Explicit Gap for the Quaquaversal Operator

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1 Introduction and Setup

Here we find an upper bound for the nontrivial eigenvalues of the linear operator defined on $L^2(SO(3))$ corresponding to the following element of $\mathbb{C}[SO(3)]$:

$$QQ := \frac{1}{8}(3 + S + ST^3 + S^2T^3 + T^4 + S^2T^2)$$

where S and T are rotations by $\pi/2$ and $\pi/3$, respectively, about orthogonal axes.

The proof follows the general strategy of Lubotzky, Phillips and Sarnak in Hecke Operators and Uniformly Distributed points on Spheres I and II (LPS I) and (LPS II).

But there are a few complications due to the relevant quadratic form:

$$\tilde{Q}(w, x, y, z) := w^2 + x^2 + 3y^2 + 3z^2$$

being slightly less nice and the relevant S-arithmetic group, $\tilde{G}(4, 6)$, the double cover of $G(4, 6) = SO_{3;Q}(\mathbb{Z}[\frac{1}{2}])$, not being free.

The group $G(4, 6)$ consisting of rotations, S and T, about orthogonal axes in $\mathbb{R}^3$ of orders 4 and 6, respectively, has a presentation given by:

$$\langle S, T | S^4 = T^6 = I, ST^3 = T^3S^{-1}, TS^2 = S^2T^{-1}\rangle \tag{1}$$

Taking the preimage of this group with respect to the double cover

$$SU(2) \twoheadrightarrow SO(3)$$

yields a group, $\tilde{G}(4, 6)$, with the following presentation:

$$\langle S, T | S^4 = T^6 = -I, ST^3 = T^3S^{-1}, TS^2 = S^2T^{-1}\rangle \tag{2}$$

(here S and T in $\tilde{G}(4, 6)$ are the preimages of S and T in $G(4, 6)$)

By the Peter-Weyl theorem the QQ operator can be block diagonalized where each block corresponds to the homogeneous polynomials, p in two complex variables of some fixed even degree. Here S and T are viewed as an element of SU(2) and acts on p via $(S \bullet p)(\vec{z}) = P(S^{-1}\vec{z})$ and $(T \bullet p)(\vec{z}) = p(T^{-1}\vec{z})$ and the action of QQ is given by extending linearly. By my dissertation the blocks of this operator can be further lower block triangulated into four subspaces and all eigenvectors with eigenvalues near 1 occur in the

$$\Lambda_{++} := \{v : T^3 v = v = S^2 v\}$$

block.

On Λ_{++}, QQ ($P_{++}QQP_{++}$ with P_{++} the orthogonal projection onto Λ_{++}) is equal to:

$$\frac{1}{8}(S + S^{-1} + 4 + T + T^{-1}) = \frac{1}{8}(4 + 2S + T^2 + T^4)$$

To get an explicit upper bound on the nontrivial eigenvalues, it therefore suffices to analyze $\Box = 2S + T^2 + T^4$ on Λ_{++}. Letting $T^+ = T + T^{-1} = T^2 + T^4$ we have $\Box = 2S + T^+$. Using the presentation (1) it is easy to see that

$$(T^+)^2 = T^+ + 2 \tag{3}$$

and furthermore that $\Box^2 = (2S + T^+)^2 = 6 + T^+ + 2(ST^+ + T^+S)$. Notice that $\Box^2 - \Box = 6 + 2(ST^+ + T^+S) - 2S$ and that S and $H = H_1 = (ST^+ + T^+S)$ are commuting hermitian operators. This suggests we can analyze the spectrum of $\Box$ using H.

To bound the spectrum of $H = H_1$ we introduce a family of Hecke Operators:

$$H_0 := I$$

$$H_n := \underbrace{ST^+ST^+ \cdots ST^+}_{n\ ST^+\text{s}} + \underbrace{T^+ST^+S \cdots T^+S}_{n\ T^+S\text{s}}$$

The H_n commute with S, are hermitian, and also satisfy the following recurrence relation:

$$HH_n = H_{n+1} + SH_n + 2S^2 H_{n-1}$$

On Λ_{++}, S has eigenvalues ± 1. Restricting to these eigenspaces we get the following recurrence relations:

$$HH_n = H_{n+1} + H_n + 2H_{n-1}$$

on the 1 eigenspace of S in Λ_{++} and

$$HH_n = H_{n+1} - H_n + 2H_{n-1}$$

on the -1 eigenspace of S in Λ_{++}.

Solving these recurrence relations we may express the H_n as chebyshev like polynomials, P_n, in H. More specifically, letting

$$l_{-1} = 0, l_0 = 1, \text{ and,}$$

$$l_n(x) = 2^{n/2} \frac{\sin((n+1)\theta)}{\sin(\theta)} \text{ with } x = 2\sqrt{2}\cos(\theta)$$

we have

$$H_n = P_n(H)$$

where $P_n(H) = l_n(H-S) + Sl_{n-1}(H-S)$ So that on the $+1$ eigenspace of S in Λ_{++},

$$H_n = P_n(H) = l_n(H-1) + l_{n-1}(H-1)$$

and on the -1 eigenspace of S in Λ_{++},

$$H_n = P_n(H) = l_n(H+1) - l_{n-1}(H+1)$$

The strategy is now to find an appropriate family of cusp forms- one for every eigenfunction, f, whose fourier coefficients can be expressed in terms of the H_n applied to f and then, therefore, the P_n applied to the eigenvalue associated with f. The Weil bounds on these fourier coefficients then translates to a bound on the eigenvalues of H. More specifically, it will turn out that for f a $\tilde{Q}$-harmonic homogeneous polynomial of degree $2m$ (in particular, an eigenfunction of QQ or H) there is a holomorphic cusp form of weight $2+2m$ with the property that the $2^{n+2^{\text{nd}}}$ fourier coefficient is a constant times $\left[2^{nm}(1+S^2)(H_n + SH_{n+1})f\right](1)$ for sufficiently large n. Applying the Ramanujan-Petersson Bounds give us:

$$\left[2^{nm}(1+S^2)(H_n + SH_{n+1})f\right](1) \ll_\epsilon (2^n)^{\frac{2m+2-1}{2}+\epsilon}$$

