= Newton 's theorem of revolving orbits =

In classical mechanics , Newton 's theorem of revolving orbits identifies the type of central force needed to multiply the angular speed of a particle by a factor k without affecting its radial motion (Figures 1 and 2) . Newton applied his theorem to understanding the overall rotation of orbits (apsidal precession , Figure 3) that is observed for the Moon and planets . The term " radial motion " signifies the motion towards or away from the center of force , whereas the angular motion is perpendicular to the radial motion .

Isaac Newton derived this theorem in Propositions 43 ? 45 of Book I of his Philosophiæ Naturalis Principia Mathematica , first published in 1687 . In Proposition 43 , he showed that the added force must be a central force , one whose magnitude depends only upon the distance r between the particle and a point fixed in space (the center) . In Proposition 44 , he derived a formula for the force , showing that it was an inverse @-@ cube force , one that varies as the inverse cube of r . In Proposition 45 Newton extended his theorem to arbitrary central forces by assuming that the particle moved in nearly circular orbit .

As noted by astrophysicist Subrahmanyan Chandrasekhar in his 1995 commentary on Newton 's Principia , this theorem remained largely unknown and undeveloped for over three centuries . Since 1997 , the theorem has been studied by Donald Lynden @-@ Bell and collaborators . Its first exact extension came in 2000 with the work of Mahomed and Vawda .

= = Historical context = =

The motion of astronomical bodies has been studied systematically for thousands of years . The stars were observed to rotate uniformly , always maintaining the same relative positions to one another . However , other bodies were observed to wander against the background of the fixed stars ; most such bodies were called planets after the Greek word " ???????? " (plan?toi) for " wanderers " . Although they generally move in the same direction along a path across the sky (the ecliptic) , individual planets sometimes reverse their direction briefly , exhibiting retrograde motion .

To describe this forward @-@ and @-@ backward motion , Apollonius of Perga (c . 262 ? c . 190 BC) developed the concept of deferents and epicycles , according to which the planets are carried on rotating circles that are themselves carried on other rotating circles , and so on . Any orbit can be described with a sufficient number of judiciously chosen epicycles , since this approach corresponds to a modern Fourier transform . Roughly 350 years later , Claudius Ptolemaeus published his Almagest , in which he developed this system to match the best astronomical observations of his era . To explain the epicycles , Ptolemy adopted the geocentric cosmology of Aristotle , according to which planets were confined to concentric rotating spheres . This model of the universe was authoritative for nearly 1500 years .

The modern understanding of planetary motion arose from the combined efforts of astronomer Tycho Brahe and physicist Johannes Kepler in the 16th century . Tycho is credited with extremely accurate measurements of planetary motions , from which Kepler was able to derive his laws of planetary motion . According to these laws , planets move on ellipses (not epicycles) about the Sun (not the Earth) . Kepler 's second and third laws make specific quantitative predictions : planets sweep out equal areas in equal time , and the square of their orbital periods equals a fixed constant times the cube of their semi @-@ major axis . Subsequent observations of the planetary orbits showed that the long axis of the ellipse (the so @-@ called line of apsides) rotates gradually with time ; this rotation is known as apsidal precession . The apses of an orbit are the points at which the orbiting body is closest or furthest away from the attracting center ; for planets orbiting the Sun , the apses correspond to the perihelion (closest) and aphelion (furthest) .

With the publication of his Principia roughly eighty years later (1687) , Isaac Newton provided a physical theory that accounted for all three of Kepler 's laws , a theory based on Newton 's laws of motion and his law of universal gravitation . In particular , Newton proposed that the gravitational force between any two bodies was a central force F (r) that varied as the inverse square of the distance r between them . Arguing from his laws of motion , Newton showed that the orbit of any particle acted upon by one such force is always a conic section , specifically an ellipse if it does not go to infinity . However , this conclusion holds only when two bodies are present (the two @-@ body problem) ; the motion of three bodies or more acting under their mutual gravitation (the n @-@ body problem) remained unsolved for centuries after Newton , although solutions to a few special cases were discovered . Newton proposed that the orbits of planets about the Sun are largely elliptical because the Sun 's gravitation is dominant ; to first approximation , the presence of the other planets can be ignored . By analogy , the elliptical orbit of the Moon about the Earth was dominated by the Earth 's gravity ; to first approximation , the Sun 's gravity and those of other bodies of the Solar System can be neglected . However , Newton stated that the gradual apsidal precession of the planetary and lunar orbits was due to the effects of these neglected interactions ; in particular , he stated that the precession of the Moon 's orbit was due to the perturbing effects of gravitational interactions with the Sun .

Newton 's theorem of revolving orbits was his first attempt to understand apsidal precession quantitatively . According to this theorem , the addition of a particular type of central force ? the inverse @-@ cube force ? can produce a rotating orbit ; the angular speed is multiplied by a factor k , whereas the radial motion is left unchanged . However , this theorem is restricted to a specific type of force that may not be relevant ; several perturbing inverse @-@ square interactions (such as those of other planets) seem unlikely to sum exactly to an inverse @-@ cube force . To make his theorem applicable to other types of forces , Newton found the best approximation of an arbitrary central force F (r) to an inverse @-@ cube potential in the limit of nearly circular orbits , that is , elliptical orbits of low eccentricity , as is indeed true for most orbits in the Solar System . To find this approximation , Newton developed an infinite series that can be viewed as the forerunner of the Taylor expansion . This approximation allowed Newton to estimate the rate of precession for arbitrary central forces . Newton applied this approximation to test models of the force causing the apsidal precession of the Moon 's orbit . However , the problem of the Moon 's motion is dauntingly complex , and Newton never published an accurate gravitational model of the Moon 's apsidal precession . After a more accurate model by Clairaut in 1747 , analytical models of the Moon 's motion were developed in the late 19th century by Hill , Brown , and Delaunay .

However , Newton 's theorem is more general than merely explaining apsidal precession . It describes the effects of adding an inverse @-@ cube force to any central force F (r) , not only to inverse @-@ square forces such as Newton 's law of universal gravitation and Coulomb 's law . Newton 's theorem simplifies orbital problems in classical mechanics by eliminating inverse @-@ cube forces from consideration . The radial and angular motions , r (t) and ?1 (t) , can be calculated without the inverse @-@ cube force ; afterwards , its effect can be calculated by multiplying the angular speed of the particle

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= = Mathematical statement = =

Consider a particle moving under an arbitrary central force F1 (r) whose magnitude depends only on the distance r between the particle and a fixed center . Since the motion of a particle under a central force always lies in a plane , the position of the particle can be described by polar coordinates (r , ?1) , the radius and angle of the particle relative to the center of force (Figure 1) . Both of these coordinates , r (t) and ?1 (t) , change with time t as the particle moves .

Imagine a second particle with the same mass m and with the same radial motion r (t) , but one whose angular speed is k times faster than that of the first particle . In other words , the azimuthal angles of the two particles are related by the equation ?2 (t) = k ?1 (t) . Newton showed that the motion of the second particle can be produced by adding an inverse @-@ cube central force to whatever force F1 (r) acts on the first particle

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where L1 is the magnitude of the first particle 's angular momentum , which is a constant of motion (conserved) for central forces .

If k2 is greater than one , F2 ? F1 is a negative number ; thus , the added inverse @-@ cube force is attractive , as observed in the green planet of Figures 1 ? 4 and 9 . By contrast , if k2 is less than one , F2 ? F1 is a positive number ; the added inverse @-@ cube force is repulsive , as observed in the green planet of Figures 5 and 10 , and in the red planet of Figures 4 and 5 .

= = = Alteration of the particle path = = =