= 1 + =
2 .
Again , there are minor variations upon this definition in the literature . Taken literally , the above definition is an application of the Recursion Theorem on the partially ordered set N2 . On the other hand , some sources prefer to use a restricted Recursion Theorem that applies only to the set of natural numbers . One then considers a to be temporarily " fixed " , applies recursion on b to define a function " a + " , and pastes these unary operations for all a together to form the full binary operation .
This recursive formulation of addition was developed by Dedekind as early as 1854 , and he would expand upon it in the following decades . He proved the associative and commutative properties , among others , through mathematical induction .

= = = Integers = = =

The simplest conception of an integer is that it consists of an absolute value (which is a natural number) and a sign (generally either positive or negative) . The integer zero is a special third case , being neither positive nor negative . The corresponding definition of addition must proceed by cases :
For an integer n , let | n | be its absolute value . Let a and b be integers . If either a or b is zero , treat it as an identity . If a and b are both positive , define a + b
= | a | + | b | . If a and b are both negative , define a + b =
? (| a | + | b |) . If a and b have different signs , define a + b to be the difference between | a | and | b | , with the sign of the term whose absolute value is larger . As an example , -6 + 4
= -2 ; because -6 and 4 have different signs , their absolute values are subtracted , and since the negative term is larger , the answer is negative .
Although this definition can be useful for concrete problems , it is far too complicated to produce elegant general proofs ; there are too many cases to consider .
A much more convenient conception of the integers is the Grothendieck group construction . The essential observation is that every integer can be expressed (not uniquely) as the difference of two natural numbers , so we may as well define an integer as the difference of two natural numbers . Addition is then defined to be compatible with subtraction :
Given two integers a ? b and c ? d , where a , b , c , and d are natural numbers , define (a ? b) + (c ? d) =
(a + c) ? (b + d) .

= = = Rational numbers (fractions) = = =

Addition of rational numbers can be computed using the least common denominator , but a conceptually simpler definition involves only integer addition and multiplication :
Define <formula>
As an example , the sum <formula> .
Addition of fractions is much simpler when the denominators are the same ; in this case , one can simply add the numerators while leaving the denominator the same : <formula> , so <formula> .
The commutativity and associativity of rational addition is an easy consequence of the laws of integer arithmetic . For a more rigorous and general discussion , see field of fractions .

= = = Real numbers = = =

A common construction of the set of real numbers is the Dedekind completion of the set of rational numbers . A real number is defined to be a Dedekind cut of rationals : a non @-@ empty set of rationals that is closed downward and has no greatest element . The sum of real numbers a and b is defined element by element :
Define <formula>

This definition was first published , in a slightly modified form , by Richard Dedekind in 1872 . The commutativity and associativity of real addition are immediate ; defining the real number 0 to be the set of negative rationals , it is easily seen to be the additive identity . Probably the trickiest part of this construction pertaining to addition is the definition of additive inverses .

Unfortunately , dealing with multiplication of Dedekind cuts is a time @-@ consuming case @-@ by @-@ case process similar to the addition of signed integers . Another approach is the metric completion of the rational numbers . A real number is essentially defined to be the a limit of a Cauchy sequence of rationals , lim an . Addition is defined term by term :

Define <formula>

This definition was first published by Georg Cantor , also in 1872 , although his formalism was slightly different . One must prove that this operation is well @-@ defined , dealing with co @-@ Cauchy sequences . Once that task is done , all the properties of real addition follow immediately from the properties of rational numbers . Furthermore , the other arithmetic operations , including multiplication , have straightforward , analogous definitions .

= = = Complex numbers = = =

Complex numbers are added by adding the real and imaginary parts of the summands . That is to say :

<formula>

Using the visualization of complex numbers in the complex plane , the addition has the following geometric interpretation : the sum of two complex numbers A and B , interpreted as points of the complex plane , is the point X obtained by building a parallelogram three of whose vertices are O , A and B. Equivalently , X is the point such that the triangles with vertices O , A , B , and X , B , A , are congruent .

= = Generalizations = =

There are many binary operations that can be viewed as generalizations of the addition operation on the real numbers . The field of abstract algebra is centrally concerned with such generalized operations , and they also appear in set theory and category theory .

= = = Addition in abstract algebra = = =

= = = = Vector addition = = = =

In linear algebra , a vector space is an algebraic structure that allows for adding any two vectors and for scaling vectors . A familiar vector space is the set of all ordered pairs of real numbers ; the ordered pair (a , b) is interpreted as a vector from the origin in the Euclidean plane to the point (a , b) in the plane . The sum of two vectors is obtained by adding their individual coordinates :

(a , b) + (c , d) = (a + c , b + d) .

This addition operation is central to classical mechanics , in which vectors are interpreted as forces .

= = = = Matrix addition = = = =

Matrix addition is defined for two matrices of the same dimensions . The sum of two m × n (pronounced " m by n ") matrices A and B , denoted by A + B , is again an m × n matrix computed by adding corresponding elements :

<formula>

For example :

<formula>

= = = = Modular arithmetic = = = =

In modular arithmetic , the set of integers modulo 12 has twelve elements ; it inherits an addition operation from the integers that is central to musical set theory . The set of integers modulo 2 has just two elements ; the addition operation it inherits is known in Boolean logic as the " exclusive or " function . In geometry , the sum of two angle measures is often taken to be their sum as real numbers modulo 2? . This amounts to an addition operation on the circle , which in turn generalizes to addition operations on many @-@ dimensional tori .

= = = = General addition = = = =

The general theory of abstract algebra allows an " addition " operation to be any associative and commutative operation on a set . Basic algebraic structures with such an addition operation include commutative monoids and abelian groups .

= = = Addition in set theory and category theory = = =

A far @-@ reaching generalization of addition of natural numbers is the addition of ordinal numbers and cardinal numbers in set theory . These give two different generalizations of addition of natural numbers to the transfinite . Unlike most addition operations , addition of ordinal numbers is not commutative . Addition of cardinal numbers , however , is a commutative operation closely related to the disjoint union operation .

In category theory , disjoint union is seen as a particular case of the coproduct operation , and general coproducts are perhaps the most abstract of all the generalizations of addition . Some coproducts , such as Direct sum and Wedge sum , are named to evoke their connection with addition .

= = Related operations = =

Addition , along with subtraction , multiplication and division , is considered one of the basic operations and is used in elementary arithmetic .

= = = Arithmetic = = =

Subtraction can be thought of as a kind of addition ? that is , the addition of an additive inverse . Subtraction is itself a sort of inverse to addition , in that adding x and subtracting x are inverse functions .

Given a set with an addition operation , one cannot always define a corresponding subtraction operation on that set ; the set of natural numbers is a simple example . On the other hand , a subtraction operation uniquely determines an addition operation , an additive inverse operation , and an additive identity ; for this reason , an additive group can be described as a set that is closed under subtraction .

Multiplication can be thought of as repeated addition . If a single term x appears in a sum n times , then the sum is the product of n and x . If n is not a natural number , the product may still make sense ; for example , multiplication by ? 1 yields the additive inverse of a number .

In the real and complex numbers , addition and multiplication can be interchanged by the exponential function :