= r3 + rs , respectively . Therefore , differences in these distances are constants , such as d1 ? d2 =

r1 ? r2 ; they depend only on the known radii of the given circles and not on the radius rs of the solution circle , which cancels out . This second formulation of Apollonius ' problem can be generalized to internally tangent solution circles (for which the center @-@ center distance equals the difference of radii) , by changing the corresponding differences of distances to sums of distances , so that the solution @-@ circle radius rs again cancels out . The re @-@ formulation in terms of center @-@ center distances is useful in the solutions below of Adriaan van Roomen and Isaac Newton , and also in hyperbolic positioning or trilateration , which is the task of locating a position from differences in distances to three known points . For example , navigation systems such as LORAN identify a receiver 's position from the differences in arrival times of signals from three fixed positions , which correspond to the differences in distances to those transmitters .

= = History = =

A rich repertoire of geometrical and algebraic methods have been developed to solve Apollonius ' problem , which has been called " the most famous of all " geometry problems . The original approach of Apollonius of Perga has been lost , but reconstructions have been offered by François Viète and others , based on the clues in the description by Pappus . The first new solution method was published in 1596 by Adriaan van Roomen , who identified the centers of the solution circles as the intersection points of two hyperbolas . Van Roomen 's method was refined in 1687 by Isaac Newton in his Principia , and by John Casey in 1881 .
Although successful in solving Apollonius ' problem , van Roomen 's method has a drawback . A prized property in classical Euclidean geometry is the ability to solve problems using only a compass and a straightedge . Many constructions are impossible using only these tools , such as dividing an angle in three equal parts . However , many such " impossible " problems can be solved by intersecting curves such as hyperbolas , ellipses and parabolas (conic sections) . For example , doubling the cube (the problem of constructing a cube of twice the volume of a given cube) cannot be done using only a straightedge and compass , but Menaechmus showed that the problem can be solved by using the intersections of two parabolas . Therefore , van Roomen 's solution ? which uses the intersection of two hyperbolas ? did not determine if the problem satisfied the straightedge @-@ and @-@ compass property .
Van Roomen 's friend François Viète , who had urged van Roomen to work on Apollonius ' problem in the first place , developed a method that used only compass and straightedge . Prior to Viète 's solution , Regiomontanus doubted whether Apollonius ' problem could be solved by straightedge and compass . Viète first solved some simple special cases of Apollonius ' problem , such as finding a circle that passes through three given points which has only one solution if the points are distinct ; he then built up to solving more complicated special cases , in some cases by shrinking or swelling the given circles . According to the 4th @-@ century report of Pappus of Alexandria , Apollonius ' own book on this problem ? entitled ?????? (Epaphaí , " Tangencies " ; Latin : De tactionibus , De contactibus) ? followed a similar progressive approach . Hence , Viète 's solution is considered to be a plausible reconstruction of Apollonius ' solution , although other reconstructions have been published independently by three different authors .
Several other geometrical solutions to Apollonius ' problem were developed in the 19th century . The most notable solutions are those of Jean @-@ Victor Poncelet (1811) and of Joseph Diaz Gergonne (1814) . Whereas Poncelet 's proof relies on homothetic centers of circles and the power of a point theorem , Gergonne 's method exploits the conjugate relation between lines and their poles in a circle . Methods using circle inversion were pioneered by Julius Petersen in 1879 ; one example is the annular solution method of HSM Coxeter . Another approach uses Lie sphere geometry , which was developed by Sophus Lie .
Algebraic solutions to Apollonius ' problem were pioneered in the 17th century by René Descartes and Princess Elisabeth of Bohemia , although their solutions were rather complex . Practical algebraic methods were developed in the late 18th and 19th centuries by several mathematicians ,

including Leonhard Euler , Nicolas Fuss , Carl Friedrich Gauss , Lazare Carnot , and Augustin Louis Cauchy .

