= 0 ; likewise , since A lies in that plane , $A \cdot L =$
0 .

The LRL vector differs from other conserved quantities in the following property . Whereas for typical conserved quantities , there is a corresponding cyclic coordinate in the three @-@ dimensional Lagrangian of the system , there does not exist such a coordinate for the LRL vector . Thus , the conservation of the LRL vector must be derived directly , e.g. , by the method of Poisson brackets , as described below . Conserved quantities of this kind are called " dynamic " , in contrast to the usual " geometric " conservation laws , e.g. , that of the angular momentum .

= = History of rediscovery = =

The LRL vector A is a constant of motion of the important Kepler problem , and is useful in describing astronomical orbits , such as the motion of the planets . Nevertheless , it has never been well known among physicists , possibly because it is less intuitive than momentum and angular momentum . Consequently , it has been rediscovered independently several times over the last three centuries .

Jakob Hermann was the first to show that A is conserved for a special case of the inverse @-@ square central force , and worked out its connection to the eccentricity of the orbital ellipse . Hermann 's work was generalized to its modern form by Johann Bernoulli in 1710 . At the end of the century , Pierre @-@ Simon de Laplace rediscovered the conservation of A , deriving it analytically , rather than geometrically . In the middle of the nineteenth century , William Rowan Hamilton derived the equivalent eccentricity vector defined below , using it to show that the momentum vector p moves on a circle for motion under an inverse @-@ square central force (Figure 3) .

At the beginning of the twentieth century , Josiah Willard Gibbs derived the same vector by vector analysis . Gibbs ' derivation was used as an example by Carle Runge in a popular German textbook on vectors , which was referenced by Wilhelm Lenz in his paper on the (old) quantum mechanical treatment of the hydrogen atom . In 1926 , the vector was used by Wolfgang Pauli to derive the spectrum of hydrogen using modern quantum mechanics , but not the Schrödinger equation ; after Pauli 's publication , it became known mainly as the Runge ? Lenz vector .

= = Mathematical definition = =

For a single particle acted on by an inverse @-@ square central force described by the equation
<formula>
the LRL vector A is defined mathematically by the formula
where
m is the mass of the point particle moving under the central force ,
p is its momentum vector ,
$L = r \times p$ is its angular momentum vector ,
k is a parameter that describes the strength of the central force ,
r is the position vector of the particle (Figure 1) , and
<formula> is the corresponding unit vector , i.e. , <formula> where r is the magnitude of r .
Since the assumed force is conservative , the total energy E is a constant of motion ,
<formula>

Furthermore , the assumed force is a central force , and thus the angular momentum vector L is also conserved and defines the plane in which the particle travels . The LRL vector A is perpendicular to the angular momentum vector L because both $p \times L$ and r are perpendicular to L. It follows that A lies in the plane of the orbit .

This definition of the LRL vector A pertains to a single point particle of mass m moving under the action of a fixed force . However , the same definition may be extended to two @-@ body problems such as Kepler 's problem , by taking m as the reduced mass of the two bodies and r as the vector between the two bodies .

A variety of alternative formulations for the same constant of motion may also be used . The most

common is to scale by mk to define the eccentricity vector
<formula>

= = Derivation of the Kepler orbits = =

The shape and orientation of the Kepler problem orbits can be determined from the LRL vector as follows . Taking the dot product of A with the position vector r gives the equation
<formula>
where ? is the angle between r and A (Figure 2) . Permuting the scalar triple product
<formula>
and rearranging yields the defining formula for a conic section , provided that A is a constant , which is the case for the inverse square force law ,
of eccentricity e ,
<formula>
and latus rectum
<formula>
The major semiaxis a of the conic section may be defined using the latus rectum and the eccentricity
<formula>
where the minus sign pertains to ellipses and the plus sign to hyperbolae .
Taking the dot product of A with itself yields an equation involving the energy E ,
<formula>
which may be rewritten in terms of the eccentricity ,
<formula>
Thus , if the energy E is negative (bound orbits) , the eccentricity is less than one and the orbit is an ellipse . Conversely , if the energy is positive (unbound orbits , also called " scattered orbits ") , the eccentricity is greater than one and the orbit is a hyperbola . Finally , if the energy is exactly zero , the eccentricity is one and the orbit is a parabola . In all cases , the direction of A lies along the symmetry axis of the conic section and points from the center of force toward the periapsis , the point of closest approach .

= = Circular momentum hodographs = =

The conservation of the LRL vector A and angular momentum vector L is useful in showing that the momentum vector p moves on a circle under an inverse @-@ square central force .
Taking the dot product of
<formula>
with itself yields
<formula>
Further choosing L along the z @-@ axis , and the major semiaxis as the x @-@ axis , yields the locus equation for p ,
.
In other words , the momentum vector p is confined to a circle of radius mk / L = L / ? centered on (0 , A / L) . The eccentricity e corresponds to the cosine of the angle ? shown in Figure 3 .
In the degenerate limit of circular orbits , and thus vanishing A , the circle centers at the origin (0 @,@ 0) . For brevity , it is also useful to introduce the variable <formula> .
This circular hodograph is useful in illustrating the symmetry of the Kepler problem .

= = Constants of motion and superintegrability = =