The Spectrum of the Quaquaversal Operator is Real

Josiah Sugarman

Abstract

In this paper we resolve a conjecture due to Draco, Sadun and Van Wieren that the spectrum of an operator associated with Conway and Radin's Quaquaversal Tiling is real.

1 Acknowledgements

I'd like to thank my advisor Alexander Gamburd for suggesting this problem and Kieran O'Reilly for many helpful conversations.

2 Introduction and Background

Let $T \subseteq \mathbb{R}^d$ be a convex polytope with nonempty interior. Let $T = \cup_{i=1}^{k} \phi_i(T)$ with $\phi_i(T)$ and $\phi_j(T)$ disjoint except perhaps on the boundary and with ϕ_i affine maps given by $\phi_i = cg_i + v_i$ where $c = (\frac{1}{k})^{\frac{1}{d}}$, $g_i \in O(d)$, and $v_i \in \mathbb{R}^d$ as in 1. We call T together with the ϕ_i a subdivision rule and the $\phi_i(T)$ are T's *daughter tiles*. Given a subdivision rule, one can 'reverse this' and instead of tiling T with k tiles congruent to cT one tiles $\phi_i^{-1}(T) \cong c^{-1}T$ with k tiles congruent to T $(\phi_i^{-1}\phi_j)$ then tiles $\phi_{i'}^{-1}\phi_i^{-1}(T) \cong c^{-2}T$ with k tiles congruent to $c^{-1}T$ (and therefore k^2 tiles congruent to T) and so on until all of R^d is tiled as in figure 2. We call the resulting tiling a *substitution tiling.*

Introduced by John Conway and Charles Radin in 1998[3], the Quaquaversal Tiling, whose subdivision rule is shown in figure 3, is a three dimensional substitution tiling with the property that the orientations of its tiles converge to a uniform distribution faster than what is possible for substitution tilings in two dimensions.

In two dimensions $SO(2)$ is abelian and $O(2)$ is 'dihedral' both groups having polynomial growth. Since the orientation of a tile that appears after n iterations of a substitution rule is given by a word of length n in the orientations of the daughter tiles it is not hard to see that the orientations of tiles must grow on the order of a polynomial in the logarithm of the radius of the sphere containing them.

The Quaquaversal Tiling remedies this issue by taking advantage of the non-commutative nature of rotations in three dimensions. Indeed the orientations of the daughter tiles in the substitution rule generate a subgroup with the property that the number of distinct words grows exponentially in the word-length rather than polynomially. This implies that the number of distinct orientations of tiles within a sphere is on the order of the radius raised to some power.

Something stronger is true. If $S = \{g_1, \ldots, g_8\}$ are the orientations of the daughter tiles with respect to the mother tile (see figure 3 for the eight rotations corresponding to the orientations of the daughter tiles), let W^N denote the set of all words of length N in $g_1, \ldots, g_8$ (the set of all orientations with respect to "common ancestor" after N iterations) then there is a λ strictly less than 1 so that for all $f \in L^2(SO(3))$:

$$\left\| \frac{1}{8^N} \sum_{w \in W^N} f(w^{-1}x) - \int_{SO(3)} f(x) d\mu(x) \right\|_2 \ll_N \lambda^N$$

where μ is the Haar measure.

This can be read as– 'The distribution of orientations approaches the uniform distribution with respect to the Haar measure in the weak sense exponentially fast'.

The expression $\frac{1}{8^N} \sum_{w \in W^N} f(w^{-1}x)$ is the N^{th} iterate of the *Quaquaversal Operator* $\mathcal{T}_z$ applied to f. Here z is the formal sum of the g_i and $\mathcal{T}_z$ is defined by:

$$(\mathcal{T}_z(f))(x) = \frac{1}{8} \sum_{g_i \in S} f(g_i^{-1}x)$$

Conway and Radin proved that the orientations of the tiles approach a uniform distribution (though not exponentially fast) by doing what was essentially showing that the eigenvalue 1 only occurs with multiplicity 1 for $\mathcal{T}_z$. Draco, Sadun, and Van Wieren[4] studied the spectrum of T_z (technically, they multiply on the right by g_i, but these operators are isomorphic) numerically and found eigenvalues larger than 0.9938 showing that the rate with which this tiling approaches uniformity is quite slow—stating that "the asymptotic roundness of the Quaquaversal Tiling, while mathematically correct, is well beyond the range of any conceivable physics"; Bourgain and Gamburd[1] showed that if a set of elements of SU(2) or SO(3) has algebraic entries and if the group it generates contains a free group then the associated Hecke operator T_z must have a spectral gap. Since the orientations of the daughter tiles are matrices with algebraic entries and the group they generate contains a free group, this proves the full exponentially rapid equidistribution result mentioned above.

Tile	Orientation	Translation
T_0	$R_\beta R_\pi Q$	$(2/5, 4/5)$
T_1	$R_\beta R_{-\pi/2} Q$	$(1/5, 2/5)$
T_2	$R_\beta R_{-\pi/2}$	$(1/5, 2/5)$
T_3	$R_\beta R_\pi$	$(6/5, 2/5)$
T_4	$R_\beta R_{-\pi/2} Q$	$(6/5, 2/5)$

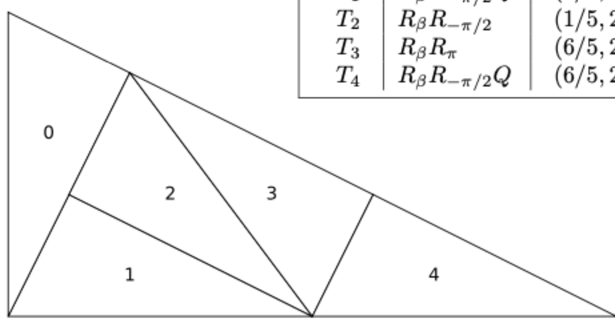


Figure 1: The substitution rule for the Pinwheel Tiling. Here $T_i = \frac{1}{\sqrt{5}}\phi_i(T)$ where $\phi_i = g_i + v_i$ where v_i is the translation vector and g_i is the relative orientation of T_i with respect to T indicated in the figure.

