= rn ? 3 and , hence , C (r) is proportional to rn . The formula above indicates that the angular motion is multiplied by a factor k =
1 / ? n , so that the apsidal angle ? equals 180 ° / ? n .
This angular scaling can be seen in the apsidal precession , i.e. , in the gradual rotation of the long axis of the ellipse (Figure 3) . As noted above , the orbit as a whole rotates with a mean angular speed ? = (k ? 1) ? , where ? equals the mean angular speed of the particle about the stationary ellipse . If the particle requires a time T to move from one apse to the other , this implies that , in the same time , the long axis will rotate by an angle ?
= ?T =
(k ? 1) ?T
= (k ? 1) × 180 ° . For an inverse @-@ square law such as Newton 's law of universal gravitation , where n equals 1 , there is no angular scaling (k =
1) , the apsidal angle ? is 180 ° , and the elliptical orbit is stationary (?
= ? =
0) .
As a final illustration , Newton considers a sum of two power laws
<formula>
which multiplies the angular speed by a factor
<formula>
Newton applies both of these formulae (the power law and sum of two power laws) to examine the apsidal precession of the Moon 's orbit .

= = Precession of the Moon 's orbit = =

The motion of the Moon can be measured accurately , and is noticeably more complex than that of the planets . The ancient Greek astronomers , Hipparchus and Ptolemy , had noted several periodic variations in the Moon 's orbit , such as small oscillations in its orbital eccentricity and the inclination of its orbit to the plane of the ecliptic . These oscillations generally occur on a once @-@ monthly or twice @-@ monthly time @-@ scale . The line of its apses precesses gradually with a period of roughly 8 @.@ 85 years , while its line of nodes turns a full circle in roughly double that time , 18 @.@ 6 years . This accounts for the roughly 18 @-@ year periodicity of eclipses , the so @-@ called Saros cycle . However , both lines experience small fluctuations in their motion , again on the monthly time @-@ scale .
In 1673 , Jeremiah Horrocks published a reasonably accurate model of the Moon 's motion in which the Moon was assumed to follow a precessing elliptical orbit . A sufficiently accurate and simple method for predicting the Moon 's motion would have solved the navigational problem of determining a ship 's longitude ; in Newton 's time , the goal was to predict the Moon 's position to 2 ' (two arc @-@ minutes) , which would correspond to a 1 ° error in terrestrial longitude . Horrocks ' model predicted the lunar position with errors no more than 10 arc @-@ minutes ; for comparison , the diameter of the Moon is roughly 30 arc @-@ minutes .
Newton used his theorem of revolving orbits in two ways to account for the apsidal precession of the Moon . First , he showed that the Moon 's observed apsidal precession could be accounted for by changing the force law of gravity from an inverse @-@ square law to a power law in which the exponent was 2 + 4 / 243 (roughly 2 @.@ 0165)
<formula>
In 1894 , Asaph Hall adopted this approach of modifying the exponent in the inverse @-@ square law slightly to explain an anomalous orbital precession of the planet Mercury , which had been observed in 1859 by Urbain Le Verrier . Ironically , Hall 's theory was ruled out by careful astronomical observations of the Moon . The currently accepted explanation for this precession involves the theory of general relativity , which (to first approximation) adds an inverse @-@ quartic force , i.e. , one that varies as the inverse fourth power of distance .
As a second approach to explaining the Moon 's precession , Newton suggested that the perturbing influence of the Sun on the Moon 's motion might be approximately equivalent to an additional linear

force
<formula>
The first term corresponds to the gravitational attraction between the Moon and the Earth , where r is the Moon 's distance from the Earth . The second term , so Newton reasoned , might represent the average perturbing force of the Sun 's gravity of the Earth @-@ Moon system . Such a force law could also result if the Earth were surrounded by a spherical dust cloud of uniform density . Using the formula for k for nearly circular orbits , and estimates of A and B , Newton showed that this force law could not account for the Moon 's precession , since the predicted apsidal angle ? was (? 180 @.@ 76 °) rather than the observed ? (? 181 @.@ 525 °) . For every revolution , the long axis would rotate 1 @.@ 5 ° , roughly half of the observed 3 @.@ 0 °

= = Generalization = =

Isaac Newton first published his theorem in 1687 , as Propositions 43 ? 45 of Book I of his Philosophiæ Naturalis Principia Mathematica . However , as astrophysicist Subrahmanyan Chandrasekhar noted in his 1995 commentary on Newton 's Principia , the theorem remained largely unknown and undeveloped for over three centuries .