Pseudoforest

In graph theory , a pseudoforest is an undirected graph in which every connected component has at most one cycle . That is , it is a system of vertices and edges connecting pairs of vertices , such that no two cycles of consecutive edges share any vertex with each other , nor can any two cycles be connected to each other by a path of consecutive edges . A pseudotree is a connected pseudoforest .

The names are justified by analogy to the more commonly studied trees and forests . (A tree is a connected graph with no cycles ; a forest is a disjoint union of trees .) Gabow and Tarjan attribute the study of pseudoforests to Dantzig 's 1963 book on linear programming , in which pseudoforests arise in the solution of certain network flow problems . Pseudoforests also form graph @-@ theoretic models of functions and occur in several algorithmic problems . Pseudoforests are sparse graphs ? they have very few edges relative to their number of vertices ? and their matroid structure allows several other families of sparse graphs to be decomposed as unions of forests and pseudoforests . The name " pseudoforest " comes from Picard & Queyranne (1982) .

Definitions and structure

We define an undirected graph to be a set of vertices and edges such that each edge has two vertices (which may coincide) as endpoints . That is , we allow multiple edges (edges with the same pair of endpoints) and loops (edges whose two endpoints are the same vertex) . A subgraph of a graph is the graph formed by any subsets of its vertices and edges such that each edge in the edge subset has both endpoints in the vertex subset . A connected component of an undirected graph is the subgraph consisting of the vertices and edges that can be reached by following edges from a single given starting vertex . A graph is connected if every vertex or edge is reachable from every other vertex or edge . A cycle in an undirected graph is a connected subgraph in which each vertex is incident to exactly two edges , or is a loop .

A pseudoforest is an undirected graph in which each connected component contains at most one cycle . Equivalently , it is an undirected graph in which each connected component has no more edges than vertices . The components that have no cycles are just trees , while the components that have a single cycle within them are called 1 @-@ trees or unicyclic graphs . That is , a 1 @-@ tree is a connected graph containing exactly one cycle . A pseudoforest with a single connected component (usually called a pseudotree , although some authors define a pseudotree to be a 1 @-@ tree) is either a tree or a 1 @-@ tree ; in general a pseudoforest may have multiple connected components as long as all of them are trees or 1 @-@ trees .

If one removes from a 1 @-@ tree one of the edges in its cycle , the result is a tree . Reversing this process , if one augments a tree by connecting any two of its vertices by a new edge , the result is a 1 @-@ tree ; the path in the tree connecting the two endpoints of the added edge , together with the added edge itself , form the 1 @-@ tree 's unique cycle . If one augments a 1 @-@ tree by adding an edge that connects one of its vertices to a newly added vertex , the result is again a 1 @-@ tree , with one more vertex ; an alternative method for constructing 1 @-@ trees is to start with a single cycle and then repeat this augmentation operation any number of times . The edges of any 1 @-@ tree can be partitioned in a unique way into two subgraphs , one of which is a cycle and the other of which is a forest , such that each tree of the forest contains exactly one vertex of the cycle .

Certain more specific types of pseudoforests have also been studied .

A 1 @-@ forest , sometimes called a maximal pseudoforest , is a pseudoforest to which no more edges can be added without causing some component of the graph to contain multiple cycles . If a pseudoforest contains a tree as one of its components , it cannot be a 1 @-@ forest , for one can add either an edge connecting two vertices within that tree , forming a single cycle , or an edge connecting that tree to some other component . Thus , the 1 @-@ forests are exactly the pseudoforests in which every component is a 1 @-@ tree .

The spanning pseudoforests of an undirected graph G are the pseudoforest subgraphs of G that have all the vertices of G. Such a pseudoforest need not have any edges , since for example the

subgraph that has all the vertices of G and no edges is a pseudoforest (whose components are trees consisting of a single vertex) .

The maximal pseudoforests of G are the pseudoforest subgraphs of G that are not contained within any larger pseudoforest of G. A maximal pseudoforest of G is always a spanning pseudoforest , but not conversely . If G has no connected components that are trees , then its maximal pseudoforests are 1 @-@ forests , but if G does have a tree component , its maximal pseudoforests are not 1 @-@ forests . Stated precisely , in any graph G its maximal pseudoforests consist of every tree component of G , together with one or more disjoint 1 @-@ trees covering the remaining vertices of G.

== Directed pseudoforests ==

Versions of these definitions are also used for directed graphs . Like an undirected graph , a directed graph consists of vertices and edges , but each edge is directed from one of its endpoints to the other endpoint . A directed pseudoforest is a directed graph in which each vertex has at most one outgoing edge ; that is , it has outdegree at most one . A directed 1 @-@ forest ? most commonly called a functional graph (see below) , sometimes maximal directed pseudoforest ? is a directed graph in which each vertex has outdegree exactly one . If D is a directed pseudoforest , the undirected graph formed by removing the direction from each edge of D is an undirected pseudoforest .

== Number of edges ==

Every pseudoforest on a set of n vertices has at most n edges , and every maximal pseudoforest on a set of n vertices has exactly n edges . Conversely , if a graph G has the property that , for every subset S of its vertices , the number of edges in the induced subgraph of S is at most the number of vertices in S , then G is a pseudoforest . 1 @-@ trees can be defined as connected graphs with equally many vertices and edges .

Moving from individual graphs to graph families , if a family of graphs has the property that every subgraph of a graph in the family is also in the family , and every graph in the family has at most as many edges as vertices , then the family contains only pseudoforests . For instance , every subgraph of a thrackle (a graph drawn so that every pair of edges has one point of intersection) is also a thrackle , so Conway 's conjecture that every thrackle has at most as many edges as vertices can be restated as saying that every thrackle is a pseudoforest . A more precise characterization is that , if the conjecture is true , then the thrackles are exactly the pseudoforests with no four @-@ vertex cycle and at most one odd cycle .

Streinu and Theran generalize the sparsity conditions defining pseudoforests : they define a graph as being (k , l) -sparse if every nonempty subgraph with n vertices has at most kn ? l edges , and (k , l) -tight if it is (k , l) -sparse and has exactly kn ? l edges . Thus , the pseudoforests are the (1 @,@ 0) -sparse graphs , and the maximal pseudoforests are the (1 @,@ 0) -tight graphs . Several other important families of graphs may be defined from other values of k and l , and when l ? k the (k , l) -sparse graphs may be characterized as the graphs formed as the edge @-@ disjoint union of l forests and k ? l pseudoforests .

Almost every sufficiently sparse random graph is pseudoforest . That is , if c is a constant with 0 < c < 1 / 2 , and Pc (n) is the probability that choosing uniformly at random among the n @-@ vertex graphs with cn edges results in a pseudoforest , then Pc (n) tends to one in the limit for large n . However , for c > 1 / 2 , almost every random graph with cn edges has a large component that is not unicyclic .

== Enumeration ==

A graph is simple if it has no self @-@ loops and no multiple edges with the same endpoints . The number of simple 1 @-@ trees with n labelled vertices is

The values for n up to 18 can be found in sequence A057500 of the On @-@ Line Encyclopedia of Integer Sequences .

The number of maximal directed pseudoforests on n vertices , allowing self @-@ loops , is nn , because for each vertex there are n possible endpoints for the outgoing edge . André Joyal used this fact to provide a bijective proof of Cayley 's formula , that the number of undirected trees on n nodes is nn ? 2 , by finding a bijection between maximal directed pseudoforests and undirected trees with two distinguished nodes . If self @-@ loops are not allowed , the number of maximal directed pseudoforests is instead (n ? 1) n .

= = Graphs of functions = =