3.7   Quadratic Models

Classifying Scatter Plots

In real life, many relationships between two variables are parabolic, as in Section 3.1, Example 5. A scatter plot can be used to give you an idea of which type of model will best fit a set of data.

Example 1   Classifying Scatter Plots

Decide whether each set of data could be better modeled by a linear model, $y = ax + b$, or a quadratic model, $y = ax^2 + bx + c$.

a. (0.9, 1.4), (1.3, 1.5), (1.3, 1.9), (1.4, 2.1), (1.6, 2.8), (1.8, 2.9), (2.1, 3.4), (2.1, 3.4), (2.5, 3.6), (2.9, 3.7), (3.2, 4.2), (3.3, 4.3), (3.6, 4.4), (4.0, 4.5), (4.2, 4.8), (4.3, 5.0)

b. (0.9, 2.5), (1.3, 4.03), (1.3, 4.1), (1.4, 4.4), (1.6, 5.1), (1.8, 6.05), (2.1, 7.48), (2.1, 7.6), (2.5, 9.8), (2.9, 12.4), (3.2, 14.3), (3.3, 15.2), (3.6, 18.1), (4.0, 19.9), (4.2, 23.0), (4.3, 23.9)

Solution

Begin by entering the data into a graphing utility, as shown in Figure 3.62.

(a)

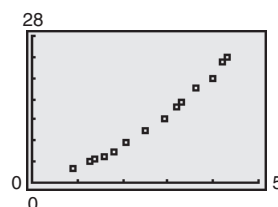

(b)

Figure 3.62

Then display the scatter plots, as shown in Figure 3.63.

(a)

(b)

Figure 3.63

From the scatter plots, it appears that the data in part (a) follow a linear pattern. So, it can be better modeled by a linear function. The data in part (b) follow a parabolic pattern. So, it can be better modeled by a quadratic function.

✓**CHECKPOINT**   Now try Exercise 3.

What you should learn

- Classify scatter plots.
- Use scatter plots and a graphing utility to find quadratic models for data.
- Choose a model that best fits a set of data.

Why you should learn it

Many real-life situations can be modeled by quadratic equations. For instance, in Exercise 15 on page 321, a quadratic equation is used to model the monthly precipitation for San Francisco, California.

Justin Sullivan/Getty Images

Fitting a Quadratic Model to Data

In Section 2.7, you created scatter plots of data and used a graphing utility to find the least squares regression lines for the data. You can use a similar procedure to find a model for nonlinear data. Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

Example 2 Fitting a Quadratic Model to Data

A study was done to compare the speed x (in miles per hour) with the mileage y (in miles per gallon) of an automobile. The results are shown in the table. (Source: Federal Highway Administration)

a. Use a graphing utility to create a scatter plot of the data.

b. Use the *regression* feature of the graphing utility to find a model that best fits the data.

c. Approximate the speed at which the mileage is the greatest.

Speed, x	Mileage, y
15	22.3
20	25.5
25	27.5
30	29.0
35	28.8
40	30.0
45	29.9
50	30.2
55	30.4
60	28.8
65	27.4
70	25.3
75	23.3

Solution

a. Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 3.64. From the scatter plot, you can see that the data appears to follow a parabolic pattern.

b. Using the *regression* feature of a graphing utility, you can find the quadratic model, as shown in Figure 3.65. So, the quadratic equation that best fits the data is given by

$$y = -0.0082x^2 + 0.746x + 13.47.$$ Quadratic model

c. Graph the data and the model in the same viewing window, as shown in Figure 3.66. Use the *maximum* feature or the *zoom* and *trace* features of the graphing utility to approximate the speed at which the mileage is greatest. You should obtain a maximum of approximately (45, 30), as shown in Figure 3.66. So, the speed at which the mileage is greatest is about 47 miles per hour.
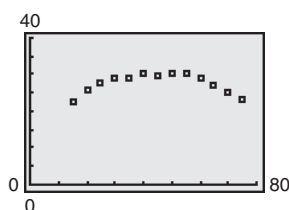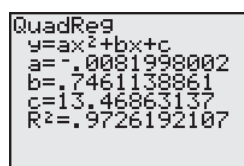
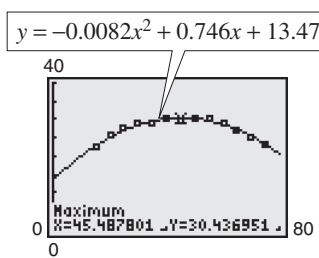**Figure 3.64** **Figure 3.65** **Figure 3.66**

