= Hilbert space =

The mathematical concept of a Hilbert space , named after David Hilbert , generalizes the notion of Euclidean space . It extends the methods of vector algebra and calculus from the two @-@ dimensional Euclidean plane and three @-@ dimensional space to spaces with any finite or infinite number of dimensions . A Hilbert space is an abstract vector space possessing the structure of an inner product that allows length and angle to be measured . Furthermore , Hilbert spaces are complete : there are enough limits in the space to allow the techniques of calculus to be used .

Hilbert spaces arise naturally and frequently in mathematics and physics , typically as infinite @-@ dimensional function spaces . The earliest Hilbert spaces were studied from this point of view in the first decade of the 20th century by David Hilbert , Erhard Schmidt , and Frigyes Riesz . They are indispensable tools in the theories of partial differential equations , quantum mechanics , Fourier analysis (which includes applications to signal processing and heat transfer) ? and ergodic theory , which forms the mathematical underpinning of thermodynamics . John von Neumann coined the term Hilbert space for the abstract concept that underlies many of these diverse applications . The success of Hilbert space methods ushered in a very fruitful era for functional analysis . Apart from the classical Euclidean spaces , examples of Hilbert spaces include spaces of square @-@ integrable functions , spaces of sequences , Sobolev spaces consisting of generalized functions , and Hardy spaces of holomorphic functions .

Geometric intuition plays an important role in many aspects of Hilbert space theory . Exact analogs of the Pythagorean theorem and parallelogram law hold in a Hilbert space . At a deeper level , perpendicular projection onto a subspace (the analog of " dropping the altitude " of a triangle) plays a significant role in optimization problems and other aspects of the theory . An element of a Hilbert space can be uniquely specified by its coordinates with respect to a set of coordinate axes (an orthonormal basis) , in analogy with Cartesian coordinates in the plane . When that set of axes is countably infinite , this means that the Hilbert space can also usefully be thought of in terms of the space of infinite sequences that are square @-@ summable . The latter space is often in the older literature referred to as the Hilbert space . Linear operators on a Hilbert space are likewise fairly concrete objects : in good cases , they are simply transformations that stretch the space by different factors in mutually perpendicular directions in a sense that is made precise by the study of their spectrum .

= = Definition and illustration = =

= = = Motivating example : Euclidean space = = =

One of the most familiar examples of a Hilbert space is the Euclidean space consisting of three @-@ dimensional vectors , denoted by ?3 , and equipped with the dot product . The dot product takes two vectors x and y , and produces a real number x · y . If x and y are represented in Cartesian coordinates , then the dot product is defined by

<formula>

The dot product satisfies the properties :