= f (1) =
0 and f and f ' are both square integrable , then the inequality holds :
<formula>
and the case of equality holds precisely when f is a multiple of sin (? x) . So ? appears as an optimal constant in Wirtinger 's inequality , and from this it follows that it is the smallest such eigenvalue (by Rayleigh quotient methods) .

The number ? serves a similar role in higher @-@ dimensional analysis , appearing as eigenvalues for other similar kinds of problems . As mentioned above , it can be characterized via its role as the best constant in the isoperimetric inequality : the area A enclosed by a plane Jordan curve of perimeter P satisfies the inequality
<formula>
and equality is clearly achieved for the circle , since in that case A
= ?r2 and P =
2?r .

Ultimately as a consequence of the isoperimetric inequality , the constant ? is associated with best constants of the Poincaré inequality . As a special case , ? appears as the optimal smallest eigenvalue of the Dirichlet energy , in dimensions 1 and 2 , which thus characterizes the role of ? in many physical phenomena as well , for example those of classical potential theory . The one @-@ dimensional case is just Wirtinger 's inequality .

The constant ? also appears as a critical spectral parameter in the Fourier transform . This is the integral transform , that takes a complex @-@ valued integrable function f on the real line to the function defined as :
<formula>

There are several different conventions for the Fourier transform , all of which involve a factor of ? that is placed somewhere . The appearance of ? is essential in these formulas , as there is there is no possibility to remove ? altogether from the Fourier transform and its inverse transform . The definition given above is the most canonical however , because it describes the unique unitary operator on L2 that is also an algebra homomorphism of L1 to L ? .

The Heisenberg uncertainty principle also contains the number ? . The uncertainty principle gives a sharp lower bound on the extent to which it is possible to localize a function both in space and in frequency : with our conventions for the Fourier transform ,
<formula>

The physical consequence , about the uncertainty in simultaneous position and momentum observations of a quantum mechanical system , is discussed below . The appearance of ? in the formulae of Fourier analysis is ultimately a consequence of the Stone ? von Neumann theorem , asserting the uniqueness of the Schrödinger representation of the Heisenberg group .

= = = Gaussian integrals = = =

The fields of probability and statistics frequently use the normal distribution as a simple model for complex phenomena ; for example , scientists generally assume that the observational error in most experiments follows a normal distribution . The Gaussian function , which is the probability density function of the normal distribution with mean ? and standard deviation ? , naturally contains ? :
<formula>

For this to be a probability density , the area under the graph of f needs to be equal to one . This follows from a change of variables in the Gaussian integral :
<formula>
which says that the area under the basic Bell curve in the figure is equal to the square root of ? .

The central limit theorem explains the central role of normal distributions , and thus of ? , in probability and statistics . This theorem is ultimately connected with the spectral characterization of ? as the eigenvalue associated with the Heisenberg uncertainty principle , and the fact that equality holds in the uncertainty principle only for the Gaussian function . Equivalently , ? is the unique constant making the Gaussian normal distribution e @-@ ?x2 equal to its own Fourier transform .

Indeed , according to Howe (1980) , the " whole business " of establishing the fundamental theorems Fourier analysis reduces to the Gaussian integral .

= = History = =

= = = Antiquity = = =

The best known approximations to ? dating before the Common Era were accurate to two decimal places ; this was improved upon in Chinese mathematics in particular by the mid first millennium , to an accuracy of seven decimal places . After this , no further progress was made until the late medieval period .

Some Egyptologists have claimed that the ancient Egyptians used an approximation of ? as 22 / 7 from as early as the Old Kingdom . This claim has met with skepticism .

The earliest written approximations of ? are found in Egypt and Babylon , both within one percent of the true value . In Babylon , a clay tablet dated 1900 ? 1600 BC has a geometrical statement that , by implication , treats ? as 25 / 8 = 3 @.@ 125 . In Egypt , the Rhind Papyrus , dated around 1650 BC but copied from a document dated to 1850 BC , has a formula for the area of a circle that treats ? as (16 / 9) 2 ? 3 @.@ 1605 .