= = Solution methods = =

= = = Intersecting hyperbolas = = =

The solution of Adriaan van Roomen (1596) is based on the intersection of two hyperbolas . Let the given circles be denoted as C1 , C2 and C3 . Van Roomen solved the general problem by solving a simpler problem , that of finding the circles that are tangent to two given circles , such as C1 and C2 . He noted that the center of a circle tangent to both given circles must lie on a hyperbola whose foci are the centers of the given circles . To understand this , let the radii of the solution circle and the two given circles be denoted as rs , r1 and r2 , respectively (Figure 3) . The distance d1 between the centers of the solution circle and C1 is either rs + r1 or rs ? r1 , depending on whether these circles are chosen to be externally or internally tangent , respectively . Similarly , the distance d2 between the centers of the solution circle and C2 is either rs + r2 or rs ? r2 , again depending on their chosen tangency . Thus , the difference d1 ? d2 between these distances is always a constant that is independent of rs . This property , of having a fixed difference between the distances to the foci , characterizes hyperbolas , so the possible centers of the solution circle lie on a hyperbola . A second hyperbola can be drawn for the pair of given circles C2 and C3 , where the internal or external tangency of the solution and C2 should be chosen consistently with that of the first hyperbola . An intersection of these two hyperbolas (if any) gives the center of a solution circle that has the chosen internal and external tangencies to the three given circles . The full set of solutions to Apollonius ' problem can be found by considering all possible combinations of internal and external tangency of the solution circle to the three given circles .

Isaac Newton (1687) refined van Roomen 's solution , so that the solution @-@ circle centers were located at the intersections of a line with a circle . Newton formulates Apollonius ' problem as a problem in trilateration : to locate a point Z from three given points A , B and C , such that the differences in distances from Z to the three given points have known values . These four points correspond to the center of the solution circle (Z) and the centers of the three given circles (A , B and C) .

Instead of solving for the two hyperbolas , Newton constructs their directrix lines instead . For any hyperbola , the ratio of distances from a point Z to a focus A and to the directrix is a fixed constant called the eccentricity . The two directrices intersect at a point T , and from their two known distance ratios , Newton constructs a line passing through T on which Z must lie . However , the ratio of distances TZ / TA is also known ; hence , Z also lies on a known circle , since Apollonius had shown that a circle can be defined as the set of points that have a given ratio of distances to two fixed points . (As an aside , this definition is the basis of bipolar coordinates .) Thus , the solutions to Apollonius ' problem are the intersections of a line with a circle .

= = = Viète 's reconstruction = = =

As described below , Apollonius ' problem has ten special cases , depending on the nature of the three given objects , which may be a circle (C) , line (L) or point (P) . By custom , these ten cases are distinguished by three letter codes such as CCP . Viète solved all ten of these cases using only compass and straightedge constructions , and used the solutions of simpler cases to solve the more complex cases .

Viète began by solving the PPP case (three points) following the method of Euclid in his Elements . From this , he derived a lemma corresponding to the power of a point theorem , which he used to solve the LPP case (a line and two points) . Following Euclid a second time , Viète solved the LLL case (three lines) using the angle bisectors . He then derived a lemma for constructing the line perpendicular to an angle bisector that passes through a point , which he used to solve the LLP problem (two lines and a point) . This accounts for the first four cases of Apollonius ' problem , those that do not involve circles .

To solve the remaining problems , Viète exploited the fact that the given circles and the solution circle may be re @-@ sized in tandem while preserving their tangencies (Figure 4) . If the solution @-@ circle radius is changed by an amount ?r , the radius of its internally tangent given circles must be likewise changed by ?r , whereas the radius of its externally tangent given circles must be changed by ? ?r . Thus , as the solution circle swells , the internally tangent given circles must swell in tandem , whereas the externally tangent given circles must shrink , to maintain their tangencies .

Viète used this approach to shrink one of the given circles to a point , thus reducing the problem to a simpler , already solved case . He first solved the CLL case (a circle and two lines) by shrinking the circle into a point , rendering it a LLP case . He then solved the CLP case (a circle , a line and a point) using three lemmas . Again shrinking one circle to a point , Viète transformed the CCL case into a CLP case . He then solved the CPP case (a circle and two points) and the CCP case (two circles and a point) , the latter case by two lemmas . Finally , Viète solved the general CCC case (three circles) by shrinking one circle to a point , rendering it a CCP case .

= = = Algebraic solutions = = =

Apollonius ' problem can be framed as a system of three equations for the center and radius of the solution circle . Since the three given circles and any solution circle must lie in the same plane , their positions can be specified in terms of the (x , y) coordinates of their centers . For example , the center positions of the three given circles may be written as (x1 , y1) , (x2 , y2) and (x3 , y3) , whereas that of a solution circle can be written as (xs , ys) . Similarly , the radii of the given circles and a solution circle can be written as r1 , r2 , r3 and rs , respectively . The requirement that a solution circle must exactly touch each of the three given circles can be expressed as three coupled quadratic equations for xs , ys and rs :

<formula>

<formula>

<formula>