and therefore,

$$\left[(1+S^2)(H_n + SH_{n+1})f\right](1) \ll_\epsilon (2^{n/2+n\epsilon})$$

If f is an eigenvalue of S (with eigenvalue $\pm 1 = \xi$) and of H (with eigenvalue λ_H), and rotating f as needed so that $f(1)$ is not equal to zero, we can write this in terms of the l_n and the $2^{n/2}$ on the right perfectly cancels the $2^{n/2}$ in the definition of l_n. Finally, using the fact that $\sin(n\theta)$ grows exponentially in n for any fixed θ with nonzero imaginary part, we can make θ sufficiently small to contradict this bound— so we must conclude that θ is real. This gives us a bound on λ_H. Specifically that $\lambda_H < 2\sqrt{2} + \xi$. Using the fact that $\Box^2 - \Box = 6 + 2H - 2S$ and therefore, since the relevant operators commute, $(\lambda_\Box)^2 - (\lambda_\Box) = 6 + 2\lambda_H - 2\xi$, where $\lambda_\Box$ is an eigenvalue of $\Box$. Then a bit of arithmetic gives us

$$\lambda_\Box \leq \frac{1 + \sqrt{1 + 4(6 + 4\sqrt{2})}}{2}$$

which translates to a bound on $QQ = (\Box + 4)/8$ on Λ_{++}. So we end up with:

$$\lambda_{QQ} \leq \frac{4 + \frac{1+\sqrt{1+4(6+4\sqrt{2})}}{2}}{8}$$

In the next section (section 2) we shall carry out the argument suggested above, in more detail. We shall assume the existence a cusp form of weight $2+2m$ whose $2^{n+2^{\text{nd}}}$ fourier coefficient is a constant times

$$\left[2^{nm}(1+S^2)(H_n + SH_{n+1})f\right](1)$$

for $\tilde{Q}$-harmonic homogeneous functions of degree $2m$ and in the first subsection we shall also assume that the H_n are equal to $P_n(H)$ to compute the bounds on the spectrum of H and then $\Box$ and finally QQ. In the second subsection we verify the recurrence relation suggested above for H_n and solve it.

After deriving and solving the recurrence relation, what is left is to construct the appropriate cusp form. This is carried out in section 3. To do this we associate the obvious quaternion algebra, B, to $\tilde{Q}$ and consider the Quaternion order $\mathcal{O} \leq B$ consisting of those elements of B with integral coordinates. We then introduce a subgroup $\tilde{G} \leq \tilde{G}(6,4) \subset B$ with the property that every element $g \in \tilde{G}$ is of the form $\frac{b}{\sqrt{\tilde{Q}(b)}}$ where $b \in \mathcal{O}$ and such that $\tilde{Q}(b)$ is a power of two. Using Schoeneberg's theorem we may then associate a cusp form to this quaternion order together with a harmonic. Choosing an appropriate harmonic closely related to the eigenfunctions of H does the trick.

2 Spectral Bound

Definition 1 ($\tilde{Q}$-harmonic polynomials)**.** *The $\tilde{Q}$-Laplacian is defined as follows:*

$$\Delta_{\tilde{Q}} f(w,x,y,z) = \frac{\partial^2 f}{\partial w^2} + \frac{\partial^2 f}{\partial x^2} + 3\frac{\partial^2 f}{\partial y^2} + 3\frac{\partial^2 f}{\partial z^2}$$

The $\tilde{Q}$-harmonic functions are the solutions to the following differential equation:

$$\Delta_{\tilde{Q}} f = 0$$

Let f be a $\tilde{Q}$-harmonic homogeneous polynomial.

For now we shall suppose the following, which shall be proved later:

Fact 1.) There is a cusp form, θ, of weight $2+2m$ whose 2^{n+2} nd fourier coefficient is given by:

$$2^{nm}\big(2(1+S^2)[H_n + SH_{n+1}]f\big)(1)$$

when n is at least 1.

Fact 2.) Define polynomials, l_n, as follows:

$$l_{-1} = 0$$

$$l_n(x) = 2^{n/2}\frac{\sin((n+1)\theta)}{\sin(\theta)} \text{ with } x = 2\sqrt{2}\cos(\theta), \text{ for } n \geq 0$$

then

$$H_n = l_n(H-S) + Sl_{n-1}(H-S), \text{ for } n \geq 0$$

The proof of the first fact will be deferred until the next section. The proof of the second fact will be carried out later in this section— in subsection 2.1

Theorem 1. *Let f be a $\tilde{Q}$-harmonic homogeneous polynomial of degree $2m$ then*

$$\left[(1+S^2)(H_n + SH_{n+1})f\right](1) \ll_\epsilon 2^{n/2+\epsilon}$$

Proof. The following bound is known for cusp forms of this type:

Lemma 1 (Deligne-Ramanujan-Petersson (DRP) Bound)**.** *Let θ be a cusp form of weight $2k$ and*

$$\theta(\tau) = \sum a_n q^n$$

then

$$|a_n| \ll_\epsilon n^{\frac{2k-1}{2}+\epsilon}$$

We have our running assumption, Fact 1:
For some constant, C,

$$\theta(\tau) = \sum C\left[(1+S^2)(H_n + SH_{n+1})f\right](1)2^{nm}q^{\frac{2^{n+2}}{4}} + \text{other terms}$$

Is a cusp form of weight $2m+2$.

The Ramanujan-Petersson Bound then gives us:

$$\begin{aligned} C\left[(1+S^2)(H_n + SH_{n+1})f\right](1)2^{nm} &\ll_\epsilon 2^{n\frac{2m+2-1}{2}+\epsilon} = 2^{nm+n/2+\epsilon} \\ \Rightarrow \left[(1+S^2)(H_n + SH_{n+1})f\right](1) &\ll_\epsilon 2^{n/2+\epsilon} \end{aligned}$$

□

Before proving the bound, let us verify that the homogenous eigenfunctions of QQ are $\tilde{Q}$-harmonic.

Theorem 2.

$$\sum g_i f = \lambda f$$

implies

$$\Delta_{\tilde{Q}} f = 0$$

That is, f is $\tilde{Q}$-harmonic.

We will show that $\Delta_{\tilde{Q}}$ commutes with the action of rotations (and more generally, elements of $\mathrm{SU}_{2;\tilde{Q}}$) on $\oplus_{k\le N} H_k$— polynomials in two complex variables of degree less than N. This will show that when f is a homogeneous eigenfunction of $\sum g_i$ with eigenvalue λ, $\sum g_i \Delta_{\tilde{Q}} f = \Delta_{\tilde{Q}} \sum g_i f = \Delta_{\tilde{Q}} \lambda f = \lambda \Delta_{\tilde{Q}} f$, but $\Delta_{\tilde{Q}}$ lowers the degree of homogeneous eigenfunctions. Therefore, $\Delta_{\tilde{Q}} \lambda f = \lambda \Delta_{\tilde{Q}} f$ is impossible unless $\Delta_{\tilde{Q}} f = 0$. Proving the theorem.