Astronomical calculations in the Shatapatha Brahmana (ca . 4th century BC) use a fractional approximation of 339 / 108 ? 3 @.@ 139 (an accuracy of 9 × 10 ? 4) . Other Indian sources by about 150 BC treat ? as ? 10 ? 3 @.@ 1622 .

= = = Polygon approximation era = = =

The first recorded algorithm for rigorously calculating the value of ? was a geometrical approach using polygons , devised around 250 BC by the Greek mathematician Archimedes . This polygonal algorithm dominated for over 1 @,@ 000 years , and as a result ? is sometimes referred to as " Archimedes ' constant " . Archimedes computed upper and lower bounds of ? by drawing a regular hexagon inside and outside a circle , and successively doubling the number of sides until he reached a 96 @-@ sided regular polygon . By calculating the perimeters of these polygons , he proved that 223 / 71 < ? < 22 / 7 (that is 3 @.@ 1408 < ? < 3 @.@ 1429) . Archimedes ' upper bound of 22 / 7 may have led to a widespread popular belief that ? is equal to 22 / 7 . Around 150 AD , Greek @-@ Roman scientist Ptolemy , in his Almagest , gave a value for ? of 3 @.@ 1416 , which he may have obtained from Archimedes or from Apollonius of Perga . Mathematicians using polygonal algorithms reached 39 digits of ? in 1630 , a record only broken in 1699 when infinite series were used to reach 71 digits .

In ancient China , values for ? included 3 @.@ 1547 (around 1 AD) , ? 10 (100 AD , approximately 3 @.@ 1623) , and 142 / 45 (3rd century , approximately 3 @.@ 1556) . Around 265 AD , the Wei Kingdom mathematician Liu Hui created a polygon @-@ based iterative algorithm and used it with a 3 @,@ 072 @-@ sided polygon to obtain a value of ? of 3 @.@ 1416 . Liu later invented a faster method of calculating ? and obtained a value of 3 @.@ 14 with a 96 @-@ sided polygon , by taking advantage of the fact that the differences in area of successive polygons form a geometric series with a factor of 4 . The Chinese mathematician Zu Chongzhi , around 480 AD , calculated that ? ? 355 / 113 (a fraction that goes by the name Milü in Chinese) , using Liu Hui 's algorithm applied to a 12 @,@ 288 @-@ sided polygon . With a correct value for its seven first decimal digits , this value of 3 @.@ 141592920 ... remained the most accurate approximation of ? available for the next 800 years .

The Indian astronomer Aryabhata used a value of 3 @.@ 1416 in his ?ryabha??ya (499 AD) . Fibonacci in c . 1220 computed 3 @.@ 1418 using a polygonal method , independent of Archimedes . Italian author Dante apparently employed the value 3 + ? 2 / 10 ? 3 @.@ 14142 .

The Persian astronomer Jamsh?d al @-@ K?sh? produced 9 sexagesimal digits , roughly the equivalent of 16 decimal digits , in 1424 using a polygon with 3 × 228 sides , which stood as the world record for about 180 years . French mathematician François Viète in 1579 achieved 9 digits with a polygon of 3 × 217 sides . Flemish mathematician Adriaan van Roomen arrived at 15 decimal places in 1593 . In 1596 , Dutch mathematician Ludolph van Ceulen reached 20 digits , a record he later increased to 35 digits (as a result , ? was called the " Ludolphian number " in Germany until the early 20th century) . Dutch scientist Willebrord Snellius reached 34 digits in 1621 , and Austrian astronomer Christoph Grienberger arrived at 38 digits in 1630 using 1040 sides , which remains the most accurate approximation manually achieved using polygonal algorithms .