✓CHECKPOINT Now try Exercise 15.

TECHNOLOGY SUPPORT For instructions on how to use the *regression* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Example 3   Fitting a Quadratic Model to Data

A basketball is dropped from a height of about 5.25 feet. The height of the basketball is recorded 23 times at intervals of about 0.02 second.* The results are shown in the table. Use a graphing utility to find a model that best fits the data. Then use the model to predict the time when the basketball will hit the ground.

Solution

Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 3.67. From the scatter plot, you can see that the data has a parabolic trend. So, using the *regression* feature of the graphing utility, you can find the quadratic model, as shown in Figure 3.68. The quadratic model that best fits the data is given by

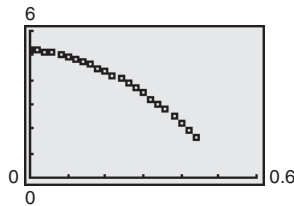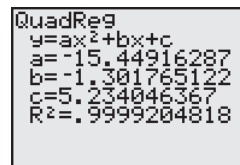$$y = -15.449x^2 - 1.30x + 5.2. \qquad \text{Quadratic model}$$

Time, x	Height, y
0.0	5.23594
0.02	5.20353
0.04	5.16031
0.06	5.09910
0.08	5.02707
0.099996	4.95146
0.119996	4.85062
0.139992	4.74979
0.159988	4.63096
0.179988	4.50132
0.199984	4.35728
0.219984	4.19523
0.23998	4.02958
0.25993	3.84593
0.27998	3.65507
0.299976	3.44981
0.319972	3.23375
0.339961	3.01048
0.359961	2.76921
0.379951	2.52074
0.399941	2.25786
0.419941	1.98058
0.439941	1.63488

Figure 3.67

Figure 3.68

Using this model, you can predict the time when the basketball will hit the ground by substituting 0 for y and solving the resulting equation for x.

$$y = -15.449x^2 - 1.30x + 5.2 \qquad \text{Write original model.}$$

$$0 = -15.449x^2 - 1.30x + 5.2 \qquad \text{Substitute 0 for } y.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-(-1.30) \pm \sqrt{(-1.30)^2 - 4(-15.449)(5.2)}}{2(-15.449)} \qquad \text{Substitute for } a, b, \text{ and } c.$$

$$\approx 0.54 \qquad \text{Choose positive solution.}$$

So, the solution is about 0.54 second. In other words, the basketball will continue to fall for about $0.54 - 0.44 = 0.1$ second more before hitting the ground.

✓*CHECKPOINT*   Now try Exercise 17.

Choosing a Model

Sometimes it is not easy to distinguish from a scatter plot which type of model will best fit the data. You should first find several models for the data, using the *Library of Parent Functions,* and then choose the model that best fits the data by comparing the y-values of each model with the actual y-values.

*Data was collected with a Texas Instruments CBL (Calculator-Based Laboratory) System.