Theorem 3.

$$\left(\sum g_i\right) \Delta_{\tilde{Q}} f = \Delta_{\tilde{Q}} \left(\sum g_i\right) f$$

Proof. In the coordinates, $x_0 = 1$, $x_1 = i$, $x_2 = J/\sqrt{3}$, and $x_3 = K/\sqrt{3}$ the $\tilde{Q}$-spherical laplacian, $\Delta_{\tilde{Q}}$, is the standard laplacian and the $\tilde{Q}$ rotations are just standard rotations. To prove this theorem it suffices to show that:

$$\Delta_{\tilde{Q}}(f \circ \rho) = \Delta_{\tilde{Q}}(f) \circ \rho$$

In the (x_0, x_1, x_2, x_3) coordinates with ρ a $\tilde{Q}$-rotation is equivalent to

$$\Delta(f \circ \rho) = \Delta(f) \circ \rho$$

with ρ a standard rotation and Δ the standard laplacian. And we know this is true. $\square$

A similar calculation shows that if f is a $\tilde{Q}$-harmonic then so is $f \circ \rho$.

Theorem 4 (bound)**.** *Let λ be an eigenvalue of H.*

If the corresponding eigenvector is an eigenvector of S with eigenvalue 1 then

$$\lambda \in [-2\sqrt{2} + 1, 2\sqrt{2} + 1]$$

If the corresponding eigenvector is an eigenvector of S with eigenvalue -1 then

$$\lambda \in [-2\sqrt{2} - 1, 2\sqrt{2} - 1]$$

Proof. Let f be a $\tilde{Q}$-harmonic homogenous polynomial of degree $2m$.

We have

$$\left|\left[(1 + S^2)(H_n + SH_{n+1})f\right](1)\right| \ll_\epsilon 2^{n/2+\epsilon}$$

Let $g \in \Lambda_{++}$ be a $\tilde{Q}$-harmonic eigenfunction of H of degree $2m$. Let ζ be an element of the $\tilde{Q}$-sphere such that $g(\zeta) \neq 0$ and let $f = g \circ R$ where R is a $\tilde{Q}$ rotation mapping ζ to 1. So we have $f(1) = g(\zeta) \neq 0$. we have:

$$|2(1 + S^2)(H_n + SH_{n+1})f(1)| = |2(1 + S^2)(H_n + SH_{n+1})g(\zeta)|$$

since

$$H_n = P_n(H) = l_n(H - S) + Sl_{n-1}(H - S)$$

we also have:

$$\begin{aligned}
&|2(1 + S^2)(H_n + SH_{n+1})g(\zeta)| = \\
&|2(1 + S^2)\left(l_n(H - S) + Sl_{n-1}(H - S) + S(l_{n+1}(H - S) + Sl_n(H - S))\right) g(\zeta)|
\end{aligned}$$

If in addition g is an eigenfunction of S with eigenvalue 1 then we also have:

$$|4\left(2l_n(H - 1) + l_{n-1}(H - 1) + l_{n+1}(H - 1))\right) g(\zeta)| \ll_\epsilon 2^{n/2+n\epsilon}$$

and therefore

$$|\left(2l_n(H - 1) + l_{n-1}(H - 1) + l_{n+1}(H - 1)\right) g(\zeta)| \ll_\epsilon 2^{n/2+n\epsilon}$$

Using $l_n(\lambda-1) = 2^{n/2}\frac{\sin((n+1)\theta)}{\sin(\theta)}$ with $\lambda - 1 = 2\sqrt{2}\cos(\theta)$ we have

$$2l_n(\lambda-1)+l_{n-1}(\lambda-1)+l_{n+1}(\lambda-1) = 2^{n/2}\frac{2\sin((n+1)\theta)+\sin(n\theta)+\sin((n+2)\theta)}{\sin(\theta)}$$

giving us,

$$\left|2^{n/2}\frac{2\sin((n+1)\theta)+\sin(n\theta)+\sin((n+2)\theta)}{\sin(\theta)}\right| \ll_\epsilon 2^{n/2+n\epsilon}$$

therefore,

$$|2\sin((n+1)\theta)+\sin(n\theta)+\sin((n+2)\theta)| \ll_\epsilon 2^{n\epsilon}$$

Since $2\sin((n+1)\theta)+\sin(n\theta)+\sin((n+2)\theta)$ grows exponentially in n if θ has a nonzero imaginary part and ϵ is arbitrary, we may conclude that θ is real and therefore,

$$\lambda - 1 \in [-2\sqrt{2}, 2\sqrt{2}]$$

So

$$\lambda \in [-2\sqrt{2}+1, 2\sqrt{2}+1]$$

If $f \in \Lambda_{++}$ is an eigenfunction of S with eigenvalue -1 then

$$|2(1+S^2)(H_n + SH_{n+1})f(1)|$$

is

$$|2(1+S^2)\left(l_n(H-S) + Sl_{n-1}(H-S) + S(l_{n+1}(H-S) + Sl_n(H-S))\right)g(\zeta)|$$

which is

$$|4\left(l_n(H+1) - l_{n-1}(H+1) - (l_{n+1}(H+1) - l_n(H+1))\right)g(\zeta)|$$

which is

$$|4\left(2l_n(H+1) - l_{n-1}(H+1) - l_{n+1}(H+1)\right)g(\zeta)|$$

so we have:

$$|(2l_n(H+1) - l_{n-1}(H+1) - l_{n+1}(H+1))g(\zeta)| \ll_\epsilon 2^{n/2+n\epsilon}$$

Using $l_n(\lambda+1) = 2^{n/2}\frac{\sin((n+1)\theta)}{\sin(\theta)}$ with $\lambda + 1 = 2\sqrt{2}\cos(\theta)$
we conclude in a similar manner that

$$\lambda \in [-2\sqrt{2}-1, 2\sqrt{2}-1]$$

□

We wish to bound the non-trivial eigenvalues of $\Box = 2S + T^+$. At this point we can use H to easily get a bound since $\Box^2 = 6 + T^+ + 2H$ applying the trivial bound to T^+ and the bound we have for H we get that the second largest eigenvalue of $\Box^2$ is bounded by $10 + 4\sqrt{2}$ and therefore, the second largest eigenvalue of $\Box$ is bounded by $\sqrt{10+4\sqrt{2}}$. This bound incurs losses due to neglecting the T^+ term.

