= Mathematical economics =

Mathematical economics is the application of mathematical methods to represent theories and analyze problems in economics . By convention , the applied methods refer to those beyond simple geometry , such as differential and integral calculus , difference and differential equations , matrix algebra , mathematical programming , and other computational methods . An advantage claimed for the approach is its allowing formulation of theoretical relationships with rigor , generality , and simplicity .

Mathematics allows economists to form meaningful , testable propositions about wide @-@ ranging and complex subjects which could less easily be expressed informally . Further , the language of mathematics allows economists to make specific , positive claims about controversial or contentious subjects that would be impossible without mathematics . Much of economic theory is currently presented in terms of mathematical economic models , a set of stylized and simplified mathematical relationships asserted to clarify assumptions and implications .

Broad applications include :

optimization problems as to goal equilibrium , whether of a household , business firm , or policy maker

static (or equilibrium) analysis in which the economic unit (such as a household) or economic system (such as a market or the economy) is modeled as not changing

comparative statics as to a change from one equilibrium to another induced by a change in one or more factors

dynamic analysis , tracing changes in an economic system over time , for example from economic growth .

Formal economic modeling began in the 19th century with the use of differential calculus to represent and explain economic behavior , such as utility maximization , an early economic application of mathematical optimization . Economics became more mathematical as a discipline throughout the first half of the 20th century , but introduction of new and generalized techniques in the period around the Second World War , as in game theory , would greatly broaden the use of mathematical formulations in economics .

This rapid systematizing of economics alarmed critics of the discipline as well as some noted economists . John Maynard Keynes , Robert Heilbroner , Friedrich Hayek and others have criticized the broad use of mathematical models for human behavior , arguing that some human choices are irreducible to mathematics .

= = History = =

The use of mathematics in the service of social and economic analysis dates back to the 17th century . Then , mainly in German universities , a style of instruction emerged which dealt specifically with detailed presentation of data as it related to public administration . Gottfried Achenwall lectured in this fashion , coining the term statistics . At the same time , a small group of professors in England established a method of " reasoning by figures upon things relating to government " and referred to this practice as Political Arithmetick . Sir William Petty wrote at length on issues that would later concern economists , such as taxation , Velocity of money and national income , but while his analysis was numerical , he rejected abstract mathematical methodology . Petty 's use of detailed numerical data (along with John Graunt) would influence statisticians and economists for some time , even though Petty 's works were largely ignored by English scholars .

The mathematization of economics began in earnest in the 19th century . Most of the economic analysis of the time was what would later be called classical economics . Subjects were discussed and dispensed with through algebraic means , but calculus was not used . More importantly , until Johann Heinrich von Thünen 's The Isolated State in 1826 , economists did not develop explicit and abstract models for behavior in order to apply the tools of mathematics . Thünen 's model of farmland use represents the first example of marginal analysis . Thünen 's work was largely theoretical , but he also mined empirical data in order to attempt to support his generalizations . In

comparison to his contemporaries , Thünen built economic models and tools , rather than applying previous tools to new problems .

Meanwhile , a new cohort of scholars trained in the mathematical methods of the physical sciences gravitated to economics , advocating and applying those methods to their subject , and described today as moving from geometry to mechanics . These included W.S. Jevons who presented paper on a " general mathematical theory of political economy " in 1862 , providing an outline for use of the theory of marginal utility in political economy . In 1871 , he published The Principles of Political Economy , declaring that the subject as science " must be mathematical simply because it deals with quantities . " Jevons expected the only collection of statistics for price and quantities would permit the subject as presented to become an exact science . Others preceded and followed in expanding mathematical representations of economic problems .

= = = Marginalists and the roots of neoclassical economics = = =

Augustin Cournot and Léon Walras built the tools of the discipline axiomatically around utility , arguing that individuals sought to maximize their utility across choices in a way that could be described mathematically . At the time , it was thought that utility was quantifiable , in units known as utils . Cournot , Walras and Francis Ysidro Edgeworth are considered the precursors to modern mathematical economics .

