= 0 point . This distribution satisfies ? (g (x)) =
0 if g is nowhere zero , and otherwise if g has a real root at x0 , then
<formula>
It is natural therefore to define the composition ? (g (x)) for continuously differentiable functions g by
<formula>
where the sum extends over all roots of g (x) , which are assumed to be simple . Thus , for example
<formula>
In the integral form the generalized scaling property may be written as
<formula>

= = = Properties in n dimensions = = =

The delta distribution in an n @-@ dimensional space satisfies the following scaling property instead :
<formula>
so that ? is a homogeneous distribution of degree ? n . Under any reflection or rotation ? , the delta function is invariant :
<formula>
As in the one @-@ variable case , it is possible to define the composition of ? with a bi @-@ Lipschitz function g : Rn ? Rn uniquely so that the identity
<formula>
for all compactly supported functions f .
Using the coarea formula from geometric measure theory , one can also define the composition of the delta function with a submersion from one Euclidean space to another one of different dimension ; the result is a type of current . In the special case of a continuously differentiable function g : Rn ? R such that the gradient of g is nowhere zero , the following identity holds
<formula>
where the integral on the right is over g ? 1 (0) , the (n ? 1) -dimensional surface defined by g (x) = 0 with respect to the Minkowski content measure . This is known as a simple layer integral .
More generally , if S is a smooth hypersurface of Rn , then we can associate to S the distribution that integrates any compactly supported smooth function g over S :
<formula>
where ? is the hypersurface measure associated to S. This generalization is associated with the potential theory of simple layer potentials on S. If D is a domain in Rn with smooth boundary S , then ?S is equal to the normal derivative of the indicator function of D in the distribution sense :
<formula>
where n is the outward normal . For a proof , see e.g. the article on the surface delta function .

= = Fourier transform = =

The delta function is a tempered distribution , and therefore it has a well @-@ defined Fourier transform . Formally , one finds
<formula>
Properly speaking , the Fourier transform of a distribution is defined by imposing self @-@ adjointness of the Fourier transform under the duality pairing <formula> of tempered distributions with Schwartz functions . Thus <formula> is defined as the unique tempered distribution satisfying
<formula>
for all Schwartz functions ? . And indeed it follows from this that <formula>
As a result of this identity , the convolution of the delta function with any other tempered distribution S is simply S :
<formula>

That is to say that ? is an identity element for the convolution on tempered distributions , and in fact the space of compactly supported distributions under convolution is an associative algebra with identity the delta function . This property is fundamental in signal processing , as convolution with a tempered distribution is a linear time @-@ invariant system , and applying the linear time @-@ invariant system measures its impulse response . The impulse response can be computed to any desired degree of accuracy by choosing a suitable approximation for ? , and once it is known , it characterizes the system completely . See LTI system theory : Impulse response and convolution .
The inverse Fourier transform of the tempered distribution f (?) = 1 is the delta function . Formally , this is expressed
<formula>
and more rigorously , it follows since
<formula>
for all Schwartz functions f .
In these terms , the delta function provides a suggestive statement of the orthogonality property of the Fourier kernel on R. Formally , one has
<formula>
This is , of course , shorthand for the assertion that the Fourier transform of the tempered distribution
<formula>
is
<formula>
which again follows by imposing self @-@ adjointness of the Fourier transform .
By analytic continuation of the Fourier transform , the Laplace transform of the delta function is found to be
<formula>

= = Distributional derivatives = =

The distributional derivative of the Dirac delta distribution is the distribution ? ? defined on compactly supported smooth test functions ? by
<formula>
The first equality here is a kind of integration by parts , for if ? were a true function then
<formula>
The k @-@ th derivative of ? is defined similarly as the distribution given on test functions by
<formula>
In particular , ? is an infinitely differentiable distribution .
The first derivative of the delta function is the distributional limit of the difference quotients :
<formula>
More properly , one has
<formula>
where ?h is the translation operator , defined on functions by ?h? (x) = ? (x + h) , and on a distribution S by
<formula>
In the theory of electromagnetism , the first derivative of the delta function represents a point magnetic dipole situated at the origin . Accordingly , it is referred to as a dipole or the doublet function .
The derivative of the delta function satisfies a number of basic properties , including :
<formula>
Furthermore , the convolution of ? ? with a compactly supported smooth function f is
<formula>
which follows from the properties of the distributional derivative of a convolution .

= = = Higher dimensions = = =

More generally , on an open set U in the n @-@ dimensional Euclidean space Rn , the Dirac delta distribution centered at a point a ? U is defined by
<formula>
for all ? ? S (U) , the space of all smooth compactly supported functions on U. If ? = (?1 , ... , ?n) is any multi @-@ index and ? ? denotes the associated mixed partial derivative operator , then the ?th derivative ? ??a of ?a is given by
<formula>
That is , the ?th derivative of ?a is the distribution whose value on any test function ? is the ?th derivative of ? at a (with the appropriate positive or negative sign) .
The first partial derivatives of the delta function are thought of as double layers along the coordinate planes . More generally , the normal derivative of a simple layer supported on a surface is a double layer supported on that surface , and represents a laminar magnetic monopole . Higher derivatives of the delta function are known in physics as multipoles .
Higher derivatives enter into mathematics naturally as the building blocks for the complete structure of distributions with point support . If S is any distribution on U supported on the set { a } consisting of a single point , then there is an integer m and coefficients c? such that
<formula>

= = Representations of the delta function = =

The delta function can be viewed as the limit of a sequence of functions
<formula>
where ?? (x) is sometimes called a nascent delta function . This limit is meant in a weak sense : either that
for all continuous functions f having compact support , or that this limit holds for all smooth functions f with compact support . The difference between these two slightly different modes of weak convergence is often subtle : the former is convergence in the vague topology of measures , and the latter is convergence in the sense of distributions .

= = = Approximations to the identity = = =

Typically a nascent delta function ?? can be constructed in the following manner . Let ? be an absolutely integrable function on R of total integral 1 , and define
<formula>
In n dimensions , one uses instead the scaling
<formula>
Then a simple change of variables shows that ?? also has integral 1 . One shows easily that (5) holds for all continuous compactly supported functions f , and so ?? converges weakly to ? in the sense of measures .