The following analysis yields a better bound, which I believe is optimal:

Recall that $\Box^2 - \Box = 6 + 2H - 2S$. Since H and S commute, suppose v is a simultaneous eigenvector of H and S with eigenvalue λ_0 with respect to S and λ with respect to H then $\lambda < 2\sqrt{2} + \lambda_0$ so that the eigenvalues of $6 + 2H - 2S$ are less than $6 + 4\sqrt{2} + 2\lambda_0 - 2\lambda_0 = 6 + 4\sqrt{2}$

To bound the spectrum of $\Box^2 - \Box$ we maximize t subject to:

$$t^2 - t \leq 6 + 4\sqrt{2}$$

$t^2 - t$ is increasing when $t > 1/2$, so it suffices to analyze

$$t^2 - t = 6 + 4\sqrt{2}$$

whose largest solution is:

$$\frac{1 + \sqrt{1 + 4(6 + 4\sqrt{2})}}{2} \approx 3.95063099$$

To bound the spectrum of $\frac{1}{8}(S + S^{-1} + 4 + T + T^{-1})$ we just add 4 and divide by 8:

$$\frac{4 + \frac{1+\sqrt{1+4(6+4\sqrt{2})}}{2}}{8} \approx 0.99382887$$

I suspect that to prove optimality one can use Kesten's theory for symmetric random walks on groups and compute the kesten measure associated with the appropriate group— or at least the support of the appropriate measure! But, I have not carried this out yet.

2.1 Recurrence Relation

Theorem 5 (Recurrence Relation)**.** *On Λ_{++} define*

$$\begin{aligned} H_0 &:= I, \\ H_1 &:= H = ST^+ + T^+S, \text{ and} \\ H_n &:= \underbrace{ST^+ST^+ \cdots ST^+}_{n\ ST^+s} + \underbrace{T^+ST^+S \cdots T^+S}_{n\ T^+Ss} \end{aligned}$$

Then

(1)

$$SH_n = H_nS$$

(2)

$$HH_n = H_{n+1} + SH_n + 2S^2 H_{n-1}$$

(3) Using the Tchebychev Polynomials, l_n, defined in Lubotzky, Phillips, and Sarnak's "Hecke Operators and Uniformly distributing points on the sphere I", with $q = 2$ we have:

$$H_n = l_n(H - S) + S l_{n-1}(H - S) =: P_n(H)$$

Proof. Recall that both T^+ and S leave Λ_{++} invariant. Also notice that $S^2 T^+ = T^+ S^2$ so (1) is clear.

Also notice on Λ_{++}, $(T^+)^2 = T^+ + 2$.

$$\begin{aligned} HH_n &= H_{n+1} + T^+ S \underbrace{ST^+ ST^+ ST^+ \cdots ST^+}_{n\ ST^+\text{s}} + ST^+ \underbrace{T^+ ST^+ S \cdots T^+ S}_{n\ T^+ S\text{s}} \\ &= H_{n+1} + S(ST^+ + 2S) \underbrace{ST^+ \cdots ST^+}_{n-1\ ST^+\text{s}} + S(T^+ S + 2S) \underbrace{T^+ S \cdots T^+ S}_{n-1\ T^+ S\text{s}} \\ &= H_{n+1} + SH_n + 2S^2 H_{n-1} \end{aligned}$$

So (2) is done.

In Lubotzky, Phillips, and Sarnak's "Hecke Operators and Uniformly distributing points on the sphere I", they introduced polynomials, l_n, defined by the following recurrence relation:

- $l_0(x) = I$
- $l_1(x) = x$
- $$x l_n(x) = l_{n+1}(x) + q l_{n-1}(x) \text{ for } n \geq 2$$

Notice that if we set l_{-1} to 0 then the three term recurrence is valid for $n \geq 1$, so let us do this. They solved this recurrence relation and found:

$$l_n(\lambda) = q^{n/2} \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

where $\lambda = 2\sqrt{q}\cos(\theta)$.

Let us set $q = 2$ so that l_n satisfies:

$$x l_n(x) = l_{n+1}(x) + 2 l_{n-1}(x) \text{ for } n \geq 2$$

and

$$l_n(\lambda) = 2^{n/2} \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

with $\lambda = 2\sqrt{2}\cos(\theta)$

Notice that $P_n(H) = l_n(H-S) + Sl_{n-1}(H-S)$ with $n \geq 0$ satisfies:

$$P_0(H) = l_0(H-S) + Sl_{-1}(H-S) = I + 0 = I$$
$$P_1(H) = l_1(H-S) + Sl_0(H-S) = H - S + S = H$$

and

$$\begin{aligned}(H-S)P_n(H) &= (H-S)l_n(H-S) + (H-S)Sl_{n-1}(H-S) \\ &= l_{n+1}(H-S) + 2l_{n-1}(H-S) + Sl_n(H-S) + 2Sl_{n-2}(H-S) \\ &= P_{n+1}(H) + 2P_{n-1}(H)\end{aligned}$$

rearranging terms we have:

$$HP_n(H) = P_{n+1}(H) + SP_n(H) + 2P_{n-1}(H)$$

This means that $P_n(H)$ satisfies the same recurrence relation and initial conditions as H_n and therefore $P_n(H) = H_n$, as needed.

□

3 Cusp Form

In this section we show that, given a $\tilde{Q}$-harmonic homogeneous polynomial, f, of degree $2m$, there is a cusp form whose $2^{n+2^{\text{nd}}}$ fourier coefficient is given by:

$$2^{nm}[2(1+S^2)(H_n + SH_{n+1})f](1)$$

To finish the proof of the bound.

