CMPSC 465 Data Structures & Algorithms
Spring 2021 Paul Medvedev and Chunhao Wang

HW 1

Due January 28, 10:00 pm

Instructions: You may work in groups of up to three people to solve the homework. You must write your own solutions and explicitly acknowledge up everyone whom you have worked with or who has given you any significant ideas about the HW solutions. You may also use books or online resources to help solve homework problems. All consulted references must be acknowledged.

You are encouraged to solve the problem sets on your own using only the textbook and lecture notes as a reference. This will give you the best chance of doing well on the exams. Relying too much on the help of group members or on online resources will hinder your performance on the exams.

Late HWs will be accepted until 11:59pm with a 20% penalty. HWs not submitted by 11:59pm will receive 0. There will be no exceptions to this policy, as we post the solutions soon after the deadline. However, you will be able to drop the three lowest HW grades.

For the full policy on HW assignments, please consult the syllabus.

1. **(0 pts.) Acknowledgements.** The assignment will receive a 0 if this question is not answered.

   (a) If you worked in a group, list the members of the group. Otherwise, write "I did not work in a group."

   (b) If you received significant ideas about the HW solutions from anyone not in your group, list their names here. Otherwise, write "I did not consult without anyone my group members".

   (c) List any resources besides the course material that you consulted in order to solve the material. If you did not consult anything, write "I did not consult any non-class materials."

2. **(20 pts.) Tribonaci numbers**
   Your manager proposed to generate IDs for customers based on what he calls Tribonaci numbers. They define Tribonaci numbers using the recurrence $R(i) = R(i-1) + R(i-2) + R(i-3)$, with the first three Tribonaci numbers defined as $R(0) = 1$, $R(1) = 2$, $R(2) = 3$. They would like to assign the i^{th} customer an ID of $R(i)$. You tell them that they are crazy and that the IDs are going to be huge; in fact, you claim that the i^{th} customer will have an ID number of at least $3^{i/2}$. Your manager, who has never taken lightly to criticism, threatens to fire you on the spot unless you can come up with a proof of your claim.

   (a) To save your job, use induction to prove that in fact $R(i) \geq 3^{i/2}$ for all $i \geq 2$.
   Solution:
   There are three base cases, which we verify

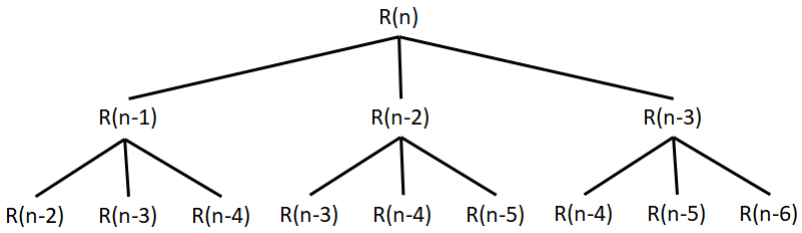
$$R(0) = 1 \geq 3^{0/2} = 1$$
$$R(1) = 2 \geq 3^{1/2} \approx 1.73$$
$$R(2) = 3 \geq 3^{2/2} = 3$$

   For the general case, we assume that $R(n) \geq 3^{n/2}$ and aim to show that $R(n+1) \geq 3^{(n+1)/2}$.

$$\begin{aligned} R(n+1) &= R(n) + R(n-1) + R(n-2) \\ &\geq 3^{n/2} + 3^{(n-1)/2} + 3^{(n-2)/2} \\ &\geq 3^{(n+1)/2}\left(3^{-1/2} + 3^{-2/2} + 3^{-3/2}\right) \\ &\geq 3^{(n+1)/2}(1.1) \\ &\geq 3^{(n+1)/2} \end{aligned}$$

(b) Your manager is perplexed. They didn't take CMPSC360 or CMPSC465 so they don't know what induction is and don't understand your proof. They propose a simple recursive algorithm to compute $R(i)$. The algorithm simply makes three recursive calls and then adds the results. You tell him that his algorithm will take exponential time, but your manager doesn't want to hear anything more about induction. They demand an intuitive picture instead of a proof. Luckily, you just covered something like this in class. Draw a recursion tree that shows the cascade of recursive invocations triggered by a single call to compute $R(i)$. Two levels deep will suffice.

Solution:



3. (10 pts.) Big Oh definitions Let f and g be functions from positive integers to positive reals. In class, we saw the following definition of big Oh: $O_1(g(n)) = \{f(n) : \exists \text{ constants } c_1 > 0, n_0 > 0 \text{ s. t. } f(n) \leq c_1 g(n), \forall n \geq n_0\}$. The textbook gives an alternate definition: $O_2(g(n)) = \{f(n) : \exists \text{ a constant } c_2 > 0 \text{ s. t. } f(n) \leq c_2 g(n) \text{ for all } n > 0\}$. Prove that the two definitions are equivalent. In other words, prove that $O_1(g(n)) = O_2(g(n))$ for all g.

Solution: There are two directions to prove. The first is that $O_2(g(n)) \subseteq O_1(g(n))$ and the second is that $O_1(g(n)) \subseteq O_2(g(n))$.

