AFRICAN INSTITUTE OF MATHEMATICAL SCIENCES AIMS-RWANDA

NUMERICAL LINEAR ALGEBRA

Dr Franck Kalala Mutombo

Professor of Mathematics

University of Lubumbashi

franck.mutombo@aims.ac.rw

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AFRICAN INSTITUTE OF MATHEMATICAL SCIENCES AIMS-RWANDA

APPLICATIONS EXAMPLES OF LINEAR ALGEBRA METHODS IN VARIOUS FIELDS

Digital Image Processing and Computer Vision

- **Singular Value Decomposition (SVD)**: Used in image compression. By applying SVD to an image matrix, it is possible to represent the image with fewer singular values, reducing storage requirements while maintaining visual quality.
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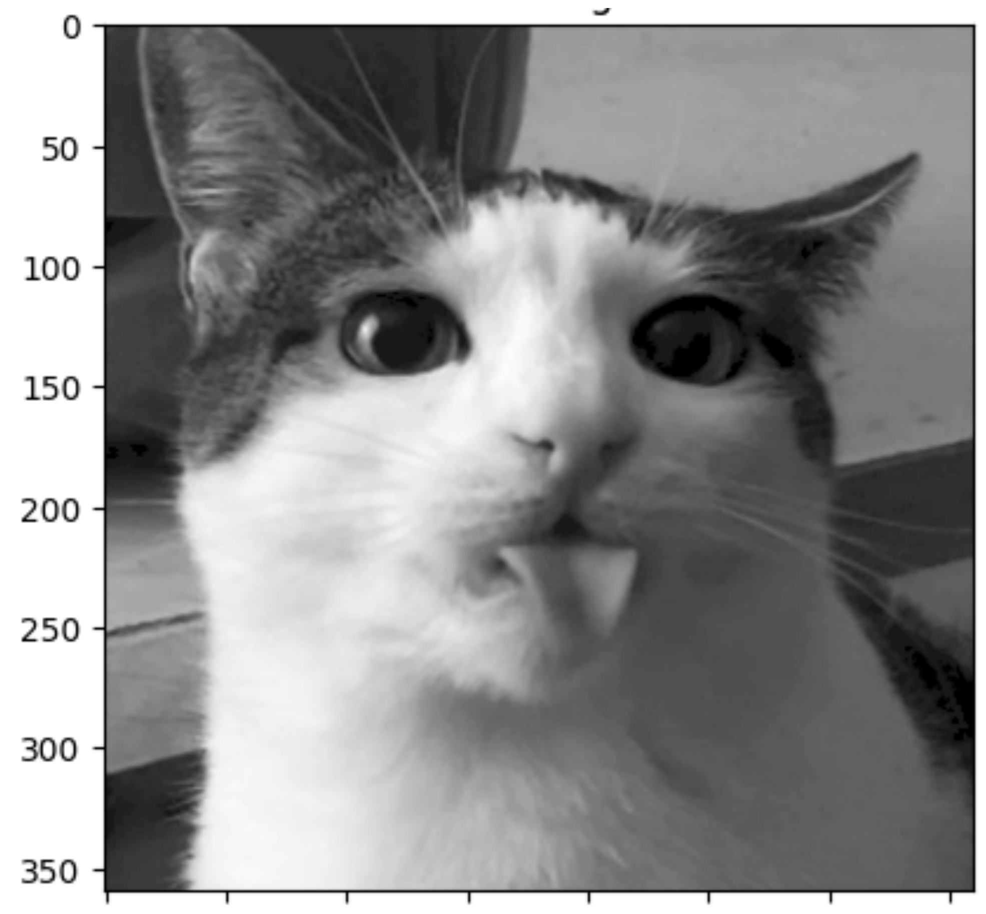
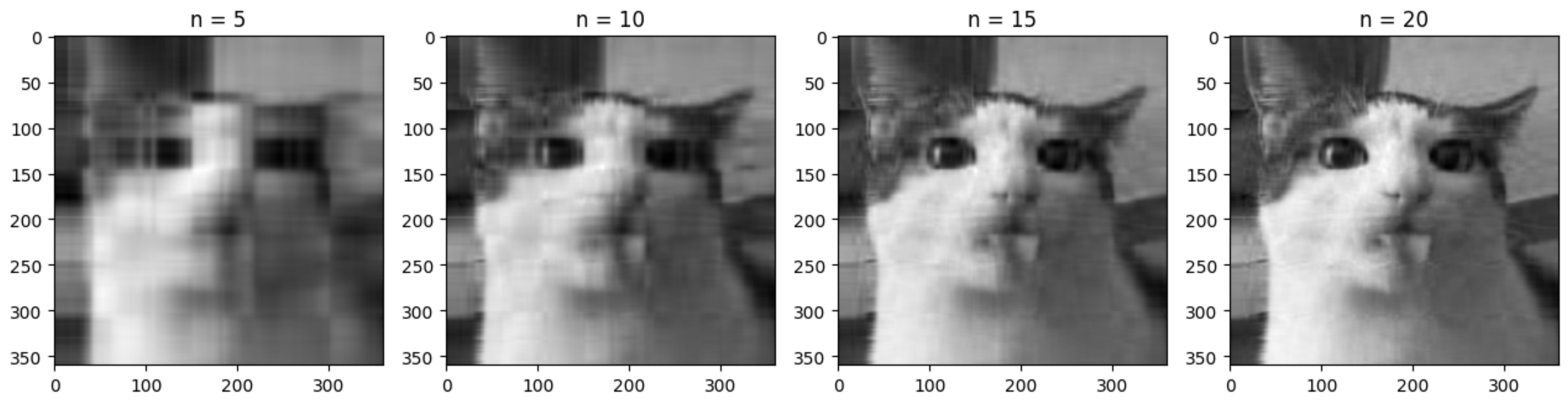


