= 5 , 3 is a generator since 31 =
3 , 32 = 9 ? 4 , 33 ? 2 , and 34 ? 1 .

Some cyclic groups have an infinite number of elements . In these groups , for every non @-@ zero element a , all the powers of a are distinct ; despite the name " cyclic group " , the powers of the elements do not cycle . An infinite cyclic group is isomorphic to (Z , +) , the group of integers under addition introduced above . As these two prototypes are both abelian , so is any cyclic group .

The study of finitely generated abelian groups is quite mature , including the fundamental theorem of finitely generated abelian groups ; and reflecting this state of affairs , many group @-@ related notions , such as center and commutator , describe the extent to which a given group is not abelian .

= = = Symmetry groups = = =

Symmetry groups are groups consisting of symmetries of given mathematical objects ? be they of geometric nature , such as the introductory symmetry group of the square , or of algebraic nature , such as polynomial equations and their solutions . Conceptually , group theory can be thought of as the study of symmetry . Symmetries in mathematics greatly simplify the study of geometrical or analytical objects . A group is said to act on another mathematical object X if every group element performs some operation on X compatibly to the group law . In the rightmost example below , an element of order 7 of the (2 @,@ 3 @,@ 7) triangle group acts on the tiling by permuting the highlighted warped triangles (and the other ones , too) . By a group action , the group pattern is connected to the structure of the object being acted on .

In chemical fields , such as crystallography , space groups and point groups describe molecular symmetries and crystal symmetries . These symmetries underlie the chemical and physical behavior of these systems , and group theory enables simplification of quantum mechanical analysis of these properties . For example , group theory is used to show that optical transitions between certain quantum levels cannot occur simply because of the symmetry of the states involved .

Not only are groups useful to assess the implications of symmetries in molecules , but surprisingly they also predict that molecules sometimes can change symmetry . The Jahn @-@ Teller effect is a distortion of a molecule of high symmetry when it adopts a particular ground state of lower symmetry from a set of possible ground states that are related to each other by the symmetry operations of the molecule .

Likewise , group theory helps predict the changes in physical properties that occur when a material undergoes a phase transition , for example , from a cubic to a tetrahedral crystalline form . An example is ferroelectric materials , where the change from a paraelectric to a ferroelectric state occurs at the Curie temperature and is related to a change from the high @-@ symmetry paraelectric state to the lower symmetry ferroelectric state , accompanied by a so @-@ called soft phonon mode , a vibrational lattice mode that goes to zero frequency at the transition .

Such spontaneous symmetry breaking has found further application in elementary particle physics , where its occurrence is related to the appearance of Goldstone bosons .

Finite symmetry groups such as the Mathieu groups are used in coding theory , which is in turn applied in error correction of transmitted data , and in CD players . Another application is differential Galois theory , which characterizes functions having antiderivatives of a prescribed form , giving group @-@ theoretic criteria for when solutions of certain differential equations are well @-@ behaved . Geometric properties that remain stable under group actions are investigated in (geometric) invariant theory .

= = = General linear group and representation theory = = =

Matrix groups consist of matrices together with matrix multiplication . The general linear group GL (n , R) consists of all invertible n @-@ by @-@ n matrices with real entries . Its subgroups are referred to as matrix groups or linear groups . The dihedral group example mentioned above can be viewed as a (very small) matrix group . Another important matrix group is the special orthogonal group SO (n) . It describes all possible rotations in n dimensions . Via Euler angles , rotation matrices are used in computer graphics .

Representation theory is both an application of the group concept and important for a deeper understanding of groups . It studies the group by its group actions on other spaces . A broad class of group representations are linear representations , i.e. the group is acting on a vector space , such as the three @-@ dimensional Euclidean space R3 . A representation of G on an n @-@ dimensional real vector space is simply a group homomorphism

? : G ? GL (n , R)

from the group to the general linear group . This way , the group operation , which may be abstractly given , translates to the multiplication of matrices making it accessible to explicit computations .

Given a group action , this gives further means to study the object being acted on . On the other hand , it also yields information about the group . Group representations are an organizing principle in the theory of finite groups , Lie groups , algebraic groups and topological groups , especially (locally) compact groups .

= = = Galois groups = = =

Galois groups were developed to help solve polynomial equations by capturing their symmetry features . For example , the solutions of the quadratic equation ax2 + bx + c = 0 are given by

<formula>

Exchanging " + " and " ? " in the expression , i.e. permuting the two solutions of the equation can be viewed as a (very simple) group operation . Similar formulae are known for cubic and quartic equations , but do not exist in general for degree 5 and higher . Abstract properties of Galois groups associated with polynomials (in particular their solvability) give a criterion for polynomials that have all their solutions expressible by radicals , i.e. solutions expressible using solely addition , multiplication , and roots similar to the formula above .

The problem can be dealt with by shifting to field theory and considering the splitting field of a polynomial . Modern Galois theory generalizes the above type of Galois groups to field extensions and establishes ? via the fundamental theorem of Galois theory ? a precise relationship between fields and groups , underlining once again the ubiquity of groups in mathematics .

= = Finite groups = =

A group is called finite if it has a finite number of elements . The number of elements is called the order of the group . An important class is the symmetric groups SN , the groups of permutations of N letters . For example , the symmetric group on 3 letters S3 is the group consisting of all possible orderings of the three letters ABC , i.e. contains the elements ABC , ACB , ... , up to CBA , in total 6 (or 3 factorial) elements . This class is fundamental insofar as any finite group can be expressed as a subgroup of a symmetric group SN for a suitable integer N (Cayley 's theorem) . Parallel to the group of symmetries of the square above , S3 can also be interpreted as the group of symmetries of an equilateral triangle .

