Finite subdivision rule

In mathematics , a finite subdivision rule is a recursive way of dividing a polygon or other two @-@ dimensional shape into smaller and smaller pieces . Subdivision rules in a sense are generalizations of fractals . Instead of repeating exactly the same design over and over , they have slight variations in each stage , allowing a richer structure while maintaining the elegant style of fractals . Subdivision rules have been used in architecture , biology , and computer science , as well as in the study of hyperbolic manifolds . Substitution tilings are a well @-@ studied type of subdivision rule .

Definition

A subdivision rule takes a tiling of the plane by polygons and turns it into a new tiling by subdividing each polygon into smaller polygons . It is finite if there are only finitely many ways that every polygon can subdivide . Each way of subdividing a tile is called a tile type . Each tile type is represented by a label (usually a letter) . Every tile type subdivides into smaller tile types . Each edge also gets subdivided according to finitely many edge types . Finite subdivision rules can only subdivide tilings that are made up of polygons labelled by tile types . Such tilings are called subdivision complexes for the subdivision rule . Given any subdivision complex for a subdivision rule , we can subdivide it over and over again to get a sequence of tilings .

For instance , binary subdivision has one tile type and one edge type :

Since the only tile type is a quadrilateral , binary subdivision can only subdivide tilings made up of quadrilaterals . This means that the only subdivision complexes are tilings by quadrilaterals . The tiling can be regular , but doesn 't have to be :

Here we start with a complex made of four quadrilaterals and subdivide it twice . All quadrilaterals are type A tiles .

Examples of finite subdivision rules

Barycentric subdivision is an example of a subdivision rule with one edge type (that gets subdivided into two edges) and one tile type (a triangle that gets subdivided into 6 smaller triangles) . Any triangulated surface is a barycentric subdivision complex .

The Penrose tiling can be generated by a subdivision rule on a set of four tile types (the curved lines in the table below only help to show how the tiles fit together) :

Certain rational maps give rise to finite subdivision rules . This includes most Lattès maps .

Every prime , non @-@ split alternating knot or link complement has a subdivision rule , with some tiles that do not subdivide , corresponding to the boundary of the link complement . The subdivision rules show what the night sky would look like to someone living in a knot complement ; because the universe wraps around itself (i.e. is not simply connected) , an observer would see the visible universe repeat itself in an infinite pattern . The subdivision rule describes that pattern .

The subdivision rule looks different for different geometries . This is a subdivision rule for the trefoil knot , which is not a hyperbolic knot :

And this is the subdivision rule for the Borromean rings , which is hyperbolic :

In each case , the subdivision rule would act on some tiling of a sphere (i.e. the night sky) , but it is easier to just draw a small part of the night sky , corresponding to a single tile being repeatedly subdivided . This is what happens for the trefoil knot :

And for the Borromean rings :

Subdivision Rules in Higher Dimensions

Subdivision rules can easily be generalized to other dimensions . For instance , barycentric subdivision is used in all dimensions . Also , binary subdivision can be generalized to other dimensions (where hypercubes get divided by every midplane) , as in the proof of the Heine @-@ Borel theorem .

= = Rigorous definition = =

A finite subdivision rule <formula> consists of the following .

1 . A finite 2 @-@ dimensional CW complex <formula> , called the subdivision complex , with a fixed cell structure such that <formula> is the union of its closed 2 @-@ cells . We assume that for each closed 2 @-@ cell <formula> of <formula> there is a CW structure <formula> on a closed 2 @-@ disk such that <formula> has at least two vertices , the vertices and edges of <formula> are contained in <formula> , and the characteristic map <formula> which maps onto <formula> restricts to a homeomorphism onto each open cell .

2 . A finite two dimensional CW complex <formula> , which is a subdivision of <formula> .

3.A continuous cellular map <formula> called the subdivision map , whose restriction to every open cell is a homeomorphism .

Each CW complex <formula> in the definition above (with its given characteristic map <formula>) is called a tile type .

An <formula> -complex for a subdivision rule <formula> is a 2 @-@ dimensional CW complex <formula> which is the union of its closed 2 @-@ cells , together with a continuous cellular map <formula> whose restriction to each open cell is a homeomorphism . We can subdivide <formula> into a complex <formula> by requiring that the induced map <formula> restricts to a homeomorphism onto each open cell . <formula> is again an <formula> -complex with map <formula> . By repeating this process , we obtain a sequence of subdivided <formula> -complexes <formula> with maps <formula> .

Binary subdivision is one example :

The subdivision complex can be created by gluing together the opposite edges of the square , making the subdivision complex <formula> into a torus . The subdivision map <formula> is the doubling map on the torus , wrapping the meridian around itself twice and the longitude around itself twice . This is a four @-@ fold covering map . The plane , tiled by squares , is a subdivision complex for this subdivision rule , with the structure map <formula> given by the standard covering map . Under subdivision , each square in the plane gets subdivided into squares of one @-@ fourth the size .

= = Quasi @-@ isometry properties = =

Subdivision rules can be used to study the quasi @-@ isometry properties of certain spaces . Given a subdivision rule <formula> and subdivision complex <formula> , we can construct a graph called the history graph that records the action of the subdivision rule . The graph consists of the dual graphs of every stage <formula> , together with edges connecting each tile in <formula> with its subdivisions in <formula> .

