Penrose tiling

A Penrose tiling is an example of non @-@ periodic tiling generated by an aperiodic set of prototiles . Penrose tilings are named after mathematician and physicist Roger Penrose , who investigated these sets in the 1970s . The aperiodicity of prototiles implies that a shifted copy of a tiling will never match the original . A Penrose tiling may be constructed so as to exhibit both reflection symmetry and fivefold rotational symmetry , as in the diagram at the right .

A Penrose tiling has many remarkable properties , most notably :

It is non @-@ periodic , which means that it lacks any translational symmetry .

It is self @-@ similar , so the same patterns occur at larger and larger scales . Thus , the tiling can be obtained through " inflation " (or " deflation ") and any finite patch from the tiling occurs infinitely many times .

It is a quasicrystal : implemented as a physical structure a Penrose tiling will produce Bragg diffraction and its diffractogram reveals both the fivefold symmetry and the underlying long range order .

Various methods to construct Penrose tilings have been discovered , including matching rules , substitutions or subdivision rules , cut and project schemes and coverings .

Background and history

Periodic and aperiodic tilings

Penrose tilings are simple examples of aperiodic tilings of the plane . A tiling is a covering of the plane by tiles with no overlaps or gaps ; the tiles normally have a finite number of shapes , called prototiles , and a set of prototiles is said to admit a tiling or tile the plane if there is a tiling of the plane using only tiles congruent to these prototiles . The most familiar tilings (e.g. , by squares or triangles) are periodic : a perfect copy of the tiling can be obtained by translating all of the tiles by a fixed distance in a given direction . Such a translation is called a period of the tiling ; more informally , this means that a finite region of the tiling repeats itself in periodic intervals . If a tiling has no periods it is said to be non @-@ periodic . A set of prototiles is said to be aperiodic if it tiles the plane but every such tiling is non @-@ periodic ; tilings by aperiodic sets of prototiles are called aperiodic tilings .

Earliest aperiodic tilings

The subject of aperiodic tilings received new interest in the 1960s when logician Hao Wang noted connections between decision problems and tilings . In particular , he introduced tilings by square plates with colored edges , now known as Wang dominoes or tiles , and posed the " Domino Problem " : to determine whether a given set of Wang dominoes could tile the plane with matching colors on adjacent domino edges . He observed that if this problem were undecidable , then there would have to exist an aperiodic set of Wang dominoes . At the time , this seemed implausible , so Wang conjectured no such set could exist .

Wang 's student Robert Berger proved that the Domino Problem was undecidable (so Wang 's conjecture was incorrect) in his 1964 thesis , and obtained an aperiodic set of 20426 Wang dominoes . He also described a reduction to 104 such prototiles ; the latter did not appear in his published monograph , but in 1968 , Donald Knuth detailed a modification of Berger 's set requiring only 92 dominoes .

The color matching required in a tiling by Wang dominoes can easily be achieved by modifying the edges of the tiles like jigsaw puzzle pieces so that they can fit together only as prescribed by the edge colorings . Raphael Robinson , in a 1971 paper which simplified Berger 's techniques and undecidability proof , used this technique to obtain an aperiodic set of just six prototiles .

Development of the Penrose tilings

The first Penrose tiling (tiling P1 below) is an aperiodic set of six prototiles , introduced by Roger Penrose in a 1974 paper , but it is based on pentagons rather than squares . Any attempt to tile the plane with regular pentagons necessarily leaves gaps , but Johannes Kepler showed , in his 1619 work Harmonices Mundi , that these gaps can be filled using pentagrams (star polygons) , decagons and related shapes . Traces of these ideas can also be found in the work of Albrecht Dürer . Acknowledging inspiration from Kepler , Penrose found matching rules (which can be imposed by decorations of the edges) for these shapes , obtaining an aperiodic set . His tiling can be viewed as a completion of Kepler 's finite Aa pattern .

Penrose subsequently reduced the number of prototiles to two , discovering the kite and dart tiling (tiling P2 below) and the rhombus tiling (tiling P3 below) . The rhombus tiling was independently discovered by Robert Ammann in 1976 . Penrose and John H. Conway investigated the properties of Penrose tilings , and discovered that a substitution property explained their hierarchical nature ; their findings were publicized by Martin Gardner in his January 1977 " Mathematical Games " column in Scientific American .

In 1981 , De Bruijn explained a method to construct Penrose tilings from five families of parallel lines as well as a " cut and project method " , in which Penrose tilings are obtained as two @-@ dimensional projections from a five @-@ dimensional cubic structure . In this approach , the Penrose tiling is viewed as a set of points , its vertices , while the tiles are geometrical shapes obtained by connecting vertices with edges .