Image of a cat

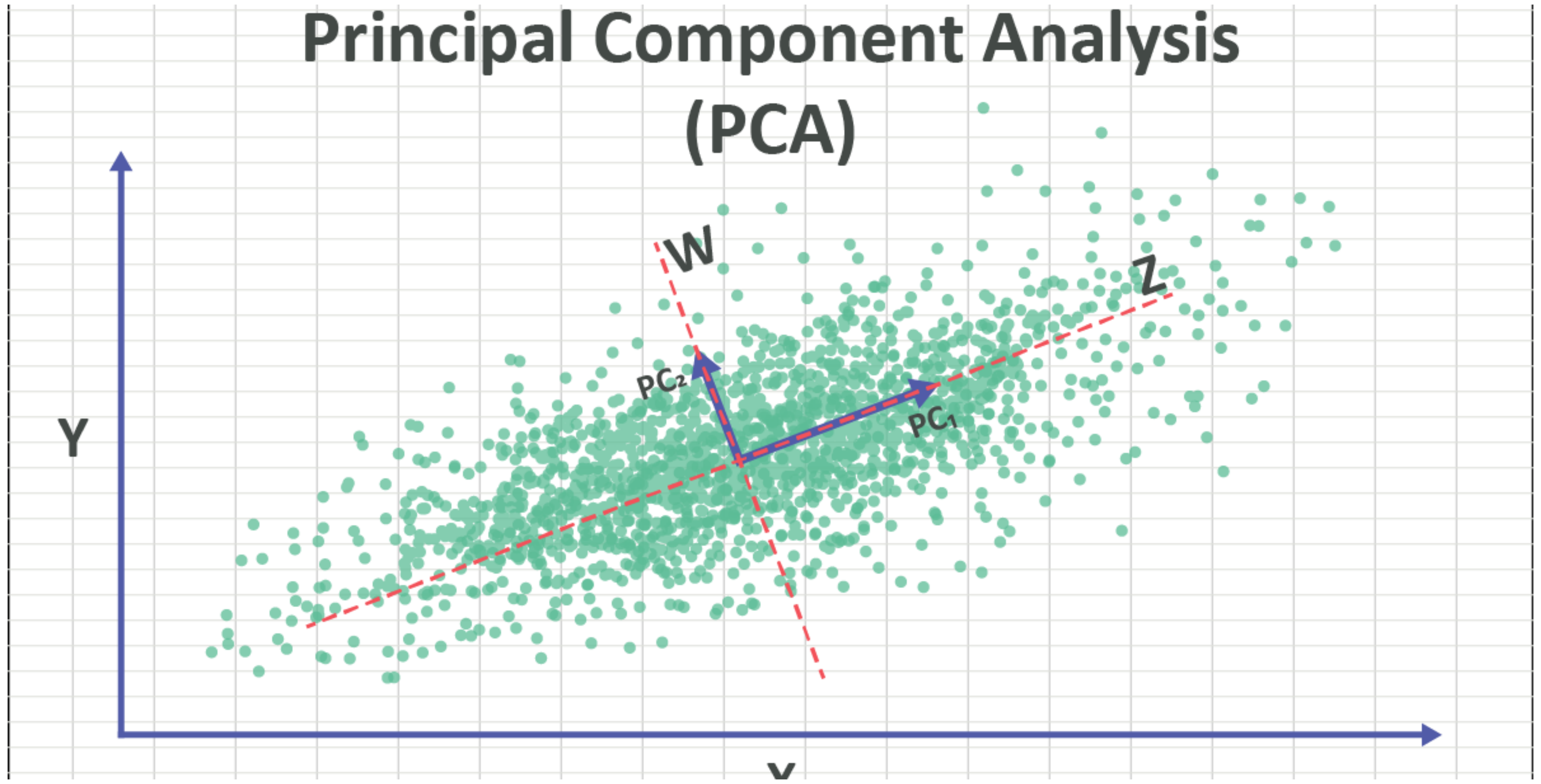


Compression using SVD

https://dmicz.github.io/machine-learning/svd-image-compression/

Digital Image Processing and Computer Vision

- **PCA (Principal Component Analysis)**: Linear algebra-based method used for dimensionality reduction, feature extraction, and noise reduction in image processing.

