= f ((x1 , ... , xN)) =
? fn (xn) .
For example , problems of linear optimization are separable . Given a separable problem with an optimal solution , we fix an optimal solution
xmin = (x1 , ... , xN) min
with the minimum value f (xmin) . For this separable problem , we also consider an optimal solution (xmin , f (xmin)) to the " convexified problem " , where convex hulls are taken of the graphs of the summand functions . Such an optimal solution is the limit of a sequence of points in the convexified problem
(xj , f (xj)) ? ? Conv (Graph (fn)) .
Of course , the given optimal @-@ point is a sum of points in the graphs of the original summands and of a small number of convexified summands , by the Shapley ? Folkman lemma .
This analysis was published by Ivar Ekeland in 1974 to explain the apparent convexity of separable problems with many summands , despite the non @-@ convexity of the summand problems . In 1973 , the young mathematician Claude Lemaréchal was surprised by his success with convex minimization methods on problems that were known to be non @-@ convex ; for minimizing nonlinear problems , a solution of the dual problem problem need not provide useful information for solving the primal problem , unless the primal problem be convex and satisfy a constraint qualification . Lemaréchal 's problem was additively separable , and each summand function was non @-@ convex ; nonetheless , a solution to the dual problem provided a close approximation to the primal problem 's optimal value . Ekeland 's analysis explained the success of methods of convex minimization on large and separable problems , despite the non @-@ convexities of the summand functions . Ekeland and later authors argued that additive separability produced an approximately convex aggregate problem , even though the summand functions were non @-@ convex . The crucial step in these publications is the use of the Shapley ? Folkman lemma . The Shapley ? Folkman lemma has encouraged the use of methods of convex minimization on other applications with sums of many functions .

= = = Probability and measure theory = = =

Convex sets are often studied with probability theory . Each point in the convex hull of a (non @-@ empty) subset Q of a finite @-@ dimensional space is the expected value of a simple random vector that takes its values in Q , as a consequence of Carathéodory 's lemma . Thus , for a non @-@ empty set Q , the collection of the expected values of the simple , Q @-@ valued random vectors equals Q 's convex hull ; this equality implies that the Shapley ? Folkman ? Starr results are useful in probability theory . In the other direction , probability theory provides tools to examine convex sets generally and the Shapley ? Folkman ? Starr results specifically . The Shapley ? Folkman ? Starr results have been widely used in the probabilistic theory of random sets , for example , to prove a law of large numbers , a central limit theorem , and a large @-@ deviations principle . These proofs of probabilistic limit theorems used the Shapley ? Folkman ? Starr results to avoid the assumption that all the random sets be convex .
A probability measure is a finite measure , and the Shapley ? Folkman lemma has applications in non @-@ probabilistic measure theory , such as the theories of volume and of vector measures . The Shapley ? Folkman lemma enables a refinement of the Brunn ? Minkowski inequality , which bounds the volume of sums in terms of the volumes of their summand @-@ sets . The volume of a set is defined in terms of the Lebesgue measure , which is defined on subsets of Euclidean space . In advanced measure @-@ theory , the Shapley ? Folkman lemma has been used to prove Lyapunov 's theorem , which states that the range of a vector measure is convex . Here , the traditional term " range " (alternatively , " image ") is the set of values produced by the function . A vector measure is a vector @-@ valued generalization of a measure ; for example , if p1 and p2 are probability measures defined on the same measurable space , then the product function p1 p2 is a vector measure , where p1 p2 is defined for every event ? by
(p1 p2) (?) = (p1 (?) , p2 (?)) .

Lyapunov 's theorem has been used in economics , in (" bang @-@ bang ") control theory , and in statistical theory . Lyapunov 's theorem has been called a continuous counterpart of the Shapley ? Folkman lemma , which has itself been called a discrete analogue of Lyapunov 's theorem .