Example 4 Choosing a Model

The table shows the amounts y (in billions of dollars) spent on admission to movie theaters in the United States for the years 1997 to 2003. Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data. Determine which model better fits the data. (Source: U.S. Bureau of Economic Analysis)

Year	Amount, y
1997	6.3
1998	6.9
1999	7.9
2000	8.6
2001	9.0
2002	9.6
2003	9.9

Solution

Let x represent the year, with $x = 7$ corresponding to 1997. Begin by entering the data into the graphing utility. Then use the *regression* feature to find a linear model (see Figure 3.69) and a quadratic model (see Figure 3.70) for the data.
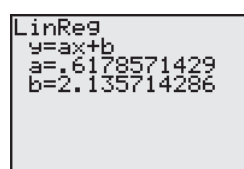
```
LinReg
 y=ax+b
 a=.6178571429
 b=2.135714286
```

```
QuadReg
 y=ax²+bx+c
 a=-.0488095238
 b=1.594047619
 c=-2.55
```

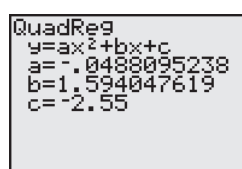**Figure 3.69**   *Linear Model*   **Figure 3.70**   *Quadratic Model*

So, a linear model for the data is given by

$$y = 0.62x + 2.1 \qquad \text{Linear model}$$

and a quadratic model for the data is given by

$$y = -0.049x^2 + 1.59x - 2.6. \qquad \text{Quadratic model}$$

Plot the data and the linear model in the same viewing window, as shown in Figure 3.71. Then plot the data and the quadratic model in the same viewing window, as shown in Figure 3.72. To determine which model fits the data better, compare the y-values given by each model with the actual y-values. The model whose y-values are closest to the actual values is the better fit. In this case, the better-fitting model is the quadratic model.
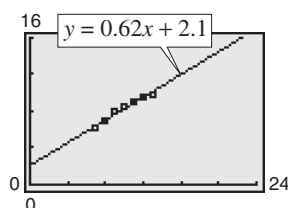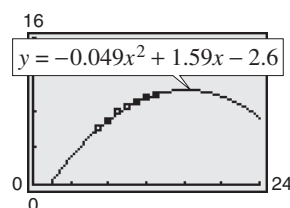

Figure 3.71

Figure 3.72

✔CHECKPOINT   Now try Exercise 18.

TECHNOLOGY TIP   When you use the regression feature of a graphing utility, the program may output an "r^2-value." This r^2-value is the **coefficient of determination** of the data and gives a measure of how well the model fits the data. The coefficient of determination for the linear model in Example 4 is $r^2 \approx 0.97629$ and the coefficient of determination for the quadratic model is $r^2 \approx 0.99456$. Because the coefficient of determination for the quadratic model is closer to 1, the quadratic model better fits the data.
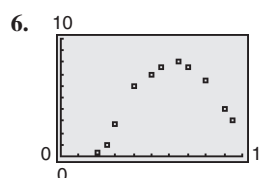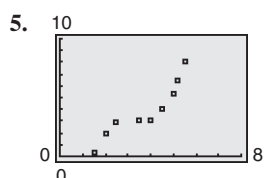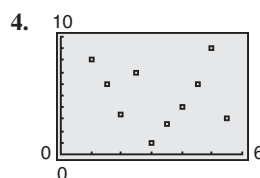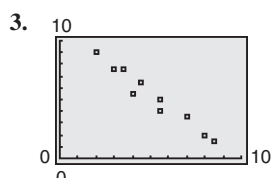
3.7   Exercises

Vocabulary Check

Fill in the blanks.

1. A scatter plot with either a positive or a negative correlation can be better modeled by a _____ equation.

2. A scatter plot that appears parabolic can be better modeled by a _____ equation.

In Exercises 1–6, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.

1.

2.

3.

4.

5.

6.

In Exercises 7–10, (a) use a graphing utility to create a scatter plot of the data, (b) determine whether the data could be better modeled by a linear model or a quadratic model, (c) use the *regression* feature of a graphing utility to find a model for the data, (d) use a graphing utility to graph the model with the scatter plot from part (a), and (e) create a table comparing the original data with the data given by the model.