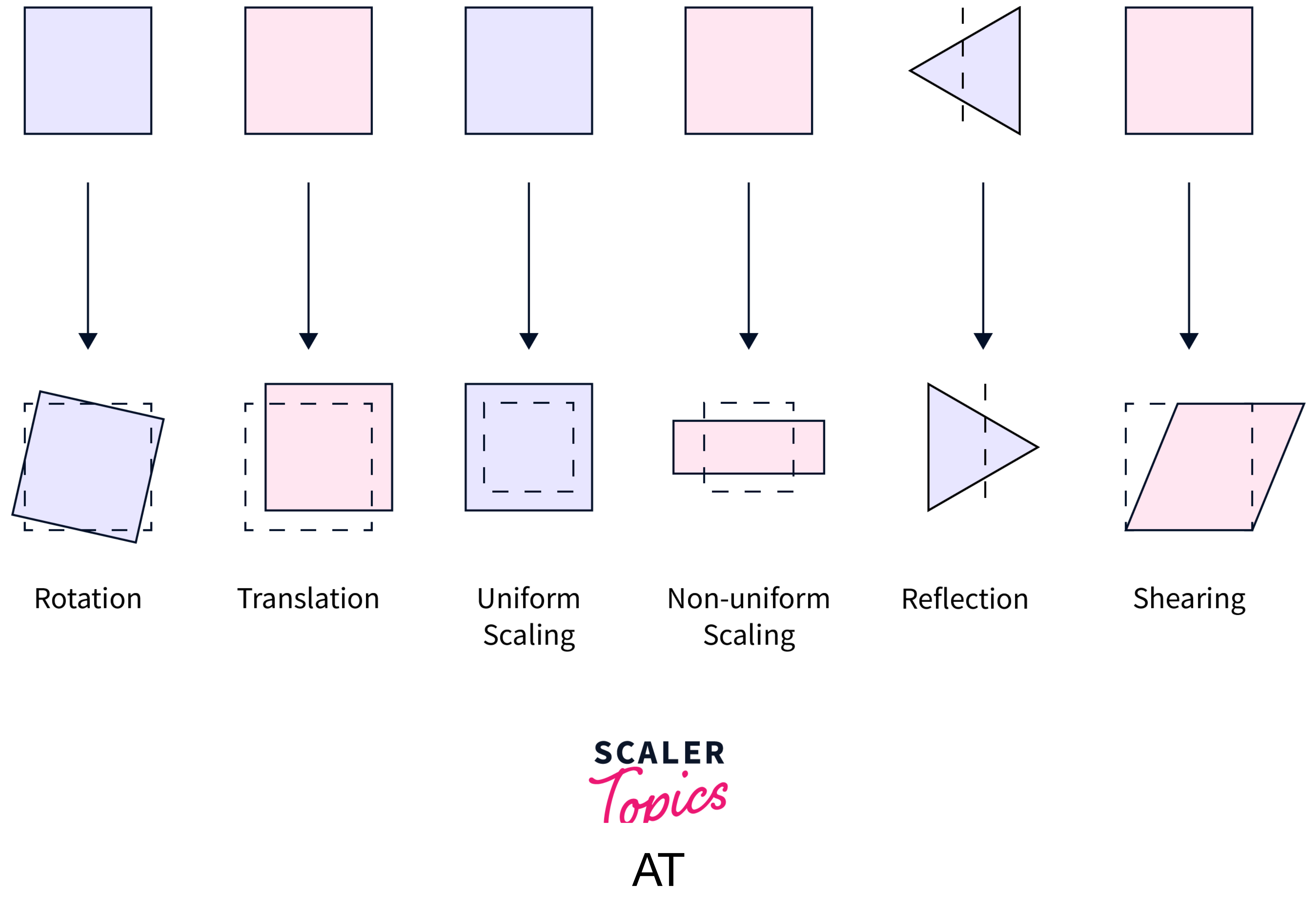


PCA

https://rasbt.github.io/mlxtend/user_guide/feature_extraction/PrincipalComponentAnalysis/

Digital Image Processing and Computer Vision

- **Affine Transformations**: In computer vision, linear algebra is used to describe image transformations such as rotations, scaling, and translations. These transformations are represented by matrices applied to pixel coordinates.



