= 5 , see Pentagonal tiling and for N =
6 , see Hexagonal tiling .
For results on tiling the plane with polyominoes , see Polyomino § Uses of polyominoes .

= = = Voronoi tilings = = =

Voronoi or Dirichlet tilings are tessellations where each tile is defined as the set of points closest to one of the points in a discrete set of defining points . (Think of geographical regions where each region is defined as all the points closest to a given city or post office .) The Voronoi cell for each defining point is a convex polygon . The Delaunay triangulation is a tessellation that is the dual graph of a Voronoi tessellation . Delaunay triangulations are useful in numerical simulation , in part because among all possible triangulations of the defining points , Delaunay triangulations maximize the minimum of the angles formed by the edges . Voronoi tilings with randomly placed points can be used to construct random tilings of the plane .

= = = Tessellations in higher dimensions = = =

Tessellation can be extended to three dimensions . Certain polyhedra can be stacked in a regular crystal pattern to fill (or tile) three @-@ dimensional space , including the cube (the only regular polyhedron to do so) , the rhombic dodecahedron , and the truncated octahedron . Naturally occurring rhombic dodecahedra are found as crystals of Andradite (a kind of Garnet) and Fluorite .
A Schwarz triangle is a spherical triangle that can be used to tile a sphere .
Tessellations in three or more dimensions are called honeycombs . In three dimensions there is just one regular honeycomb , which has eight cubes at each polyhedron vertex . Similarly , in three dimensions there is just one quasiregular honeycomb , which has eight tetrahedra and six octahedra at each polyhedron vertex . However , there are many possible semiregular honeycombs in three dimensions . Uniform polyhedra can be constructed using the Wythoff construction .
The Schmitt @-@ Conway biprism is a convex polyhedron with the property of tiling space only aperiodically .

= = = Tessellations in non @-@ Euclidean geometries = = =

It is possible to tessellate in non @-@ Euclidean geometries such as hyperbolic geometry . A uniform tiling in the hyperbolic plane (which may be regular , quasiregular or semiregular) is an edge @-@ to @-@ edge filling of the hyperbolic plane , with regular polygons as faces ; these are vertex @-@ transitive (transitive on its vertices) , and isogonal (there is an isometry mapping any vertex onto any other) .
A uniform honeycomb in hyperbolic space is a uniform tessellation of uniform polyhedral cells . In 3 @-@ dimensional hyperbolic space there are nine Coxeter group families of compact convex uniform honeycombs , generated as Wythoff constructions , and represented by permutations of rings of the Coxeter diagrams for each family .

= = In art = =

In architecture , tessellations have been used to create decorative motifs since ancient times . Mosaic tilings often had geometric patterns . Later civilisations also used larger tiles , either plain or individually decorated . Some of the most decorative were the Moorish wall tilings of Islamic architecture , using Girih and Zellige tiles in buildings such as the Alhambra and La Mezquita .
Tessellations frequently appeared in the graphic art of M. C. Escher ; he was inspired by the Moorish use of symmetry in places such as the Alhambra when he visited Spain in 1936 . Escher made four " Circle Limit " drawings of tilings that use hyperbolic geometry . For his woodcut " Circle Limit IV " (1960) , Escher prepared a pencil and ink study showing the required geometry . Escher explained that " No single component of all the series , which from infinitely far away rise like rockets

perpendicularly from the limit and are at last lost in it , ever reaches the boundary line . "

Tessellated designs often appear on textiles , whether woven , stitched in or printed . Tessellation patterns have been used to design interlocking motifs of patch shapes in quilts .

Tessellations are also a main genre in origami (paper folding) , where pleats are used to connect molecules such as twist folds together in a repeating fashion .

= = In manufacturing = =

Tessellation is used in manufacturing industry to reduce the wastage of material (yield losses) such as sheet metal when cutting out shapes for objects like car doors or drinks cans .

= = In nature = =

The honeycomb provides a well @-@ known example of tessellation in nature with its hexagonal cells .

In botany , the term " tessellate " describes a checkered pattern , for example on a flower petal , tree bark , or fruit . Flowers including the Fritillary and some species of Colchicum are characteristically tessellate .

Many patterns in nature are formed by cracks in sheets of materials . These patterns can be described by Gilbert tessellations , also known as random crack networks . The Gilbert tessellation is a mathematical model for the formation of mudcracks , needle @-@ like crystals , and similar structures . The model , named after Edgar Gilbert , allows cracks to form starting from randomly scattered over the plane ; each crack propagates in two opposite directions along a line through the initiation point , its slope chosen at random , creating a tessellation of irregular convex polygons . Basaltic lava flows often display columnar jointing as a result of contraction forces causing cracks as the lava cools . The extensive crack networks that develop often produce hexagonal columns of lava . One example of such an array of columns is the Giant 's Causeway in Northern Ireland . Tessellated pavement , a characteristic example of which is found at Eaglehawk Neck on the Tasman Peninsula of Tasmania , is a rare sedimentary rock formation where the rock has fractured into rectangular blocks .

Other natural patterns occur in foams ; these are packed according to Plateau 's laws , which require minimal surfaces . Such foams present a problem in how to pack cells as tightly as possible : in 1887 , Lord Kelvin proposed a packing using only one solid , the bitruncated cubic honeycomb with very slightly curved faces . In 1993 , Denis Weaire and Robert Phelan proposed the Weaire ? Phelan structure , which uses less surface area to separate cells of equal volume than Kelvin 's foam .

= = In puzzles and recreational mathematics = =

Tessellations have given rise to many types of tiling puzzle , from traditional jigsaw puzzles (with irregular pieces of wood or cardboard) and the tangram to more modern puzzles which often have a mathematical basis . For example , polyiamonds and polyominoes are figures of regular triangles and squares , often used in tiling puzzles . Authors such as Henry Dudeney and Martin Gardner have made many uses of tessellation in recreational mathematics . For example , Dudeney invented the hinged dissection , while Gardner wrote about the rep @-@ tile , a shape that can be dissected into smaller copies of the same shape . Inspired by Gardner 's articles in Scientific American , the amateur mathematician Marjorie Rice found four new tessellations with pentagons . Squaring the square is the problem of tiling an integral square (one whose sides have integer length) using only other integral squares . An extension is squaring the plane , tiling it by squares whose sizes are all natural numbers without repetitions ; James and Frederick Henle proved that this was possible .

= = Examples = =