= = = Infinite series = = =

The calculation of ? was revolutionized by the development of infinite series techniques in the 16th and 17th centuries . An infinite series is the sum of the terms of an infinite sequence . Infinite series allowed mathematicians to compute ? with much greater precision than Archimedes and others who used geometrical techniques . Although infinite series were exploited for ? most notably by European mathematicians such as James Gregory and Gottfried Wilhelm Leibniz , the approach was first discovered in India sometime between 1400 and 1500 AD . The first written description of an infinite series that could be used to compute ? was laid out in Sanskrit verse by Indian astronomer Nilakantha Somayaji in his Tantrasamgraha , around 1500 AD . The series are presented without proof , but proofs are presented in a later Indian work , Yuktibh??? , from around 1530 AD . Nilakantha attributes the series to an earlier Indian mathematician , Madhava of Sangamagrama , who lived c . 1350 ? c . 1425 . Several infinite series are described , including series for sine , tangent , and cosine , which are now referred to as the Madhava series or Gregory ? Leibniz series . Madhava used infinite series to estimate ? to 11 digits around 1400 , but that value was improved on around 1430 by the Persian mathematician Jamsh?d al @-@ K?sh? , using a polygonal algorithm .

The first infinite sequence discovered in Europe was an infinite product (rather than an infinite sum , which are more typically used in ? calculations) found by French mathematician François Viète in 1593 :

<formula> A060294

The second infinite sequence found in Europe , by John Wallis in 1655 , was also an infinite product :

<formula>

The discovery of calculus , by English scientist Isaac Newton and German mathematician Gottfried Wilhelm Leibniz in the 1660s , led to the development of many infinite series for approximating ? . Newton himself used an arcsin series to compute a 15 digit approximation of ? in 1665 or 1666 , later writing " I am ashamed to tell you to how many figures I carried these computations , having no other business at the time . "

In Europe , Madhava 's formula was rediscovered by Scottish mathematician James Gregory in 1671 , and by Leibniz in 1674 :

<formula>

This formula , the Gregory ? Leibniz series , equals ? / 4 when evaluated with z = 1 . In 1699 , English mathematician Abraham Sharp used the Gregory ? Leibniz series for <formula> to compute ? to 71 digits , breaking the previous record of 39 digits , which was set with a polygonal algorithm . The Gregory ? Leibniz for <formula> series is simple , but converges very slowly (that is , approaches the answer gradually) , so it is not used in modern ? calculations .

In 1706 John Machin used the Gregory ? Leibniz series to produce an algorithm that converged much faster :

<formula>

Machin reached 100 digits of ? with this formula . Other mathematicians created variants , now known as Machin @-@ like formulae , that were used to set several successive records for calculating digits of ? . Machin @-@ like formulae remained the best @-@ known method for calculating ? well into the age of computers , and were used to set records for 250 years , culminating in a 620 @-@ digit approximation in 1946 by Daniel Ferguson ? the best approximation

achieved without the aid of a calculating device .
A record was set by the calculating prodigy Zacharias Dase , who in 1844 employed a Machin @-@ like formula to calculate 200 decimals of ? in his head at the behest of German mathematician Carl Friedrich Gauss . British mathematician William Shanks famously took 15 years to calculate ? to 707 digits , but made a mistake in the 528th digit , rendering all subsequent digits incorrect .

= = = = Rate of convergence = = = =

Some infinite series for ? converge faster than others . Given the choice of two infinite series for ? , mathematicians will generally use the one that converges more rapidly because faster convergence reduces the amount of computation needed to calculate ? to any given accuracy . A simple infinite series for ? is the Gregory ? Leibniz series :
<formula>
As individual terms of this infinite series are added to the sum , the total gradually gets closer to ? , and ? with a sufficient number of terms ? can get as close to ? as desired . It converges quite slowly , though ? after 500 @,@ 000 terms , it produces only five correct decimal digits of ? .
An infinite series for ? (published by Nilakantha in the 15th century) that converges more rapidly than the Gregory ? Leibniz series is :
<formula>
The following table compares the convergence rates of these two series :
After five terms , the sum of the Gregory ? Leibniz series is within 0 @.@ 2 of the correct value of ? , whereas the sum of Nilakantha 's series is within 0 @.@ 002 of the correct value of ? . Nilakantha 's series converges faster and is more useful for computing digits of ? . Series that converge even faster include Machin 's series and Chudnovsky 's series , the latter producing 14 correct decimal digits per term .

