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SYSTEMS OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION AND ITS VARIANTS

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SYSTEM OF LINEAR EQUATIONS

Gaussian Elimination

$$\left[\begin{array}{ccc|c} a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & & \vdots & \vdots \\ a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{array}\right].$$

The operations are

$$m_{i2} = a_{i2}^{(1)} / a_{22}^{(1)} \qquad i = 3, \ldots, n$$

and

$$\begin{aligned} a_{ij}^{(2)} &= a_{ij}^{(1)} - m_{i2} a_{2j}^{(1)} && j = 3, \ldots, n, \; i = 3, \ldots, n \\ b_i^{(2)} &= b_i^{(1)} - m_{i2} b_2^{(1)} && i = 3, \ldots, n. \end{aligned}$$

As in the first step, there is no need to calculate $a_{i2}^{(2)}$ explicitly for $i = 3, \ldots, n$, because the multipliers m_{i2} were chosen so that $a_{i2}^{(2)} = 0$.

In order to carry out this step, we need $a_{22}^{(1)} \neq 0$. That this is so follows from the assumption that

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After the second step the augmented matrix will have been transformed to

$$\left[\begin{array}{c|c|c c c|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
\hline
0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\
\hline
0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} & b_3^{(2)} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} & b_n^{(2)}
\end{array}\right].$$

In a computer implementation the zeros will be replaced by the multipliers $m_{21}, \ldots,$ m_{n1} and $m_{32}, \ldots, m_{n2}$. Since the second step is the same as the first, but on a matrix with one less row and one less column, the flop count for the second step is about $2(n-1)^2$.

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The third step is identical to the previous two, except that it operates on the smaller matrix

$$\left[\begin{array}{ccc|c} a_{33}^{(2)} & \cdots & a_{3n}^{(2)} & b_3^{(2)} \\ \vdots & & \vdots & \vdots \\ a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} & b_n^{(2)} \end{array}\right].$$

In order to carry out the step, we need to have $a_{33}^{(2)} \neq 0$. This is guaranteed by the assumption that A_3 is nonsingular. After the first two steps, A_3 will have been transformed to

$$\hat{A}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$

The non singularity of A_3 implies the non singularity of $\hat{A}_3$

This implies that $a_{33}^{(2)} \neq 0$

via two elementary row operations of type 1.

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After $n-1$ steps the system will be reduced to $[U \mid y]$, where U is upper triangular. For each k, the possibility of carrying out step k is guaranteed by the assumption that

A_k is nonsingular. In the end we know that U is nonsingular because A is. Thus the system $Ux = y$ can be solved by back substitution to yield x, the unique solution of $Ax = b$.

The total flop count for the reduction to triangular form is approximately

$$2n^2 + 2(n-1)^2 + 2(n-2)^2 + \ldots = 2\sum_{k=1}^{n} k^2.$$

We can use the approximation $$2\sum_{k=1}^{n} k^2 \approx 2\int_0^n k^2\, dk = \tfrac{2}{3}n^3.$$

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The additional cost of the back substitution is n^2 flops, which is relatively insignificant for large n. Thus the total cost of solving $Ax = b$ by this method is about $\frac{2}{3}n^3$ flops. This is about twice the cost of solving a positive definite system by Cholesky's method, which saves a factor of 2 by exploiting the symmetry of the coefficient matrix.

Exemple

Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 9 \\ 9 \\ 16 \end{bmatrix}.$$

We can easily show that the A is non singular and that the solution exist

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The augmented matrix is:

$$[A \mid b] = \left[\begin{array}{rrr|r} 2 & 1 & 1 & 9 \\ 2 & 2 & -1 & 9 \\ 4 & -1 & 6 & 16 \end{array}\right].$$

The multipliers for the first step are $m_{21} = a_{21}/a_{11} = 1$ and $m_{31} = a_{31}/a_{11} = 2$.

Thus we subtract 1 times the first row from the second row and 2 times the first row from the third row to create zeros in the $(2, 1)$ and $(3, 1)$ positions. Performing these row operations, we obtain

$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 9 \\ 0 & 1 & -2 & 0 \\ 0 & -3 & 4 & -2 \end{array}\right]$$

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The multiplier for the second step is $m_{32} = a_{32}^{(1)}/a_{22}^{(1)} = -3$. Thus we subtract -3 times the second row from the third row to obtain

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right].$$

After two steps the reduction is complete. If we save the multipliers in place of the zeros, the array looks like this:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 1 & 1 & -2 & 0 \\ 2 & -3 & -2 & -2 \end{array}\right].$$

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We can now solve the system by solving

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ -2 \end{bmatrix}$$

by back substitution. Doing so, we find that $x_3 = 1$, $x_2 = 2$, and $x_1 = 3$.