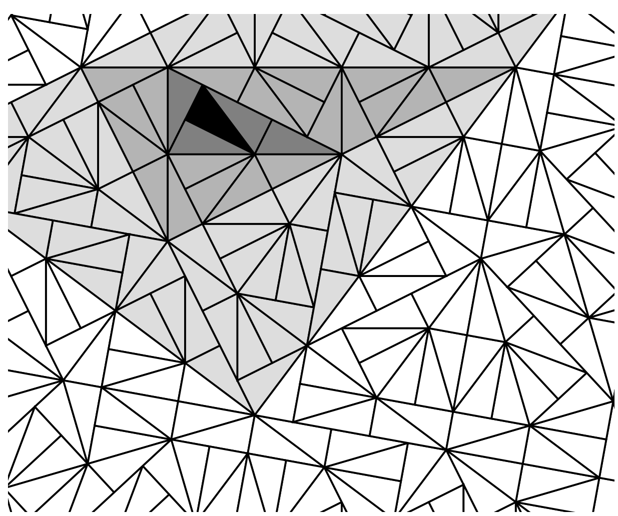
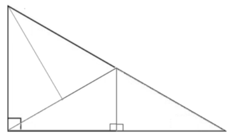
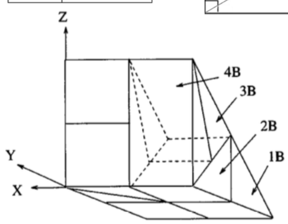


Figure 2: This is a patch of the Pinwheel Tiling. Tiles are of the form $\phi_3^{-n}\omega T$ with $|w| = n$. Darker tiles correspond to tiles of the form $\phi_3^{-n}\omega T$ with smaller n. These are the tiles that appear after n iterations of the substitution-inflation rules.

Tile	Rotation
$1B$	R_x^0
$2B$	R_z^π
$3B$	$\mathrm{R}_x^{4\pi/3}\mathrm{R}_y^\pi$
$4B$	$\mathrm{R}_x^{4\pi/3}$





Tile	Rotation
$1A$	R_x^0
$2A$	$\mathrm{R}_z^\pi\mathrm{R}_y^{\pi/2}$
$3A$	$\mathrm{R}_y^{\pi/2}$
$4A$	R_x^0

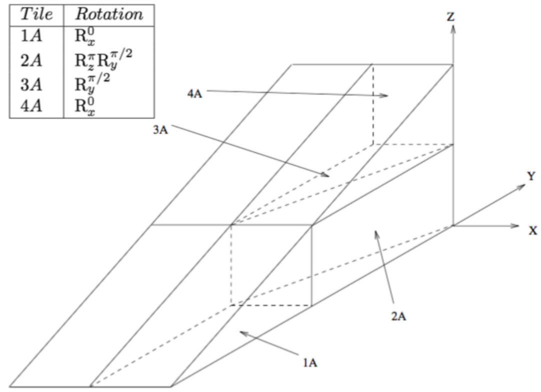


Figure 3: Subdivision Rule for the Quaquaversal Tiling. Adapted from [4]

While studying the spectrum of this operator numerically, Draco, Sadun, and van Wieren [4] observed and conjectured that the spectrum is real. The main result of this paper is the proof of this conjecture given in section 4. In outline, the proof proceeds as follows: The Peter-Weyl theorem allows us to decompose $L^2(SO(3))$ into finite dimensional irreducible representations. The operator associated with the Quaquaversal Tiling then decomposes into finite dimensional operators on each of these irreducible representations. We find a well chosen partition of each irreducible representation (as a vector space) that this operator is lower block triangular with respect to and verify that the blocks along the main diagonal are Hermitian.

2.1 Definitions and Notation

If k is a positive integer then we shall let $[k]$ denote the set $\{0, \ldots, k-1\}$ so that $[0]$ denotes the empty set (this is in addition to and does not replace the usual, $\emptyset$)

Let $u \in \mathbb{R}^3$ be a nonzero vector and choose an orientation. We define R_u^θ to be the counterclockwise rotation by θ fixing u. More specifically θ is measured in the following manner. Since $u \neq 0$, we may choose perpendicular (to u as well as each other) vectors u_0 and u_1 in $u^\perp$ such that $u_0 \times u_1 = u$ then the angle θ is measured from u_0 to u_1.

In two dimensions there is naturally no need to specify an axis and we let R_θ denote the counterclockwise rotation by θ.

Note that $R_u^\theta = R_{-u}^{-\theta}$. The symbols R_x^θ, R_y^θ, and R_z^θ are shorthand for $R_{(1,0,0)}^\theta$, $R_{(0,1,0)}^\theta$, and $R_{(0,0,1)}^\theta$ respectively.

