= 0 and g ? h =
0 , there is a Toda bracket ? f , g , h ? of these elements (Toda 1962) . The Toda bracket is not quite an element of a stable homotopy group , because it is only defined up to addition of products of certain other elements . Hiroshi Toda used the composition product and Toda brackets to label many of the elements of homotopy groups . There are also higher Toda brackets of several elements , defined when suitable lower Toda brackets vanish . This parallels the theory of Massey products in cohomology . Every element of the stable homotopy groups of spheres can be expressed using composition products and higher Toda brackets in terms of certain well known elements , called Hopf elements (Cohen 1968) .

= = Computational methods = =

If X is any finite simplicial complex with finite fundamental group , in particular if X is a sphere of dimension at least 2 , then its homotopy groups are all finitely generated abelian groups . To compute these groups , they are often factored into their p @-@ components for each prime p , and calculating each of these p @-@ groups separately . The first few homotopy groups of spheres can be computed using ad hoc variations of the ideas above ; beyond this point , most methods for computing homotopy groups of spheres are based on spectral sequences (Ravenel 2003) . This is usually done by constructing suitable fibrations and taking the associated long exact sequences of homotopy groups ; spectral sequences are a systematic way of organizing the complicated information that this process generates .

" The method of killing homotopy groups " , due to Cartan and Serre (1952a , 1952b) involves repeatedly using the Hurewicz theorem to compute the first non @-@ trivial homotopy group and then killing (eliminating) it with a fibration involving an Eilenberg @-@ MacLane space . In principle this gives an effective algorithm for computing all homotopy groups of any finite simply connected simplicial complex , but in practice it is too cumbersome to use for computing anything other than the first few nontrivial homotopy groups as the simplicial complex becomes much more complicated every time one kills a homotopy group .

The Serre spectral sequence was used by Serre to prove some of the results mentioned previously . He used the fact that taking the loop space of a well behaved space shifts all the homotopy groups down by 1 , so the nth homotopy group of a space X is the first homotopy group of its (n ? 1) -fold repeated loop space , which is equal to the first homology group of the (n ? 1) -fold loop space by the Hurewicz theorem . This reduces the calculation of homotopy groups of X to the calculation of homology groups of its repeated loop spaces . The Serre spectral sequence relates the homology of a space to that of its loop space , so can sometimes be used to calculate the homology of loop spaces . The Serre spectral sequence tends to have many non @-@ zero differentials , which are hard to control , and too many ambiguities appear for higher homotopy groups . Consequently , it has been superseded by more powerful spectral sequences with fewer non @-@ zero differentials , which give more information .

The EHP spectral sequence can be used to compute many homotopy groups of spheres ; it is based on some fibrations used by Toda in his calculations of homotopy groups (Mahowald 2001 , Toda 1962) .

The classical Adams spectral sequence has E2 term given by the Ext groups ExtA (p) ? , ? (Zp , Zp) over the mod p Steenrod algebra A (p) , and converges to something closely related to the p @-@ component of the stable homotopy groups . The initial terms of the Adams spectral sequence are themselves quite hard to compute : this is sometimes done using an auxiliary spectral sequence called the May spectral sequence (Ravenel 2003 , pp. 67 ? 74) .

The Adams ? Novikov spectral sequence is a more powerful version of the Adams spectral sequence replacing ordinary cohomology mod p with a generalized cohomology theory , such as complex cobordism or , more usually , a piece of it called Brown ? Peterson cohomology . The initial term is again quite hard to calculate ; to do this one can use the chromatic spectral sequence (Ravenel 2003 , Chapter 5) .

A variation of this last approach uses a backwards version of the Adams ? Novikov spectral sequence for Brown ? Peterson cohomology : the limit is known , and the initial terms involve unknown stable homotopy groups of spheres that one is trying to find . Kochman (1990) used this approach to calculate the 2 @-@ components of the first 64 stable homotopy groups ; unfortunately there was a mistake in his calculations for the 54th stem and beyond , which was corrected by Kochman & Mahowald (1995) .

The computation of the homotopy groups of S2 has been reduced to a combinatorial group theory question . Berrick et al . (2006) identify these homotopy groups as certain quotients of the Brunnian braid groups of S2 . Under this correspondence , every nontrivial element in ?n (S2) for n > 2 may be represented by a Brunnian braid over S2 that is not Brunnian over the disk D2 . For example , the Hopf map S3 ? S2 corresponds to the Borromean rings .

= = Applications = =