7. $(0, 2.1)$, $(1, 2.4)$, $(2, 2.5)$, $(3, 2.8)$, $(4, 2.9)$, $(5, 3.0)$, $(6, 3.0)$, $(7, 3.2)$, $(8, 3.4)$, $(9, 3.5)$, $(10, 3.6)$

8. $(-2, 11.0)$, $(-1, 10.7)$, $(0, 10.4)$, $(1, 10.3)$, $(2, 10.1)$, $(3, 9.9)$, $(4, 9.6)$, $(5, 9.4)$, $(6, 9.4)$, $(7, 9.2)$, $(8, 9.0)$

9. $(0, 3480)$, $(5, 2235)$, $(10, 1250)$, $(15, 565)$, $(20, 150)$, $(25, 12)$, $(30, 145)$, $(35, 575)$, $(40, 1275)$, $(45, 2225)$, $(50, 3500)$, $(55, 5010)$

10. $(0, 6140)$, $(2, 6815)$, $(4, 7335)$, $(6, 7710)$, $(8, 7915)$, $(10, 7590)$, $(12, 7975)$, $(14, 7700)$, $(16, 7325)$, $(18, 6820)$, $(20, 6125)$, $(22, 5325)$

In Exercises 11–14, (a) use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data, (b) determine the coefficient of determination for each model, and (c) use the coefficient of determination to determine which model fits the data better.

11. $(1, 4.0)$, $(2, 6.5)$, $(3, 8.8)$, $(4, 10.6)$, $(5, 13.9)$, $(6, 15.0)$, $(7, 17.5)$, $(8, 20.1)$, $(9, 24.0)$, $(10, 27.1)$

12. $(0, 0.1)$, $(1, 2.0)$, $(2, 4.1)$, $(3, 6.3)$, $(4, 8.3)$, $(5, 10.5)$, $(6, 12.6)$, $(7, 14.5)$, $(8, 16.8)$, $(9, 19.0)$

13. $(-6, 10.7)$, $(-4, 9.0)$, $(-2, 7.0)$, $(0, 5.4)$, $(2, 3.5)$, $(4, 1.7)$, $(6, -0.1)$, $(8, -1.8)$, $(10, -3.6)$, $(12, -5.3)$

14. $(-20, 805)$, $(-15, 744)$, $(-10, 704)$, $(-5, 653)$, $(0, 587)$, $(5, 551)$, $(10, 512)$, $(15, 478)$, $(20, 436)$, $(25, 430)$

15. *Meteorology*   The table shows the monthly normal precipitation P (in inches) for San Francisco, California. (Source: U.S. National Oceanic and Atmospheric Administration)

Month	Precipitation, P
January	4.45
February	4.01
March	3.26
April	1.17
May	0.38
June	0.11
July	0.03
August	0.07
September	0.20
October	1.40
November	2.49
December	2.89

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the month, with $t = 1$ corresponding to January.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model with the scatter plot from part (a).

(d) Use the graph from part (c) to determine in which month the normal precipitation in San Francisco is the least.

16. *Sales*   The table shows the sales S (in millions of dollars) for jogging and running shoes from 1998 to 2004. (Source: National Sporting Goods Association)

Year	Sales, S (in millions of dollars)
1998	1469
1999	1502
2000	1638
2001	1670
2002	1733
2003	1802
2004	1838

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 8$ corresponding to 1998.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model with the scatter plot from part (a).

(d) Use the model to find when sales of jogging and running shoes will exceed 2 billion dollars.

(e) Is this a good model for predicting future sales? Explain.

17. *Sales*   The table shows college textbook sales S (in millions of dollars) in the United States from 2000 to 2005. (Source: Book Industry Study Group, Inc.)

Year	Textbook sales, S (in millions of dollars)
2000	4265.2
2001	4570.7
2002	4899.1
2003	5085.9
2004	5478.6
2005	5703.2

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model with the scatter plot from part (a).

(d) Use the model to find when the sales of college textbooks will exceed 10 billion dollars.

(e) Is this a good model for predicting future sales? Explain.

18. *Media*   The table shows the numbers S of FM radio stations in the United States from 1997 to 2003. (Source: Federal Communications Commission)

Year	FM stations, S
1997	5542
1998	5662
1999	5766
2000	5892
2001	6051
2002	6161
2003	6207

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data and identify the coefficient of determination.