Let G be a semigroup (in particular, a group or a monoid) and $\{g_1, \ldots, g_k\}$ be a multi-set of elements in G. A word in $\{g_1, \ldots, g_k\}$ is an element of the form $g_{i_1}^{m_1} g_{i_2}^{m_2} \cdots g_{i_n}^{m_n}$ where the m_n are positive integers. If w is a word in $\{g_1, \ldots, g_k\}$ then the length of w, denoted $|w|$, is the smallest $m_1 + \cdots + m_n$ with $w = g_{i_1}^{m_1} g_{i_2}^{m_2} \cdots g_{i_n}^{m_n}$ and $W^N\{g_1, \ldots, g_k\}$ denotes words in the free semigroup on k generators of length N evaluated at $g_1, \ldots, g_k$. When there is no ambiguity we denote $W^N\{g_1, \ldots, g_k\}$ simply W^N.

3 Statement of the Draco, Sadun, van Wieren Conjecture

In this section we observe that $\mathcal{T}_z$ can be block diagonalized with an infinite number of blocks of finite size. Then we carefully state the conjecture that this operator has a real spectrum, which we prove in the next section.

3.1 Peter-Weyl Theorem

Given a compact group G, as usual let $L^2(G)$ be the space of L^2 functions with respect to the Haar measure. This is a Hilbert space in the usual way with a representation defined via $(g \cdot f)(z) = f(g^{-1}z)$. We call this representation the *left regular representation* or just the *regular representation*

Theorem (Peter-Weyl[6][2])**.**

Definition 1. *A matrix coefficient is a function $f : G \to \mathbb{C}$ of the form $\ell \circ \rho$ where $\ell : GL_n \subset \mathbb{R}^{n^2} \to \mathbb{C}$ is a linear functional and $\rho : G \to U(n) \leq GL_n$ is a finite dimensional unitary representation. We say a matrix coefficient is irreducible if ρ is.*

The left regular representation decomposes into finite dimensional irreducible representations:

$$L^2(G, \mu) = \oplus_\rho V_\rho^{m_\rho}$$

This direct sum is indexed by every *irreducible representation, ρ. These irreducible representations are necessarily finite dimensional with dimension m_ρ, which is also their multiplicity in the decomposition of the regular representation. The $V_\rho^{m_\rho}$ are the matrix coefficients coming from ρ.*

The compact group SU(2) has irreducible representations given by:

$$H_{n+1} := S^n H_2$$

where n ranges over $\mathbb{Z}_{\geq 0}$, H_2 is the standard two dimensional representation, $g \cdot v = gv$, and S^n denotes the n^{th} symmetric power functor[7]. If V is a representation of G then $S^n V$ inherits a representation of G via:

$$g \bullet (v \circ \cdots \circ v) = gv \circ \cdots \circ gv$$

when $G = \mathrm{SU}(2)$ the $S^n H_2$ are exactly the irreducible representations of SU(2) (the zeroth symmetric power is the trivial representation) and their dimension is equal to n. The compact group, SO(3), is (homomorphically) double covered by SU(2) and we have the short exact sequence:

$$1 \to \{\pm 1\} \to \mathrm{SU}(2) \to \mathrm{SO}(3) \to 1$$

Every irreducible representation of SO(3) lifts to an irreducible representation of SU(2) and those that factor through SO(3) are exactly the odd dimensional symmetric powers, $H_{2\ell+1}$

This together with the Peter-Weyl theorem tells us that:

$$L^2(\mathrm{SO}(3)) = \oplus_{\ell\in\mathbb{Z}} H_{2\ell+1}^{2\ell+1}$$

This decomposition implies that the operators on $L^2(\mathrm{SO}(3))$ induced by elements of the group ring, $\sum_i \lambda_i g_i \in \mathbb{C}[\mathrm{SO}(3)]$ may be decomposed into direct sums of finite dimensional operators on $H_{2\ell+1}$ with multiplicity $2\ell+1$ given by $\sum_i \lambda_i \rho_{2\ell+1}(g_i)$[5]. Here, of course $\rho_{2\ell+1} : G \to U(H_{2\ell+1})$ is the irreducible unitary representation of dimension $2\ell+1$.

In particular, we conclude that the element of the group ring corresponding to the Quaquaversal Tiling,

$$z = \left(I + I + I + R_x^{\pi/2} + R_x^{\pi/2} R_y^{\pi} + R_z^{\pi} + R_y^{4\pi/3} + R_y^{4\pi/3} R_x^{\pi}\right)$$

has for every $\ell \in \mathbb{Z}_{\geq 0}$, a finite dimensional linear operator $z_\ell : \mathbb{R}^{2\ell+1} \to \mathbb{R}^{2\ell+1}$ given by:

$$z_\ell := \pi_{2\ell+1}\left(3I + R_x^{\pi/2} + R_x^{\pi/2} R_y^{\pi} + R_z^{\pi} + R_y^{4\pi/3} + R_y^{4\pi/3} R_x^{\pi}\right)$$

and T_z decomposes into a direct sum of these finite dimensional operators.

In the course of their numerical experiments, Draco, Sadun, and Van Wieren[4] observed and conjectured that T_z or, by the above discussion equivalently, that the infinite set of finite dimensional linear operators, $\{z_\ell \in U(2\ell+1)\}_{\ell\in\mathbb{Z}_{\geq 0}}$ has a real spectrum. Let us now restate this as one contiguous theorem and then prove it in the next section.

