= Shapley ? Folkman lemma =

The Shapley ? Folkman lemma is a result in convex geometry with applications in mathematical economics that describes the Minkowski addition of sets in a vector space . Minkowski addition is defined as the addition of the sets ' members : for example , adding the set consisting of the integers zero and one to itself yields the set consisting of zero , one , and two :

{ 0 , 1 } + { 0 , 1 }
= { 0 + 0 , 0 + 1 , 1 + 0 , 1 + 1 } =
{ 0 , 1 , 2 } .

The Shapley ? Folkman lemma and related results provide an affirmative answer to the question , " Is the sum of many sets close to being convex ? " A set is defined to be convex if every line segment joining two of its points is a subset in the set : For example , the solid disk <formula> is a convex set but the circle <formula> is not , because the line segment joining two distinct points <formula> is not a subset of the circle . The Shapley ? Folkman lemma suggests that if the number of summed sets exceeds the dimension of the vector space , then their Minkowski sum is approximately convex .

The Shapley ? Folkman lemma was introduced as a step in the proof of the Shapley ? Folkman theorem , which states an upper bound on the distance between the Minkowski sum and its convex hull . The convex hull of a set Q is the smallest convex set that contains Q. This distance is zero if and only if the sum is convex . The theorem 's bound on the distance depends on the dimension D and on the shapes of the summand @-@ sets , but not on the number of summand @-@ sets N , when N > D. The shapes of a subcollection of only D summand @-@ sets determine the bound on the distance between the Minkowski average of N sets

1 ? N (Q1 + Q2 + ... + QN)

and its convex hull . As N increases to infinity , the bound decreases to zero (for summand @-@ sets of uniformly bounded size) . The Shapley ? Folkman theorem 's upper bound was decreased by Starr 's corollary (alternatively , the Shapley ? Folkman ? Starr theorem) .

The lemma of Lloyd Shapley and Jon Folkman was first published by the economist Ross M. Starr , who was investigating the existence of economic equilibria while studying with Kenneth Arrow . In his paper , Starr studied a convexified economy , in which non @-@ convex sets were replaced by their convex hulls ; Starr proved that the convexified economy has equilibria that are closely approximated by " quasi @-@ equilibria " of the original economy ; moreover , he proved that every quasi @-@ equilibrium has many of the optimal properties of true equilibria , which are proved to exist for convex economies . Following Starr 's 1969 paper , the Shapley ? Folkman ? Starr results have been widely used to show that central results of (convex) economic theory are good approximations to large economies with non @-@ convexities ; for example , quasi @-@ equilibria closely approximate equilibria of a convexified economy . " The derivation of these results in general form has been one of the major achievements of postwar economic theory " , wrote Roger Guesnerie . The topic of non @-@ convex sets in economics has been studied by many Nobel laureates , besides Lloyd Shapley who won the prize in 2012 : Arrow (1972) , Robert Aumann (2005) , Gérard Debreu (1983) , Tjalling Koopmans (1975) , Paul Krugman (2008) , and Paul Samuelson (1970) ; the complementary topic of convex sets in economics has been emphasized by these laureates , along with Leonid Hurwicz , Leonid Kantorovich (1975) , and Robert Solow (1987) .

The Shapley ? Folkman lemma has applications also in optimization and probability theory . In optimization theory , the Shapley ? Folkman lemma has been used to explain the successful solution of minimization problems that are sums of many functions . The Shapley ? Folkman lemma has also been used in proofs of the " law of averages " for random sets , a theorem that had been proved for only convex sets .

= = Introductory example = =

For example , the subset of the integers { 0 , 1 , 2 } is contained in the interval of real numbers [0 ,

2] , which is convex . The Shapley ? Folkman lemma implies that every point in [0 , 2] is the sum of an integer from { 0 , 1 } and a real number from [0 , 1] .
The distance between the convex interval [0 , 2] and the non @-@ convex set { 0 , 1 , 2 } equals one @-@ half
1 / 2
= | 1 ? 1 / 2 | =
| 0 ? 1 / 2 |
= | 2 ? 3 / 2 | =
| 1 ? 3 / 2 | .
However , the distance between the average Minkowski sum
1 / 2 ({ 0 , 1 } + { 0 , 1 }) = { 0 , 1 / 2 , 1 }
and its convex hull [0 , 1] is only 1 / 4 , which is half the distance (1 / 2) between its summand { 0 , 1 } and [0 , 1] . As more sets are added together , the average of their sum " fills out " its convex hull : The maximum distance between the average and its convex hull approaches zero as the average includes more summands .