= = = = Augustin Cournot = = = =

Cournot , a professor of mathematics , developed a mathematical treatment in 1838 for duopoly ? a market condition defined by competition between two sellers . This treatment of competition , first published in Researches into the Mathematical Principles of Wealth , is referred to as Cournot duopoly . It is assumed that both sellers had equal access to the market and could produce their goods without cost . Further , it assumed that both goods were homogeneous . Each seller would vary her output based on the output of the other and the market price would be determined by the total quantity supplied . The profit for each firm would be determined by multiplying their output and the per unit Market price . Differentiating the profit function with respect to quantity supplied for each firm left a system of linear equations , the simultaneous solution of which gave the equilibrium quantity , price and profits . Cournot 's contributions to the mathematization of economics would be neglected for decades , but eventually influenced many of the marginalists . Cournot 's models of duopoly and Oligopoly also represent one of the first formulations of non @-@ cooperative games . Today the solution can be given as a Nash equilibrium but Cournot 's work preceded modern game theory by over 100 years .

= = = = Léon Walras = = = =

While Cournot provided a solution for what would later be called partial equilibrium , Léon Walras attempted to formalize discussion of the economy as a whole through a theory of general competitive equilibrium . The behavior of every economic actor would be considered on both the production and consumption side . Walras originally presented four separate models of exchange , each recursively included in the next . The solution of the resulting system of equations (both linear and non @-@ linear) is the general equilibrium . At the time , no general solution could be expressed for a system of arbitrarily many equations , but Walras 's attempts produced two famous results in economics . The first is Walras ' law and the second is the principle of tâtonnement . Walras ' method was considered highly mathematical for the time and Edgeworth commented at length about this fact in his review of Éléments d 'économie politique pure (Elements of Pure Economics) .

Walras ' law was introduced as a theoretical answer to the problem of determining the solutions in general equilibrium . His notation is different from modern notation but can be constructed using more modern summation notation . Walras assumed that in equilibrium , all money would be spent

on all goods : every good would be sold at the market price for that good and every buyer would expend their last dollar on a basket of goods . Starting from this assumption , Walras could then show that if there were n markets and n @-@ 1 markets cleared (reached equilibrium conditions) that the nth market would clear as well . This is easiest to visualize with two markets (considered in most texts as a market for goods and a market for money) . If one of two markets has reached an equilibrium state , no additional goods (or conversely , money) can enter or exit the second market , so it must be in a state of equilibrium as well . Walras used this statement to move toward a proof of existence of solutions to general equilibrium but it is commonly used today to illustrate market clearing in money markets at the undergraduate level .

Tâtonnement (roughly , French for groping toward) was meant to serve as the practical expression of Walrasian general equilibrium . Walras abstracted the marketplace as an auction of goods where the auctioneer would call out prices and market participants would wait until they could each satisfy their personal reservation prices for the quantity desired (remembering here that this is an auction on all goods , so everyone has a reservation price for their desired basket of goods) .

Only when all buyers are satisfied with the given market price would transactions occur . The market would " clear " at that price ? no surplus or shortage would exist . The word tâtonnement is used to describe the directions the market takes in groping toward equilibrium , settling high or low prices on different goods until a price is agreed upon for all goods . While the process appears dynamic , Walras only presented a static model , as no transactions would occur until all markets were in equilibrium . In practice very few markets operate in this manner .

= = = = Francis Ysidro Edgeworth = = = =

Edgeworth introduced mathematical elements to Economics explicitly in Mathematical Psychics : An Essay on the Application of Mathematics to the Moral Sciences , published in 1881 . He adopted Jeremy Bentham 's felicific calculus to economic behavior , allowing the outcome of each decision to be converted into a change in utility . Using this assumption , Edgeworth built a model of exchange on three assumptions : individuals are self @-@ interested , individuals act to maximize utility , and individuals are " free to recontract with another independently of ... any third party . "

Given two individuals , the set of solutions where the both individuals can maximize utility is described by the contract curve on what is now known as an Edgeworth Box . Technically , the construction of the two @-@ person solution to Edgeworth 's problem was not developed graphically until 1924 by Arthur Lyon Bowley . The contract curve of the Edgeworth box (or more generally on any set of solutions to Edgeworth 's problem for more actors) is referred to as the core of an economy .