Theorem 1. *The quaquaversal tiling is a substitution tiling with associated element of the group ring given by:*

$$z = 1/8(1 + 1 + 1 + S^2T^3 + T^4 + T^4S^2 + S + ST^3)$$

where $S = R_x^{\pi/2}$ *and* $T = R_y^{\pi/3}$ *are rotations about orthogonal vectors,* $\vec{x}$*, and* $\vec{y}$*, by* $\pi/2$ *and* $\pi/3$*, respectively. The associated operator*

$$\pi(z) : L^2(\mathrm{SO}(3)) \to L^2(\mathrm{SO}(3))$$

or equivalently the infinite family of finite dimensional operators

$$\pi_{2k+1}(z) : \mathcal{H}_{2k+1} \to \mathcal{H}_{2k+1}$$

have a real spectrum.

4 Proof of the Draco, Sadun, Van Wieren Conjecture

In this chapter we prove that the spectrum of the Quaquaversal Operator is real. To do this we find a partition that this operator is block lower triangular with respect to and then check that the operator has hermitian blocks along the main diagonal. Indeed, the operators on $L^2(SO(3))$ defined by $[R_x^\pi(f)](O) = f(R_x^{-\pi}O)$ and $[R_y^\pi(f)](O) = f(R_y^{-\pi}O)$ are commuting and the relevant partition is given by their simultaneous eigenspaces. It will turn out that operators of the form $R_x^\theta + R_x^\theta R_y^\pi$ and $R_y^\theta + R_y^\theta R_x^\pi$ are both lower block triangular with respect to this partition and have Hermitian blocks along the main diagonal.

In this section our discussion takes place in an arbitrary, yet fixed, unitary, irreducible representation of SO(3), $\pi_{2\ell+1} : \mathrm{SO}(3) \to U(H_{2\ell+1})$. We will write $\pi_{2\ell+1}(R_u^\theta)$ and the like as R_u^θ, suppressing the representation in the notation for the remainder of this chapter.

Given a pair of orthogonal vectors (u, w) in $\mathbb{R}^3$ we define $\Lambda_{\alpha;\beta}$ by

$$\{v : R_u^\pi v = \alpha v \text{ and } R_w^\pi v = \beta v\}$$

when we wish to make explicit the dependence on (u, w) we write $\Lambda_{\alpha;\beta}(u, w)$. Notice that $\Lambda_{\alpha;\beta}(u, w) = \Lambda_{\beta;\alpha}(w, u)$.

Since $(R_u^\pi)^2 = (R_w^\pi)^2 = I$ these operators can only have eigenvalues ± 1. This means that $\Lambda_{\alpha;\beta}$ defines at most four nontrivial subspaces $\Lambda_{\pm 1;\pm 1}$. Since R_x^π and R_y^π are unitary, they are diagonalizable. In fact, they each have an orthonormal basis of eigenvectors. Furthermore, $R_x^\pi R_y^\pi = R_y^\pi R_x^\pi$ and therefore they are commuting diagonalizable maps. It is a theorem that commuting diagonal maps have a basis of simultaneous eigenvectors. Altogether this implies that $H_{2\ell+1}$ decomposes in the following manner:

$$H_{2\ell+1} = \Lambda_{+1;+1} \oplus \Lambda_{+1;-1} \oplus \Lambda_{-1;+1} \oplus \Lambda_{-1;-1}$$

We shall denote this partition in this order $\Pi(u, v)$.

Since the 1-eigenspace and the -1-eigenspace are orthogonal for R_u^π and for R_w^π the $\Lambda_{\alpha;\beta}$ are actually orthogonal to one another. We shall let $P_{\alpha;\beta}$ be the orthogonal projection onto $\Lambda_{\alpha,\beta}$. Notice that it has kernel

$$\oplus_{(\alpha',\beta') \neq (\alpha,\beta)} \Lambda_{\alpha';\beta'}$$

I will omit the 1's so that, for example, $P_{-+} = P_{-1;+1}$.

Lemma 1.

1. *$AB = BA$ implies $Av = \lambda v \Rightarrow A(Bv) = \lambda(Bv)$*
2. *$AB = -BA$ implies $Av = \lambda v \Rightarrow A(Bv) = -\lambda(Bv)$*

Proof. Let $Av = \lambda v$ and $AB = BA$ then $ABv = BAv = \lambda Bv$

Let $Av = \lambda v$ and $AB = BA$ then $ABv = -BAv = -\lambda Bv$

□

Lemma 2. *Let $u, v \in \mathbb{R}^2$ be perpendicular, then:*

1. $$(R_u^\theta + R_u^{-\theta})R_v^\pi = R_v^\pi(R_u^\theta + R_u^{-\theta})$$

2. $$(R_u^\theta - R_u^{-\theta})R_v^\pi = -R_v^\pi(R_u^\theta - R_u^{-\theta})$$

3. $$(R_u^\theta + R_u^{-\theta})R_u^\pi = R_u^\pi(R_u^\theta + R_u^{-\theta})$$

4. $$(R_u^\theta - R_u^{-\theta})R_u^\pi = R_u^\pi(R_u^\theta - R_u^{-\theta})$$

Proof. The first two statements follow from the relation:

$$R_u^\theta R_v^\pi = R_v^\pi R_u^{-\theta}$$

$$(R_u^\theta + R_u^{-\theta})R_v^\pi = R_u^\theta R_v^\pi + R_u^{-\theta} R_v^\pi = R_v^\pi R_u^{-\theta} + R_v^\pi R_u^\theta = R_v^\pi(R_u^\theta + R_u^{-\theta})$$

$$(R_u^\theta - R_u^{-\theta})R_v^\pi = R_u^\theta R_v^\pi - R_u^{-\theta} R_v^\pi = R_v^\pi R_u^{-\theta} - R_v^\pi R_u^\theta = -R_v^\pi(R_u^\theta - R_u^{-\theta})$$

