AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES, AIMS RWANDA

AIMS | African Institute for Mathematical Sciences RWANDA

SYSTEMS OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION AND ITS VARIANTS

Prof. Franck Kalala Mutombo
University of Lubumbashi

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

We want now to solve the $A\mathbf{x} = \mathbf{b}$

We now drop the the assumption that the leading principal submatrices are singular.

We will develop an algorithm that uses elementary row operations of types 1 and 2

Either to solve the system $A\mathbf{x} = \mathbf{b}$ or (in theory, at least) to determine that it is singular.

The algorithm is identical to the algorithm that we developed previously except that at each step a row interchange can be made.

We now consider Gaussian Elimination with row interchange also known as

Gaussian Elimination with Pivoting

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Let us consider the 4th step of the algorithm. After k-1 steps, the array that originally contained A has been transformed to the form

$$\left[\begin{array}{cccc|cccc}
u_{11} & u_{12} & \cdots & u_{1,k-1} & u_{1k} & & \cdots & u_{1n} \\
m_{21} & u_{22} & \cdots & u_{2,k-1} & u_{2k} & & \cdots & u_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
m_{k-1,1} & & & u_{k-1,k-1} & u_{k-1,k} & & \cdots & u_{k-1,n} \\
\hline
m_{k1} & \cdots & & m_{k,k-1} & a_{k,k}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{k,n}^{(k-1)} \\
 & & & & a_{k+1,k}^{(k-1)} & a_{k+1,k+1}^{(k-1)} & \cdots & a_{k+1,n}^{(k-1)} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
m_{n1} & \cdots & & m_{n,k-1} & a_{n,k}^{(k-1)} & a_{n,k+1}^{(k-1)} & \cdots & a_{n,n}^{(k-1)}
\end{array}\right]$$

The u_{ij} are entries that will not undergo any further changes (eventual entries of the matrix U). The m_{ij} are stored multipliers (eventual entries of the matrix L.) The $a_{ij}^{(k-1)}$ are the entries that are still active.

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

$$\left[\begin{array}{cccc|cccc}
u_{11} & u_{12} & \cdots & u_{1,k-1} & u_{1k} & & \cdots & u_{1n} \\
m_{21} & u_{22} & \cdots & u_{2,k-1} & u_{2k} & & \cdots & u_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
m_{k-1,1} & & & u_{k-1,k-1} & u_{k-1,k} & & \cdots & u_{k-1,n} \\
\hline
m_{k1} & \cdots & & m_{k,k-1} & a_{k,k}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{k,n}^{(k-1)} \\
 & & & & a_{k+1,k}^{(k-1)} & a_{k+1,k+1}^{(k-1)} & \cdots & a_{k+1,n}^{(k-1)} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
m_{n1} & \cdots & & m_{n,k-1} & a_{n,k}^{(k-1)} & a_{n,k+1}^{(k-1)} & \cdots & a_{n,n}^{(k-1)}
\end{array}\right]$$

To calculate the multipliers for the kth step, we should divide by $a_{kk}^{(k-1)}$. If $a_{kk}^{(k-1)} = 0$, we will have to use a type 2 row operation (row interchange) to get a nonzero entry into the (k,k) position. In fact, even if $a_{kk}^{(k-1)} \neq 0$, we may still choose to do a row interchange.

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Consider the following possibility. If $|a_{kk}^{(k-1)}|$ is very small, it may be that $a_{kk}^{(k-1)}$ should be exactly zero and is nonzero only because of roundoff errors made on previous steps. If we now calculate multipliers by dividing by this number, we will surely get erroneous results. For this and other reasons, we will always carry out row interchanges in such a way as to avoid having a small entry in the (k, k) position.

$$\left[\begin{array}{cccc|cccc}
u_{11} & u_{12} & \cdots & u_{1,k-1} & u_{1k} & & \cdots & u_{1n} \\
m_{21} & u_{22} & \cdots & u_{2,k-1} & u_{2k} & & \cdots & u_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
m_{k-1,1} & & & u_{k-1,k-1} & u_{k-1,k} & & \cdots & u_{k-1,n} \\
\hline
m_{k1} & \cdots & & m_{k,k-1} & a_{k,k}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{k,n}^{(k-1)} \\
 & & & & a_{k+1,k}^{(k-1)} & a_{k+1,k+1}^{(k-1)} & \cdots & a_{k+1,n}^{(k-1)} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
m_{n1} & \cdots & & m_{n,k-1} & a_{n,k}^{(k-1)} & a_{n,k+1}^{(k-1)} & \cdots & a_{n,n}^{(k-1)}
\end{array}\right]$$

