Tessellation

A tessellation of a flat surface is the tiling of a plane using one or more geometric shapes , called tiles , with no overlaps and no gaps . In mathematics , tessellations can be generalized to higher dimensions and a variety of geometries .

A periodic tiling has a repeating pattern . Some special kinds include regular tilings with regular polygonal tiles all of the same shape , and Semiregular tilings with regular tiles of more than one shape and with every corner identically arranged . The patterns formed by periodic tilings can be categorized into 17 wallpaper groups . A tiling that lacks a repeating pattern is called " non @-@ periodic " . An aperiodic tiling uses a small set of tile shapes that cannot form a repeating pattern . In the geometry of higher dimensions , a space @-@ filling or honeycomb is also called a tessellation of space .

A real physical tessellation is a tiling made of materials such as cemented ceramic squares or hexagons . Such tilings may be decorative patterns , or may have functions such as providing durable and water @-@ resistant pavement , floor or wall coverings . Historically , tessellations were used in Ancient Rome and in Islamic art such as in the decorative tiling of the Alhambra palace . In the twentieth century , the work of M. C. Escher often made use of tessellations , both in ordinary Euclidean geometry and in hyperbolic geometry , for artistic effect . Tessellations are sometimes employed for decorative effect in quilting . Tessellations form a class of patterns in nature , for example in the arrays of hexagonal cells found in honeycombs .

History

Tessellations were used by the Sumerians (about 4000 BC) in building wall decorations formed by patterns of clay tiles .

Decorative mosaic tilings made of small squared blocks called tesserae were widely employed in classical antiquity , sometimes displaying geometric patterns .

In 1619 Johannes Kepler made an early documented study of tessellations . He wrote about regular and semiregular tessellations in his Harmonices Mundi ; he was possibly the first to explore and to explain the hexagonal structures of honeycomb and snowflakes .

Some two hundred years later in 1891 , the Russian crystallographer Yevgraf Fyodorov proved that every periodic tiling of the plane features one of seventeen different groups of isometries . Fyodorov 's work marked the unofficial beginning of the mathematical study of tessellations . Other prominent contributors include Shubnikov and Belov (1964) , and Heinrich Heesch and Otto Kienzle (1963) .

Etymology

In Latin , tessella is a small cubical piece of clay , stone or glass used to make mosaics . The word " tessella " means " small square " (from tessera , square , which in turn is from the Greek word ??????? for four) . It corresponds with the everyday term tiling , which refers to applications of tessellations , often made of glazed clay .

Overview

Tessellation or tiling in two dimensions is a topic in geometry that studies how shapes , known as tiles , can be arranged to fill a plane without any gaps , according to a given set of rules . These rules can be varied . Common ones are that there must be no gaps between tiles , and that no corner of one tile can lie along the edge of another . The tessellations created by bonded brickwork do not obey this rule . Among those that do , a regular tessellation has both identical regular tiles and identical regular corners or vertices , having the same angle between adjacent edges for every tile . There are only three shapes that can form such regular tessellations : the equilateral triangle , square , and regular hexagon . Any one of these three shapes can be duplicated infinitely to fill a plane with no gaps .

Many other types of tessellation are possible under different constraints . For example , there are eight types of semi @-@ regular tessellation , made with more than one kind of regular polygon but still having the same arrangement of polygons at every corner . Irregular tessellations can also be made from other shapes such as pentagons , polyominoes and in fact almost any kind of geometric shape . The artist M. C. Escher is famous for making tessellations with irregular interlocking tiles , shaped like animals and other natural objects . If suitable contrasting colours are chosen for the tiles of differing shape , striking patterns are formed , and these can be used to decorate physical surfaces such as church floors .

More formally , a tessellation or tiling is a cover of the Euclidean plane by a countable number of closed sets , called tiles , such that the tiles intersect only on their boundaries . These tiles may be polygons or any other shapes . Many tessellations are formed from a finite number of prototiles in which all tiles in the tessellation are congruent to the given prototiles . If a geometric shape can be used as a prototile to create a tessellation , the shape is said to tessellate or to tile the plane . The Conway criterion is a sufficient but not necessary set of rules for deciding if a given shape tiles the plane periodically without reflections : some tiles fail the criterion but still tile the plane . No general rule has been found for determining if a given shape can tile the plane or not , which means there are many unsolved problems concerning tessellations . For example , the types of convex pentagon that can tile the plane remains an unsolved problem .

