Problem of Apollonius

In Euclidean plane geometry , Apollonius 's problem is to construct circles that are tangent to three given circles in a plane (Figure 1) . Apollonius of Perga (ca . 262 BC ? ca . 190 BC) posed and solved this famous problem in his work ?????? (Epaphaí , " Tangencies ") ; this work has been lost , but a 4th @-@ century report of his results by Pappus of Alexandria has survived . Three given circles generically have eight different circles that are tangent to them (Figure 2) and each solution circle encloses or excludes the three given circles in a different way : in each solution , a different subset of the three circles is enclosed (its complement is excluded) and there are 8 subsets of a set whose cardinality is 3 , since 8 = 23 .

In the 16th century , Adriaan van Roomen solved the problem using intersecting hyperbolas , but this solution does not use only straightedge and compass constructions . François Viète found such a solution by exploiting limiting cases : any of the three given circles can be shrunk to zero radius (a point) or expanded to infinite radius (a line) . Viète 's approach , which uses simpler limiting cases to solve more complicated ones , is considered a plausible reconstruction of Apollonius ' method . The method of van Roomen was simplified by Isaac Newton , who showed that Apollonius ' problem is equivalent to finding a position from the differences of its distances to three known points . This has applications in navigation and positioning systems such as LORAN .

Later mathematicians introduced algebraic methods , which transform a geometric problem into algebraic equations . These methods were simplified by exploiting symmetries inherent in the problem of Apollonius : for instance solution circles generically occur in pairs , with one solution enclosing the given circles that the other excludes (Figure 2) . Joseph Diaz Gergonne used this symmetry to provide an elegant straightedge and compass solution , while other mathematicians used geometrical transformations such as reflection in a circle to simplify the configuration of the given circles . These developments provide a geometrical setting for algebraic methods (using Lie sphere geometry) and a classification of solutions according to 33 essentially different configurations of the given circles .

Apollonius ' problem has stimulated much further work . Generalizations to three dimensions ? constructing a sphere tangent to four given spheres ? and beyond have been studied . The configuration of three mutually tangent circles has received particular attention . René Descartes gave a formula relating the radii of the solution circles and the given circles , now known as Descartes ' theorem . Solving Apollonius ' problem iteratively in this case leads to the Apollonian gasket , which is one of the earliest fractals to be described in print , and is important in number theory via Ford circles and the Hardy ? Littlewood circle method .

Statement of the problem

The general statement of Apollonius ' problem is to construct one or more circles that are tangent to three given objects in a plane , where an object may be a line , a point or a circle of any size . These objects may be arranged in any way and may cross one another ; however , they are usually taken to be distinct , meaning that they do not coincide . Solutions to Apollonius ' problem are sometimes called Apollonius circles , although the term is also used for other types of circles associated with Apollonius .

The property of tangency is defined as follows . First , a point , line or circle is assumed to be tangent to itself ; hence , if a given circle is already tangent to the other two given objects , it is counted as a solution to Apollonius ' problem . Two distinct geometrical objects are said to intersect if they have a point in common . By definition , a point is tangent to a circle or a line if it intersects them , that is , if it lies on them ; thus , two distinct points cannot be tangent . If the angle between lines or circles at an intersection point is zero , they are said to be tangent ; the intersection point is called a tangent point or a point of tangency . (The word " tangent " derives from the Latin present participle , tangens , meaning " touching " .) In practice , two distinct circles are tangent if they intersect at only one point ; if they intersect at zero or two points , they are not tangent . The same holds true for a line and a circle . Two distinct lines cannot be tangent in the plane , although two

parallel lines can be considered as tangent at a point at infinity in inversive geometry (see below) .
The solution circle may be either internally or externally tangent to each of the given circles . An external tangency is one where the two circles bend away from each other at their point of contact ; they lie on opposite sides of the tangent line at that point , and they exclude one another . The distance between their centers equals the sum of their radii . By contrast , an internal tangency is one in which the two circles curve in the same way at their point of contact ; the two circles lie on the same side of the tangent line , and one circle encloses the other . In this case , the distance between their centers equals the difference of their radii . As an illustration , in Figure 1 , the pink solution circle is internally tangent to the medium @-@ sized given black circle on the right , whereas it is externally tangent to the smallest and largest given circles on the left .