CMPSC 465 Data Structures & Algorithms
Spring 2021 Paul Medvedev and Chunhao Wang

HW 7

Due March 18, 10:00 pm

Instructions: You may work in groups of up to three people to solve the homework. You must write your own solutions and explicitly acknowledge up everyone whom you have worked with or who has given you any significant ideas about the HW solutions. You may also use books or online resources to help solve homework problems. All consulted references must be acknowledged.

You are encouraged to solve the problem sets on your own using only the textbook and lecture notes as a reference. This will give you the best chance of doing well on the exams. Relying too much on the help of group members or on online resources will hinder your performance on the exams.

Late HWs will be accepted until 11:59pm with a 20% penalty. HWs not submitted by 11:59pm will receive 0. There will be no exceptions to this policy, as we post the solutions soon after the deadline. However, you will be able to drop the three lowest HW grades.

For the full policy on HW assignments, please consult the syllabus.

1. **(0 pts.) Acknowledgements.** The assignment will receive a 0 if this question is not answered.

   (a) If you worked in a group, list the members of the group. Otherwise, write "I did not work in a group."

   (b) If you received significant ideas about the HW solutions from anyone not in your group, list their names here. Otherwise, write "I did not consult without anyone my group members".

   (c) List any resources besides the course material that you consulted in order to solve the material. If you did not consult anything, write "I did not consult any non-class materials."

2. **(15 pts.) Shortest bitonic paths**

   Suppose that you have a directed graph $G = (V, E)$ with an edge weight function w and a source vertex $s \in V$. The weights can be negative, but there are no negative weight cycles. Furthermore, assume that all edge weights are distinct (i.e. no two edges have the same weight). The single source shortest path problem is to find the shortest path distances from s to every vertex in V.

   (a) Suppose that you also guaranteed that for all $v \in V$, a shortest path from s to v has increasing edge weights. Give an algorithm to find the shortest path distances from s to every vertex in V. Analyze the running time of your algorithm and explain why it is correct. For full credit, your algorithm should run in time $O(V + E \log E)$.

   (b) A sequence is *bitonic* if it monotonically increases and then monotonically decreases, or if by a circular shift it monotonically increases and then monotonically decreases. For example the sequences $(1,4,6,8,3,-2)$, $(9,2,-4,-10,-5)$, and $(1,2,3,4)$ are bitonic, but $(1,3,12,4,2,10)$ is not bitonic. Now, suppose that the sequences of edge weights along the shortest paths are no longer guaranteed to be increasing, but instead are guaranteed to be bitonic. Give a single source shortest path algorithm, explain why it is correct, and analyze its running time. For full credit, your algorithm should run in time $O(V + E \log E)$.

3. (10 pts.) Dijkstra's on negative Consider the behavior of Dijkstra's Algorithm on directed graphs with negative edges.

(a) Give an example where Dijkstra's does not produce the correct algorithm. Be sure to draw the directed graph with edge weights and identify the start vertex. Label each vertex with the distance produced by Dijkstra's as well as with the correct distance. (You can use drawing/handwriting for this answer)

(b) Identify the part in the proof of Dijkstra's correctness (in the handout) which does not necessarily hold when the input graph contains negative edges. Elaborate why, in more detail then in the handout.

4. (10 pts.) Greedy choice

Recall that in the 0-1 knapsack problem, you are given a backpack with capacity W. There are n items and the i-th item has weight w_i and value v_i. The goal is to pick some items with total weight at most W that maximize the total value. More formally, you need to find an assignment $(x_1, \ldots, x_n)$ with $x_i \in \{0, 1\}$ (i.e., x_i is either 0 or 1), where $x_i = 1$ means the i-th item is picked and $x_i = 0$ otherwise. The goal is to maximize $\sum_{i=1}^{n} x_i v_i$ subject to the constraint $\sum_{i=1}^{n} x_i w_i \le W$. As we have discussed in the class, this 0-1 knapsack problem does not have the greedy choice property.

Now consider the *fractional knapsack problem*, where you can take a fraction of an item. More formally, you need to find an assignment $(x_1, \ldots, x_n)$ with $x_i \in [0, 1]$ (i.e., x_i can be any real number inclusively between 0 and 1). The goal is to maximize $\sum_{i=1}^{n} x_i v_i$ subject to the constraint $\sum_{i=1}^{n} x_i w_i \le W$. The greedy heuristic for each step is to find the item with the largest value per pound and take as much of it as possible. Suppose the j-th item has the largest value-per-pound. Prove that there exists an optimal solution that contains as much of the j-th item as possible. (This shows the greedy choice property.)