= = = Irrationality and transcendence = = =

Not all mathematical advances relating to ? were aimed at increasing the accuracy of approximations . When Euler solved the Basel problem in 1735 , finding the exact value of the sum of the reciprocal squares , he established a connection between ? and the prime numbers that later contributed to the development and study of the Riemann zeta function :
<formula>
Swiss scientist Johann Heinrich Lambert in 1761 proved that ? is irrational , meaning it is not equal to the quotient of any two whole numbers . Lambert 's proof exploited a continued @-@ fraction representation of the tangent function . French mathematician Adrien @-@ Marie Legendre proved in 1794 that ?2 is also irrational . In 1882 , German mathematician Ferdinand von Lindemann proved that ? is transcendental , confirming a conjecture made by both Legendre and Euler . Hardy and Wright states that " the proofs were afterwards modified and simplified by Hilbert , Hurwitz , and other writers " .

= = = Adoption of the symbol ? = = =

The earliest known use of the Greek letter ? to represent the ratio of a circle 's circumference to its diameter was by Welsh mathematician William Jones in his 1706 work Synopsis Palmariorum Matheseos ; or , a New Introduction to the Mathematics . The Greek letter first appears there in the phrase " 1 / 2 Periphery (?) " in the discussion of a circle with radius one . Jones may have chosen ? because it was the first letter in the Greek spelling of the word periphery . However , he writes that his equations for ? are from the " ready pen of the truly ingenious Mr. John Machin " , leading to speculation that Machin may have employed the Greek letter before Jones . It had indeed been used earlier for geometric concepts . William Oughtred used ? and ? , the Greek letter equivalents of p and d , to express ratios of periphery and diameter in the 1647 and later editions of Clavis Mathematicae .

After Jones introduced the Greek letter in 1706 , it was not adopted by other mathematicians until Euler started using it , beginning with his 1736 work Mechanica . Before then , mathematicians sometimes used letters such as c or p instead . Because Euler corresponded heavily with other mathematicians in Europe , the use of the Greek letter spread rapidly . In 1748 , Euler used ? in his widely read work Introductio in analysin infinitorum (he wrote : " for the sake of brevity we will write this number as ? ; thus ? is equal to half the circumference of a circle of radius 1 ") and the practice was universally adopted thereafter in the Western world .

= = Modern quest for more digits = =

= = = Computer era and iterative algorithms = = =

The development of computers in the mid @-@ 20th century again revolutionized the hunt for digits of ? . American mathematicians John Wrench and Levi Smith reached 1 @,@ 120 digits in 1949 using a desk calculator . Using an inverse tangent (arctan) infinite series , a team led by George Reitwiesner and John von Neumann that same year achieved 2 @,@ 037 digits with a calculation that took 70 hours of computer time on the ENIAC computer . The record , always relying on an arctan series , was broken repeatedly (7 @,@ 480 digits in 1957 ; 10 @,@ 000 digits in 1958 ; 100 @,@ 000 digits in 1961) until 1 million digits were reached in 1973 .

Two additional developments around 1980 once again accelerated the ability to compute ? . First , the discovery of new iterative algorithms for computing ? , which were much faster than the infinite series ; and second , the invention of fast multiplication algorithms that could multiply large numbers very rapidly . Such algorithms are particularly important in modern ? computations , because most of the computer 's time is devoted to multiplication . They include the Karatsuba algorithm , Toom ? Cook multiplication , and Fourier transform @-@ based methods .

The iterative algorithms were independently published in 1975 ? 1976 by American physicist Eugene Salamin and Australian scientist Richard Brent . These avoid reliance on infinite series . An iterative algorithm repeats a specific calculation , each iteration using the outputs from prior steps as its inputs , and produces a result in each step that converges to the desired value . The approach was actually invented over 160 years earlier by Carl Friedrich Gauss , in what is now termed the arithmetic ? geometric mean method (AGM method) or Gauss ? Legendre algorithm . As modified by Salamin and Brent , it is also referred to as the Brent ? Salamin algorithm .