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Exercise 1

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 13 \\ -2 \\ 24 \\ -14 \end{bmatrix}.$$

(a) Calculate the appropriate (four) determinants to show that A can be transformed to (nonsingular) upper-triangular form by operations of type 1 only. (By the way, this is strictly an academic exercise. In practice one never calculates these determinants in advance.)

(b) Carry out the row operations of type 1 to transform the system $Ax = b$ to an equivalent system $Ux = y$, where U is upper triangular. Save the multipliers [redacted].

(c) Carry out the back substitution on the system $Ux = y$ to obtain the solution of $Ax = b$. Don't forget to check your work.

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Interpretation of the Multipliers

Suppose we have solved A$\mathbf{x}$ = b and we want now to solve $A\mathbf{x} = \hat{\mathbf{b}}$, the matrix A is the same

We do not need to do again from scratch the operations since the coefficient matrix is the same as before. All multipliers and row operations will be the same.

If the multipliers have been saved we can performed row operations on the $\hat{\mathbf{b}}$ only and save computation time.

Interpretation of the Multipliers

Let see how this works. The operations on $\mathbf{b}$

$$
\begin{array}{lcll}
b_i^{(1)} & = & b_i - m_{i1} b_1 & i = 2, 3, \ldots, n \\
b_i^{(2)} & = & b_i^{(1)} - m_{i2} b_2^{(1)} & i = 3, 4, \ldots, n \\
\vdots & & & \vdots \\
b_i^{(j)} & = & b_i^{(j-1)} - m_{ij} b_j^{(j-1)} & i = j+1 \ldots, n \\
\vdots & & & \vdots \\
b_i^{(n-1)} & = & b_i^{(n-2)} - m_{i,n-1} b_{n-1}^{(n-2)} & i = n.
\end{array}
$$

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Interpretation of the Multipliers

At the end of the operations b has been transformed to

$$\begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(n-1)} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = y.$$

2

The same operations applied to $\hat{b}$ would yield a vector $\hat{y}$. The system $Ax = \hat{b}$ could then be solved by solving the equivalent upper-triangular system $Ux = \hat{y}$, where U is the upper-triangular matrix that was obtained in the original reduction of A.

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Interpretation of the Multipliers

Suppose we want to solve the system $A\mathbf{x} = \hat{\mathbf{b}}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} \quad \text{and} \quad \hat{b} = \begin{bmatrix} 7 \\ 3 \\ 20 \end{bmatrix}.$$

The is the same matrix as from previous example and the multipliers are

$$m_{21} = 1, m_{31} = 2, \text{ and } m_{32} = -3.$$

This means that A can transform to upper triangular matrix by subtracting

1 time first row from second, 2 times first row from third row, -3 times the new

second row from the third.

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Rather than performing these operations on the augmented matrix $[A \mid \hat{b}]$, we apply them to the column $\hat{b}$ only and get

$$\hat{b}_2^{(1)} \quad = \quad \hat{b}_2 - m_{21}\hat{b}_1 = 3 - 1 \cdot 7 = -4$$

$$\begin{aligned} \hat{b}_3^{(1)} &= \hat{b}_3 - m_{31}\hat{b}_1 = 20 - 2 \cdot 7 = 6 \\ \hat{b}_3^{(2)} &= \hat{b}_3^{(1)} - m_{32}\hat{b}_2^{(1)} = 6 - (-3)(-4) = -6. \end{aligned}$$

After these transformations, the new right-hand side is

$$\hat{y} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}.$$

Interpretation of the Multipliers

Now we can get the solution by solving $Ux = y$ by back substitution, where

$$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix},$$

Doing so, we find that $x_3 = 3$, $x_2 = 2$, and $x_1 = 1$.