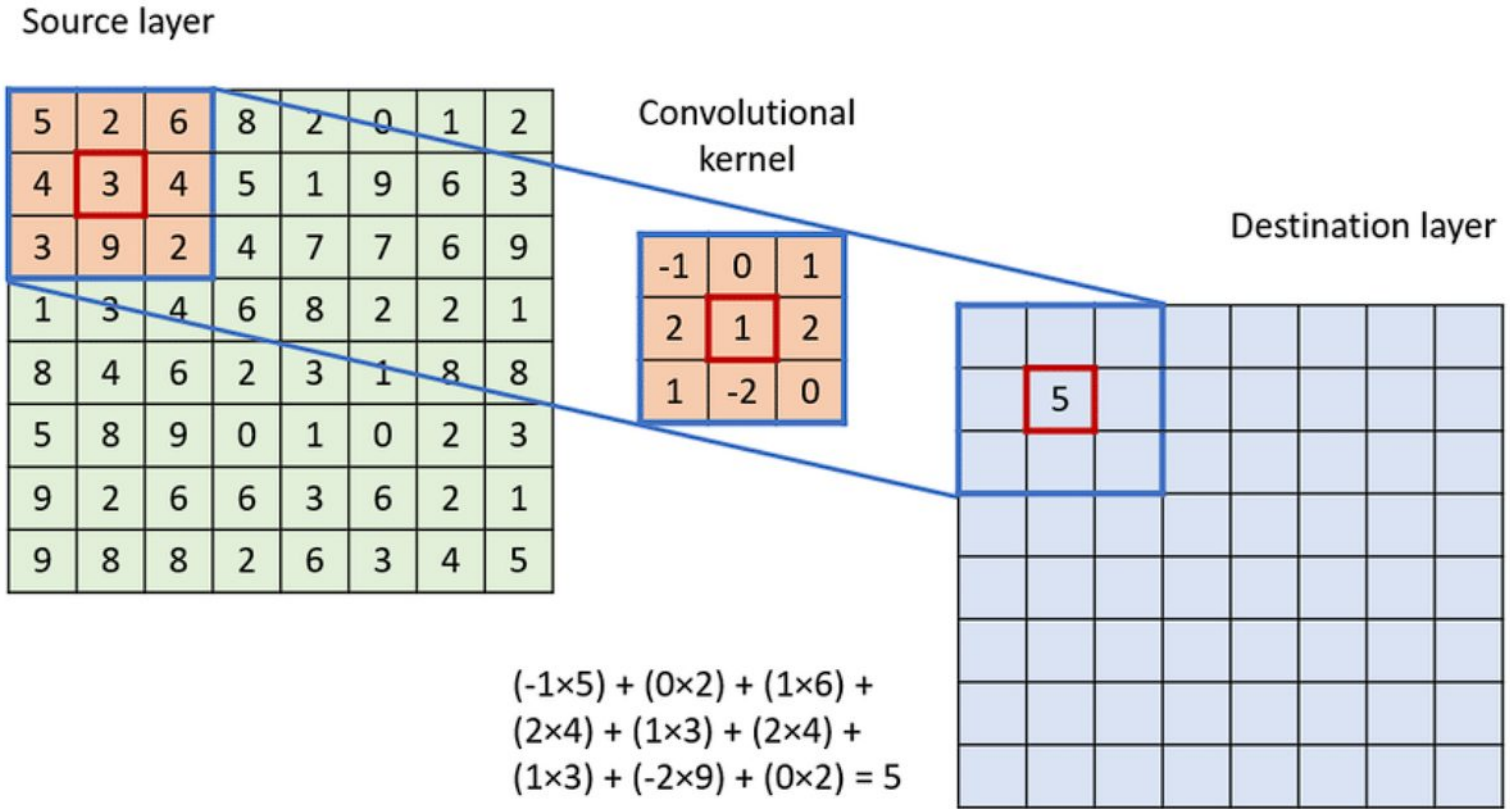
Digital Signal Processing and Communication Science

- **Fourier Transform (FT)**: Linear algebra plays a role in the Discrete Fourier Transform (DFT), which is used to analyze the frequency content of digital signals.

- The Fast Fourier Transform (FFT), an optimized DFT algorithm, is based on matrix factorization.

Digital Signal Processing and Communication Science

- **Convolution Operations**: Convolution, a fundamental operation in signal processing and communications, can be expressed as matrix multiplication, making linear algebra techniques essential for understanding filtering and signal reconstruction.