The quasi @-@ isometry properties of the history graph can be studied using subdivision rules . For instance , the history graph is quasi @-@ isometric to hyperbolic space exactly when the subdivision rule is conformal , as described in the combinatorial Riemann mapping theorem .

= = Applications = =

Islamic Girih tiles in Islamic architecture are self @-@ similar tilings that can be modeled with finite subdivision rules . In 2007 , Peter J. Lu of Harvard University and Professor Paul J. Steinhardt of Princeton University published a paper in the journal Science suggesting that girih tilings possessed properties consistent with self @-@ similar fractal quasicrystalline tilings such as Penrose tilings (presentation 1974 , predecessor works starting in about 1964) predating them by five centuries .

Subdivision surfaces in computer graphics use subdivision rules to refine a surface to any given level of precision . These subdivision surfaces (such as the Catmull @-@ Clark subdivision surface) take a polygon mesh (the kind used in 3D animated movies) and refines it to a mesh with more polygons by adding and shifting points according to different recursive formulas . Although many points get shifted in this process , each new mesh is combinatorially a subdivision of the old mesh (meaning that for every edge and vertex of the old mesh , you can identify a corresponding edge and vertex in the new one , plus several more edges and vertices) .

Subdivision rules were applied by Cannon , Floyd and Parry (2000) to the study of large @-@ scale growth patterns of biological organisms . Cannon , Floyd and Parry produced a mathematical growth model which demonstrated that some systems determined by simple finite subdivision rules can results in objects (in their example , a tree trunk) whose large @-@ scale form oscillates wildly over time even though the local subdivision laws remain the same . Cannon , Floyd and Parry also applied their model to the analysis of the growth patterns of rat tissue . They suggested that the " negatively curved " (or non @-@ euclidean) nature of microscopic growth patterns of biological organisms is one of the key reasons why large @-@ scale organisms do not look like crystals or polyhedral shapes but in fact in many cases resemble self @-@ similar fractals . In particular they suggested that such " negatively curved " local structure is manifested in highly folded and highly connected nature of the brain and the lung tissue .

= = Cannon 's conjecture = =

Cannon , Floyd , and Parry first studied finite subdivision rules in an attempt to prove the following conjecture :

Cannon 's conjecture : Every Gromov hyperbolic group with a 2 @-@ sphere at infinity acts geometrically on hyperbolic 3 @-@ space .

Here , a geometric action is a cocompact , properly discontinuous action by isometries . This conjecture was partially solved by Grigori Perelman in his proof of the Geometrization conjecture , which states (in part) than any Gromov hyperbolic group that is a 3 @-@ manifold group must act geometrically on hyperbolic 3 @-@ space . However , it still remains to show that a Gromov hyperbolic group with a 2 @-@ sphere at infinity is a 3 @-@ manifold group .

Cannon and Swenson showed that a hyperbolic group with a 2 @-@ sphere at infinity has an associated subdivision rule . If this subdivision rule is conformal in a certain sense , the group will be a 3 @-@ manifold group with the geometry of hyperbolic 3 @-@ space .

= = Combinatorial Riemann Mapping Theorem = =

Subdivision rules give a sequence of tilings of a surface , and tilings give an idea of distance , length , and area (by letting each tile have length and area 1) . In the limit , the distances that come from these tilings may converge in some sense to an analytic structure on the surface . The Combinatorial Riemann Mapping Theorem gives necessary and sufficient conditions for this to occur .

Its statement needs some background . A tiling <formula> of a ring <formula> (i.e. , a closed annulus) gives two invariants , <formula> and <formula> , called approximate moduli . These are similar to the classical modulus of a ring . They are defined by the use of weight functions . A weight function <formula> assigns a non @-@ negative number called a weight to each tile of <formula> . Every path in <formula> can be given a length , defined to be the sum of the weights of all tiles in the path . Define the height <formula> of <formula> under <formula> to be the infimum of the length of all possible paths connecting the inner boundary of <formula> to the outer boundary . The circumference <formula> of <formula> under <formula> is the infimum of the length of all possible paths circling the ring (i.e. not nullhomotopic in R) . The area <formula> of <formula> under <formula> is defined to be the sum of the squares of all weights in <formula> . Then define

<formula>

<formula> .

Note that they are invariant under scaling of the metric .

A sequence <formula> of tilings is conformal (<formula>) if mesh approaches 0 and :

For each ring <formula> , the approximate moduli <formula> and <formula> , for all <formula> sufficiently large , lie in a single interval of the form <formula> ; and

Given a point <formula> in the surface , a neighborhood <formula> of <formula> , and an integer <formula> , there is a ring <formula> in <formula> separating x from the complement of <formula> , such that for all large <formula> the approximate moduli of <formula> are all greater than <formula> .

= = = Statement of theorem = = =

If a sequence <formula> of tilings of a surface is conformal (<formula>) in the above sense , then there is a conformal structure on the surface and a constant <formula> depending only on <formula> in which the classical moduli and approximate moduli (from <formula> for <formula> sufficiently large) of any given annulus are <formula> -comparable , meaning that they lie in a single interval <formula> .

= = = Consequences = = =

The Combinatorial Riemann Mapping Theorem implies that a group <formula> acts geometrically on <formula> if and only if it is Gromov hyperbolic , it has a sphere at infinity , and the natural subdivision rule on the sphere gives rise to a sequence of tilings that is conformal in the sense above . Thus , Cannon 's conjecture would be true if all such subdivision rules were conformal .