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Interpretation of the Multipliers

Transformation of b to y

$$
\begin{aligned}
b_i^{(1)} &= b_i - m_{i1}y_1 && i = 2, 3, \ldots, n \\
b_i^{(2)} &= b_i^{(1)} - m_{i2}y_2 && i = 3, 4, \ldots, n \\
&\vdots && \vdots \\
b_i^{(n-1)} &= b_i^{(n-2)} - m_{i,n-1}y_{n-1} && i = n.
\end{aligned}
$$

These equations can be used to derived $y_1, y_2, \cdots, y_n$

$$y_2 = b_2^{(1)} = b_2 - m_{21}y_1$$

By the first two equations we have $y_3 = b_3^{(2)} = b_3^{(1)} - m_{32}y_2 = b_3 - m_{31}y_1 - m_{32}y_2.$

Similarly $y_4 = b_4 - m_{41}y_1 - m_{42}y_2 - m_{43}y_3.$

In general: $$y_i = b_i - \sum_{j=1}^{i-1} m_{ij}y_j \qquad i = 1, 2, \ldots, n.$$

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Equation $y_i = b_i - \sum_{j=1}^{i-1} m_{ij} y_j \qquad i = 1, 2, \ldots, n.$ Can be written as:

$$\sum_{j=1}^{i-1} m_{ij} y_j + y_i = b_i, \qquad i = 1, \ldots, n.$$

Or matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \ddots & \vdots \\ m_{31} & m_{32} & 1 & \ddots & \\ \vdots & & & \ddots & 0 \\ m_{n1} & m_{n2} & m_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \ddots & \vdots \\ m_{31} & m_{32} & 1 & \ddots & \\ \vdots & & & \ddots & 0 \\ m_{n1} & m_{n2} & m_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

We see that y is just a solution of a Linear Ly = b, L is lower triangular.

In fact L is unit is lower triangular which means that its main diagonal entries are all ones

This system can solved by using forward substitution

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Summary

A brief summary of what we have done so far will lead to an interesting and important conclusion: L and U can be interpreted as factors of A. In order to solve the system

$$Ax = b,$$

we reduced it to the form

$$Ux = y,$$

where U is upper triangular, and y is the solution of a unit lower-triangular system

$$Ly = b.$$

Combining these last two equations, we find that $LUx = b$. Thus $LUx = b = Ax$.

These equations hold for any choice of b, and hence for any choice of x. (For a given x, the appropriate b is obtained by the calculation $b = Ax$.) Since the equation $LUx = Ax$ holds for all $x \in \mathbb{R}^n$, it must be that

$$A = LU.$$

Summary

We conclude that Gaussian elimination without row interchanges (saving the multipliers) can be viewed as a process of decomposing A into a product, $A = LU$, where L is lower triangular and U is upper triangular. In fact the usual procedure is not to form an augmented matrix $[A \mid b]$ but to do row operations on A alone. A is reduced, saving multipliers, to the form

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ m_{21} & u_{22} & u_{23} & \cdots & u_{2n} \\ m_{31} & m_{32} & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & m_{n3} & & u_{nn} \end{bmatrix},$$

which contains all information about L and U. The system $LUx = b$ is then solved by first solving $Ly = b$ for y by forward substitution and then solving $Ux = y$ by back substitution.

Example

Solve the system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}.$$

$A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}.$$

Solving $Ly = b$ by forward substitution, we get $y = [3,\ -3,\ -4]^T$. Solving $Ux = y$ by back substitution, we get $x = [0,\ 1,\ 2]^T$. $\square$

Exercise 2

Solver the linear system $A\mathbf{x} = \hat{\mathbf{b}}$ as in Exercise 1 and $\hat{\mathbf{b}} = [12,\ -8,\ 21,\ -26]^T$

Use the L and U that you calculated in Exercise 1.

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Theorem

[redacted] *(LU Decomposition Theorem) Let* A *be an* $n \times n$ *matrix whose leading principal submatrices are all nonsingular. Then* A *can be decomposed in exactly one way into a product*

$$A = LU,$$

such that L *is unit lower triangular and* U *is upper triangular.*

Proof. We have already shown that L and U exist.[8] It remains only to show that they are unique. Our uniqueness proof will yield a second algorithm for calculating the LU decomposition. Look at the equation $A = LU$ in detail.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}.$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}.$$

