Research Statement

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Given a group, G, and a ring, R, the group ring $R[G]$ consists of formal linear combinations of G with coefficients in R. It is a ring with multiplication given by:

$$\left(\sum_{i\in S} r_i g_i\right)\left(\sum_{j\in S'} r_j g_j\right) = \sum_{i\in S, j\in S'} r_i r_j g_i g_j$$

Given an element of $z \in \mathbb{C}[G]$, $z = \sum \lambda_i g_i$, and any representation, $\rho : G \to GL(V)$ we get an operator on V defined by:

$$v \mapsto \sum \lambda_i \rho(g_i) v$$

When the coefficients of z are non-negative real numbers we normalize and get an associated probability measure $\frac{\lambda_i}{\sum_j \lambda_j}\delta_{g_i} = \nu_z$. If z_1 and z_2 are measures then $\nu_{z_1 z_2} = \nu_{z_1} * \nu_{z_2}$.

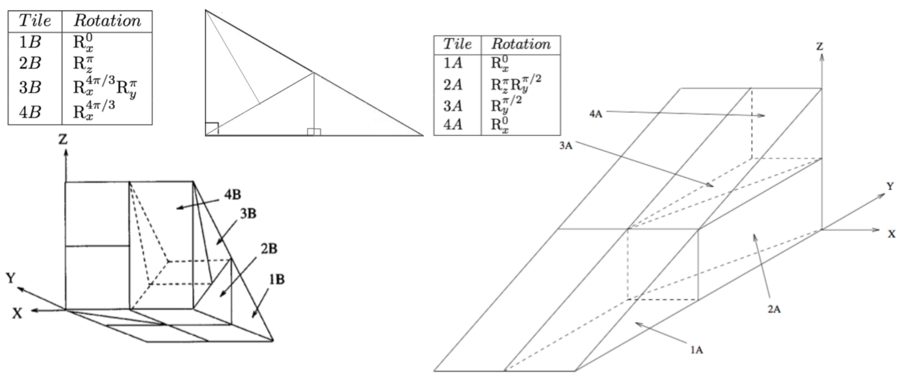
In my dissertation, I studied the spectral theory of a specific element of $\mathbb{Q}[SO(3)]$ viewed as an operator on the regular representation, $L^2(SO(3))$. I showed that the spectrum of this operator is real, resolving a 2006 conjecture of Draco, Sadun, and van Wieren from their paper "Growth Rates in the Quaquaversal Tiling". Later I found an explicit spectral gap for this operator resolving the other conjecture from this paper. I shall refer to this operator as the Quaquaversal operator and the element as the Quaquaversal element.

This operator is related to an interesting tiling of $\mathbb{R}^3$ introduced by Conway and Radin called the Quaquaversal Tiling. The Quaquaversal tiling is a substitution tiling with the property that the orientations of its tiles approach equidistribution faster than what is possible for substitution tilings in two dimensions. Indeed, the orientations of the tiles in the Quaquaversal Tiling equidistribute exponentially rapidly. In two dimensions the group of orientations, SO(2), is abelian and therefore has polynomial growth. A simple consequence of this is that the number of distinct orientations of the tiles in the interior of a bounding sphere grows as a polynomial in the logarithm of the radius of that sphere, so that rapid equidistribution is impossible. This argument fails for SO(3) which has exponential growth.

The fact that the orientations in the Quaquaversal tiling exhibit rapid equidistribution has some physical significance. A material modeled by such a tiling would have a "deterministic structure" but would have an isotropic diffraction pattern similar to a glass. Such materials would form an interesting class of solids that experimental scientists have not yet considered (at least they had not yet considered them when they were introduced in 1998).

1 The Quaquaversal Tiling

The Quaquaversal tiling is the tiling generated by the substitution rule pictured below.



Tile	*Rotation*
$1B$	R_x^0
$2B$	R_z^{π}
$3B$	$R_x^{4\pi/3}R_y^{\pi}$
$4B$	$R_x^{4\pi/3}$

Tile	*Rotation*
$1A$	R_x^0
$2A$	$R_z^{\pi}R_y^{\pi/2}$
$3A$	$R_y^{\pi/2}$
$4A$	R_x^0

Here we have a triangular mother prism subdivided into 8 daughter prisms each congruent to the mother prism dilated by a factor of 1/2. We can use this substitution rule to get a tiling of $\mathbb{R}^3$ by repeatedly subdividing and then dilating by a factor of 2 about an appropriate point. Each daughter prism can be expressed as a rotation followed by a dilation followed by a translation of the mother prism. If we take the formal sum of just the rotations and divide by 8 so that the sum of the coefficients is 1 we get the element of $\mathbb{Q}[SO(3)]$ that we called the Quaquaversal element (and we call these rotations the relative orientations of the daughter tiles with respect to their mother tile). As we described before this element acts on $L^2(\mathrm{SO}(3))$ and we call the resulting operator the Quaquaversal operator.