Edgeworth devoted considerable effort to insisting that mathematical proofs were appropriate for all schools of thought in economics . While at the helm of The Economic Journal , he published several articles criticizing the mathematical rigor of rival researchers , including Edwin Robert Anderson Seligman , a noted skeptic of mathematical economics . The articles focused on a back and forth over tax incidence and responses by producers . Edgeworth noticed that a monopoly producing a good that had jointness of supply but not jointness of demand (such as first class and economy on an airplane , if the plane flies , both sets of seats fly with it) might actually lower the price seen by the consumer for one of the two commodities if a tax were applied . Common sense and more traditional , numerical analysis seemed to indicate that this was preposterous . Seligman insisted that the results Edgeworth achieved were a quirk of his mathematical formulation . He suggested that the assumption of a continuous demand function and an infinitesimal change in the tax resulted in the paradoxical predictions . Harold Hotelling later showed that Edgeworth was correct and that the same result (a " diminution of price as a result of the tax ") could occur with a discontinuous demand function and large changes in the tax rate .

= = Modern mathematical economics = =

From the later @-@ 1930s , an array of new mathematical tools from the differential calculus and

differential equations , convex sets , and graph theory were deployed to advance economic theory in a way similar to new mathematical methods earlier applied to physics . The process was later described as moving from mechanics to axiomatics .

= = = Differential calculus = = =

Vilfredo Pareto analyzed microeconomics by treating decisions by economic actors as attempts to change a given allotment of goods to another , more preferred allotment . Sets of allocations could then be treated as Pareto efficient (Pareto optimal is an equivalent term) when no exchanges could occur between actors that could make at least one individual better off without making any other individual worse off . Pareto 's proof is commonly conflated with Walrassian equilibrium or informally ascribed to Adam Smith 's Invisible hand hypothesis . Rather , Pareto 's statement was the first formal assertion of what would be known as the first fundamental theorem of welfare economics . These models lacked the inequalities of the next generation of mathematical economics .

In the landmark treatise Foundations of Economic Analysis (1947) , Paul Samuelson identified a common paradigm and mathematical structure across multiple fields in the subject , building on previous work by Alfred Marshall . Foundations took mathematical concepts from physics and applied them to economic problems . This broad view (for example , comparing Le Chatelier 's principle to tâtonnement) drives the fundamental premise of mathematical economics : systems of economic actors may be modeled and their behavior described much like any other system . This extension followed on the work of the marginalists in the previous century and extended it significantly . Samuelson approached the problems of applying individual utility maximization over aggregate groups with comparative statics , which compares two different equilibrium states after an exogenous change in a variable . This and other methods in the book provided the foundation for mathematical economics in the 20th century .

= = = Linear models = = =

Restricted models of general equilibrium were formulated by John von Neumann in 1937 . Unlike earlier versions , the models of von Neumann had inequality constraints . For his model of an expanding economy , von Neumann proved the existence and uniqueness of an equilibrium using his generalization of Brouwer 's fixed point theorem . Von Neumann 's model of an expanding economy considered the matrix pencil A - ? B with nonnegative matrices A and B ; von Neumann sought probability vectors p and q and a positive number ? that would solve the complementarity equation

pT (A - ? B) q = 0 @,@

along with two inequality systems expressing economic efficiency . In this model , the (transposed) probability vector p represents the prices of the goods while the probability vector q represents the " intensity " at which the production process would run . The unique solution ? represents the rate of growth of the economy , which equals the interest rate . Proving the existence of a positive growth rate and proving that the growth rate equals the interest rate were remarkable achievements , even for von Neumann . Von Neumann 's results have been viewed as a special case of linear programming , where von Neumann 's model uses only nonnegative matrices . The study of von Neumann 's model of an expanding economy continues to interest mathematical economists with interests in computational economics .

