= Ker ( ? ) is a closed vector subspace of H , not equal to H , hence there exists a non @-@ zero vector v orthogonal to F. The vector u is a suitable scalar multiple ?v of v. The requirement that ? ( v ) =
? v , u ? yields
<formula>
This correspondence ? ? u is exploited by the bra ? ket notation popular in physics . It is common in physics to assume that the inner product , denoted by ? x | y ? , is linear on the right ,
<formula>
The result ? x | y ? can be seen as the action of the linear functional ? x | ( the bra ) on the vector | y ? ( the ket ) .
The Riesz representation theorem relies fundamentally not just on the presence of an inner product , but also on the completeness of the space . In fact , the theorem implies that the topological dual of any inner product space can be identified with its completion . An immediate consequence of the Riesz representation theorem is also that a Hilbert space H is reflexive , meaning that the natural map from H into its double dual space is an isomorphism .

= = = Weakly convergent sequences = = =

In a Hilbert space H , a sequence { xn } is weakly convergent to a vector x ? H when
<formula>
for every v ? H.
For example , any orthonormal sequence { fn } converges weakly to 0 , as a consequence of Bessel 's inequality . Every weakly convergent sequence { xn } is bounded , by the uniform boundedness principle .
Conversely , every bounded sequence in a Hilbert space admits weakly convergent subsequences ( Alaoglu 's theorem ) . This fact may be used to prove minimization results for continuous convex functionals , in the same way that the Bolzano ? Weierstrass theorem is used for continuous functions on Rd . Among several variants , one simple statement is as follows :
If f : H ? R is a convex continuous function such that f ( x ) tends to + ? when | | x | | tends to ? , then f admits a minimum at some point x0 ? H.
This fact ( and its various generalizations ) are fundamental for direct methods in the calculus of variations . Minimization results for convex functionals are also a direct consequence of the slightly more abstract fact that closed bounded convex subsets in a Hilbert space H are weakly compact , since H is reflexive . The existence of weakly convergent subsequences is a special case of the Eberlein ? ?mulian theorem .

= = = Banach space properties = = =

Any general property of Banach spaces continues to hold for Hilbert spaces . The open mapping theorem states that a continuous surjective linear transformation from one Banach space to another is an open mapping meaning that it sends open sets to open sets . A corollary is the bounded inverse theorem , that a continuous and bijective linear function from one Banach space to another is an isomorphism ( that is , a continuous linear map whose inverse is also continuous ) . This theorem is considerably simpler to prove in the case of Hilbert spaces than in general Banach spaces . The open mapping theorem is equivalent to the closed graph theorem , which asserts that a function from one Banach space to another is continuous if and only if its graph is a closed set . In the case of Hilbert spaces , this is basic in the study of unbounded operators ( see closed operator ) .
The ( geometrical ) Hahn ? Banach theorem asserts that a closed convex set can be separated from any point outside it by means of a hyperplane of the Hilbert space . This is an immediate consequence of the best approximation property : if y is the element of a closed convex set F closest to x , then the separating hyperplane is the plane perpendicular to the segment xy passing through its midpoint .

## Operators on Hilbert spaces

### Bounded operators

The continuous linear operators A : H1 ? H2 from a Hilbert space H1 to a second Hilbert space H2 are bounded in the sense that they map bounded sets to bounded sets . Conversely , if an operator is bounded , then it is continuous . The space of such bounded linear operators has a norm , the operator norm given by

<formula>

The sum and the composite of two bounded linear operators is again bounded and linear . For y in H2 , the map that sends x ? H1 to ? Ax , y ? is linear and continuous , and according to the Riesz representation theorem can therefore be represented in the form

<formula>

for some vector A * y in H1 . This defines another bounded linear operator A * : H2 ? H1 , the adjoint of A. One can see that A * * = A.

The set B ( H ) of all bounded linear operators on H , together with the addition and composition operations , the norm and the adjoint operation , is a C * -algebra , which is a type of operator algebra .