The spectrum of this operator controls the rate with which the orientations of tiles approach a uniform distribution.

The largest eigenvalue is 1 with eigenvectors given by the constant functions. And, even though this operator is not normal, the orthogonal complement of the constant functions is an invariant subspace. If T is the Quaquaversal Operator then this together with the spectral radius formula tells us $||T^n(f) - \int f d\mu|| \ll \rho^{n+\epsilon}$ where μ is the Haar measure and ρ is the spectral radius of T restricted to the orthogonal complement of the constant functions. Furthermore, ρ is the smallest number for which this inequality holds. On the other hand, given an ϵ, there is always an f such that $\rho^{n-\epsilon} \ll ||T^n(f) - \int f d\mu||$.

Putting this all together we see that ρ determines the rate with which the orientations of the tiles approach uniformity in the weak sense.

In their 2006 paper, Draco, Sadun, and van Wieren computed many non-trivial eigenvalues numerically and found some as large as 0.9938. Giving us the following lower bound on ρ:

$$0.9938 \leq \rho$$

They conjectured the following upper bound for ρ:

Conjecture 1.
$$\rho \leq 2^{-1/112} \approx 0.9938303$$

They also conjectured that the spectrum of this operator was real, which I was able to prove by introducing a special partition of $L^2(SO(3))$ (as a vector space). The Quaquaversal Operator is block lower triangular with respect to this partition with Hermitian blocks along the main diagonal. This partition was and still is quite mysterious to me and I am putting in some effort and slowly understanding its arithmetic significance.

While work of Bourgain and Gamburd [1] shows that the Quaquaversal Operator has a non-zero spectral gap. Their argument does not provide an explicit gap, so it is not quite sufficient to resolve the other conjecture. To prove this I used the work of Serre [12], which shows that $G(6,4)$, the group generated by rotations of order 4 and 6 about orthogonal axes (Let's call these S and T, respectively) is $SO_{3;Q}(\mathbb{Z}[1/2])$ where Q is the quadratic form defined by $Q(x,y,z) = 3x^2 + 3y^2 + z^2$. All the eigenvalues that are near 1 live in one block and on this block the operator coincides with $S + S^{-1} + T + T^{-1}$. Modulo a few complications, this allows us to carry out the strategy of Lubotzky, Phillips, and Sarnak [6] [7] in order to find an explicit upper bound on ρ equal to

$$\rho \leq \frac{4 + \frac{1+\sqrt{1+4(6+4\sqrt{2})}}{2}}{8} \approx 0.9938289$$

Since $0.9938289 < 0.9938303$ this proves the other conjecture.

2 Dite and Kart Tiling

A tiling is a collection of tiles each congruent to some element from a set of *prototiles* such that they intersect only on their boundaries. Normally we consider tilings whose union is $\mathbb{R}^n$ for some n. In general, a tiling together with one of its prototiles, P, has an associated group given by the set of relative orientations of the copies of P with respect to one another.

Radin and Sadun introduced another Quaquaversal like tiling they called the Dite and Kart Tiling. This tiling is similar to the Quaquaversal tiling in many ways. They both are three dimensional substitution tilings with the property that, fixing any prototile, the number of distinct orientations for tiles in a sphere grows on the order of $o(r^\alpha)$ for some $\alpha \in \mathbb{R}^+$ where r is the radius of the bounding sphere. Both of these tilings have associated operators whose spectral properties explain equidistribution phenomena. That is, the spectral radius of the operator restricted to the orthogonal complement of the Perron-Frobenius eigenvector controls the rate with which the distribution of orientations in the tiling approaches a limiting distribution.

However, these tilings differ in a number of key ways. The Quaquaversal Tiling has a single prototile, whereas the Dite and Kart tiling has eight prototiles. This makes the corresponding operator and "element" more complicated for the Dite and Kart tiling as well as complicating the limiting distribution for the orientations of the prototiles. In the case of the Quaquaversal Tiling the corresponding element lives in $\mathbb{Q}[\mathrm{SO}(3)]$ and in the case of the Dite and Kart tiling, the corresponding element lives in $M_8(\mathbb{Q}[\mathrm{SO}(3)])$, its operator acts on $L^2(\mathrm{SO}(3))^8$, and the corresponding limiting distribution for the Dite and Kart comes from the stationary measure of a certain associated transition matrix rather than the uniform distribution as is the case for the Quaquaversal tiling.