The iterative algorithms were widely used after 1980 because they are faster than infinite series algorithms : whereas infinite series typically increase the number of correct digits additively in successive terms , iterative algorithms generally multiply the number of correct digits at each step . For example , the Brent @-@ Salamin algorithm doubles the number of digits in each iteration . In 1984 , the Canadian brothers John and Peter Borwein produced an iterative algorithm that quadruples the number of digits in each step ; and in 1987 , one that increases the number of digits five times in each step . Iterative methods were used by Japanese mathematician Yasumasa Kanada to set several records for computing ? between 1995 and 2002 . This rapid convergence comes at a price : the iterative algorithms require significantly more memory than infinite series .

= = = Motivations for computing ? = = =

For most numerical calculations involving ? , a handful of digits provide sufficient precision . According to Jörg Arndt and Christoph Haenel , thirty @-@ nine digits are sufficient to perform most cosmological calculations , because that is the accuracy necessary to calculate the circumference of the observable universe with a precision of one atom . Despite this , people have worked strenuously to compute ? to thousands and millions of digits . This effort may be partly ascribed to the human compulsion to break records , and such achievements with ? often make headlines around the world . They also have practical benefits , such as testing supercomputers , testing numerical analysis algorithms (including high @-@ precision multiplication algorithms) ; and within pure mathematics itself , providing data for evaluating the randomness of the digits of ? .

= = = Rapidly convergent series = = =

Modern ? calculators do not use iterative algorithms exclusively . New infinite series were discovered in the 1980s and 1990s that are as fast as iterative algorithms , yet are simpler and less memory intensive . The fast iterative algorithms were anticipated in 1914 , when the Indian mathematician Srinivasa Ramanujan published dozens of innovative new formulae for ? , remarkable for their elegance , mathematical depth , and rapid convergence . One of his formulae , based on modular equations , is
<formula>
This series converges much more rapidly than most arctan series , including Machin 's formula . Bill Gosper was the first to use it for advances in the calculation of ? , setting a record of 17 million digits in 1985 . Ramanujan 's formulae anticipated the modern algorithms developed by the Borwein brothers and the Chudnovsky brothers . The Chudnovsky formula developed in 1987 is
<formula>
It produces about 14 digits of ? per term , and has been used for several record @-@ setting ? calculations , including the first to surpass 1 billion (109) digits in 1989 by the Chudnovsky brothers , 2 @.@ 7 trillion (2 @.@ 7 × 1012) digits by Fabrice Bellard in 2009 , and 10 trillion (1013) digits in 2011 by Alexander Yee and Shigeru Kondo . For similar formulas , see also the Ramanujan ? Sato series .
In 2006 , Canadian mathematician Simon Plouffe used the PSLQ integer relation algorithm to generate several new formulas for ? , conforming to the following template :
<formula>
where q is e? (Gelfond 's constant) , k is an odd number , and a , b , c are certain rational numbers that Plouffe computed .

= = = Monte Carlo methods = = =

Monte Carlo methods , which evaluate the results of multiple random trials , can be used to create approximations of ? . Buffon 's needle is one such technique : If a needle of length ? is dropped n times on a surface on which parallel lines are drawn t units apart , and if x of those times it comes to rest crossing a line (x > 0) , then one may approximate ? based on the counts :
<formula>
Another Monte Carlo method for computing ? is to draw a circle inscribed in a square , and randomly place dots in the square . The ratio of dots inside the circle to the total number of dots will approximately equal ? / 4 .
Another way to calculate ? using probability is to start with a random walk , generated by a sequence of (fair) coin tosses : independent random variables Xk such that Xk ? { ? 1 @,@ 1 } with equal probabilities . The associated random walk is
<formula>
so that , for each n , Wn is drawn from a standard binomial distribution . As n varies Wn defines a (discrete) stochastic process . Then ? can be calculated by
<formula>
This Monte Carlo method is independent of any relation to circles , and is a consequence of the central limit theorem , discussed above .
These Monte Carlo methods for approximating ? are very slow compared to other methods , and do not provide any information on the exact number of digits that are obtained . Thus they are never used to approximate ? when speed or accuracy is desired .