A theorem of Serre tells us that there is a basis such that:

Lemma 2.
$$G(4,6) = SO_{3;Q}\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$$

where
$$Q(x,y,z) = x^2 + 3y^2 + 3z^2$$

It is therefore natural to define the following quaternion algebra, B, as follows:

$B := \{w + xi + yJ + zK : \text{ where } i^2 = -1, J^2 = -3, iJ = K = -Ji \text{ and } w, x, y, z \in \mathbb{R}\}$

In B we have the following quadratic form, which is a multiplicative norm:

$$\tilde{Q}(b) = b\bar{b} = (w + xi + yJ + zK)(w - xi - yJ - zK) = w^2 + x^2 + 3y^2 + 3z^2$$

sometimes we will write

$$\tilde{Q}(w,x,y,z) := w^2 + x^2 + 3y^2 + 3z^2$$

$\tilde{Q}$ restricts to Q on the purely imaginary part of B and there is a surjective homomorphism $B^* = B \setminus 0$ to $SO_{3;Q}$ defined by:

$$b \mapsto \left[v \mapsto \frac{b}{\sqrt{\tilde{Q}(b)}} v \frac{\bar{b}}{\sqrt{\tilde{Q}(\bar{b})}} = \frac{bv\bar{b}}{\tilde{Q}(b)} \right]$$

with kernel given by the reals. In other words, $B \backslash 0$ acts as $SO_{3;Q}$ via conjugation with stabilizer equal to its center, the reals. It is not hard to see that the preimage of any element g of $SO_{3;Q}$ under this map has exactly two elements with norm 1, $\pm\tilde{b} \in B$. The elements, b of B, whose norm, $\tilde{Q}(b)$, is equal to 1 form a group of elements with norm 1 double cover $SO_{3;Q}$ via:

$$\tilde{b} \mapsto [v \mapsto \tilde{b} v \tilde{b}^{-1}]$$

and we shall denote this group $SU_{2;\tilde{Q}}$. We clearly have:

$$B^* \to SU_{2;\tilde{Q}}$$

via

$$b \mapsto \frac{b}{\sqrt{\tilde{Q}(b)}}$$

The rotation of order 4 about one axis corresponds to $S = \frac{1+i}{\sqrt{2}} \in SU_{2;\tilde{Q}}$ and a rotation of order 6 about a perpendicular axis is given by $T = \frac{J+3}{2\sqrt{3}} \in SU_{2;\tilde{Q}}$. The images of these elements generate $G(4,6)$ and Serre's result tells us that the images of S and T under the above map generate the entirety of $SO_{3;Q}\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$. We shall denote the group generated by S and T themselves $\tilde{G}(4,6)$ naturally this double covers $G(4,6)$.

Let us now define the quaternion order, $\mathcal{O}$, as follows:

$$\mathcal{O} = \{w + xi + yJ + zK \in B : w, x, y, z \in \mathbb{Z}\}$$

Notice that S and T have norm 1 and are real multiples of elements in $\mathcal{O}$. This implies they are of the form $\frac{b}{\sqrt{\tilde{Q}(b)}}$ with $b \in \mathcal{O}$. Since these generate $\tilde{G}(4,6)$, every element of $\tilde{G}(4,6)$ can be expressed as $\frac{b}{\sqrt{\tilde{Q}(b)}}$ for some $b \in \mathcal{O}$. We define a map:

$$\tilde{G}(4,6) \to \mathcal{O}$$

mapping g to such a b with $\tilde{Q}(b)$ minimal. So we define the *height* of an element $g \in \tilde{G}(4,6)$ denotes $N(g)$ as the smallest positive integer such that $\sqrt{N(g)}g \in \mathcal{O}$, so multiplying by $N(g)$ "just clears the denominators". And, we have a correspondence between $\tilde{G}(4,6)$ and a certain subset of $\mathcal{O}$ via:

$$\sqrt{N(g)}g = b \leftrightarrow \frac{b}{\sqrt{\tilde{Q}(b)}} = g$$

It is not hard to see that the height of g is the norm of the corresponding b and that b cannot be divisible by an integer besides ± 1— if it were then we could find a corresponding element of $\mathcal{O}$ with smaller norm. It is also not hard to check that if m is an integer such that $\sqrt{m}g \in \mathcal{O}$ then $m = d^2 N(g)$ for some $d \in \mathbb{Z}$.

Lemma 3 ($\tilde{G}(4,6)$ Canonical Form)**.** *Every element of $\tilde{G}(4,6)$ can be written uniquely in the following form:*

$$\pm S^{a_0} T^{b_0}$$

or

$$\pm S^{a_0} T^{b_0} S T^{b_1} \cdots S T^{b_n} E, \textit{ for } n > 0 \textit{ and } E = 1 \textit{ or } S$$

Where $a_0 \in \{0,1,2,3\}$, $b_0 \in \{0,3\}$, and for $j > 0$, $b_j \in \{2,4\}$

This follows from the following lemma due to Sadun, Draco, and Van Wieren in their paper "Growth Rates in the Quaquaversal Tiling"

Lemma 4. *Every element of $G(4,6)$ can be written uniquely in one of the following forms:*

(I)

$$S^{a_0} T^{b_0}$$

(II)

$$S^{a_0} T^{b_0} S T^{b_1} \cdots S T^{b_n} E, \textit{ for } n > 0 \textit{ and } E = 1 \textit{ or } S$$

Where $a_0 \in \{0,1,2,3\}$, $b_0 \in \{0,3\}$, and for $j > 0$, $b_j \in \{2,4\}$.

Proof of $\tilde{G}(4,6)$ canonical form. To see this, just project the element of $\tilde{G}(4,6)$ onto $G(4,6)$. Expressing this in canonical form pulls back to the given preimage up to multiplication by ± 1 as needed. □

The Hecke-Schoeneberg Theorem states:

Theorem 6 (Hecke-Schoeneberg Theorem)**.** *Let $Q(x) = \frac{1}{2} x^t A x$ be an integral quadratic form in $r = 2k$ variables (A has integer entries with even numbers along the main diagonal) and let N be its level (the smallest integer such that NA^{-1} has integral entries with even numbers along the main diagonal). Let $P(n)$ be a Q-harmonic homogeneous polynomial of degree m and let h satisfy $Ah \equiv 0(N)$ then*

$$\sum_{n \equiv h(N)} P(n) e^{2\pi i \tau Q(n)/N^2}$$

is a modular form of weight $m+k$ and level $\Gamma_0(N)$ and a cusp form when $m > 0$.

