= Mayer ? Vietoris sequence =

In mathematics , particularly algebraic topology and homology theory , the Mayer ? Vietoris sequence is an algebraic tool to help compute algebraic invariants of topological spaces , known as their homology and cohomology groups . The result is due to two Austrian mathematicians , Walther Mayer and Leopold Vietoris . The method consists of splitting a space into subspaces , for which the homology or cohomology groups may be easier to compute . The sequence relates the (co) homology groups of the space to the (co) homology groups of the subspaces . It is a natural long exact sequence , whose entries are the (co) homology groups of the whole space , the direct sum of the (co) homology groups of the subspaces , and the (co) homology groups of the intersection of the subspaces .

The Mayer ? Vietoris sequence holds for a variety of cohomology and homology theories , including singular homology and singular cohomology . In general , the sequence holds for those theories satisfying the Eilenberg ? Steenrod axioms , and it has variations for both reduced and relative (co) homology . Because the (co) homology of most spaces cannot be computed directly from their definitions , one uses tools such as the Mayer ? Vietoris sequence in the hope of obtaining partial information . Many spaces encountered in topology are constructed by piecing together very simple patches . Carefully choosing the two covering subspaces so that , together with their intersection , they have simpler (co) homology than that of the whole space may allow a complete deduction of the (co) homology of the space . In that respect , the Mayer ? Vietoris sequence is analogous to the Seifert ? van Kampen theorem for the fundamental group , and a precise relation exists for homology of dimension one .

= = Background , motivation , and history = =

Like the fundamental group or the higher homotopy groups of a space , homology groups are important topological invariants . Although some (co) homology theories are computable using tools of linear algebra , many other important (co) homology theories , especially singular (co) homology , are not computable directly from their definition for nontrivial spaces . For singular (co) homology , the singular (co) chains and (co) cycles groups are often too big to handle directly . More subtle and indirect approaches become necessary . The Mayer ? Vietoris sequence is such an approach , giving partial information about the (co) homology groups of any space by relating it to the (co) homology groups of two of its subspaces and their intersection .

The most natural and convenient way to express the relation involves the algebraic concept of exact sequences : sequences of objects (in this case groups) and morphisms (in this case group homomorphisms) between them such that the image of one morphism equals the kernel of the next . In general , this does not allow (co) homology groups of a space to be completely computed . However , because many important spaces encountered in topology are topological manifolds , simplicial complexes , or CW complexes , which are constructed by piecing together very simple patches , a theorem such as that of Mayer and Vietoris is potentially of broad and deep applicability .

Mayer was introduced to topology by his colleague Vietoris when attending his lectures in 1926 and 1927 at a local university in Vienna . He was told about the conjectured result and a way to its solution , and solved the question for the Betti numbers in 1929 . He applied his results to the torus considered as the union of two cylinders . Vietoris later proved the full result for the homology groups in 1930 but did not express it as an exact sequence . The concept of an exact sequence only appeared in print in the 1952 book Foundations of Algebraic Topology by Samuel Eilenberg and Norman Steenrod where the results of Mayer and Vietoris were expressed in the modern form .

= = Basic versions for singular homology = =

Let X be a topological space and A , B be two subspaces whose interiors cover X. (The interiors of A and B need not be disjoint .) The Mayer ? Vietoris sequence in singular homology for the triad (X

, A , B) is a long exact sequence relating the singular homology groups (with coefficient group the integers Z) of the spaces X , A , B , and the intersection A ? B. There is an unreduced and a reduced version .

= = = Unreduced version = = =

For unreduced homology , the Mayer ? Vietoris sequence states that the following sequence is exact :
<formula>
Here the maps i : A ? B ? A , j : A ? B ? B , k : A ? X , and l : B ? X are inclusion maps and <formula> denotes the direct sum of abelian groups .

= = = Boundary map = = =

The boundary maps ? * lowering the dimension may be made explicit as follows . An element in Hn (X) is the homology class of an n @-@ cycle x which , by barycentric subdivision for example , can be written as the sum of two n @-@ chains u and v whose images lie wholly in A and B , respectively . Thus ? x
= ? (u + v) =
0 so that ? u
= ? ? v. This implies that the images of both these boundary (n ? 1) -cycles are contained in the intersection A ? B. Then ? * ([x]) is the class of ? u in Hn ? 1 (A ? B) . Choosing another decomposition x =
u ? + v ? does not affect [? u] , since ? u + ? v
= ? x =
? u ? + ? v ? , which implies ? u ? ? u ?
= ? (v ? ? v) , and therefore ? u and ? u ? lie in the same Homology class ; nor does choosing a different representative x ? , since then ? x ? =
? x = 0 . Notice that the maps in the Mayer ? Vietoris sequence depend on choosing an order for A and B. In particular , the boundary map changes sign if A and B are swapped .

= = = Reduced version = = =

For reduced homology there is also a Mayer ? Vietoris sequence , under the assumption that A and B have non @-@ empty intersection . The sequence is identical for positive dimensions and ends as :
<formula>

= = = Analogy with the Seifert ? van Kampen theorem = = =

There is an analogy between the Mayer ? Vietoris sequence (especially for homology groups of dimension 1) and the Seifert ? van Kampen theorem . Whenever A ? B is path @-@ connected the reduced Mayer ? Vietoris sequence yields the isomorphism
<formula>
where , by exactness ,
<formula>
This is precisely the abelianized statement of the Seifert ? van Kampen theorem . Compare with the fact that H1 (X) is the abelianization of the fundamental group ?1 (X) when X is path @-@ connected .

= = Basic applications = =

= = = k @-@ sphere = = =

To completely compute the homology of the k @-@ sphere X = Sk , let A and B be two hemispheres of X with intersection homotopy equivalent to a (k ? 1) -dimensional equatorial sphere . Since the k @-@ dimensional hemispheres are homeomorphic to k @-@ discs , which are contractible , the homology groups for A and B are trivial . The Mayer ? Vietoris sequence for reduced homology groups then yields
<formula>
Exactness immediately implies that the map ? * is an isomorphism . Using the reduced homology of the 0 @-@ sphere (two points) as a base case , it follows
<formula>
where ? is the Kronecker delta . Such a complete understanding of the homology groups for spheres is in stark contrast with current knowledge of homotopy groups of spheres , especially for the case n > k about which little is known .

= = = Klein bottle = = =

A slightly more difficult application of the Mayer ? Vietoris sequence is the calculation of the homology groups of the Klein bottle X. One uses the decomposition of X as the union of two Möbius strips A and B glued along their boundary circle (see illustration on the right) . Then A , B and their intersection A ? B are homotopy equivalent to circles , so the nontrivial part of the sequence yields
<formula>
and the trivial part implies vanishing homology for dimensions greater than 2 . The central map ? sends 1 to (2 , ? 2) since the boundary circle of a Möbius band wraps twice around the core circle . In particular ? is injective so homology of dimension 2 also vanishes . Finally , choosing (1 , 0) and (1 , ? 1) as a basis for Z2 , it follows
<formula>

= = = Wedge sums = = =