{\sim p \qquad} \\ \therefore\ q \end{array}$$

9.
$$\begin{array}{l} p \to q \\ \underline{q \to p} \\ \therefore\ p \wedge q \end{array}$$

10.
$$\begin{array}{l} p \to q \\ \underline{q \to r \qquad\quad} \\ \therefore \sim p \to \sim r \end{array}$$

11.
$$\begin{array}{l} p \to q \\ \underline{q \wedge r \quad} \\ \therefore\ p \vee r \end{array}$$

12.
$$\begin{array}{l} p \leftrightarrow q \\ \underline{q \to r \qquad\quad} \\ \therefore \sim r \to \sim p \end{array}$$

13.
$$\begin{array}{l} q \to \sim p \\ \underline{q \wedge r \qquad} \\ \therefore\ r \to p \end{array}$$

14.
$$\begin{array}{l} \sim p \wedge q \\ \underline{p \leftrightarrow r} \\ \therefore\ p \wedge r \end{array}$$

In exercises 15–22, determine which of the standard argument forms matches the given argument, and indicate whether this is a valid argument.

15. If it is cold, my windows frost over. My windows are not frosted over, so it is not cold.

16. You must eat well or you will not be healthy. I eat well, therefore I am healthy.

17. We must build a hydroelectric plant or a nuclear plant. We won't build a nuclear plant, so we must build a hydroelectric plant.

18. If we open the window, we will hear the birds. We hear the birds. Therefore, we opened the window.

19. If I'm tired, I'm cranky. I'm tired. Therefore, I'm cranky.

20. If everyone obeyed the law, no jails would be needed. Not everyone obeys the law, so some jails are needed.

21. If you pass this course, you will graduate. If you graduate, you will get a job. Therefore, if you get a job, you must have passed this course.

22. If you study the old masters, your art will improve. If your art improves, you will get accepted to the art institute. Therefore, if you study the old masters, you will get accepted to the art institute.