The Penrose tilings

The three types of Penrose tiling P1 ? P3 are described individually below . They have many common features : in each case , the tiles are constructed from shapes related to the pentagon (and hence to the golden ratio) , but the basic tile shapes need to be supplemented by matching rules in order to tile aperiodically ; these rules may be described using labeled vertices or edges , or patterns on the tile faces ? alternatively the edge profile can be modified (e.g. by indentations and protrusions) to obtain an aperiodic set of prototiles .

The original pentagonal Penrose tiling (P1)

Penrose 's first tiling uses pentagons and three other shapes : a five @-@ pointed " star " (a pentagram) , a " boat " (roughly 3 / 5 of a star) and a " diamond " (a thin rhombus) . To ensure that all tilings are non @-@ periodic , there are matching rules that specify how tiles may meet each other , and there are three different types of matching rule for the pentagonal tiles . It is common to indicate the three different types of pentagonal tiles using three different colors , as in the figure above right .

Kite and dart tiling (P2)

Penrose 's second tiling uses quadrilaterals called the " kite " and " dart " , which may be combined to make a rhombus . However , the matching rules prohibit such a combination . Both the kite and dart are composed of two triangles , called Robinson triangles , after 1975 notes by Robinson .

The kite is a quadrilateral whose four interior angles are 72 , 72 , 72 , and 144 degrees . The kite may be bisected along its axis of symmetry to form a pair of acute Robinson triangles (with angles of 36 , 72 and 72 degrees) .

The dart is a non @-@ convex quadrilateral whose four interior angles are 36 , 72 , 36 , and 216 degrees . The dart may be bisected along its axis of symmetry to form a pair of obtuse Robinson triangles (with angles of 36 , 36 and 108 degrees) , which are smaller than the acute triangles .

The matching rules can be described in several ways . One approach is to color the vertices (with two colors , e.g. , black and white) and require that adjacent tiles have matching vertices . Another is to use a pattern of circular arcs (as shown above left in green and red) to constrain the placement of tiles : when two tiles share an edge in a tiling , the patterns must match at these edges .

These rules often force the placement of certain tiles : for example , the concave vertex of any dart is necessarily filled by two kites . The corresponding figure (center of the top row in the lower image on the left) is called an " ace " by Conway ; although it looks like an enlarged kite , it does not tile in the same way . Similarly the concave vertex formed when two kites meet along a short edge is necessarily filled by two darts (bottom right) . In fact , there are only seven possible ways for the tiles to meet at a vertex ; two of these figures ? namely , the " star " (top left) and the " sun " (top right) ? have 5 @-@ fold dihedral symmetry (by rotations and reflections) , while the remainder have a single axis of reflection (vertical in the image) . All of these vertex figures , apart from the ace and the sun , force the placement of additional tiles .

= = = Rhombus tiling (P3) = = =

The third tiling uses a pair of rhombuses (often referred to as " rhombs " in this context) with equal sides but different angles . Ordinary rhombus @-@ shaped tiles can be used to tile the plane periodically , so restrictions must be made on how tiles can be assembled : no two tiles may form a parallelogram , as this would allow a periodic tiling , but this constraint is not sufficient to force aperiodicity , as figure 1 above shows .

There are two kinds of tile , both of which can be decomposed into Robinson triangles .

The thin rhomb t has four corners with angles of 36 , 144 , 36 , and 144 degrees . The t rhomb may be bisected along its short diagonal to form a pair of acute Robinson triangles .

The thick rhomb T has angles of 72 , 108 , 72 , and 108 degrees . The T rhomb may be bisected along its long diagonal to form a pair of obtuse Robinson triangles ; in contrast to the P2 tiling , these are larger than the acute triangles .

The matching rules distinguish sides of the tiles , and entail that tiles may be juxtaposed in certain particular ways but not in others . Two ways to describe these matching rules are shown in the image on the right . In one form , tiles must be assembled such that the curves on the faces match in color and position across an edge . In the other , tiles must be assembled such that the bumps on their edges fit together .

There are 54 cyclically ordered combinations of such angles that add up to 360 degrees at a vertex , but the rules of the tiling allow only seven of these combinations to appear (although one of these arises in two ways) .

The various combinations of angles and facial curvature allow construction of arbitrarily complex tiles , such as the Penrose chickens .

= = Features and constructions = =

= = = The golden ratio and local pentagonal symmetry = = =

Several properties and common features of the Penrose tilings involve the golden ratio ?
= (1 + ? 5) / 2 (approximately 1 @.@ 618) . This is the ratio of chord lengths to side lengths in a regular pentagon , and satisfies ? =
1 + 1 / ? .