We use this to attach a family of modular forms to $\mathcal{O}$. One for every $\tilde{Q}$-harmonic homogenous polynomial of even degree. These will be cusp forms when the degree of these polynomials is positive. They are defined as follows:

Lemma 5. *Let f be a $\tilde{Q}$-harmonic homogeneous polynomial of degree $2m$ then*

$$\theta(\tau) := \frac{1}{6^{2m}} \sum_{\substack{n=6b \text{ with} \\ b\in\mathcal{O}\ b\not\equiv 0(2)}} f(n) q^{\tilde{Q}(n)/144} = \sum_{\substack{b\in\mathcal{O} \\ b\not\equiv 0(2)}} f(b) q^{\tilde{Q}(b)/4}$$

$$= \sum_{\substack{b\in\mathcal{O} \\ b\not\equiv 0(2)}} \tilde{Q}(b)^m f\left(\frac{b}{\sqrt{\tilde{Q}(b)}}\right) q^{\tilde{Q}(b)/4} = \sum_{\nu\in\mathbb{N}} \nu^m \left(\sum_{\substack{b\not\equiv 0(2) \\ \tilde{Q}(b)=\nu}} f\left(\frac{b}{\sqrt{\nu}}\right)\right) q^{\nu/4}$$

Is a cusp form of level 12 *and weight* $2m+2$

Proof.

$$\frac{1}{6^{2m}} \sum_{\substack{n=6b \text{ with} \\ b\in\mathcal{O}\ b\not\equiv 0(2)}} f(n) q^{\tilde{Q}(n)/144} = \frac{1}{6^{2m}} \sum_{h\not\equiv 0(2)} \left(\sum_{n\equiv 6h(12)} f(n) q^{\tilde{Q}(n)/144} \right)$$

on the right of the equals sign is a sum of 15 modular forms taken over those elements of $F_2^4 \not\equiv 0(2)$. To see this, note that $6h$ satisfies the hypotheses of the 'h' in the Hecke-Schoeneberg theorem when we take the quadratic form, Q in the theorem to be $\tilde{Q}$. □

The crucial feature of this cusp form is that for sufficiently large n, if $\nu = 2^{n+2}$, then

$$\sum_{\substack{b\not\equiv 0(2) \\ \tilde{Q}(b)=\nu}} f\left(\frac{b}{\sqrt{\nu}}\right)$$

—which is essentially the coefficient of $q^{\nu/4} = q^{2^{n+2}/4} = q^{2^n}$, is equal to

$$\left[2(1+S^2)(H_n + SH_{n+1})f\right](1)$$

The remainder of this article verifies this fact.

It will be convenient to compute lifts of T, T^2, T^3, T^4 and S in B. These can be given by:

$$\begin{array}{ccc} T & = & \frac{J+3}{2\sqrt{3}} \\ T^2 & = & \frac{1+J}{2} \\ T^3 & = & \frac{J}{\sqrt{3}} \\ T^4 & = & \frac{-1+J}{2} \\ S & = & \frac{1+i}{\sqrt{2}} \end{array}$$

$$T^6 = -1 = S^4$$

Theorem 7. *Define the group $\tilde{G}$ as follows:*

$$\left\{ \frac{b}{\sqrt{\tilde{Q}(b)}} : b \in \mathcal{O} \text{ and } \exists n \in \mathbb{N}, \tilde{Q}(b) = 2^n \right\}$$

then

1. $$\tilde{G} \leq \tilde{G}(4,6)$$

2. *The elements of $\tilde{G}$ have the following canonical form:*

$$\pm S^{a_0} \tag{4}$$

or

$$\pm S^{a_0} T^{b_1} \cdots S T^{b_n} S^{c_0}, \text{ for } n > 0 \text{ and } c_0 = 0 \text{ or } 1 \tag{5}$$

Where $a_0 \in \{0,1,2,3\}$ and for $j > 0$, $b_j \in \{2,4\}$.

3. *The elements of the form $\sqrt{N(g)}g = b$ with $g \in \tilde{G}$ are exactly those $b \in \mathcal{O}$ such that $\tilde{Q}(b)$ is a power of two but b is not divisible by two. We shall denote this set $\mathcal{O}_2$.*

Proof. To show the first result we just have to check that conjugation by elements of $\tilde{G}$ acts like an element of $SO_{3,Q}$, but this is clear by the following simple calculation. Let $\frac{b}{\sqrt{\tilde{Q}(b)}} \in \tilde{G}$ with $b \in \mathcal{O}$ and $\tilde{Q}(b)$ a power of two. If v has integral coordinates then

$$\frac{b}{\sqrt{\tilde{Q}(b)}} v \frac{\bar{b}}{\sqrt{\tilde{Q}(b)}} = \frac{bv\bar{b}}{\tilde{Q}(b)}$$

has coordinates in $\mathbb{Z}[1/2]$ as needed.

To show that the canonical form of an element of $\tilde{G}$ is given as indicated. We check that the elements of $\tilde{G}$ are those elements of $\tilde{G}(4,6)$ with no T^3 in their canonical form ($b_0 = 0$).

Using that T^2 and T^4 are element of $\mathcal{O}$ over 2, S is an element of $\mathcal{O}$ over $\sqrt{2}$, and T^3 is an element of $\mathcal{O}$ over $2\sqrt{3}$ and using the canonical form for $\tilde{G}(4,6)$ we can see that every element g of $\tilde{G}(4,6)$ can be multiplied by $(\sqrt{3})^{b_0}\sqrt{2}^N$, for some $N \in \mathbb{Z}_{\geq 0}$ to obtain an element $\left((\sqrt{3})^{b_0}\sqrt{2}^N\right) g \in \mathcal{O}$ where the b_0 is 1 when T^3 appears in the canonical form and b_0 is 0 otherwise. If $b_0 = 1$ then $3 * 2^N = d^2 N(g)$ for some $d\mathbb{Z}$, which implies that $3|N(g)$ so that $b_0 = 0$. On the other hand, if $b_0 = 0$ then clearly $d^2 N(g) = 2^N$, so clearly $N(g)$ is a power of 2 as needed to show the elements of $\tilde{G}$ are exactly those with $b_0 = 0$ in their canonical forms.

The last statement is clear. If there is a $b \in \mathcal{O}$ such that $g = \frac{b}{\sqrt{\tilde{Q}(b)}}$ with $\tilde{Q}(b)$ a power of two then we could divide the numerator and denominator by 2, in which case $\tilde{Q}(b)$ is still a power of two.

$\square$

Theorem 8. *The Hecke operator which sums over elements of $\tilde{G}$ with a given height applied to $f \in \Lambda_{++}$ is given by:*

$N(g)$	$f \mapsto \sum f(g)$
1	$2(1+S^2)H_0$
2	$2(1+S^2)SH_0$
4	$2(1+S^2)SH_1$
2^n *for* $n \geq 3$	$2(1+S^2)(H_n + SH_{n+1})$

Lemma 6. *The number of elements $b \in \mathcal{O}_2$ with a given $\tilde{Q}(b)$ is equal to the number of elements $g \in \tilde{G}$ with the same $N(g)$ and is equal to:*

$N(g)$	$\tilde{Q}(b)$	*number of elements*
1	1	4
2	2	4
4	4	16
$2^\alpha,\ \alpha \geq 3$	$2^\alpha,\ \alpha \geq 3$	$4(3 * 2^{\alpha-1})$

Proof. Let n be a positive integer and let $\#(n)$ be the number of representations of n as $a^2 + b^2 + 3c^2 + 3d^2$ (order matters). It is well known (See, for example, the paper "Nineteen Quadratic forms by Alaca, Alaca, Lemire, and Williams") that if

$$n = 2^\alpha 3^\beta N$$

with N an integer such that $\gcd(N, 6) = 1$ then

$$\#(n) = 4(2^{\alpha+1} - 3)\sigma(N), \qquad \text{if } \alpha > 0$$
$$\#(n) = 4\sigma(N), \qquad \text{if } \alpha = 0$$

where σ is the sum of the divisors.

