= 1 / r ( cf . Bertrand 's theorem ) and ? =
cos ? , with the angle ? defined by
<formula>
and ? is the Lorentz factor . As before , we may obtain a conserved binormal vector B by taking the cross product with the conserved angular momentum vector
<formula>
These two vectors may likewise be combined into a conserved dyadic tensor W ,
<formula>
In illustration , the LRL vector for a non @-@ relativistic , isotropic harmonic oscillator can be calculated . Since the force is central ,
<formula>
the angular momentum vector is conserved and the motion lies in a plane .
The conserved dyadic tensor can be written in a simple form
<formula>
although it should be noted that p and r are not necessarily perpendicular .
The corresponding Runge ? Lenz vector is more complicated ,
<formula>
where
<formula>
is the natural oscillation frequency and
<formula>

= = Proofs that the Laplace ? Runge ? Lenz vector is conserved in Kepler problems = =

The following are arguments showing that the LRL vector is conserved under central forces that obey an inverse @-@ square law .

= = = Direct proof of conservation = = =

A central force <formula> acting on the particle is
<formula>
for some function <formula> of the radius <formula> . Since the angular momentum <formula> is conserved under central forces , <formula> and
<formula>
where the momentum <formula> and where the triple cross product has been simplified using Lagrange 's formula
<formula>
The identity
<formula>
yields the equation
<formula>
For the special case of an inverse @-@ square central force <formula> , this equals
<formula>
Therefore , A is conserved for inverse @-@ square central forces
<formula>
A shorter proof is obtained by using the relation of angular momentum to angular velocity , <formula> , which holds for a particle traveling in a plane perpendicular to <formula> . Specifying to inverse @-@ square central forces , the time derivative of <formula> is
<formula>
where the last equality holds because a unit vector can only change by rotation , and <formula> is the orbital velocity of the rotating vector . Thus , A is seen to be a difference of two vectors with equal time derivatives .
As described below , this LRL vector A is a special case of a general conserved vector <formula>

that can be defined for all central forces . However , since most central forces do not produce closed orbits ( see Bertrand 's theorem ) , the analogous vector <formula> rarely has a simple definition and is generally a multivalued function of the angle ? between r and <formula> .

= = = Hamilton ? Jacobi equation in parabolic coordinates = = =

The constancy of the LRL vector can also be derived from the Hamilton ? Jacobi equation in parabolic coordinates ( ? , ? ) , which are defined by the equations
<formula>
<formula>
where r represents the radius in the plane of the orbit
<formula>
The inversion of these coordinates is
<formula>
<formula>
Separation of the Hamilton ? Jacobi equation in these coordinates yields the two equivalent equations
<formula>
<formula>
where ? is a constant of motion . Subtraction and re @-@ expression in terms of the Cartesian momenta px and py shows that ? is equivalent to the LRL vector
<formula>

= = = Noether 's theorem = = =

The connection between the rotational symmetry described above and the conservation of the LRL vector can be made quantitative by way of Noether 's theorem . This theorem , which is used for finding constants of motion , states that any infinitesimal variation of the generalized coordinates of a physical system
<formula>
that causes the Lagrangian to vary to first order by a total time derivative
<formula>
corresponds to a conserved quantity ?
<formula>
In particular , the conserved LRL vector component As corresponds to the variation in the coordinates
<formula>
where i equals 1 , 2 and 3 , with xi and pi being the ith components of the position and momentum vectors r and p , respectively ; as usual , ?is represents the Kronecker delta . The resulting first @-@ order change in the Lagrangian is
<formula>
Substitution into the general formula for the conserved quantity ? yields the conserved component As of the LRL vector ,
<formula>

= = = Lie transformation = = =

The Noether theorem derivation of the conservation of the LRL vector A is elegant , but has one drawback : the coordinate variation ?xi involves not only the position r , but also the momentum p or , equivalently , the velocity v. This drawback may be eliminated by instead deriving the conservation of A using an approach pioneered by Sophus Lie . Specifically , one may define a Lie transformation in which the coordinates r and the time t are scaled by different powers of a parameter ? ( Figure 9 ) ,

<formula>
This transformation changes the total angular momentum L and energy E ,
<formula>
but preserves their product EL2 . Therefore , the eccentricity e and the magnitude A are preserved , as may be seen from the equation for A2
<formula>
The direction of A is preserved as well , since the semiaxes are not altered by a global scaling . This transformation also preserves Kepler 's third law , namely , that the semiaxis a and the period T form a constant T2 / a3 .

= = Alternative scalings , symbols and formulations = =

Unlike the momentum and angular momentum vectors p and L , there is no universally accepted definition of the Laplace ? Runge ? Lenz vector ; several different scaling factors and symbols are used in the scientific literature . The most common definition is given above , but another common alternative is to divide by the constant mk to obtain a dimensionless conserved eccentricity vector
<formula>
where v is the velocity vector . This scaled vector e has the same direction as A and its magnitude equals the eccentricity of the orbit . Other scaled versions are also possible , e.g. , by dividing A by m alone
<formula>
or by p0
<formula>
which has the same units as the angular momentum vector L. In rare cases , the sign of the LRL vector may be reversed , i.e. , scaled by ? 1 . Other common symbols for the LRL vector include a , R , F , J and V. However , the choice of scaling and symbol for the LRL vector do not affect its conservation .
An alternative conserved vector is the binormal vector B studied by William Rowan Hamilton
<formula>