= = Preliminaries = =

The Shapley ? Folkman lemma depends upon the following definitions and results from convex geometry .

= = = Real vector spaces = = =

A real vector space of two dimensions can be given a Cartesian coordinate system in which every point is identified by an ordered pair of real numbers , called " coordinates " , which are conventionally denoted by x and y . Two points in the Cartesian plane can be added coordinate @-@ wise
(x1 , y1) + (x2 , y2)
= (x1 + x2 , y1 + y2) ;
further , a point can be multiplied by each real number ? coordinate @-@ wise
? (x , y) =
(?x , ?y) .
More generally , any real vector space of (finite) dimension D can be viewed as the set of all D @-@ tuples of D real numbers { (v1 , v2 , . . . , vD) } on which two operations are defined : vector addition and multiplication by a real number . For finite @-@ dimensional vector spaces , the operations of vector addition and real @-@ number multiplication can each be defined coordinate @-@ wise , following the example of the Cartesian plane .

= = = Convex sets = = =

In a real vector space , a non @-@ empty set Q is defined to be convex if , for each pair of its points , every point on the line segment that joins them is a subset of Q. For example , a solid disk <formula> is convex but a circle <formula> is not , because it does not contain a line segment joining its points <formula> ; the non @-@ convex set of three integers { 0 , 1 , 2 } is contained in the interval [0 , 2] , which is convex . For example , a solid cube is convex ; however , anything that is hollow or dented , for example , a crescent shape , is non @-@ convex . The empty set is convex , either by definition or vacuously , depending on the author .
More formally , a set Q is convex if , for all points v0 and v1 in Q and for every real number ? in the unit interval [0 @,@ 1] , the point
(1 ? ?) v0 + ?v1
is a member of Q.
By mathematical induction , a set Q is convex if and only if every convex combination of members of Q also belongs to Q. By definition , a convex combination of an indexed subset { v0 , v1 , . . . , vD

} of a vector space is any weighted average ?0v0 + ?1v1 + . . . + ?DvD , for some indexed set of non @-@ negative real numbers { ?d } satisfying the equation ?0 + ?1 + . . . + ?D = 1 .

The definition of a convex set implies that the intersection of two convex sets is a convex set . More generally , the intersection of a family of convex sets is a convex set . In particular , the intersection of two disjoint sets is the empty set , which is convex .

= = = Convex hull = = =

For every subset Q of a real vector space , its convex hull Conv (Q) is the minimal convex set that contains Q. Thus Conv (Q) is the intersection of all the convex sets that cover Q. The convex hull of a set can be equivalently defined to be the set of all convex combinations of points in Q. For example , the convex hull of the set of integers { 0 @,@ 1 } is the closed interval of real numbers [0 @,@ 1] , which contains the integer end @-@ points . The convex hull of the unit circle is the closed unit disk , which contains the unit circle .

= = = Minkowski addition = = =

In a real vector space , the Minkowski sum of two (non @-@ empty) sets Q1 and Q2 is defined to be the set Q1 + Q2 formed by the addition of vectors element @-@ wise from the summand sets

Q1 + Q2

= { q1 + q2 : q1 ? Q1 and q2 ? Q2 } .

For example

{ 0 , 1 } + { 0 , 1 } =

{ 0 + 0 , 0 + 1 , 1 + 0 , 1 + 1 }

= { 0 , 1 , 2 } .

By the principle of mathematical induction , the Minkowski sum of a finite family of (non @-@ empty) sets

{ Qn : Qn ? Ø and 1 ? n ? N }

is the set formed by element @-@ wise addition of vectors

? Qn =

{ ? qn : qn ? Qn } .

= = = Convex hulls of Minkowski sums = = =

Minkowski addition behaves well with respect to " convexification " ? the operation of taking convex hulls . Specifically , for all subsets Q1 and Q2 of a real vector space , the convex hull of their Minkowski sum is the Minkowski sum of their convex hulls . That is ,