= m / n . For example , if k =
1 / 3 (green planet in Figure 5 , green orbit in Figure 10) , the resulting orbit is called the third subharmonic of the original orbit . Although such orbits are unlikely to occur in nature , they are helpful for illustrating Newton 's theorem .

Limit of nearly circular orbits

In Proposition 45 of his Principia , Newton applies his theorem of revolving orbits to develop a method for finding the force laws that govern the motions of planets . Johannes Kepler had noted that the orbits of most planets and the Moon seemed to be ellipses , and the long axis of those ellipses can determined accurately from astronomical measurements . The long axis is defined as the line connecting the positions of minimum and maximum distances to the central point , i.e. , the line connecting the two apses . For illustration , the long axis of the planet Mercury is defined as the line through its successive positions of perihelion and aphelion . Over time , the long axis of most orbiting bodies rotates gradually , generally no more than a few degrees per complete revolution , because of gravitational perturbations from other bodies , oblateness in the attracting body , general relativistic effects , and other effects . Newton 's method uses this apsidal precession as a sensitive probe of the type of force being applied to the planets .

Newton 's theorem describes only the effects of adding an inverse @-@ cube central force . However , Newton extends his theorem to an arbitrary central forces F (r) by restricting his attention to orbits that are nearly circular , such as ellipses with low orbital eccentricity (? ? 0 @.@ 1) , which is true of seven of the eight planetary orbits in the solar system . Newton also applied his theorem to the planet Mercury , which has an eccentricity ? of roughly 0 @.@ 21 , and suggested that it may pertain to Halley 's comet , whose orbit has an eccentricity of roughly 0 @.@ 97 .

A qualitative justification for this extrapolation of his method has been suggested by Valluri , Wilson and Harper . According to their argument , Newton considered the apsidal precession angle ? (the angle between the vectors of successive minimum and maximum distance from the center) to be a smooth , continuous function of the orbital eccentricity ? . For the inverse @-@ square force , ? equals 180 ° ; the vectors to the positions of minimum and maximum distances lie on the same line . If ? is initially not 180 ° at low ? (quasi @-@ circular orbits) then , in general , ? will equal 180 ° only for isolated values of ? ; a randomly chosen value of ? would be very unlikely to give ? = 180 ° . Therefore , the observed slow rotation of the apsides of planetary orbits suggest that the force of gravity is an inverse @-@ square law .

Quantitative formula

To simplify the equations , Newton writes F (r) in terms of a new function C (r)
<formula>
where R is the average radius of the nearly circular orbit . Newton expands C (r) in a series ? now known as a Taylor expansion ? in powers of the distance r , one of the first appearances of such a series . By equating the resulting inverse @-@ cube force term with the inverse @-@ cube force for revolving orbits , Newton derives an equivalent angular scaling factor k for nearly circular orbits
<formula>
In other words , the application of an arbitrary central force F (r) to a nearly circular elliptical orbit can accelerate the angular motion by the factor k without affecting the radial motion significantly . If an elliptical orbit is stationary , the particle rotates about the center of force by 180 ° as it moves from one end of the long axis to the other (the two apses) . Thus , the corresponding apsidal angle ? for a general central force equals k × 180 ° , using the general law ?2 = k ?1 .

Examples