The next two statements follow from the relation:

$$R_u^\theta R_u^\pi = R_u^\pi R_u^\theta$$

$$(R_u^\theta + R_u^{-\theta})R_u^\pi = R_u^\theta R_u^\pi + R_u^{-\theta} R_u^\pi = R_u^\pi R_u^\theta + R_u^\pi R_u^{-\theta} = R_u^\pi(R_u^\theta + R_u^{-\theta})$$

$$(R_u^\theta - R_u^{-\theta})R_u^\pi = R_u^\theta R_u^\pi - R_u^{-\theta} R_u^\pi = R_u^\pi R_u^\theta - R_u^\pi R_u^{-\theta} = R_u^\pi(R_u^\theta - R_u^{-\theta})$$

□

Corollary 1. *Let u and v be perpendicular vectors and define $\Lambda_{\alpha,\beta}$ with respect to (u, v) then the hermitian part of R_u^θ,*

$$[R_u^\theta]^+ := \frac{R_u^\theta + R_u^{-\theta}}{2}$$

maps $\Lambda_{\alpha,\beta}$ onto $\Lambda_{\alpha,\beta}$

The skew hermitian part of R_u^θ,

$$[R_u^\theta]^- := \frac{R_u^\theta - R_u^{-\theta}}{2}$$

maps $\Lambda_{\alpha,\beta}$ onto $\Lambda_{\alpha,-\beta}$

Therefore,

$$R_u^\theta = [R_u^\theta]^+ + [R_u^\theta]^-$$

has the following block decomposition with respect to

$$\Pi(u, v) = \Lambda_{+1;+1} \oplus \Lambda_{+1;-1} \oplus \Lambda_{-1;+1} \oplus \Lambda_{-1;-1}$$

$$\begin{pmatrix} P_{++}[R_u^\theta]^+P_{++} & P_{++}[R_u^\theta]^-P_{+-} & 0 & 0 \\ P_{+-}[R_u^\theta]^-P_{++} & P_{-+}[R_u^\theta]^+P_{+-} & 0 & 0 \\ 0 & 0 & P_{-+}[R_u^\theta]^+P_{-+} & P_{-+}[R_u^\theta]^-P_{--} \\ 0 & 0 & P_{--}[R_u^\theta]^-P_{-+} & P_{--}[R_u^\theta]^+P_{--} \end{pmatrix}$$

Proof. The second lemma states that the hermitian part of R_u^θ commutes with both R_u^π and R_v^π. The first lemma tells us this implies that both eigenspaces are preserved and therefore,

$$[R_u^\theta]^+ : \Lambda_{\alpha,\beta} \to \Lambda_{\alpha,\beta}$$

The second lemma states that the skew-hermitian part of R_u^θ commutes with both R_u^π and anticommutes with R_v^π. The first lemma tells us this implies that the R_u^π eigenspace is preserved and the R_v^π λ-eigenspace maps onto the R_v^π $-\lambda$-eigenspace. Therefore,

$$[R_u^\theta]^- : \Lambda_{\alpha,\beta} \to \Lambda_{\alpha,-\beta}$$

□

Corollary 2. *$R_u^\theta + R_u^\theta R_v^\pi = 2R_u^\theta\left(\frac{(I+R_v^\pi)}{2}\right)$ has the following block decomposition with respect to*

$$\Pi(u,v) = \Lambda_{+1;+1} \oplus \Lambda_{+1;-1} \oplus \Lambda_{-1;+1} \oplus \Lambda_{-1;-1}$$

$$2\begin{pmatrix} P_{++}[R_u^\theta]^+P_{++} & 0 & 0 & 0 \\ P_{+-}[R_u^\theta]^-P_{++} & 0 & 0 & 0 \\ 0 & 0 & P_{-+}[R_u^\theta]^+P_{-+} & 0 \\ 0 & 0 & P_{--}[R_u^\theta]^-P_{-+} & 0 \end{pmatrix}$$

Proof. To show this we check that $\frac{(I+R_v^\pi)}{2}$ projects onto the 1-eigenspace of R_v^π $(\Lambda_{1,1} \oplus \Lambda_{-1,1})$ with kernel the -1-eigenspace $(\Lambda_{1,-1} \oplus \Lambda_{-1,-1})$.

Since the space decomposes into eigenspaces of R_v^π with eigenvalues ± 1. We just check the two cases.

If $R_v^\pi v = v$ then

$$\frac{(I+R_v^\pi)}{2}v = \frac{v+v}{2} = v$$

If $R_v^\pi v = -v$ then

$$\frac{(I+R_v^\pi)}{2}v = \frac{v-v}{2} = 0$$

□

We can now compute the block decomposition of the Quaquaversal operator:

$$z_\ell := \pi_{2\ell+1}\left(3I + R_x^{\pi/2} + R_x^{\pi/2}R_y^\pi + R_z^\pi + R_y^{4\pi/3} + R_y^{4\pi/3}R_x^\pi\right)$$

The second corollary gives us the decomposition of $R_x^{\pi/2} + R_x^{\pi/2}R_y^\pi$ with respect to $\Pi(x,y)$ and the decomposition of $R_y^{4\pi/3} + R_y^{4\pi/3}R_x^\pi$ with respect to $\Pi(y,x)$.