= = = Spigot algorithms = = =

Two algorithms were discovered in 1995 that opened up new avenues of research into ? . They are called spigot algorithms because , like water dripping from a spigot , they produce single digits of ? that are not reused after they are calculated . This is in contrast to infinite series or iterative algorithms , which retain and use all intermediate digits until the final result is produced .

American mathematicians Stan Wagon and Stanley Rabinowitz produced a simple spigot algorithm in 1995 . Its speed is comparable to arctan algorithms , but not as fast as iterative algorithms .

Another spigot algorithm , the BBP digit extraction algorithm , was discovered in 1995 by Simon Plouffe :

<formula>

This formula , unlike others before it , can produce any individual hexadecimal digit of ? without calculating all the preceding digits . Individual binary digits may be extracted from individual hexadecimal digits , and octal digits can be extracted from one or two hexadecimal digits . Variations of the algorithm have been discovered , but no digit extraction algorithm has yet been found that rapidly produces decimal digits . An important application of digit extraction algorithms is to validate new claims of record ? computations : After a new record is claimed , the decimal result is converted to hexadecimal , and then a digit extraction algorithm is used to calculate several random hexadecimal digits near the end ; if they match , this provides a measure of confidence that the entire computation is correct .

Between 1998 and 2000 , the distributed computing project PiHex used Bellard 's formula (a modification of the BBP algorithm) to compute the quadrillionth (1015th) bit of ? , which turned out to be 0 . In September 2010 , a Yahoo ! employee used the company 's Hadoop application on one thousand computers over a 23 @-@ day period to compute 256 bits of ? at the two @-@ quadrillionth (2 × 1015th) bit , which also happens to be zero .

= = Use = =

Because ? is closely related to the circle , it is found in many formulae from the fields of geometry and trigonometry , particularly those concerning circles , spheres , or ellipses . Other branches of science , such as statistics , physics , Fourier analysis , and number theory , also include ? in some of their important formulae .

= = = Geometry and trigonometry = = =

? appears in formulae for areas and volumes of geometrical shapes based on circles , such as ellipses , spheres , cones , and tori . Below are some of the more common formulae that involve ? .

The circumference of a circle with radius r is 2?r .

The area of a circle with radius r is ?r2 .

The volume of a sphere with radius r is 4 / 3?r3 .

The surface area of a sphere with radius r is 4?r2 .

The formulae above are special cases of the volume of the n @-@ dimensional ball and the surface area of its boundary , the (n ? 1) -dimensional sphere , given below .

Definite integrals that describe circumference , area , or volume of shapes generated by circles typically have values that involve ? . For example , an integral that specifies half the area of a circle of radius one is given by :

<formula>

In that integral the function ? 1 ? x2 represents the top half of a circle (the square root is a consequence of the Pythagorean theorem) , and the integral ? 1

? 1 computes the area between that half of a circle and the x axis .

The trigonometric functions rely on angles , and mathematicians generally use radians as units of measurement. ? plays an important role in angles measured in radians , which are defined so that a complete circle spans an angle of 2? radians . The angle measure of 180 ° is equal to ? radians , and 1 ° = ? / 180 radians .

Common trigonometric functions have periods that are multiples of ? ; for example , sine and cosine have period 2? , so for any angle ? and any integer k ,

<formula>

= = = Topology = = =

The constant ? appears in the Gauss ? Bonnet formula which relates the differential geometry of surfaces to their topology . Specifically , if a compact surface ? has Gauss curvature K , then
<formula>
where ? (?) is the Euler characteristic , which is an integer . An example is the surface area of a sphere S of curvature 1 (so that its radius of curvature , which coincides with its radius , is also 1 .) The Euler characteristic of a sphere can be computed from its homology groups , and is found to be equal to two . Thus we have
<formula>
reproducing the formula for the surface area of a sphere of radius 1 .
The constant appears in many other integral formulae in topology , in particular those involving characteristic classes via the Chern ? Weil homomorphism .

