= 0) and p (l =
1) atomic orbitals . Such mixing cannot be done with ordinary three @-@ dimensional translations or rotations , but is equivalent to a rotation in a higher dimension .
For negative energies ? i.e. , for bound systems ? the higher symmetry group is SO (4) , which preserves the length of four @-@ dimensional vectors
<formula>
In 1935 , Vladimir Fock showed that the quantum mechanical bound Kepler problem is equivalent to the problem of a free particle confined to a three @-@ dimensional unit sphere in four @-@ dimensional space . Specifically , Fock showed that the Schrödinger wavefunction in the momentum space for the Kepler problem was the stereographic projection of the spherical harmonics on the sphere . Rotation of the sphere and reprojection results in a continuous mapping of the elliptical orbits without changing the energy ; quantum mechanically , this corresponds to a mixing of all orbitals of the same energy quantum number n . Valentine Bargmann noted subsequently that the Poisson brackets for the angular momentum vector L and the scaled LRL vector D formed the Lie algebra for SO (4) . Simply put , the six quantities D and L correspond to the six conserved angular momenta in four dimensions , associated with the six possible simple rotations in that space (there are six ways of choosing two axes from four) . This conclusion does not imply that our universe is a three @-@ dimensional sphere ; it merely means that this particular physics problem (the two @-@ body problem for inverse @-@ square central forces) is mathematically equivalent to a free particle on a three @-@ dimensional sphere .
For positive energies ? i.e. , for unbound , " scattered " systems ? the higher symmetry group is SO (3 @,@ 1) , which preserves the Minkowski length of 4 @-@ vectors
<formula>
Both the negative- and positive @-@ energy cases were considered by Fock and Bargmann and have been reviewed encyclopedically by Bander and Itzykson .
The orbits of central @-@ force systems ? and those of the Kepler problem in particular ? are also symmetric under reflection . Therefore , the SO (3) , SO (4) and SO (3 @,@ 1) groups cited above are not the full symmetry groups of their orbits ; the full groups are O (3) , O (4) and O (3 @,@ 1) , respectively . Nevertheless , only the connected subgroups , SO (3) , SO (4) and SO (3 @,@ 1) , are needed to demonstrate the conservation of the angular momentum and LRL vectors ; the reflection symmetry is irrelevant for conservation , which may be derived from the Lie algebra of the group .

= = Rotational symmetry in four dimensions = =

The connection between the Kepler problem and four @-@ dimensional rotational symmetry SO (4) can be readily visualized . Let the four @-@ dimensional Cartesian coordinates be denoted (w , x , y , z) where (x , y , z) represent the Cartesian coordinates of the normal position vector r . The three @-@ dimensional momentum vector p is associated with a four @-@ dimensional vector <formula> on a three @-@ dimensional unit sphere
<formula>
where <formula> is the unit vector along the new w @-@ axis . The transformation mapping p to ? can be uniquely inverted ; for example , the x @-@ component of the momentum equals
<formula>
and similarly for py and pz . In other words , the three @-@ dimensional vector p is a stereographic projection of the four @-@ dimensional <formula> vector , scaled by p0 (Figure 8) .
Without loss of generality , we may eliminate the normal rotational symmetry by choosing the Cartesian coordinates such that the z @-@ axis is aligned with the angular momentum vector L and the momentum hodographs are aligned as they are in Figure 7 , with the centers of the circles on the y @-@ axis . Since the motion is planar , and p and L are perpendicular , pz
= ?z =
0 and attention may be restricted to the three @-@ dimensional vector <formula>
= (?w , ?x , ?y) . The family of Apollonian circles of momentum hodographs (Figure 7)

correspond to a family of great circles on the three @-@ dimensional <formula> sphere , all of which intersect the ?x @-@ axis at the two foci ?x =
± 1 , corresponding to the momentum hodograph foci at px = ± p0 . These great circles are related by a simple rotation about the ?x @-@ axis (Figure 8) . This rotational symmetry transforms all the orbits of the same energy into one another ; however , such a rotation is orthogonal to the usual three @-@ dimensional rotations , since it transforms the fourth dimension ?w . This higher symmetry is characteristic of the Kepler problem and corresponds to the conservation of the LRL vector .

An elegant action @-@ angle variables solution for the Kepler problem can be obtained by eliminating the redundant four @-@ dimensional coordinates <formula> in favor of elliptic cylindrical coordinates (? , ? , ?)

<formula>

<formula>

<formula>

<formula>

where sn , cn and dn are Jacobi 's elliptic functions .

= = Generalizations to other potentials and relativity = =

The Laplace ? Runge ? Lenz vector can also be generalized to identify conserved quantities that apply to other situations .

In the presence of a uniform electric field E , the generalized Laplace ? Runge ? Lenz vector <formula> is

<formula>

where q is the charge of the orbiting particle . Although <formula> is not conserved , it gives rise to a conserved quantity , namely <formula> .

Further generalizing the Laplace ? Runge ? Lenz vector to other potentials and special relativity , the most general form can be written as

<formula>