The order of an element a in a group G is the least positive integer n such that a n = e , where a n represents

<formula>

i.e. application of the operation ? to n copies of a . (If ? represents multiplication , then an corresponds to the nth power of a .) In infinite groups , such an n may not exist , in which case the order of a is said to be infinity . The order of an element equals the order of the cyclic subgroup generated by this element .

More sophisticated counting techniques , for example counting cosets , yield more precise statements about finite groups : Lagrange 's Theorem states that for a finite group G the order of any finite subgroup H divides the order of G. The Sylow theorems give a partial converse .

The dihedral group (discussed above) is a finite group of order 8 . The order of r1 is 4 , as is the order of the subgroup R it generates (see above) . The order of the reflection elements fv etc. is 2 . Both orders divide 8 , as predicted by Lagrange 's theorem . The groups Fp × above have order p ? 1 .

= = = Classification of finite simple groups = = =

Mathematicians often strive for a complete classification (or list) of a mathematical notion . In the context of finite groups , this aim leads to difficult mathematics . According to Lagrange 's theorem , finite groups of order p , a prime number , are necessarily cyclic (abelian) groups Zp . Groups of order p2 can also be shown to be abelian , a statement which does not generalize to order p3 , as the non @-@ abelian group D4 of order 8 = 23 above shows . Computer algebra systems can be used to list small groups , but there is no classification of all finite groups . An intermediate step is the classification of finite simple groups . A nontrivial group is called simple if its only normal subgroups are the trivial group and the group itself . The Jordan ? Hölder theorem exhibits finite simple groups as the building blocks for all finite groups . Listing all finite simple groups was a major achievement in contemporary group theory . 1998 Fields Medal winner Richard Borcherds succeeded in proving the monstrous moonshine conjectures , a surprising and deep relation between the largest finite simple sporadic group ? the " monster group " ? and certain modular functions , a piece of classical complex analysis , and string theory , a theory supposed to unify the description of many physical phenomena .

= = Groups with additional structure = =

Many groups are simultaneously groups and examples of other mathematical structures . In the language of category theory , they are group objects in a category , meaning that they are objects (that is , examples of another mathematical structure) which come with transformations (called morphisms) that mimic the group axioms . For example , every group (as defined above) is also a set , so a group is a group object in the category of sets .

= = = Topological groups = = =

Some topological spaces may be endowed with a group law . In order for the group law and the topology to interweave well , the group operations must be continuous functions , that is , g ? h , and g ? 1 must not vary wildly if g and h vary only little . Such groups are called topological groups , and they are the group objects in the category of topological spaces . The most basic examples are the reals R under addition , (R ? { 0 } , ·) , and similarly with any other topological field such as the complex numbers or p @-@ adic numbers . All of these groups are locally compact , so they have Haar measures and can be studied via harmonic analysis . The former offer an abstract formalism of invariant integrals . Invariance means , in the case of real numbers for example :
<formula>
for any constant c . Matrix groups over these fields fall under this regime , as do adele rings and adelic algebraic groups , which are basic to number theory . Galois groups of infinite field extensions such as the absolute Galois group can also be equipped with a topology , the so @-@ called Krull topology , which in turn is central to generalize the above sketched connection of fields and groups to infinite field extensions . An advanced generalization of this idea , adapted to the needs of algebraic geometry , is the étale fundamental group .

= = = Lie groups = = =

Lie groups (in honor of Sophus Lie) are groups which also have a manifold structure , i.e. they are spaces looking locally like some Euclidean space of the appropriate dimension . Again , the additional structure , here the manifold structure , has to be compatible , i.e. the maps corresponding to multiplication and the inverse have to be smooth .

A standard example is the general linear group introduced above : it is an open subset of the space of all n @-@ by @-@ n matrices , because it is given by the inequality
det (A) ? 0 @,@
where A denotes an n @-@ by @-@ n matrix .
Lie groups are of fundamental importance in modern physics : Noether 's theorem links continuous symmetries to conserved quantities . Rotation , as well as translations in space and time are basic symmetries of the laws of mechanics . They can , for instance , be used to construct simple models ? imposing , say , axial symmetry on a situation will typically lead to significant simplification in the equations one needs to solve to provide a physical description . Another example are the Lorentz transformations , which relate measurements of time and velocity of two observers in motion relative to each other . They can be deduced in a purely group @-@ theoretical way , by expressing the transformations as a rotational symmetry of Minkowski space . The latter serves ? in the absence of significant gravitation ? as a model of space time in special relativity . The full symmetry group of Minkowski space , i.e. including translations , is known as the Poincaré group . By the above , it plays a pivotal role in special relativity and , by implication , for quantum field theories . Symmetries that vary with location are central to the modern description of physical interactions with the help of gauge theory .

= = Generalizations = =

In abstract algebra , more general structures are defined by relaxing some of the axioms defining a group . For example , if the requirement that every element has an inverse is eliminated , the resulting algebraic structure is called a monoid . The natural numbers N (including 0) under addition form a monoid , as do the nonzero integers under multiplication (Z ? { 0 } , ·) , see above . There is a general method to formally add inverses to elements to any (abelian) monoid , much the same way as (Q ? { 0 } , ·) is derived from (Z ? { 0 } , ·) , known as the Grothendieck group . Groupoids are similar to groups except that the composition a ? b need not be defined for all a and b . They arise in the study of more complicated forms of symmetry , often in topological and analytical structures , such as the fundamental groupoid or stacks . Finally , it is possible to generalize any of these concepts by replacing the binary operation with an arbitrary n @-@ ary one (i.e. an operation taking n arguments) . With the proper generalization of the group axioms this gives rise to an n @-@ ary group . The table gives a list of several structures generalizing groups .

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