For the first direction, let $f(n) \in O_2(g(n))$. This means that there exists $c_2 > 0$ such that $f(n) \leq c_2 g(n)$ for all $n > 0$. If we let $n_0 = 1$ and $c_1 = c_2$, then n_0 and c_2 satisfy the definition of $f(n) \in O_1(g(n))$, i.e. $f(n) \leq c_1$ for all $n \geq n_0$.

For the second direction, let $f(n) \in O_1(g(n))$. This means that there exists $c_1 > 0$ and $n_0 > 0$ such that $f(n) \leq c_1 g(n)$ for all $n \geq n_0$. Let $c_2 = \max\{c_1, \max_{0<n<n_0} \frac{f(n)}{g(n)}\}$. We need to show that for all $n > 0$, $f(n) \leq c_2 g(n)$, or, equivalently, $\frac{f(n)}{g(n)} \leq c_2$. First, consider the case that $0 < n < n_0$. Then,

$$\frac{f(n)}{g(n)} \leq \max_{0<n<n_0} \frac{f(n)}{g(n)} \leq \max\{c_1, \max_{0<n<n_0} \frac{f(n)}{g(n)}\} = c_2.$$

Second, consider the case that $n \geq n_0$. Then,

$$\frac{f(n)}{g(n)} \leq c_1 \leq \max\{c_1, \max_{0<n<n_0} \frac{f(n)}{g(n)}\} = c_2.$$

This completes the second direction.

Note that other solutions are possible, especially for the second direction.

4. (20 pts.) Analyze Running Time. For each pseudo-code below, give the asymptotic running time in Θ notation. (You may assume that standard arithmetic operations take $\Theta(1)$ time.) You may assume that n is a power of two if it simplifies your analysis. Briefly justify your answer.

1.
```
for i := 1 to n do
    j := i;
    while j < n do
        j := j + 5;
    end
end
```

2.
```
for i := 1 to n do
    for j := 4i to n do
        s := s + 2;
    end
end
```

3.
```
s := 0;
i := n;
while i ≥ 1 do
    i := i div 2;
    for j := 1 to i do
        s := s + 1
    end
end
```

4.
```
for i := 1 to n do
    j := i^2;
    while j ≤ n do
        j := j + 1
    end
end
```

Solution

1. The first assignment statement is executed n times. The second assignment statement is executed roughly $\sum_{i=1}^{n} \frac{n-i}{5}$ times. If we let $i' = n - i + 1$, then this becomes $\frac{1}{5}\sum_{i'=1}^{n} i' = O(n^2)$. Thus, the total running time is $n + O(n^2) = O(n^2)$.
2. The outer for loop condition is evaluated n times. The inner for loop condition is evaluated n plus the number of times that the assignment statement is evaluated. The assignment statement is not evaluated when $i > n/4$. Therefore, it is executed roughly $\sum_{i=1}^{n/4}(n-4i) = n^2/4 - 4\sum_{i=1}^{n/4} i = n^2/4 - 4(n/4)(n/4+1)/2 = \Theta(n^2)$ times.
3. For simplicity, round n up to the nearest power of 2. Remember that $\log n$ is the number of times one divides a number by 2 before reaching 1. The running time is

$$\sum_{k=1}^{\log n} n \cdot 2^{-k} = \Theta(n).$$

4. When $i \leq \sqrt{n}$, the inner loop runs for $n - i^2$ steps; otherwise, it stops immediately and takes time $\Theta(1)$. So the running time is

$$\Theta\left(n + \sum_{i=1}^{\sqrt{n}} (n - i^2)\right) = \Theta\left(n + n\sqrt{n} - \sum_{i=1}^{\sqrt{n}} i^2\right) \tag{1}$$

$$= \Theta\left(n + n\sqrt{n} - \frac{\sqrt{n}(\sqrt{n}+1)(2\sqrt{n}+1)}{6}\right) \tag{2}$$

$$= \Theta\left(n + n\sqrt{n} - \frac{2n\sqrt{n}}{6} - O(n)\right) \tag{3}$$

$$= \Theta(n\sqrt{n}) \tag{4}$$

5. (10 pts.) Tighter Analysis. We are going to analyze the running time of an n-bit counter that counts from 0 to $2^n - 1$. For example, when $n = 5$, the counter has the following values:

Step	Value	# Bit-Flips
0	`00000`	–
1	`00001`	1
2	`00010`	2
3	`00011`	1
4	`00100`	3
⋮	⋮	
30	`11110`	2
31	`11111`	1

Note that we only need to flip the last bit to go from 0 to 1, but we need to flip the last two bits to go from 1 to 2. It is easy to see that to go from 0 to $2^n - 1$, it takes at most $O(n2^n)$ bit-flips. Show that, in fact, it takes $\Theta(2^n)$ bit flips.

(*Hint:* How many times is the last bit flipped? How many times is the second last bit flipped?)

Solution The number of times the i-th-from-last bit is flipped is 2^{n-i}. There are about $\log n$ bits, so the total number of bit flips is $\sum_{i=1}^{\log n} 2^{n-i} = \Theta(2^n)$.