Using the fact that $\Lambda_{\alpha;\beta}(x,y) = \Lambda_{\beta;\alpha}(y,x)$ allows us to write $R_y^{4\pi/3} + R_y^{4\pi/3}R_x^{\pi}$ with respect to $\Pi(x,y)$. Adding these we see that:

$$R_x^{\pi/2} + R_x^{\pi/2}R_y^{\pi} + R_y^{4\pi/3} + R_y^{4\pi/3}R_x^{\pi}$$

is

$$2\begin{pmatrix} P_{++}[R_x^{\pi/2} + R_y^{4\pi/3}]^+P_{++} & 0 & 0 & 0 \\ P_{+-}[R_x^{\pi/2}]^-P_{++} & P_{+-}[R_y^{3\pi/4}]^+P_{+-} & 0 & 0 \\ P_{-+}[R_y^{3\pi/4}]^-P_{++} & 0 & P_{-+}[R_x^{\pi/2}]^+P_{-+} & 0 \\ 0 & P_{--}[R_y^{3\pi/4}]^-P_{+-} & P_{--}[R_x^{\pi/2}]^-P_{-+} & 0 \end{pmatrix}$$

Since the blocks along the main diagonal are all of the form PHP where P is an orthogonal projection and H is hermitian, $(PHP)^* = P^*H^*P^* = PHP$, so these blocks are hermitian. Since $R_z^{\pi} = R_x^{\pi}R_y^{\pi}$, on $\Lambda_{\alpha;\beta}$, $R_z^{\pi} = R_x^{\pi}R_y^{\pi} = \alpha I\beta I = \alpha\beta I$ this tells us that we must add $(3+\alpha\beta)I$ along the main diagonals of the block decomposition to get the block decomposition of

$$z_\ell := \pi_{2\ell+1}\left(3I + R_x^{\pi/2} + R_x^{\pi/2}R_y^{\pi} + R_z^{\pi} + R_y^{4\pi/3} + R_y^{4\pi/3}R_x^{\pi}\right)$$

with respect to $\Pi(x,y)$:

$$2\begin{pmatrix} P_{++}[R_x^{\pi/2} + 2 + R_y^{4\pi/3}]^+P_{++} & 0 & 0 & 0 \\ P_{+-}[R_x^{\pi/2}]^-P_{++} & P_{+-}[1 + R_y^{3\pi/4}]^+P_{+-} & 0 & 0 \\ P_{-+}[R_y^{3\pi/4}]^-P_{++} & 0 & P_{-+}[1 + R_x^{\pi/2}]^+P_{-+} & 0 \\ 0 & P_{--}[R_y^{3\pi/4}]^-P_{+-} & P_{--}[R_x^{\pi/2}]^-P_{-+} & 2 \end{pmatrix}$$

Since we simply added real multiples of the identity to the hermitian blocks along the main diagonal they are still hermitian. Therefore we have a block lower triangular matrix with hermitian blocks along the main diagonal and so the spectrum of this operator is real as needed.

5 Spectral Analysis of the Blocks

As before, $R_u^{\theta+} = \frac{R_u^{\theta}+R_u^{-\theta}}{2}$ and $R_u^{\theta-} = \frac{R_u^{\theta}-R_u^{-\theta}}{2}$. And we will write $S = R_y^{\pi/2}$ and $T = R_x^{\pi/3}$, so that $S^+ = \frac{S+S^*}{2}$, $S^- = \frac{S-S^*}{2}$, and $S = S^+ + S^-$; and $T^+ = \frac{T+T^*}{2}$, $T^- = \frac{T-T^*}{2}$, and $T = T^+ + T^-$. It is not difficult to compute the eigenvalues of three of the four blocks along the main diagonal, the $\Lambda_{-1,-1}$ block, the $\Lambda_{1,-1}$ block, and the $\Lambda_{-1,1}$ block. The last block, $\Lambda_{+1,+1}$, is quite difficult to analyze and is the most interesting as it contains all the eigenvalues near 1.

The easiest blocks are $1/8(2I) = (1/4)I$ on $\Lambda_{-1,1}$ and $1/8(4I) = (1/2)I$ on $\Lambda_{-1,-1}$. Since these are just multiples of the identity we see that the blocks have eigenvalues $1/4$ with multiplicity equal to the dimension of $\Lambda_{-1,1}$ and $1/2$ with multiplicity equal to the dimension of $\Lambda_{-1,-1}$, respectively. Since the trace of a projection is equal to the dimension of the image we just need to find projections onto each of these subspaces and compute their traces. Let us verify that

$$P_{\alpha,\beta} = \frac{1}{4}(I + \alpha R_x^\pi + \beta R_y^\pi + \alpha\beta R_z^\pi)$$

is a projection onto $\Lambda_{\alpha,\beta}$.

Let $v \in \Lambda_{\alpha',\beta'}$ with α', β' both in $\{\pm 1\}$. Then

$$P_{\alpha,\beta} v = 1/4(1 + \alpha\alpha' + \beta\beta' + \alpha\beta\alpha'\beta')v$$

If $(\alpha, \beta, \alpha', \beta') \in \{\pm 1\}^4$ then $(1 + \alpha\alpha' + \beta\beta' + \alpha\beta\alpha'\beta')$ vanishes whenever $(\alpha, \beta) \neq (\alpha', \beta')$ and is equal to 4 otherwise. Therefore, $P_{\alpha,\beta}$ projects onto $\Lambda_{\alpha,\beta}$ as needed.