(c) Use a graphing utility to graph the model with the scatter plot from part (a).

(d) Use the *regression* feature of a graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(e) Use a graphing utility to graph the quadratic model with the scatter plot from part (a).

(f) Which model is a better fit for the data?

(g) Use each model to find when the number of FM stations will exceed 7000.

19. *Entertainment*   The table shows the amounts A (in dollars) spent per person on the Internet in the United States from 2000 to 2005. (Source: Veronis Suhler Stevenson)

Year	Amount, A (in dollars)
2000	49.64
2001	68.94
2002	84.76
2003	96.35
2004	107.02
2005	117.72

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) A cubic model for the data is $S = 0.25444t^3 - 3.0440t^2 + 22.485t + 49.55$ which has an r^2-value of 0.99992. Use a graphing utility to graph this model with the scatter plot from part (a). Is the cubic model a good fit for the data? Explain.

(c) Use the *regression* feature of a graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(d) Use a graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data? Explain.

(e) Which model is a better fit for the data? Explain.

(f) The projected amounts A^* spent per person on the Internet for the years 2006 to 2008 are shown in the table. Use the models from parts (b) and (c) to predict the amount spent for the same years. Explain why your values may differ from those in the table.

Year	2006	2007	2008
A^*	127.76	140.15	154.29

20. *Entertainment*   The table shows the amounts A (in hours) of time per person spent watching television and movies, listening to recorded music, playing video games, and reading books and magazines in the United States from 2000 to 2005.   (Source: Veronis Suhler Stevenson)

Year	Amount, A (in hours)
2000	3492
2001	3540
2002	3606
2003	3663
2004	3757
2005	3809

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) A cubic model for the data is $A = -1.500t^3 + 13.61t^2 + 33.2t + 3493$ which has an r^2-value of 0.99667. Use a graphing utility to graph this model with the scatter plot from part (a). Is the cubic model a good fit for the data? Explain.

(c) Use the *regression* feature of a graphing utility to find a quadratic model for the data and identify the coefficient of determination.
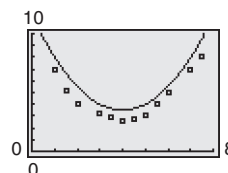
(d) Use a graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data? Explain.

(e) Which model is a better fit for the data? Explain.

(f) The projected amounts A^* of time spent per person for the years 2006 to 2008 are shown in the table. Use the models from parts (b) and (c) to predict the number of hours for the same years. Explain why your values may differ from those in the table.

Year	2006	2007	2008
A^*	3890	3949	4059

Synthesis

True or False?   In Exercises 21 and 22, determine whether the statement is true or false. Justify your answer.

21. The graph of a quadratic model with a negative leading coefficient will have a maximum value at its vertex.

22. The graph of a quadratic model with a positive leading coefficient will have a minimum value at its vertex.

23. *Writing*   Explain why the parabola shown in the figure is not a good fit for the data.

Skills Review

In Exercises 24–27, find (a) $f \circ g$ and (b) $g \circ f$.

24. $f(x) = 2x - 1$, $g(x) = x^2 + 3$

25. $f(x) = 5x + 8$, $g(x) = 2x^2 - 1$

26. $f(x) = x^3 - 1$, $g(x) = \sqrt[3]{x + 1}$

27. $f(x) = \sqrt[3]{x + 5}$, $g(x) = x^3 - 5$

In Exercises 28–31, determine algebraically whether the function is one-to-one. If it is, find its inverse function. Verify your answer graphically.

28. $f(x) = 2x + 5$ **29.** $f(x) = \dfrac{x - 4}{5}$

30. $f(x) = x^2 + 5, x \geq 0$ **31.** $f(x) = 2x^2 - 3, x \geq 0$

In Exercises 32–35, plot the complex number in the complex plane.

32. $1 - 3i$ **33.** $-2 + 4i$

34. $-5i$ **35.** $8i$