Consequently , the ratio of the lengths of long sides to short sides in the (isosceles) Robinson triangles is ? : 1 . It follows that the ratio of long side lengths to short in both kite and dart tiles is also ? : 1 , as are the length ratios of sides to the short diagonal in the thin rhomb t , and of long diagonal to sides in the thick rhomb T. In both the P2 and P3 tilings , the ratio of the area of the larger Robinson triangle to the smaller one is ? : 1 , hence so are the ratios of the areas of the kite to the dart , and of the thick rhomb to the thin rhomb . (Both larger and smaller obtuse Robinson triangles can be found in the pentagon on the left : the larger triangles at the top ? the halves of the thick

rhomb ? have linear dimensions scaled up by ? compared to the small shaded triangle at the base , and so the ratio of areas is ?2 : 1 .)
Any Penrose tiling has local pentagonal symmetry , in the sense that there are points in the tiling surrounded by a symmetric configuration of tiles : such configurations have fivefold rotational symmetry about the center point , as well as five mirror lines of reflection symmetry passing through the point , a dihedral symmetry group . This symmetry will generally preserve only a patch of tiles around the center point , but the patch can be very large : Conway and Penrose proved that whenever the colored curves on the P2 or P3 tilings close in a loop , the region within the loop has pentagonal symmetry , and furthermore , in any tiling , there are at most two such curves of each color that do not close up .
There can be at most one center point of global fivefold symmetry : if there were more than one , then rotating each about the other would yield two closer centers of fivefold symmetry , which leads to a mathematical contradiction . There are only two Penrose tilings (of each type) with global pentagonal symmetry : for the P2 tiling by kites and darts , the center point is either a " sun " or " star " vertex .

= = = Inflation and deflation = = =

Many of the common features of Penrose tilings follow from a hierarchical pentagonal structure given by substitution rules : this is often referred to as inflation and deflation , or composition and decomposition , of tilings or (collections of) tiles . The substitution rules decompose each tile into smaller tiles of the same shape as those used in the tiling (and thus allow larger tiles to be " composed " from smaller ones) . This shows that the Penrose tiling has a scaling self @-@ similarity , and so can be thought of as a fractal .
Penrose originally discovered the P1 tiling in this way , by decomposing a pentagon into six smaller pentagons (one half of a net of a dodecahedron) and five half @-@ diamonds ; he then observed that when he repeated this process the gaps between pentagons could all be filled by stars , diamonds , boats and other pentagons . By iterating this process indefinitely he obtained one of the two P1 tilings with pentagonal symmetry .

= = = = Robinson triangle decompositions = = = =

The substitution method for both P2 and P3 tilings can be described using Robinson triangles of different sizes . The Robinson triangles arising in P2 tilings (by bisecting kites and darts) are called A @-@ tiles , while those arising in the P3 tilings (by bisecting rhombs) are called B @-@ tiles . The smaller A @-@ tile , denoted AS , is an obtuse Robinson triangle , while the larger A @-@ tile , AL , is acute ; in contrast , a smaller B @-@ tile , denoted BS , is an acute Robinson triangle , while the larger B @-@ tile , BL , is obtuse .
Concretely , if AS has side lengths (1 , 1 , ?) , then AL has side lengths (? , ? , 1) . B @-@ tiles can be related to such A @-@ tiles in two ways :
If BS has the same size as AL then BL is an enlarged version ?AS of AS , with side lengths (? , ? , ?2 = 1 + ?) ? this decomposes into an AL tile and AS tile joined along a common side of length 1 .
If instead BL is identified with AS , then BS is a reduced version (1 / ?) AL of AL with side lengths (1 / ? , 1 / ? , 1) ? joining a BS tile and a BL tile along a common side of length 1 then yields (a decomposition of) an AL tile .
In these decompositions , there appears to be an ambiguity : Robinson triangles may be decomposed in two ways , which are mirror images of each other in the (isosceles) axis of symmetry of the triangle . In a Penrose tiling , this choice is fixed by the matching rules ? furthermore , the matching rules also determine how the smaller triangles in the tiling compose to give larger ones .
It follows that the P2 and P3 tilings are mutually locally derivable : a tiling by one set of tiles can be used to generate a tiling by another ? for example a tiling by kites and darts may be subdivided into A @-@ tiles , and these can be composed in a canonical way to form B @-@ tiles and hence

rhombs . The P2 and P3 tilings are also both mutually locally derivable with the P1 tiling (see figure 2 above) .

The decomposition of B @-@ tiles into A @-@ tiles may be written