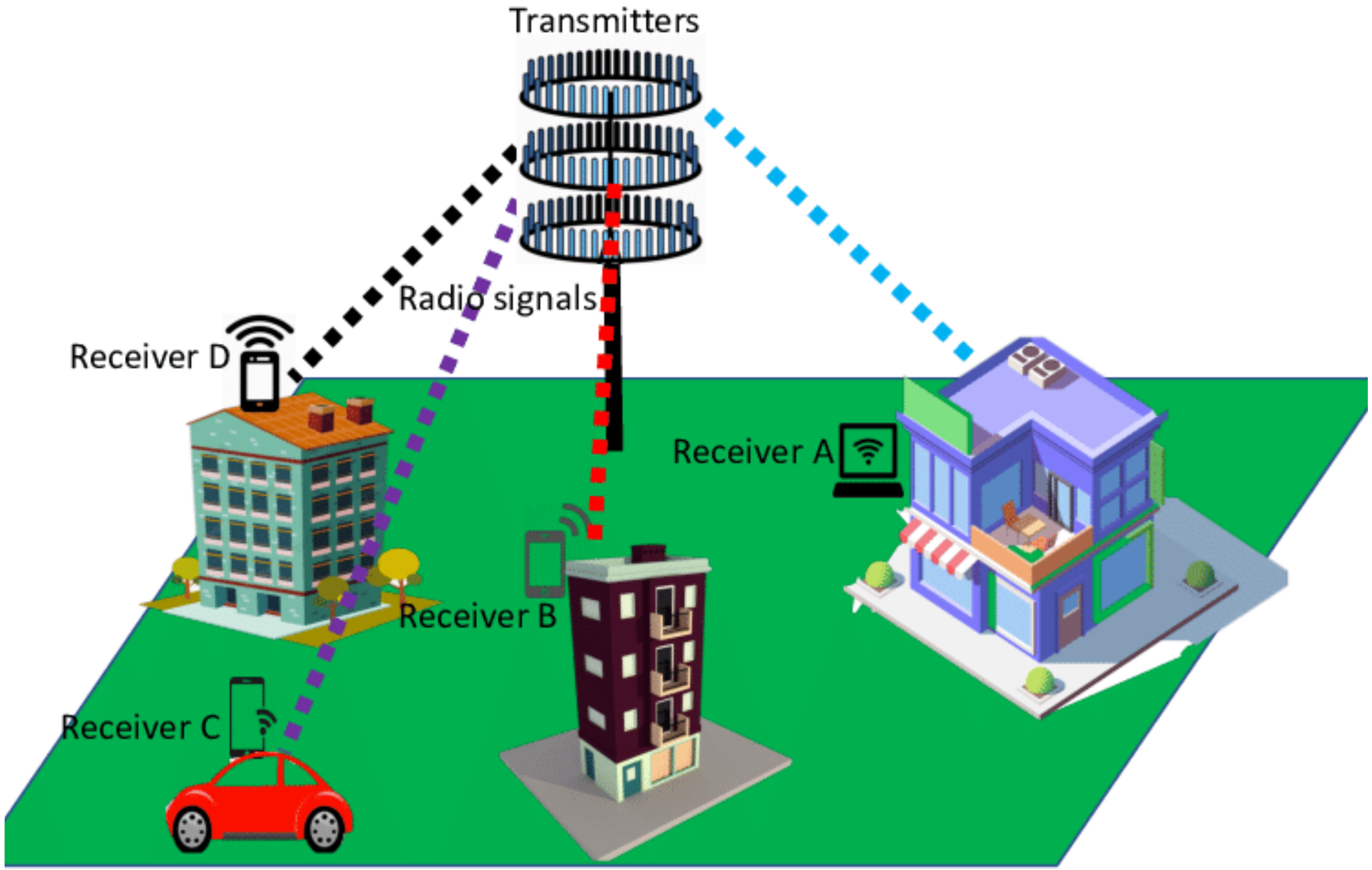


Convolution Illustration

https://medium.com/@bdhuma/6-basic-things-to-know-about-convolution-daef5e1bc411

Digital Signal Processing and Communication Science

MIMO Systems (Multiple Input, Multiple Output): In modern communication systems, linear algebra is used to model and solve the transmission and reception of signals in multi-antenna systems. Techniques like matrix diagonalization and eigenvalue decomposition help optimize signal transmission.