The first row of L is known completely, and it has only one nonzero entry. Multiplying the first row of L by the jth column of U, we find that

$$a_{1j} = 1u_{1j} + 0u_{2j} + 0u_{3j} + \cdots + 0u_{nj}.$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}.$$

$$a_{1j} = 1u_{1j} + 0u_{2j} + 0u_{3j} + \cdots + 0u_{nj}.$$

That is, $u_{1j} = a_{1j}$. Thus the first row of U is uniquely determined. Now that we know the first row of U, we see that the first column of U is also known, since its only nonzero entry is u_{11}. Multiplying the ith row of L by the first column of U, we find that

$$a_{i1} = l_{i1}u_{11}, \qquad i = 2, \ldots, n.$$

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The assumption that A is nonsingular implies that U is also nonsingular. (Why?)

Hence $u_{kk} \neq 0$ for $k = 1, \ldots, n$, and, in particular, $u_{11} \neq 0$.
determines the first column of L uniquely:

$$l_{i1} = \frac{a_{i1}}{u_{11}}, \qquad i = 2, \ldots, n.$$

Thus the first column of L is uniquely determined.

Thus the first column of L is uniquely determined. Now that the first row of U and first column of L have been determined, it is not hard to show that the second row of U is also uniquely determined. As an exercise, determine a formula for u_{2j} $(j \geq 2)$ in terms of a_{2j} and entries of the first row of U and column of L. Once u_{22} is known, it is possible to determine the second column of L. Do this also as an exercise.

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Now suppose the first $k-1$ rows of U and columns of L have been shown to be uniquely determined. We will show that the kth row of U and column of L are uniquely determined; this will prove uniqueness by induction. The kth row of L is $[l_{k1}\ l_{k2}\ \cdots\ l_{k,k-1}\ 1\ 0\ \cdots\ 0]$. Since $l_{k1}, \ldots, l_{k,k-1}$ are all in the first $k-1$ columns of L, they are uniquely determined. Multiplying the kth row of L by the jth column of U ($j \geq k$), we find that

$$a_{kj} = \sum_{m=1}^{k-1} l_{km} u_{mj} + u_{kj}.$$

All of the u_{mj} (aside from u_{kj}) lie in the first $k-1$ rows of U and are therefore known (i.e., uniquely determined). Therefore u_{kj} is uniquely determined by [redacted]):

All of the l_{im} (aside from l_{ik}) lie in the first $k-1$ columns of L and are therefore uniquely determined. Furthermore $u_{kk} \neq 0$. Thus [redacted] determines l_{ik} uniquely:

$$l_{ik} = u_{kk}^{-1}\left(a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk}\right), \qquad i = k+1, \ldots, n.$$ [redacted]

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Example

Gaussian Elimination

$$A = \begin{bmatrix} 2 & 4 & 2 & 3 \\ -2 & -5 & -3 & -2 \\ 4 & 7 & 6 & 8 \\ 6 & 10 & 1 & 12 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -3 \\ 3 \\ -1 \\ -16 \end{bmatrix}.$$

We will calculate L and U such that $A = LU$ by two different methods. First let's do Gaussian elimination by row operations.

Step 1:

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & -1 & 2 & 2 \\ 3 & -2 & -5 & 3 \end{bmatrix}$$

Step 2:

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & -3 & 1 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & -1 & 2 \end{bmatrix}$$

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Now let's try the inner-product formulation.

Step 1:

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -5 & -3 & -2 \\ 2 & 7 & 6 & 8 \\ 3 & 10 & 1 & 12 \end{bmatrix}$$

The first row of U and column of L have been calculated. The rest of the matrix remains untouched.

Step 2:

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & 1 & 6 & 8 \\ 3 & 2 & 1 & 12 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & -1 & 12 \end{bmatrix}$$

Now only u_{44} remains to be calculated.

Step 4:

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & -1 & 2 \end{bmatrix}$$

Both reductions yield the same result. You might find it instructive to try the inner-product reduction by the erasure method. Begin with the entries of A entered in pencil. As you calculate each entry of L or U, erase the corresponding entry of A and replace it with the new result. Do the arithmetic in your head.

Now that we have the LU decomposition of A, we perform forward substitution on

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -1 \\ 16 \end{bmatrix}$$

to get $y = [-3,\ 0,\ 5,\ -2]^T$. We then perform back substitution on

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 5 \\ -2 \end{bmatrix}$$

to get $x = [4,\ -3,\ 2,\ -1]^T$.