Additionally, the relative orientations of the tiles in the Quaquaversal tiling form the group $G(6,4)$, the group of rotations generated by rotations of order 6 and 4 about orthogonal axes. As mentioned before this group is S-arithmetic and therefore one can compute an explicit spectral gap. One should be able to show that this spectral gap is optimal in the following sense: It is always the case that elements of the group ring acts on $l^2(G(6,4))$ and the spectral radius of this l^2 always provides a lower bound on the spectral radius of the operator on $L^2(SO(3))$. Optimality is that the spectral radii for these operators actually agree.

In the case of the Dite and Kart tiling the group of orientations is $G(10,4)$, the group generated by rotations about orthogonal axes of order 10 and 4. This is a thin group and, while it still seems likely that optimal expansion occurs, one must use more modern methods such as the work of Bourgain and Gamburd to even show that there is a gap between the largest nontrivial eigenvalue and 1. And it is even more difficult to get explicit spectral bounds for thin groups, let alone optimal explicit bounds. Further complications occur because the operator for the Dite and Kart acts on $L^2(\mathrm{SO}(3))^8$ and is more complicated than one that comes from an element of the group ring.

In summary, a substitution tiling may have one protitile or many and the group of relative orientations may be S-arithmetic or thin (or neither).

Tilings which have a single prototile have a simpler operator to analyze, as one can take a formal sum of the orientations of the daughter tiles to get an element of the group ring $\mathbb{Q}[SO(3)]$ (or more precisely, $\mathbb{Q}[G]$, where G is the group of relative orientations) which gives rise to an operator that acts on $L^2(SO(3))$. For Substitution Tilings with k protitiles one must keep track of which prototile is a daughter of which other prototile and one gets a matrix of relative orientations $M_k(\mathbb{Q}[G])$ (or $M_k(\mathbb{Q}[G])$, if you normalize). I will refer to this as the tiling's "formal element" for lack of a better word. This then acts on $L^2(SO(3))^k$. This difficulty seems mostly artificial and I am working on extending the basic spectral results for $\mathbb{Q}[SO(3)]$ to $M_k(\mathbb{Q}[G])$.

For tilings that have an S-arithmetic group of relative orientations, G, the methods of Lubotzky, Phillips, and Sarnak are mostly applicable [6] [7] [8]. One gets an associated Quaternion Order. And one can attach a family of cusp forms to this order on which the the group acts. From the fourier coefficient bounds on these cusp forms that one can obtain from the Ramanujan-Petersson conjectures proved by Deligne in this setting one can often translate these bounds to explicit spectral bounds for the relevant operator.

There is still the question of whether or not we can upgrade 'often translate' the fourier coefficient bounds to spectral bounds to 'always translate' and whether or not we can upgrade 'explicit spectral bounds' to 'explicit and optimal spectral bounds' in the sentence above. Here optimal means the following: The formal element also acts on $(l^2(G))^k$, where k is the number of prototiles. And, at least in the easier to analyze case where the element is Hermitian, the spectrum of this operator provides a lower bound for the non-trivial part of the spectrum on $(L^2(SO(3)))^k$. We say that the bound is optimal if the supremum of the eigenvalues on $(l^2(G))^k$ coincides with the supremum of the spectrum of $(L^2(SO(3)))^k$ on the orthogonal complement of the trivial representation.

S-arithmetic groups are well understood and while I can't prove optimal expansion in full generality at the moment, it is not unlikely that I can do this by absorbing the literature and applying standard methods.

This is not the case for thin groups. It is likely that one can achieve non-explicit spectral bounds for thin groups without too much difficulty by using the theorem of Bourgain and Gamburd[1] that sets of matrices in SU(2) with algebraic entries that generate a group containing a free group have a spectral gap (the result is immediate when there is 1 prototile, but some work has to be done when there is more than 1). There is quite a bit of literature discussing spectral bounds for thin groups and it is likely that with some effort one can use these methods and write an explicit bound, but it seems quite difficult to get a bound which is both explicit and optimal and I think that this would be a bit of an acheivement.

3 Applications

More generally, my work involves studying the spectra of operators coming from $\mathbb{C}[G]$ or $M_n(\mathbb{C}[G])$, viewed as an operator acting on a function space. Especially $L^2(\tilde{G})$ where $\tilde{G}$ is a classical group that G embeds into, usually $SO(3)$, $SU(2)$, or $U(n)$ but also $l^2(G)$. These have quite a few applications, which I discuss below.