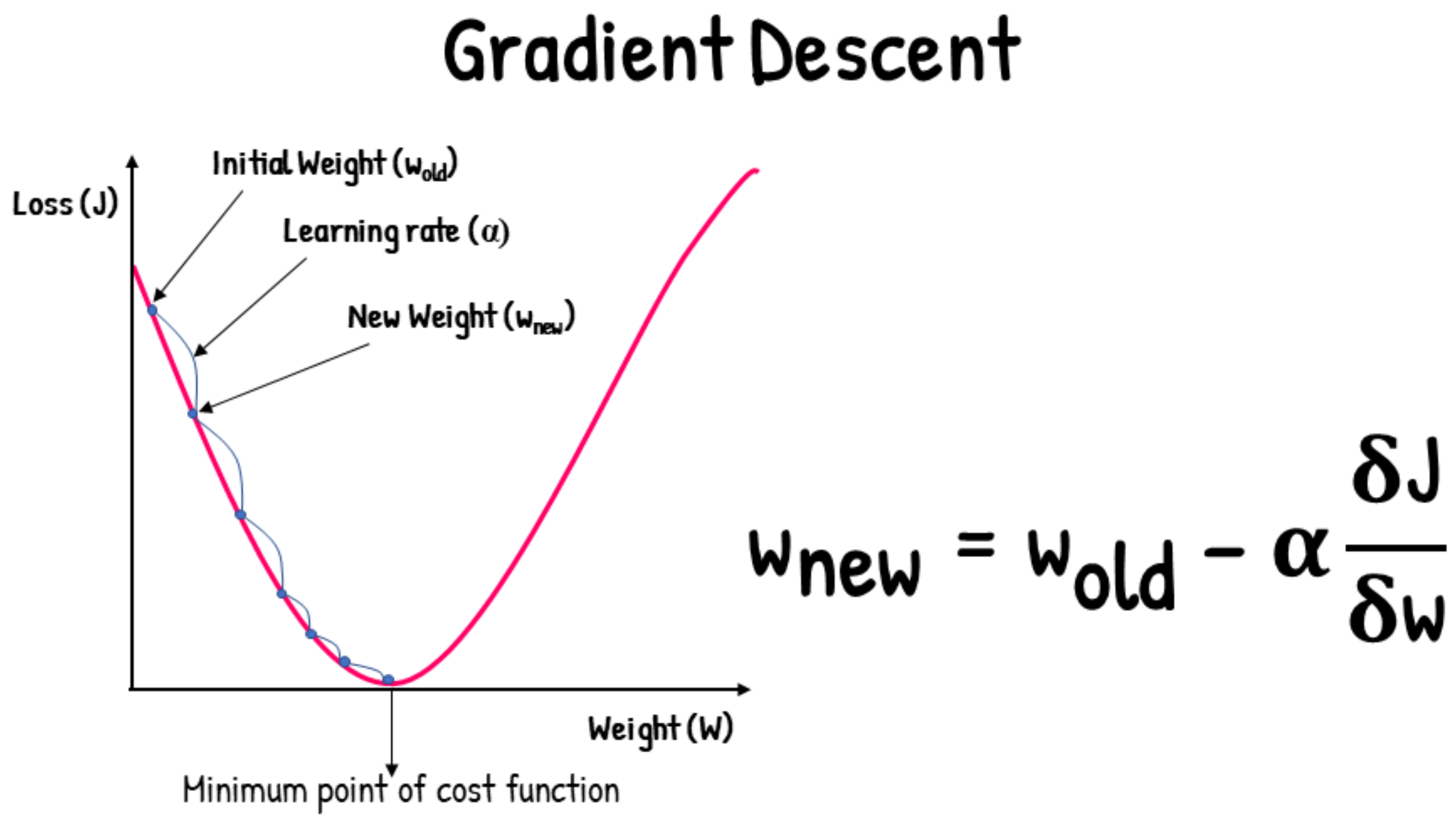


Numerical Solution of Mathematical Optimization Problems

- **Gradient Descent and Linear Programming**: In optimization problems, linear algebra is used in iterative algorithms such as gradient descent, where matrix and vector operations are essential for calculating search directions and step sizes.

- **Convex Optimization**: Linear algebra techniques, such as matrix factorization, are widely used in solving convex optimization problems, especially in quadratic programming where minimizing a quadratic function subject to linear constraints requires linear algebra methods

- **Interior-Point Methods**: These optimization algorithms for linear and nonlinear programming involve solving systems of linear equations as part of their iterative steps.



Gradient Descent

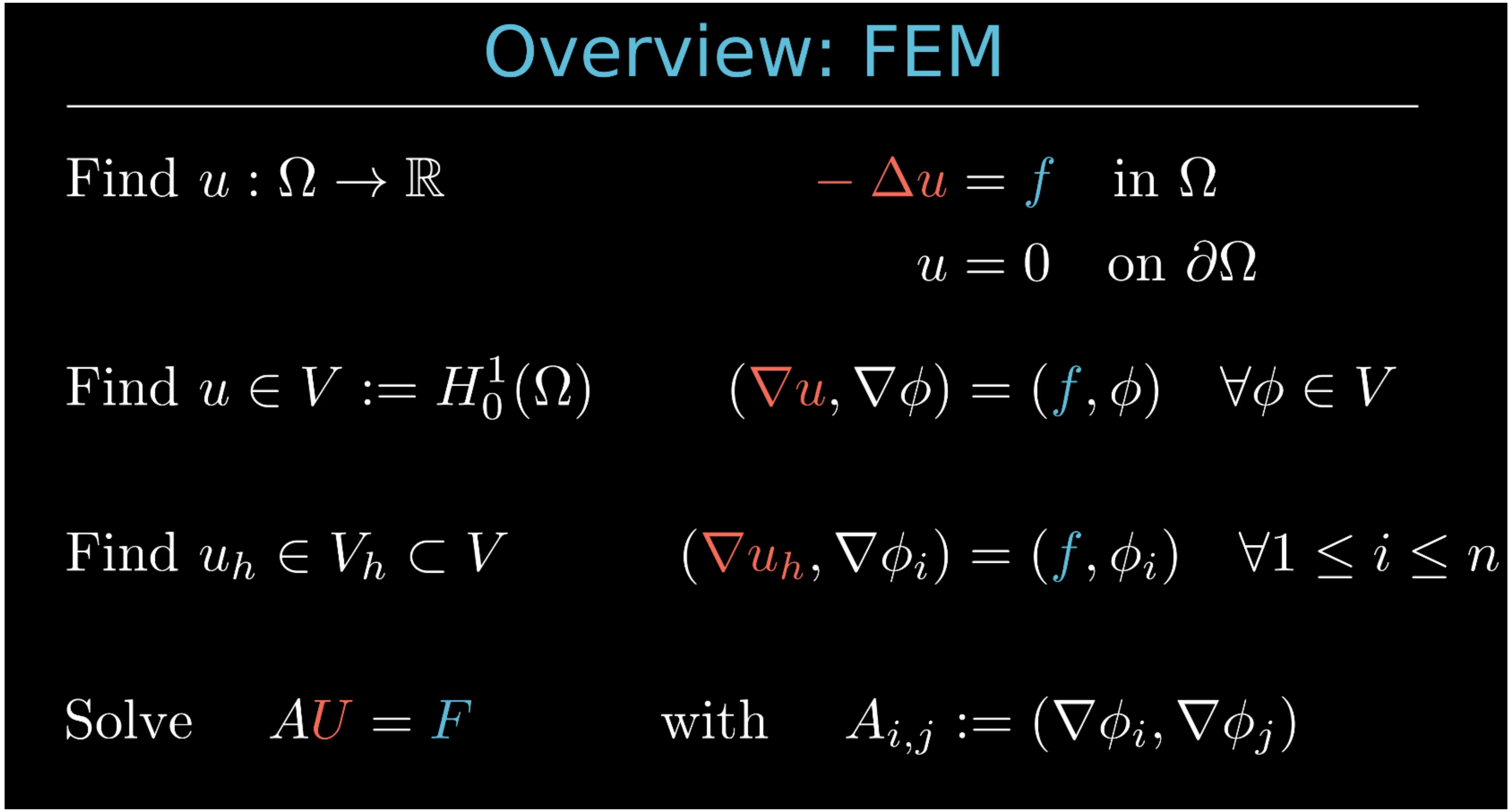
https://www.geeksforgeeks.org/gradient-descent-in-linear-regression/