In particular,

$$\#(1) = 4$$

And if $\alpha > 0$ and $n = 2^\alpha$ then

$$\#(n) = 4(2^{\alpha+1} - 3)$$

corresponding to those elements in the Quaternion order, $\mathcal{O}$, whose norm is equal to 2^α. Since those elements with norm 2^α divisible by 2 are exactly elements of the form $2z$ with $\tilde{Q}(z) = 2^{\alpha-2}$ there are $4(2^{\alpha+1} - 3) - 4(2^{\alpha-1} - 3) = 4(3 * 2^{\alpha-1})$ elements whose norm is 2^α, but are not divisible by 2 when $\alpha \geq 3$. When $\alpha = 2$, there are 16 elements with norm 2^α, not divisible by 2. □

Suppose that $g \in \tilde{G}$, we will show that applying right multiplication by $M = ST^2$ or $M = ST^4$ can only multiply the $N(g)$ by a factor of 1/8, 1/2, 2, or 8 and that in nearly every case $N(g)$ is either multiplied by 1/2 or 2. More specifically:

Lemma 7. *Let $g \in \tilde{G}$ and let M be ST^2 or ST^4 then*

Case 1:

$$N(g) = 1 \Rightarrow g = \pm 1 \text{ or } g = \pm i = \pm S^2 \Rightarrow N(gM) = 8$$

Case 2:

$$N(gM) = 1 \Rightarrow gM = \pm 1 \text{ or } gM = \pm i = \pm S^2 \Rightarrow N(g) = 8$$

Case 3: Otherwise, $N(gM) = 2N(g)$ *or* $N(gM) = N(g)/2$

Proof. Let $g \in \tilde{G}$ and let M be ST^2, ST^4, or one of their inverses.

We already know that $N(g)$ and $N(gM)$ are both powers of two. We also have:

$$\sqrt{N(g)}g\sqrt{N(h)}h = \sqrt{N(g)}\sqrt{N(h)}gh \in \mathcal{O}$$

which implies

$$N(gh) \leq N(g)N(h)$$

calculating

$$\sqrt{8}ST^2 = 2\sqrt{2}ST^2 = 2\sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{1+J}{2}\right) = 1 + i + J + K = \sqrt{N(ST^2)}ST^2$$

and

$$\sqrt{8}ST^4 = 2\sqrt{2}ST^4 = 2\sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{-1+J}{2}\right) = -1-i+J+K = \sqrt{N(ST^4)}ST^4$$

we see that $N(ST^2) = N(ST^4) = 8$ and similarly for their inverses.

So we have $N(gM) \leq 8N(g)$. Letting $g = gM$ and $M = M^{-1}$ we have $N(gMM^{-1}) = N(g) \leq 8N(gM)$ and therefore, $\frac{1}{8}N(g) \leq N(gM)$.

Since M has a $\sqrt{2}$ in the denominator, $2^k M\sqrt{N(g)}g \notin \mathcal{O}$ for all $k \in \mathbb{Z}$ therefore, the following cannot be equal to $N(gM)$:

$$\frac{N(g)}{4}, N(g), \text{or } 4N(g)$$

Let $z = a + bi + cJ + dK \in \mathcal{O}_2$ with $\tilde{Q}(z) = ||z||^2 = 2^\alpha$. Viewing z as a vector in coordinates $(1, i, J, K)$. We first show that every element of $\mathcal{O}_2$ has an even number of odd entries except ± 1 or $\pm i$.

We have:

1.) $\tilde{Q}(z) = 2^0 \Rightarrow z \in \{\pm 1, \pm i\}$ (in which case there is exactly one odd entry)

2.) $\tilde{Q}(z) = 2^1 \Rightarrow z \in \{\pm 1 + i, \pm 1 - i\}$ (in which case there is exactly two odd entries)

3.) For $\tilde{Q}(z) = 2^\alpha$ with $\alpha \geq 2$, either all of a, b, c, d are odd or exactly one of a, b is and exactly one of c, d is, so these have four and two odd entries, respectively.

The first two cases are easy to check. For the third case, consider the equation

$$a^2 + b^2 + 3c^2 + 3d^2 = 2^\alpha$$

modulo 4. We have:

$$a^2 + b^2 - c^2 - d^2 \equiv 0$$

The only odd square mod 4 is 1. They cannot all be even since $2 \nmid z$. So, the only possibilities are either all of a, b, c, and d are odd or exactly one of a and b and exactly on of c and d as needed.

Notice that if $z = a + bi + cJ + dK$ with all of a, b, c, and d odd then the matrix associated with the operator

$$R_z : B \to B$$

$$v \mapsto vz$$

has all odd entries in the basis $\{1, i, J, K\}$.

This means, in particular, since $\sqrt{N(ST^2)}ST^2$, $\sqrt{N(ST^4)}ST^4$ and their conjugates, $\sqrt{N((ST^4)^{-1})}(ST^4)^{-1}$ and $\sqrt{N((ST^2)^{-1})}(ST^2)^{-1}$, have four odd entries when viewed as vectors in the basis $\{1, i, J, K\}$, letting M be one of ST^2, ST^4, or their inverses, tells us the matrix corresponding to right multiplication by M,

$$R_{\sqrt{N(M)}M}$$

has all odd entries.

