= 1) or externally (s =
? 1) . For example , in Figures 1 and 4 , the pink solution is internally tangent to the medium @-@ sized given circle on the right and externally tangent to the smallest and largest given circles on the left ; if the given circles are ordered by radius , the signs for this solution are " ? + ? " . Since the three signs may be chosen independently , there are eight possible sets of equations (2 × 2 × 2 = 8) , each set corresponding to one of the eight types of solution circles .

The general system of three equations may be solved by the method of resultants . When multiplied out , all three equations have xs2 + ys2 on the left @-@ hand side , and rs2 on the right @-@ hand side . Subtracting one equation from another eliminates these quadratic terms ; the remaining linear terms may be re @-@ arranged to yield formulae for the coordinates xs and ys

<formula>

<formula>

where M , N , P and Q are known functions of the given circles and the choice of signs . Substitution of these formulae into one of the initial three equations gives a quadratic equation for rs , which can be solved by the quadratic formula . Substitution of the numerical value of rs into the linear formulae yields the corresponding values of xs and ys .

The signs s1 , s2 and s3 on the right @-@ hand sides of the equations may be chosen in eight possible ways , and each choice of signs gives up to two solutions , since the equation for rs is quadratic . This might suggest (incorrectly) that there are up to sixteen solutions of Apollonius ' problem . However , due to a symmetry of the equations , if (rs , xs , ys) is a solution , with signs si , then so is (? rs , xs , ys) , with opposite signs ? si , which represents the same solution circle . Therefore , Apollonius ' problem has at most eight independent solutions (Figure 2) . One way to avoid this double @-@ counting is to consider only solution circles with non @-@ negative radius .

The two roots of any quadratic equation may be of three possible types : two different real numbers , two identical real numbers (i.e. , a degenerate double root) , or a pair of complex conjugate roots . The first case corresponds to the usual situation ; each pair of roots corresponds to a pair of solutions that are related by circle inversion , as described below (Figure 6) . In the second case , both roots are identical , corresponding to a solution circle that transforms into itself under inversion . In this case , one of the given circles is itself a solution to the Apollonius problem , and the number of distinct solutions is reduced by one . The third case of complex conjugate radii does not correspond to a geometrically possible solution for Apollonius ' problem , since a solution circle cannot have an imaginary radius ; therefore , the number of solutions is reduced by two . Interestingly , Apollonius ' problem cannot have seven solutions , although it may have any other number of solutions from zero to eight .

= = = Lie sphere geometry = = =

The same algebraic equations can be derived in the context of Lie sphere geometry . That geometry represents circles , lines and points in a unified way , as a five @-@ dimensional vector X
= (v , cx , cy , w , sr) , where c =
(cx , cy) is the center of the circle , and r is its (non @-@ negative) radius . If r is not zero , the sign s may be positive or negative ; for visualization , s represents the orientation of the circle , with counterclockwise circles having a positive s and clockwise circles having a negative s . The parameter w is zero for a straight line , and one otherwise .

In this five @-@ dimensional world , there is a bilinear product similar to the dot product :

<formula>