. Numerical Solution of Partial Differential Equations (PDEs)

- **Finite Difference and Finite Element Methods (FEM)**: Solving PDEs numerically often leads to large systems of linear equations. Matrix operations and factorizations like LU or Cholesky are used to solve these systems efficiently.

- **Matrix Assembly in FEM**: In finite element analysis, linear algebra is used to assemble stiffness matrices that approximate the PDEs, converting differential equations into solvable algebraic equations.

- **Multigrid Methods**: These are iterative solvers for large linear systems derived from PDE discretizations, relying heavily on matrix representations and linear algebra techniques to improve convergence rates.



FEM

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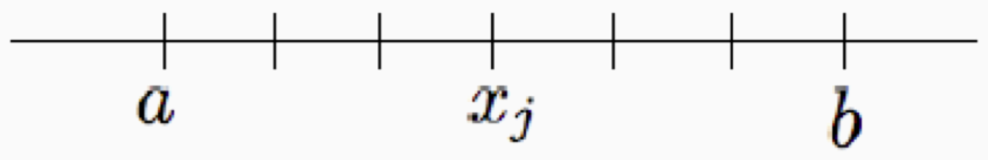
Numerical Methods for BVP: Finite Differences

Problem:

$$y'' = p(x)y' + q(x)y + f(x)\ , \quad a \le x \le b$$

$$y(a) = \alpha\ , \quad y(b) = \beta$$



Grid: $x_j = a + jh\ , \quad h = (b-a)/(N+1)$ Note: N interior grid points

Centered Difference Formulas: (From Taylor expansions)

$$y''(x_j) = \frac{1}{h^2}\left[y(x_{j+1}) - 2y(x_j) + y(x_{j-1})\right] - \frac{h^2}{24} y^{(4)}(\xi_j)$$

$$y'(x_j) = \frac{1}{2h}\left[y(x_{j+1}) - y(x_{j-1})\right] - \frac{h^2}{6} y'''(\eta_j)$$

System:

$$\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{h^2} = p(x_j)\left[\frac{y(x_{j+1}) - y(x_{j-1})}{2h}\right] + q(x_j)y(x_j)$$

$$+ f(x_j) - \frac{h^2}{12}\left[2p(x_j)y'''(\eta_j) - y^{(4)}(\xi_j)\right]$$

FD

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Finite Difference System: Define $y_0 = \alpha$, $y_{N+1} = \beta$ and consider

$$\left(\frac{2y_j - y_{j+1} - y_{j-1}}{h^2}\right) + p(x_j)\left(\frac{y_{j+1} - y_{j-1}}{2h}\right) + q(x_j)y_j = -f(x_j)$$

$$\Rightarrow -\left(1 + \frac{h}{2}p(x_j)\right) y_{j-1} + (2 + h^2 q(x_j))y_j - \left(1 - \frac{h}{2}p(x_j)\right) y_{j+1} = -h^2 f(x_j)$$

for $j = 1, 2, \cdots, N$

Matrix System:

$$\begin{bmatrix} 2+h^2q(x_1) & -1+\frac{h}{2}p(x_1) & 0 & & \\ -1-\frac{h}{2}p(x_2) & 2+h^2q(x_2) & -1+\frac{h}{2}p(x_2) & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1+\frac{h}{2}p(x_{N-1}) \\ & & 0 & -1-\frac{h}{2}p(x_N) & 2+h^2q(x_N) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix} = \begin{bmatrix} -h^2 f(x_1) + \left(1+\frac{h}{2}p(x_1)\right)\alpha \\ -h^2 f(x_2) \\ \vdots \\ -h^2 f(x_{N-1}) \\ -h^2 f(x_N) + \left(1-\frac{h}{2}p(x_N)\right)\beta \end{bmatrix}$$