Mathematically , tessellations can be extended to spaces other than the Euclidean plane . The Swiss geometer Ludwig Schläfli pioneered this by defining polyschemes , which mathematicians nowadays call polytopes . These are the analogues to polygons and polyhedra in spaces with more dimensions . He further defined the Schläfli symbol notation to make it easy to describe polytopes . For example , the Schläfli symbol for an equilateral triangle is { 3 } , while that for a square is { 4 } . The Schläfli notation makes it possible to describe tilings compactly . For example , a tiling of regular hexagons has three six @-@ sided polygons at each vertex , so its Schläfli symbol is { 6 @,@ 3 } .

Other methods also exist for describing polygonal tilings . When the tessellation is made of regular polygons , the most common notation is the vertex configuration , which is simply a list of the number of sides of the polygons around a vertex . The square tiling has a vertex configuration of 4 @.@ 4 @.@ 4 @.@ 4 , or 44 . The tiling of regular hexagons is noted 6 @.@ 6 @.@ 6 , or 63 .

= = In mathematics = =

= = = Introduction to tessellations = = =

Mathematicians use some technical terms when discussing tilings . An edge is the intersection between two bordering tiles ; it is often a straight line . A vertex is the point of intersection of three or more bordering tiles . Using these terms , an isogonal or vertex @-@ transitive tiling is a tiling where every vertex point is identical ; that is , the arrangement of polygons about each vertex is the same . The fundamental region is a shape such as a rectangle that is repeated to form the tessellation . For example , a regular tessellation of the plane with squares has a meeting of four squares at every vertex .

The sides of the polygons are not necessarily identical to the edges of the tiles . An edge @-@ to @-@ edge tiling is any polygonal tessellation where adjacent tiles only share one full side , i.e. , no tile shares a partial side or more than one side with any other tile . In an edge @-@ to @-@ edge tiling , the sides of the polygons and the edges of the tiles are the same . The familiar " brick wall " tiling is not edge @-@ to @-@ edge because the long side of each rectangular brick is shared with two bordering bricks .

A normal tiling is a tessellation for which every tile is topologically equivalent to a disk , the intersection of any two tiles is a single connected set or the empty set , and all tiles are uniformly bounded . This means that a single circumscribing radius and a single inscribing radius can be used for all the tiles in the whole tiling ; the condition disallows tiles that are pathologically long or thin .

A monohedral tiling is a tessellation in which all tiles are congruent ; it has only one prototile . A particularly interesting type of monohedral tessellation is the spiral monohedral tiling . The first spiral monohedral tiling was discovered by Heinz Voderberg in 1936 ; the Voderberg tiling has a unit tile that is a nonconvex enneagon . The Hirschhorn tiling , published by Michael D. Hirschhorn and D. C. Hunt in 1985 , is a pentagon tiling using irregular pentagons : regular pentagons cannot tile the Euclidean plane as the internal angle of a regular pentagon , 3? / 5 , is not a divisor of 2? .
An isohedral tiling is a special variation of a monohedral tiling in which all tiles belong to the same transitivity class , that is , all tiles are transforms of the same prototile under the symmetry group of the tiling . If a prototile admits a tiling , but no such tiling is isohedral , then the prototile is called anisohedral and forms anisohedral tilings .
A regular tessellation is a highly symmetric , edge @-@ to @-@ edge tiling made up of regular polygons , all of the same shape . There are only three regular tessellations : those made up of equilateral triangles , squares , or regular hexagons . All three of these tilings are isogonal and monohedral .
A semi @-@ regular (or Archimedean) tessellation uses more than one type of regular polygon in an isogonal arrangement . There are eight semi @-@ regular tilings (or nine if the mirror @-@ image pair of tilings counts as two) . These can be described by their vertex configuration ; for example , a semi @-@ regular tiling using squares and regular octagons has the vertex configuration 4 @.@ 82 (each vertex has one square and two octagons) . Many non @-@ edge @-@ to @-@ edge tilings of the Euclidean plane are possible , including the family of Pythagorean tilings , tessellations that use two (parameterised) sizes of square , each square touching four squares of the other size .