3.1 $G(m, n)$

In "Subgroups of $SO(3)$ associated with tilings"[11] Radin and Sadun studied $G(m, n)$, the group of orientations generated by three dimensional rotations about orthogonal axes of order m and n. In this paper they computed presentations and canonical forms for these groups and described a strategy for constructing Quaquaversal like tilings whose group of relative orientations were of the form $G(m, 4)$. In "Growth Rates in the Quaquaversal Tiling" Draco, Sadun, and van Wieren said that "slow growth in complexity" seemed to be a typical feature for three dimensional substitution tilings. So, presumably, these tilings have an operator with a small spectral gap.

It is likely that, using the combinatorics of the presentation and Kesten's theory of random walks on groups [5], one can bound the spectral radius of the $l^2(G)$ operator providing a lower bounds for the spectral gap of the $L^2(SO(3))$ operator on the nontrivial eigenvalues. If there are no surprises then this should show that a large class of Quaquaversal-like tilings, while technically having an exponentially rapid rate with which the orientations of their tiles approach a uniform distribution, the base for the rate of convergence must be quite close to 1.

3.2 Equidistribution Theorems for Random Quantum circuits

A quantum gate with one qubit can be identified with an element of $PU(1)$. Quantum gates are built by composing elements from a fixed set of quantum gates called 'Elementary Quantum Gates'. The particular class of gates considered elementary vary from model to model. Given a generic quantum gate one hopes to find short or 'inexpensive' nearby circuits. It is therefore natural to ask what a random quantum circuit 'looks like' for various stochastic processes where the circuit lengths or 'cost' increases as we run the process especially whether or not such a process exhibits rapid equidistribution.

For example, since $SO(3) \cong PU(1)$, a formal sum of elements of $SO(3)$ like the one coming from the Quaquaversal tiling gives rise to a stochastic process that generates quantum gates. Here if $z = \frac{1}{k}(g_1 + \cdots + g_k)$, albeit with respect to a slightly odd set of elementary gates. Indeed, taking the relative orientations of the daughter tiles as elementary gates, the orientations of the tiles in the subtitution tiling after n generations are the circuits of length n.

Some less artificial examples come from the 'Super Golden Gates' of Parzanchevski and Sarnak[10]. In this paper Parzanchevski and Sarnak discuss particularly nice elementary gate sets they call the 'Super Golden Gates'. These gates are all of the form $C \cup T$, where C is a finite group and T is an involution. These elementary gate sets exhibit both strong equidistribution properties and a decent algorithm for approximating an arbitrary gate with a circuit built from the elementary gates that has a nearly optimal T-count (apparently this is an appropriate notion of 'cost' in this setting). As luck would have it, many of these gates are used in practice and considered practical.

The spectral gap of the following element when viewed as an operator on $L^2(PU(1))$:

$$z = \frac{1}{|C \cup T|} \sum_{g \in C \cup T} g$$

controls the rate with which the following random process equidistributes: Build a circuit by repeatedly attaching a random gate sampled uniformly at random from $C \cup T$. Naturally, I expect this process to exhibit optimal equidistribution in the sense discussed above. It would be interesting to actually compute the base for the exponential rate of convergence. I suspect these are close to 1 like the Quaquaversal element.

3.3 Equidistribution Results for the Product Replacement Graph on SU(2)

The product replacement graph on k generators is a $4k$-regular graph where the vertices are generating k-tuples and the edges are obtained by replacing an entry g_i with an entry of the form $g_i g_j^{\pm 1}$ or $g_j^{\pm 1} g_i$ and g_j is some other entry [2] Random walks on the product replacement graph have been used extensively in computational group theory to sample from nearly uniform random distributions of generating sets or elements in blackbox groups. My advisor, Alexander Gamburd, had suggested that this procedure may be useful for generating random elements of SU(2). A result of Lubotzky shows that if $Aut(F_n)$ has property T (the trivial representation is isolated from the rest of the unitary representations) then the product replacement graph expands on its connected components. Building on work of Ozawa [9] who found a computable strategy for detecting property T, Kaluba, Kielak, Nowak, and Ozawa [3] [4] were able to show that $Aut(F_n)$ for $n \geq 5$ have property T. This example is quite similar to those discussed above and one direction I would like to go with my research is revisiting this result to see to what extent Lubotzky's argument can be extended to the compact setting in general and to SU(2) in particular.

References

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