= 1 and v =
2 , whereby a positive b has two square roots) ; in this case the principal root is defined to be the positive one .
Thus we have (? 27) 1 / 3
= ? 3 and (? 27) 2 / 3 =
9 . The number 4 has two 3 / 2th roots , namely 8 and ? 8 ; however , by convention 43 / 2 denotes the principal root , which is 8 . Since there is no real number x such that x2
= ? 1 , the definition of bu / v when b is negative and v is even must use the imaginary unit i , as described more fully in the section § Powers of complex numbers .
Care needs to be taken when applying the power identities with negative nth roots . For instance , ? 27 =
(? 27) ((2 / 3) ? (3 / 2))
= ((? 27) 2 / 3) 3 / 2 =
93 / 2 = 27 is clearly wrong . The problem here occurs in taking the positive square root rather than the negative one at the last step , but in general the same sorts of problems occur as described for complex numbers in the section § Failure of power and logarithm identities .

= = Real exponents = =

The identities and properties shown above for integer exponents are true for positive real numbers with non @-@ integer exponents as well . However the identity
<formula>
cannot be extended consistently to cases where b is a negative real number (see § Real exponents with negative bases) . The failure of this identity is the basis for the problems with complex number powers detailed under § Failure of power and logarithm identities .
Exponentiation to real powers of positive real numbers can be defined either by extending the rational powers to reals by continuity , or more usually as given in § Powers via logarithms below .

= = = Limits of rational exponents = = =

Since any irrational number can be expressed as the limit of a sequence of rational numbers , exponentiation of a positive real number b with an arbitrary real exponent x can be defined by continuity with the rule
<formula>
where the limit as r gets close to x is taken only over rational values of r . This limit only exists for positive b . The (? , ?) -definition of limit is used , this involves showing that for any desired accuracy of the result bx one can choose a sufficiently small interval around x so all the rational powers in the interval are within the desired accuracy .
For example , if x
= ? , the nonterminating decimal representation ? =
3 @.@ 14159 ... can be used (based on strict monotonicity of the rational power) to obtain the intervals bounded by rational powers
<formula> , <formula> , <formula> , <formula> , <formula> , <formula> , ...
The bounded intervals converge to a unique real number , denoted by <formula> . This technique can be used to obtain the power of a positive real number b for any irrational exponent . The function fb (x) = bx is thus defined for any real number x .

= = = The exponential function = = =

The important mathematical constant e , sometimes called Euler 's number , is approximately equal to 2 @.@ 718 and is the base of the natural logarithm . Although exponentiation of e could , in principle , be treated the same as exponentiation of any other real number , such exponentials turn out to have particularly elegant and useful properties . Among other things , these properties allow

exponentials of e to be generalized in a natural way to other types of exponents , such as complex numbers or even matrices , while coinciding with the familiar meaning of exponentiation with rational exponents .
As a consequence , the notation ex usually denotes a generalized exponentiation definition called the exponential function , exp (x) , which can be defined in many equivalent ways , for example by :

<formula>
Among other properties , exp satisfies the exponential identity
<formula>
The exponential function is defined for all integer , fractional , real , and complex values of x . In fact , the matrix exponential is well @-@ defined for square matrices (in which case this exponential identity only holds when x and y commute) , and is useful for solving systems of linear differential equations .
Since exp (1) is equal to e and exp (x) satisfies this exponential identity , it immediately follows that exp (x) coincides with the repeated @-@ multiplication definition of ex for integer x , and it also follows that rational powers denote (positive) roots as usual , so exp (x) coincides with the ex definitions in the previous section for all real x by continuity .

= = = Powers via logarithms = = =

The natural logarithm ln (x) is the inverse of the exponential function ex . It is defined for b > 0 , and satisfies
<formula>
If bx is to preserve the logarithm and exponent rules , then one must have
<formula>
for each real number x .
This can be used as an alternative definition of the real number power bx and agrees with the definition given above using rational exponents and continuity . The definition of exponentiation using logarithms is more common in the context of complex numbers , as discussed below .

= = = Real exponents with negative bases = = =