= = = Wallpaper groups = = =

Tilings with translational symmetry in two independent directions can be categorized by wallpaper groups , of which 17 exist . It has been claimed that all seventeen of these groups are represented in the Alhambra palace in Granada , Spain . Though this is disputed , the variety and sophistication of the Alhambra tilings have surprised modern researchers . Of the three regular tilings two are in the p6m wallpaper group and one is in p4m . Tilings in 2D with translational symmetry in just one direction can be categorized by the seven frieze groups describing the possible frieze patterns . Orbifold notation can be used to describe wallpaper groups of the Euclidean plane .

= = = Aperiodic tilings = = =

Penrose tilings , which use two different quadrilaterals , are the best known example of tiles that forcibly create non @-@ periodic patterns . They belong to a general class of aperiodic tilings , which use tiles that cannot tessellate periodically . The recursive process of substitution tiling is a method of generating aperiodic tilings . One class that can be generated in this way is the rep @-@ tiles ; these tilings have surprising self @-@ replicating properties . Pinwheel tilings are non @-@ periodic , using a rep @-@ tile construction ; the tiles appear in infinitely many orientations . It might be thought that a non @-@ periodic pattern would be entirely without symmetry , but this is not so . Aperiodic tilings , while lacking in translational symmetry , do have symmetries of other types , by infinite repetition of any bounded patch of the tiling and in certain finite groups of rotations or reflections of those patches . A substitution rule , such as can be used to generate some Penrose patterns using assemblies of tiles called rhombs , illustrates scaling symmetry . A Fibonacci word can be used to build an aperiodic tiling , and to study quasicrystals , which are structures with aperiodic order .
Wang tiles are squares coloured on each edge , and placed so that abutting edges of adjacent tiles have the same colour ; hence they are sometimes called Wang dominoes . A suitable set of Wang dominoes can tile the plane , but only aperiodically . This is known because any Turing machine can be represented as a set of Wang dominoes that tile the plane if and only if the Turing machine does not halt . Since the halting problem is undecidable , the problem of deciding whether a Wang

domino set can tile the plane is also undecidable .
Truchet tiles are square tiles decorated with patterns so they do not have rotational symmetry ; in 1704 , Sébastien Truchet used a square tile split into two triangles of contrasting colours . These can tile the plane either periodically or randomly .

= = = Tessellations and colour = = =

Sometimes the colour of a tile is understood as part of the tiling ; at other times arbitrary colours may be applied later . When discussing a tiling that is displayed in colours , to avoid ambiguity one needs to specify whether the colours are part of the tiling or just part of its illustration . This affects whether tiles with the same shape but different colours are considered identical , which in turn affects questions of symmetry . The four colour theorem states that for every tessellation of a normal Euclidean plane , with a set of four available colours , each tile can be coloured in one colour such that no tiles of equal colour meet at a curve of positive length . The colouring guaranteed by the four @-@ colour theorem does not generally respect the symmetries of the tessellation . To produce a colouring which does , it is necessary to treat the colours as part of the tessellation . Here , as many as seven colours may be needed , as in the picture at right .

= = = Tessellations with polygons = = =

Next to the various tilings by regular polygons , tilings by other polygons have also been studied .
Any triangle or quadrilateral (even non @-@ convex) can be used as a prototile to form a monohedral tessellation , often in more than one way . Copies of an arbitrary quadrilateral can form a tessellation with translational symmetry and 2 @-@ fold rotational symmetry with centres at the midpoints of all sides . For an asymmetric quadrilateral this tiling belongs to wallpaper group p2 . As fundamental domain we have the quadrilateral . Equivalently , we can construct a parallelogram subtended by a minimal set of translation vectors , starting from a rotational centre . We can divide this by one diagonal , and take one half (a triangle) as fundamental domain . Such a triangle has the same area as the quadrilateral and can be constructed from it by cutting and pasting .