= 0 , so that r1 =
r2 . In this case , the original force is not scaled , and its argument is unchanged ; the inverse @-@ cube force is added , but the inverse @-@ square term is not . Also , the path of the second particle is r2 = g (?2 / k) , consistent with the formula given above .

= = Derivations = =

= = = Newton 's derivation = = =

Newton 's derivation is found in Section IX of his Principia , specifically Propositions 43 ? 45 . His derivations of these Propositions are based largely on geometry .
Proposition 43 ; Problem 30
It is required to make a body move in a curve that revolves about the center of force in the same manner as another body in the same curve at rest .
Newton 's derivation of Proposition 43 depends on his Proposition 2 , derived earlier in the Principia . Proposition 2 provides a geometrical test for whether the net force acting on a point mass (a particle) is a central force . Newton showed that a force is central if and only if the particle sweeps out equal areas in equal times as measured from the center .
Newton 's derivation begins with a particle moving under an arbitrary central force F1 (r) ; the motion of this particle under this force is described by its radius r (t) from the center as a function of time , and also its angle ?1 (t) . In an infinitesimal time dt , the particle sweeps out an approximate right triangle whose area is
<formula>
Since the force acting on the particle is assumed to be a central force , the particle sweeps out equal angles in equal times , by Newton 's Proposition 2 . Expressed another way , the rate of sweeping out area is constant
<formula>
This constant areal velocity can be calculated as follows . At the apapsis and periapsis , the positions of closest and furthest distance from the attracting center , the velocity and radius vectors are perpendicular ; therefore , the angular momentum L1 per mass m of the particle (written as h1) can be related to the rate of sweeping out areas
<formula>
Now consider a second particle whose orbit is identical in its radius , but whose angular variation is multiplied by a constant factor k
<formula>
The areal velocity of the second particle equals that of the first particle multiplied by the same factor k
<formula>
Since k is a constant , the second particle also sweeps out equal areas in equal times . Therefore , by Proposition 2 , the second particle is also acted upon by a central force F2 (r) . This is the conclusion of Proposition 43 .
Proposition 44
The difference of the forces , by which two bodies may be made to move equally , one in a fixed , the other in the same orbit revolving , varies inversely as the cube of their common altitudes .
To find the magnitude of F2 (r) from the original central force F1 (r) , Newton calculated their difference F2 (r) ? F1 (r) using geometry and the definition of centripetal acceleration . In Proposition 44 of his Principia , he showed that the difference is proportional to the inverse cube of the radius , specifically by the formula given above , which Newtons writes in terms of the two constant areal velocities , h1 and h2
<formula>
Proposition 45 ; Problem 31
To find the motion of the apsides in orbits approaching very near to circles .

In this Proposition , Newton derives the consequences of his theorem of revolving orbits in the limit of nearly circular orbits . This approximation is generally valid for planetary orbits and the orbit of the Moon about the Earth . This approximation also allows Newton to consider a great variety of central force laws , not merely inverse @-@ square and inverse @-@ cube force laws .

= = = Modern derivation = = =

Modern derivations of Newton 's theorem have been published by Whittaker (1937) and Chandrasekhar (1995) . By assumption , the second angular speed is k times faster than the first
<formula>
Since the two radii have the same behavior with time , r (t) , the conserved angular momenta are related by the same factor k
<formula>
The equation of motion for a radius r of a particle of mass m moving in a central potential V (r) is given by Lagrange 's equations
<formula>
Applying the general formula to the two orbits yields the equation
<formula>
which can be re @-@ arranged to the form
<formula>
This equation relating the two radial forces can be understood qualitatively as follows . The difference in angular speeds (or equivalently , in angular momenta) causes a difference in the centripetal force requirement ; to offset this , the radial force must be altered with an inverse @-@ cube force .
Newton 's theorem can be expressed equivalently in terms of potential energy , which is defined for central forces
<formula>
The radial force equation can be written in terms of the two potential energies
<formula>
Integrating with respect to the distance r , Newtons 's theorem states that a k @-@ fold change in angular speed results from adding an inverse @-@ square potential energy to any given potential energy V1 (r)
<formula>