= = = Vector calculus = = =

Vector calculus is a branch of calculus that is concerned with the properties of vector fields , and has many physical applications such as to electricity and magnetism . The Newtonian potential for a point source Q situated at the origin of a three dimensional Cartesian coordinate system is
<formula>
which represents the potential energy of a unit mass (or charge) placed a distance | x | from the source , and k is a dimensional constant . The field , denoted here by E , which may be the (Newtonian) gravitational field or the (Coulomb) electric field , is the negative gradient of the potential :
<formula>
Special cases include Coulomb 's law and Newton 's law of universal gravitation . Gauss ' law states that the outward flux of the field through any smooth , simple , closed , orientable surface S containing the origin is equal to 4?kQ :
<formula> <formula> <formula>
It is standard to absorb this factor of 4? into the constant k , but this argument shows why it must appear somewhere . Furthermore , 4? is the surface area of the unit sphere , but we have not assumed that S is the sphere . However , as a consequence of the divergence theorem , because the region away from the origin is vacuum (source @-@ free) it is only the homology class of the surface S in R3 \ { 0 } that matters in computing the integral , so it can be replaced by any convenient surface in the same homology class , in particular a sphere , where spherical coordinates can be used to calculate the integral .
A consequence of the Gauss law is that the negative Laplacian of the potential V is equal to 4?kQ times the Dirac delta function :
<formula>
More general distributions of matter (or charge) are obtained from this by convolution , giving the Poisson equation
<formula>
where ? is the distribution function .
The constant ? also plays an analogous role in four @-@ dimensional potentials associated with Einstein 's equations , a fundamental formula which forms the basis of the general theory of relativity and describes the fundamental interaction of gravitation as a result of spacetime being curved by matter and energy :
<formula>
where R?? is the Ricci curvature tensor , R is the scalar curvature , g?? is the metric tensor , ? is the cosmological constant , G is Newton 's gravitational constant , c is the speed of light in vacuum , and T?? is the stress ? energy tensor . The left @-@ hand side of Einstein 's equation is a non @-@

linear analog of the Laplacian of the metric tensor (and reduces to that in the weak field limit) , and the right hand side is the analog of the distribution function , times 8? .

= = = Cauchy 's integral formula = = =

One of the key tools in complex analysis is contour integration of a function over a positively oriented (rectifiable) Jordan curve ? . A form of Cauchy 's integral formula states that if a point z0 is interior to ? , then
<formula>
Although the curve ? is not a circle , and hence does not have any obvious connection to the constant ? , a standard proof of this result uses Morera 's theorem , which implies that the integral is invariant under homotopy of the curve , so that it can be deformed to a circle and then integrated explicitly in polar coordinates . More generally , it is true that if a rectifiable closed curve ? does not contain z0 , then the above integral is 2?i times the winding number of the curve .
The general form of Cauchy 's integral formula establishes the relationship between the values of a complex analytic function f (z) on the Jordan curve ? and the value of f (z) at any interior point z0 of ? :
<formula>
provided f (z) is analytic in the region enclosed by ? and extends continuously to ? . Cauchy 's integral formula is a special case of the residue theorem , that if g (z) is a meromorphic function the region enclosed by ? and is continuous in a neighborhood of ? , then
<formula>
where the sum is of the residues at the poles of g (z) .

= = = The gamma function and Stirling 's approximation = = =

The factorial function n ! is the product of all of the positive integers through n . The gamma function extends the concept of factorial (normally defined only for non @-@ negative integers) to all complex numbers , except the negative real integers . When the gamma function is evaluated at half @-@ integers , the result contains ? ; for example <formula> and <formula> .
The gamma function is defined by its Weierstrass product development :
<formula>