The trace of R_y^π is $(-1)^k$, so we have:

$$\dim \Lambda_{\alpha,\beta} = \operatorname{tr} P_{\alpha\beta} = 1/4(2k + 1 + \alpha(-1)^k + \beta(-1)^k + \alpha\beta(-1)^k)$$

and therefore,

$$\dim \Lambda_{1,1} = \operatorname{tr} P_{11} = 1/4(2k + 1 + 3(-1)^k)$$

and,

$$\dim \Lambda_{-1,1} = \dim \Lambda_{1,-1} = \dim \Lambda_{-1,-1} = 1/4(2k + 1 - (-1)^k) \tag{1}$$

Since z is $(1/4)I$ on $\Lambda_{-1,1}$ and $(1/2)I$ on $\Lambda_{-1,-1}$ these blocks have eigenvalues 1/4 and 1/2 respectively with multiplicity $1/4(2k + 1 - (-1)^k)$.

It is a bit more difficult to compute the $\Lambda_{1,-1}$ block. The Quaquaversal Operator, z, is $1/8(2I - 2T^+) = 1/4(I - T^+)$ on this block.

Let η be a primitive sixth root of unity. We will show that the 1/2-eigenvectors of T^+ in the $\Lambda_{1,-1}$ block are exactly those vectors $u = v_\eta + S^2 v_\eta$ where $v_\eta \in H_{2k+1}$ satisfies $Tv_\eta = \eta v_\eta$. It will turn out that the only eigenvalues of T^+ in the $\Lambda_{1,-1}$ block are 1/2 and -1. Since we already know that the dimension of this block is $1/4(2k + 1 - (-1)^k)$, computing the multiplicity of the 1/2 eigenvalue will allow us to determine the multiplicity of -1 by subtraction.

Let us now verify that the $v_\eta + S^2 v_\eta$ are indeed 1/2-eigenvectors of T^+ in the $\Lambda_{1,-1}$ block.

The $\Lambda_{1,-1}$ block consists of exactly those vectors, v, satisfying:

1.) $R_y^\pi v = T^3 v = -v$

2.) $R_x^\pi v = S^2 v = v$

Recall that $S^2 T = T^{-1} S^2$.

$$T^3(v_\eta + S^2 v_\eta) = \eta^3 v_\eta + T^3 S^2 v_\eta = -v_\eta + S^2 T^{-3} v_\eta = -v_\eta + S^2 \bar{\eta}^3 v_\eta = -(v_\eta + S^2 v_\eta)$$

$$S^2(v_\eta + S^2 v_\eta) = S^2 v_\eta + S^4 v_\eta = v_\eta + S^2 v_\eta$$

We have now verified that $v_\eta + S^2 v_\eta$ is in $\Lambda_{1,-1}$. Let us now verify that $T^+(v_\eta + S^2 v_\eta) = 1/2(v_\eta + S^2 v_\eta)$. Using the fact that $\frac{\eta+\bar{\eta}}{2} = \frac{(1+\sqrt{3}i)/2+(1-\sqrt{3}i)/2}{2} = 1/2$ we get:

$$T^+(v_\eta + S^2 v_\eta) = \frac{T + T^{-1}}{2}(v_\eta + S^2 v_\eta) = \frac{\eta + \bar{\eta}}{2} v_\eta + \frac{T + T^{-1}}{2} S^2 v_\eta = \frac{1}{2} v_\eta + S^2 \frac{T^{-1} + T}{2} v_\eta = \frac{1}{2}(v_\eta + S^2 v_\eta) \quad (2)$$

as needed.

We shall now prove the converse—that the elements of $\Lambda_{1,-1}$ which are 1/2-eigenvectors of T^+ are of the form $v_\eta + S^2 v_\eta$.

Since T is diagonalizable and $T^3 = -I$ on $\Lambda_{1,-1}$, we know that any $v \in \Lambda_{1,-1}$ can be uniquely written as $v = v_\eta + v_{\bar{\eta}} + v_{-1}$ with $T v_\gamma = \gamma v_\gamma$. Furthermore, $v_{\bar{\eta}} = S^2 v_\eta$. To see this, notice that S^2 swaps the η-eigenspace of T with the $\bar{\eta}$-eigenspace. Indeed,

$$T(S^2 v) = S^2 T^{-1} v = S^2 \bar{\eta} v = \bar{\eta}(S^2 v)$$

So,

$$v_\eta + v_{\bar{\eta}} + v_{-1} = v = S^2 v = S^2 v_{\bar{\eta}} + S^2 v_\eta + S^2 v_{-1}$$

Since the representation is unique, we know that $S^2 v_\eta = v_{\bar{\eta}}$ as needed.

If v is an eigenvector of T^+, then it is either of the form $v_\eta + v_{\bar{\eta}}$ or v_{-1}. Indeed,

$$T^+(v_\eta + v_{\bar{\eta}} - v_{-1}) = \frac{\eta + \bar{\eta}}{2}(v_\eta + v_{\bar{\eta}}) - v_{-1} = \frac{(1+\sqrt{3}i)/2 + (1-\sqrt{3}i)/2}{2}(v_\eta + v_{\bar{\eta}}) - v_{-1} = 1/2(v_\eta + v_{\bar{\eta}}) - v_{-1} \quad (3)$$

So to be an eigenvector either v_η and $v_{\bar{\eta}}$ are both zero or v_{-1} is zero. In the first case the eigenvalue is -1 and in the second case it is 1/2.