Let M be ST^2 or ST^4. Since the product of a vector with an even number of odd entries by a matrix with all odd entries is an even vector we know that the only time $N(gh)$ is equal to $8N(g)$ is when $g \in \{\pm 1, \pm i\}$. This also tells us that $N(gh)$ is equal to $N(g)/8$ only when $gh \in \{\pm 1, \pm i\}$. $\square$

Proof of Theorem 8. We now consider elements in the canonical forms:

$$\pm S^{a_0} \tag{6}$$

or

$$\pm S^{a_0} T^{b_1} \cdots ST^{b_n} E, \text{ for } n > 0 \tag{7}$$

The first case consists of the four elements, g, with $N(g) = 1$ given by ± 1 and $\pm i = \pm S^2$ as well as the four elements with $N(g) = 2$, $\pm(i \pm 1) = (\pm S^{\pm 1})$. The elements with $N(g) = 1$ are

$$\pm E \text{ with } E = 1 \text{ or } S^2$$

Recalling that $H_0 = I$, and using the fact that the $f \in \Lambda_{++}$ consists of homogenous polynomials of even degree the action of ± 1 fixes f ($f(-z) = f(z)$) is not hard to see that the corresponding Hecke operator is given by:

$$2(1 + S^2)H_0$$

The elements with $N(g) = 2$ are

$$\pm ES \text{ with } E = 1 \text{ or } S^2$$

So it is similarly not hard to see that the corresponding Hecke operator is given by:

$$2(1 + S^2)SH_0$$

Those elements whose canonical forms are in the second case therefore consist of all elements, $g \in \tilde{G}$, with $N(g) = 2^\alpha$ and $\alpha \geq 2$.

When $N(g) = 4$ there are 16 elements given by $g = \pm 1 \pm J$, $\pm i \pm J$, $\pm i \pm J$, $\pm i \pm K$. It is not hard to check that these are elements of the form:

$$\pm Eg \text{ where } g \text{ is of the form } T^{a_0} \text{ or } ST^{a_0}S \text{ with } a_0 = 2 \text{ or } 4$$

Recalling that H_1 is the Hecke operator that sums over $\{ST^{a_0} \text{ or } T^{a_0}S\}$, it is not hard to check using the relations that the corresponding Hecke operator is:

$$2(1 + S^2)SH_1$$

Suppose that $2^\alpha = N(g) \geq 8$ there are $4(3 * 2^{\alpha-1})$ of these. There are 4 units, ± 1 and $\pm S^2 = \pm i$. Using the canonical form, it is not hard to see that left multiplication by a unit and conjugation by S results in 8 distinct elements so left multiplication by a unit and conjugation by S partitions the set of $g \in \tilde{G}$ of height 2^α into $3 * 2^{\alpha-2}$ equivalence classes. Furthermore, using the canonical form it is not hard to see that each class has a unique representative of the form:

$$T^{b_1}ST^{b_1} \cdots T^{b_{n+1}}$$

or

$$ST^{b_1}ST^{b_2} \cdots T^{b_n}$$

with $a_i \in \{2, 4\}$.

We will argue by induction that these have height 2^{n+2}.

First verify that elements of the form ST^{b_1} and $T^{b_1}ST^{b_2}$ have height 8 as needed for our base case when $n = 1$.

Now assume the theorem is true for all $m < n$ for some $n > 1$. We wish to show that:

$$T^{b_1}ST^{b_2} \cdots ST^{b_{n+1}} \tag{8}$$

and

$$ST^{b_1}ST^{b_2} \cdots T^{b_n} \tag{9}$$

have height 2^{n+2}.

Notice that by the inductive assumption elements of the form:

$$T^{b_1} \cdots ST^{b_{m+1}} \tag{10}$$

or

$$ST^{b_1}ST^{b_2} \cdots T^{b_m} \tag{11}$$

have height 2^{m+2}.

Notice that there are 2^{m+1} elements of form $(T^{b_1} \cdots ST^{b_{m+1}})$ and 2^m elements of form $(ST^{b_1}ST^{b_2} \cdots T^{b_m})$ which adds up to $3 * 2^m = 3 * 2^{(m+2)-2}$. So these are all the elements whose norm is equal to 2^{m+2} up to left multiplication by a unit and conjugation by S. Collectively these exhaust those elements of norm 2^α for $3 \leq \alpha \leq n+1$.

Recall that right multiplication by ST^{b_j} (for $b_j \in \{2,4\}$) multiplies or divides the height of these elements by 2. Now consider an element of the form $ST^{b_1}ST^{b_2} \cdots T^{b_{n-1}}$. This has height 2^{n+1}. So multiplication on the right by ST^{b_n} must have either have height 2^n or 2^{n+2} and gives us an element of the form $T^{b_1}ST^{b_1} \cdots ST^{b_n}$. Since this element is not of form $T^{b_1} \cdots ST^{b_{m+1}}$ or $ST^{b_1}ST^{b_2} \cdots T^{b_m}$ with $m < n$ it cannot have height less than or equal to 2^n and therefore must have height 2^{n+2} as needed.

Similarly, elements of the form

$$T^{b_1}ST^{b_2} \cdots T^{b_n}$$

when multiplied on the right by $ST^{b_{n+1}}$ are of the form

$$T^{b_1}ST^{b_2} \cdots T^{b_{n+1}}$$

with height 2^{n+2}.

So now we have that elements of the form

$$T^{b_1}ST^{b_2} \cdots ST^{b_{n+1}} \tag{Form 1}$$

and

$$ST^{b_1}ST^{b_2} \cdots T^{b_n} \tag{Form 2}$$

are exactly those elements with height 2^{n+2} up to conjugation by S and multiplication by a unit.

H_n is the Hecke operator that sums over elements of the form:

$$ST^{b_1}ST^{b_2} \cdots T^{b_n} \cup T^{b_1}ST^{b_2} \cdots T^{b_n}S$$

those elements to the right of the $\cup$ are the conjugates of those on the left. Multiplying these elements on the left by the units fills out the equivalence classes of those elements in (Form 2).

SH_{n+1} is the Hecke operator that sums over elements of the form:

$$S^2T^{b_1}ST^{b_2} \cdots ST^{b_{n+1}} \cup ST^{b_1}ST^{b_2} \cdots ST^{b_{n+1}}S$$

the elements to the left of $\cup$ are of the form S^2 (a unit) times elements of (Form 1). The elements to the right are conjugates of those on the left of the $\cup$. Multiplying these on the left by a unit fills out the equivalence classes of those elements in (Form 1).

Together these elements exhaust those of height 2^{n+2}.

On Λ_{++}, left multiplication by a unit has the following hecke operator:

$$2(1 + S^2)$$

This is because the units are ± 1 and $\pm S$, but $f \in \Lambda_{++}$ are homogeneous polynomials of even degree so that multiplication by ± 1 fixes these elements. Putting this all together we have that the Hecke Operator that sums over those elements of height 2^{n+2} for $n \geq 1$ is given by:

$$2(1 + S^2)[H_n + SH_{n+1}]$$

$\square$