= = = = Input @-@ output economics = = = =

In 1936 , the Russian ? born economist Wassily Leontief built his model of input @-@ output analysis from the ' material balance ' tables constructed by Soviet economists , which themselves followed earlier work by the physiocrats . With his model , which described a system of production and demand processes , Leontief described how changes in demand in one economic sector would influence production in another . In practice , Leontief estimated the coefficients of his simple

models , to address economically interesting questions . In production economics , " Leontief technologies " produce outputs using constant proportions of inputs , regardless of the price of inputs , reducing the value of Leontief models for understanding economies but allowing their parameters to be estimated relatively easily . In contrast , the von Neumann model of an expanding economy allows for choice of techniques , but the coefficients must be estimated for each technology .

= = = Mathematical optimization = = =

In mathematics , mathematical optimization (or optimization or mathematical programming) refers to the selection of a best element from some set of available alternatives . In the simplest case , an optimization problem involves maximizing or minimizing a real function by selecting input values of the function and computing the corresponding values of the function . The solution process includes satisfying general necessary and sufficient conditions for optimality . For optimization problems , specialized notation may be used as to the function and its input (s) . More generally , optimization includes finding the best available element of some function given a defined domain and may use a variety of different computational optimization techniques .

Economics is closely enough linked to optimization by agents in an economy that an influential definition relatedly describes economics qua science as the " study of human behavior as a relationship between ends and scarce means " with alternative uses . Optimization problems run through modern economics , many with explicit economic or technical constraints . In microeconomics , the utility maximization problem and its dual problem , the expenditure minimization problem for a given level of utility , are economic optimization problems . Theory posits that consumers maximize their utility , subject to their budget constraints and that firms maximize their profits , subject to their production functions , input costs , and market demand .

Economic equilibrium is studied in optimization theory as a key ingredient of economic theorems that in principle could be tested against empirical data . Newer developments have occurred in dynamic programming and modeling optimization with risk and uncertainty , including applications to portfolio theory , the economics of information , and search theory .

Optimality properties for an entire market system may be stated in mathematical terms , as in formulation of the two fundamental theorems of welfare economics and in the Arrow ? Debreu model of general equilibrium (also discussed below) . More concretely , many problems are amenable to analytical (formulaic) solution . Many others may be sufficiently complex to require numerical methods of solution , aided by software . Still others are complex but tractable enough to allow computable methods of solution , in particular computable general equilibrium models for the entire economy .

Linear and nonlinear programming have profoundly affected microeconomics , which had earlier considered only equality constraints . Many of the mathematical economists who received Nobel Prizes in Economics had conducted notable research using linear programming : Leonid Kantorovich , Leonid Hurwicz , Tjalling Koopmans , Kenneth J. Arrow , and Robert Dorfman , Paul Samuelson , and Robert Solow . Both Kantorovich and Koopmans acknowledged that George B. Dantzig deserved to share their Nobel Prize for linear programming . Economists who conducted research in nonlinear programming also have won the Nobel prize , notably Ragnar Frisch in addition to Kantorovich , Hurwicz , Koopmans , Arrow , and Samuelson .

= = = = Linear optimization = = = =

Linear programming was developed to aid the allocation of resources in firms and in industries during the 1930s in Russia and during the 1940s in the United States . During the Berlin airlift (1948) , linear programming was used to plan the shipment of supplies to prevent Berlin from starving after the Soviet blockade .

= = = = Nonlinear programming = = = =

Extensions to nonlinear optimization with inequality constraints were achieved in 1951 by Albert W. Tucker and Harold Kuhn , who considered the nonlinear optimization problem :