= Homotopy groups of spheres =

In the mathematical field of algebraic topology , the homotopy groups of spheres describe how spheres of various dimensions can wrap around each other . They are examples of topological invariants , which reflect , in algebraic terms , the structure of spheres viewed as topological spaces , forgetting about their precise geometry . Unlike homology groups , which are also topological invariants , the homotopy groups are surprisingly complex and difficult to compute .

The n @-@ dimensional unit sphere ? called the n @-@ sphere for brevity , and denoted as Sn ? generalizes the familiar circle (S1) and the ordinary sphere (S2) . The n @-@ sphere may be defined geometrically as the set of points in a Euclidean space of dimension n + 1 located at a unit distance from the origin . The i @-@ th homotopy group ?i (Sn) summarizes the different ways in which the i @-@ dimensional sphere Si can be mapped continuously into the n @-@ dimensional sphere Sn . This summary does not distinguish between two mappings if one can be continuously deformed to the other ; thus , only equivalence classes of mappings are summarized . An " addition " operation defined on these equivalence classes makes the set of equivalence classes into an abelian group .

The problem of determining ?i (Sn) falls into three regimes , depending on whether i is less than , equal to , or greater than n . For 0 < i < n , any mapping from Si to Sn is homotopic (i.e. , continuously deformable) to a constant mapping , i.e. , a mapping that maps all of Si to a single point of Sn . When i = n , every map from Sn to itself has a degree that measures how many times the sphere is wrapped around itself . This degree identifies ?n (Sn) with the group of integers under addition . For example , every point on a circle can be mapped continuously onto a point of another circle ; as the first point is moved around the first circle , the second point may cycle several times around the second circle , depending on the particular mapping . However , the most interesting and surprising results occur when i > n . The first such surprise was the discovery of a mapping called the Hopf fibration , which wraps the 3 @-@ sphere S3 around the usual sphere S2 in a non @-@ trivial fashion , and so is not equivalent to a one @-@ point mapping .

The question of computing the homotopy group ?n + k (Sn) for positive k turned out to be a central question in algebraic topology that has contributed to development of many of its fundamental techniques and has served as a stimulating focus of research . One of the main discoveries is that the homotopy groups ?n + k (Sn) are independent of n for n ? k + 2 . These are called the stable homotopy groups of spheres and have been computed for values of k up to 64 . The stable homotopy groups form the coefficient ring of an extraordinary cohomology theory , called stable cohomotopy theory . The unstable homotopy groups (for n < k + 2) are more erratic ; nevertheless , they have been tabulated for k < 20 . Most modern computations use spectral sequences , a technique first applied to homotopy groups of spheres by Jean @-@ Pierre Serre . Several important patterns have been established , yet much remains unknown and unexplained .

= = Background = =

The study of homotopy groups of spheres builds on a great deal of background material , here briefly reviewed . Algebraic topology provides the larger context , itself built on topology and abstract algebra , with homotopy groups as a basic example .

= = = n @-@ sphere = = =

An ordinary sphere in three @-@ dimensional space ? the surface , not the solid ball ? is just one example of what a sphere means in topology . Geometry defines a sphere rigidly , as a shape . Here are some alternatives .