Combining these two facts we see that the 1/2-eigenvectors of T^+ in $\Lambda_{-1,1}$ are of the form $v_\eta + S^2 v_\eta$, as needed.

We have just shown that the space of 1/2-eigenvectors of T^+ in $\Lambda_{-1,1}$ is $I + S^2$ applied to the space of η-eigenvectors of T. As $I + S^2$ is injective, indeed if $v_\eta + S^2 v_\eta = v'_\eta + S^2 v'_\eta$ then $v_\eta - v'_\eta + S^2 v_\eta - S^2 v'_\eta = 0$ implies $v_\eta - v'_\eta = 0$ since $v_\eta - v'_\eta$ and $S^2 v_\eta - S^2 v'_\eta$ live in disjoint subspaces. On the η-eigenvectors of T, the dimension of this space is the dimension the space of 1/2-eigenvectors of T^+ in $\Lambda_{-1,1}$ is equal to the multiplicity of η-eigenvectors of T in H_{2k+1}. T has eigenvalues given by $\{\eta^j\}_{j=-k}^{j=k}$ so the η occurs with multiplicity $\lfloor \frac{k+5}{6} \rfloor + \lfloor \frac{k+1}{6} \rfloor$. The remaining eigenvalue of T^+ on this block is -1, so to compute it's multiplicity we just subtract this from the dimension of the block and get

$$1/4\left(2k + 1 - (-1)^k\right) - \left(\left\lfloor \frac{k+5}{6} \right\rfloor + \left\lfloor \frac{k+1}{6} \right\rfloor\right)$$

So, since $z = 1/4(I - T^+)$ on this block. We have $1/4(1 - 1/2) = 1/8$ with multiplicity $\lfloor \frac{k+5}{6} \rfloor + \lfloor \frac{k+1}{6} \rfloor$ and $1/4(1 - (-1)) = 1/2$ with multiplicity $1/4\left(2k + 1 - (-1)^k\right) - \left(\lfloor \frac{k+5}{6} \rfloor + \lfloor \frac{k+1}{6} \rfloor\right)$

It is much more difficult to analyze the spectrum of z on $\Lambda_{1,1}$. We have shown that z on this block is equal to $1/8(S^+ + 4 + T^+)$ with $S^2 = 1 = T^3$. To analyze this block it suffices to study the spectrum of $(S^+ + T^+)$ and rescale and translate. Both S^+ and T^+ leave $\Lambda_{1,1}$ invariant. Let us now describe the eigenstructure of T^+ and S^+ on this space.

Since $T^3 v = v$, $v = v_1 + v_\eta + v_{\bar{\eta}}$ with $Tv_\gamma = \gamma v_\gamma$ and η a primitive third root of unity. Applying T^+ to v we get:

$$T^+ v = 2v_1 + (\eta + \bar{\eta})(v_\eta + v_{\bar{\eta}}) = 2v_1 + (-1)(v_\eta + v_{\bar{\eta}})$$

This tells us that the eigenvalues of T^+ are 2 and -1. In the first case the eigenvector is of the form v_1, a 1-eigenvector of T. In the second case it is of the form $v_\eta + v_{\bar{\eta}}$.

Since $S^2 v = v$, $v = v_1 + v_{-1}$ with $Sv_\gamma = \gamma v_\gamma$. Applying S^+ to v we get:

$$S^+ v = 2v_1 + (-2)v_{-1}$$

This tells us that the eigenvalues of S^+ are 2 and -2. In the first case the eigenvector is of the form v_1. In the second case it is of the form v_{-1}. In both cases it is an eigenvector of S.

If v is a simultaneous eigenvector of S^+ and T^+ then $(T^+ + S^+)$ is $(2 + \pm 2)v$ or $(-1 + \pm 2)$, so this can be $-3, 0, 1$, or 4. The eigenvalues 4 and 0 can only occurs when v is also a simultaneous eigenvector for T and S this means that the eigenspace is the trivial representation so the eigenvalue must be 4 and 0 is impossible. The other two possibilities, -3 and 1 both occur with relative abundance. We conjecture that all integral eigenvalues in this block come from simultaneous eigenvectors of S and T and that the remaining eigenvalues are irrational and conjugate to one another.

References

[1] Jean Bourgain and Alex Gamburd. On the spectral gap for finitely-generated subgroups of SU(2). *Inventiones mathematicae*, 171(1):83–121, 2008.

[2] Daniel Bump. *Lie groups*, volume 8. Springer, 2004.

[3] John H Conway and Charles Radin. Quaquaversal tilings and rotations. *Inventiones mathematicae*, 132(1):179–188, 1998.

[4] Brimstone Draco, Lorenzo Sadun, and Douglas Van Wieren. Growth rates in the quaquaversal tiling. *Discrete & Computational Geometry*, 23(3):419–435, 2000.

[5] Alex Gamburd, Peter Sarnak, and Dmitry Jakobson. Spectra of elements in the group ring of SU(2). *Journal of the European Mathematical Society*, 1(1):51–85, 1999.

[6] Anthony W Knapp. *Representation theory of semisimple groups.* Princeton university press, 2016.

[7] Hermann Weyl. *The classical groups.* Princeton university press, 2016.