FD

Document Data Representation and Web-Page Ranking

- **Latent Semantic Indexing (LSI)**: In natural language processing, LSI is an application of SVD to a term-document matrix to identify patterns and relationships between documents, improving search relevance.

https://www.geeksforgeeks.org/latent-semantic-analysis/

Document Data Representation and Web-Page Ranking

- **Google's PageRank Algorithm**: Linear algebra, specifically eigenvalue problems, is used in Google's PageRank algorithm to rank web pages. The ranking is based on the eigenvector of the web link matrix, representing the importance of each page in the network.
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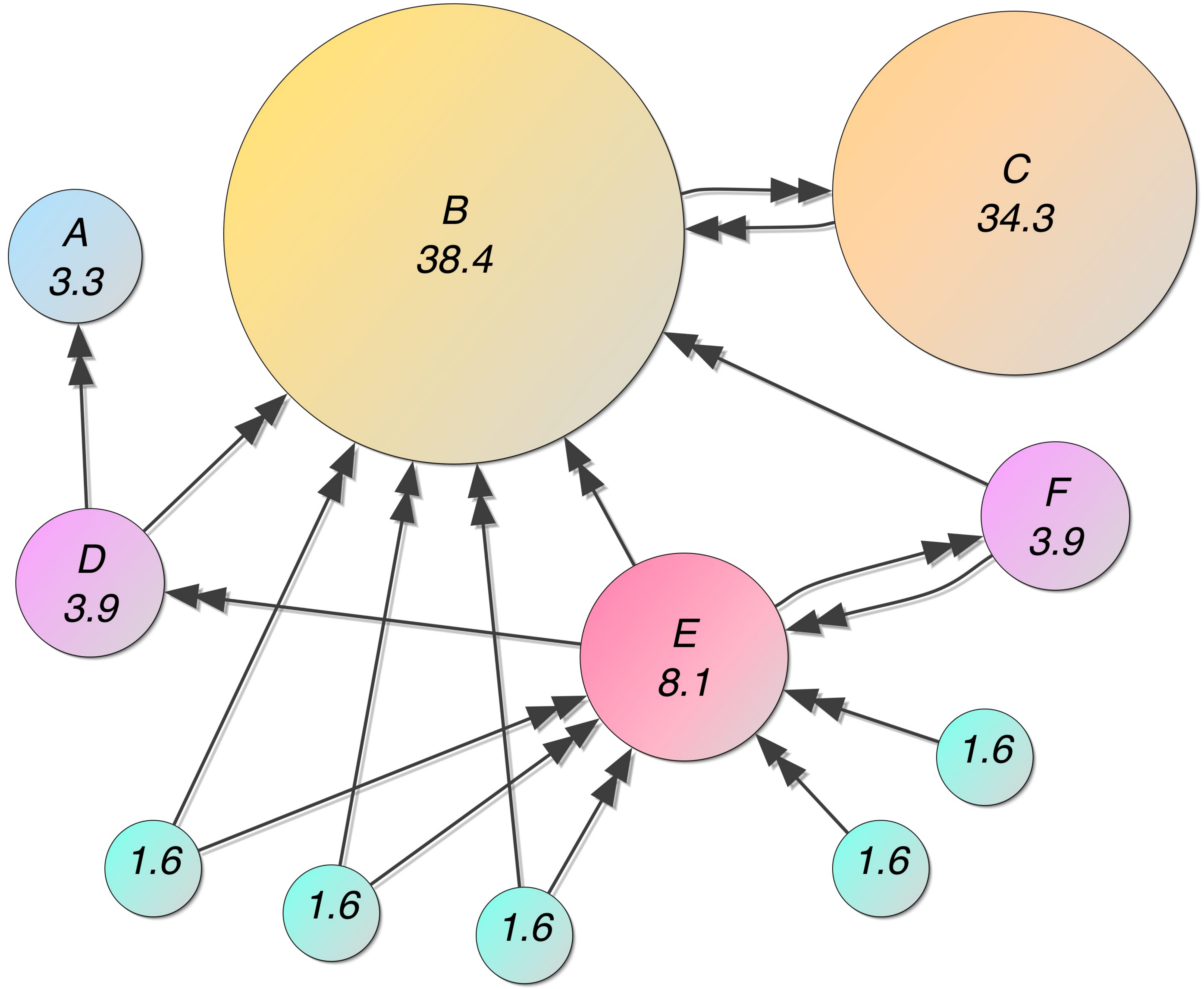

Web-Page illustration

Document Data Representation and Web-Page Ranking

- **TF-IDF and Matrix Representations**: In information retrieval, linear algebra helps model document-term relationships via term frequency-inverse document frequency (TF-IDF) matrices, which are used to rank documents in search queries.

https://maelfabien.github.io/machinelearning/NLP_2/#

MORE APPLICATIONS OF (NUMERICAL) LINEAR ALGEBRA IN MACHINE LEARNING

GO HERE

https://drive.google.com/file/d/1KikIQXcDiV_R_OM-eSWKYYKLfKZKVFC7/view?usp=sharing