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

$$
\left[\begin{array}{cccc|cccc}
u_{11} & u_{12} & \cdots & u_{1,k-1} & u_{1k} & & \cdots & u_{1n} \\
m_{21} & u_{22} & \cdots & u_{2,k-1} & u_{2k} & & \cdots & u_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
m_{k-1,1} & & & u_{k-1,k-1} & u_{k-1,k} & & \cdots & u_{k-1,n} \\
\hline
m_{k1} & \cdots & & m_{k,k-1} & a_{k,k}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{k,n}^{(k-1)} \\
 & & & & a_{k+1,k}^{(k-1)} & a_{k+1,k+1}^{(k-1)} & \cdots & a_{k+1,n}^{(k-1)} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
m_{n1} & \cdots & & m_{n,k-1} & a_{n,k}^{(k-1)} & a_{n,k+1}^{(k-1)} & \cdots & a_{n,n}^{(k-1)}
\end{array}\right]
$$

Returning to the description of our algorithm, we examine the entries $a_{kk}^{(k-1)}$, $a_{k+1,k}^{(k-1)}, \ldots, a_{nk}^{(k-1)}$. If all are zero, then A is singular [redacted]. Set a flag to warn that this is the case. At this point we can either stop or go on to step $k+1$.

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

$$\left[\begin{array}{cccc|cccc}
u_{11} & u_{12} & \cdots & u_{1,k-1} & u_{1k} & & \cdots & u_{1n} \\
m_{21} & u_{22} & \cdots & u_{2,k-1} & u_{2k} & & \cdots & u_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
m_{k-1,1} & & & u_{k-1,k-1} & u_{k-1,k} & & \cdots & u_{k-1,n} \\
\hline
m_{k1} & \cdots & & m_{k,k-1} & a_{k,k}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{k,n}^{(k-1)} \\
 & & & & a_{k+1,k}^{(k-1)} & a_{k+1,k+1}^{(k-1)} & \cdots & a_{k+1,n}^{(k-1)} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
m_{n1} & \cdots & & m_{n,k-1} & a_{n,k}^{(k-1)} & a_{n,k+1}^{(k-1)} & \cdots & a_{n,n}^{(k-1)}
\end{array}\right]$$

If not all of $a_{kk}^{(k-1)}, \ldots, a_{nk}^{(k-1)}$ are zero, let $a_{mk}^{(k-1)}$ be the one whose absolute value is greatest. Interchange rows m and k, including the stored multipliers. Keep a record of the row interchange. This is easily done in an integer array of length n. Store the number m in position k of the array to indicate that at step k, rows k and m were interchanged. Now subtract the appropriate multiples of the new kth row from rows $k+1, \ldots, n$ to produce zeros in positions $(k+1, k), \ldots, (n, k)$. Of course, we actually store the multipliers $m_{k+1,k}, \ldots, m_{nk}$ in those positions instead of zeros. This concludes the description of step k.

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

$$
\left[\begin{array}{cccc|cccc}
u_{11} & u_{12} & \cdots & u_{1,k-1} & u_{1k} & & \cdots & u_{1n} \\
m_{21} & u_{22} & \cdots & u_{2,k-1} & u_{2k} & & \cdots & u_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
m_{k-1,1} & & & u_{k-1,k-1} & u_{k-1,k} & & \cdots & u_{k-1,n} \\
\hline
m_{k1} & \cdots & & m_{k,k-1} & a^{(k-1)}_{k,k} & a^{(k-1)}_{k,k+1} & \cdots & a^{(k-1)}_{k,n} \\
 & & & & a^{(k-1)}_{k+1,k} & a^{(k-1)}_{k+1,k+1} & \cdots & a^{(k-1)}_{k+1,n} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
m_{n1} & \cdots & & m_{n,k-1} & a^{(k-1)}_{n,k} & a^{(k-1)}_{n,k+1} & \cdots & a^{(k-1)}_{n,n}
\end{array}\right]
$$

The eventual (k, k) entry, by which we divide to form the multipliers, is called the *pivot* for step k. The kth row, multiples of which are subtracted from each of the remaining rows at step k, is called the *pivotal row* for step k. Our strategy, which is to make the pivot at step k (i.e., at *each* step) as far from zero as possible in order to protect against disasters due to roundoff errors, is called *partial pivoting*.

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

$$
\left[\begin{array}{cccc|cccc}
u_{11} & u_{12} & \cdots & u_{1,k-1} & u_{1k} & & \cdots & u_{1n} \\
m_{21} & u_{22} & \cdots & u_{2,k-1} & u_{2k} & & \cdots & u_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
m_{k-1,1} & & & u_{k-1,k-1} & u_{k-1,k} & & \cdots & u_{k-1,n} \\
\hline
m_{k1} & \cdots & & m_{k,k-1} & a_{k,k}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{k,n}^{(k-1)} \\
 & & & & a_{k+1,k}^{(k-1)} & a_{k+1,k+1}^{(k-1)} & \cdots & a_{k+1,n}^{(k-1)} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
m_{n1} & \cdots & & m_{n,k-1} & a_{n,k}^{(k-1)} & a_{n,k+1}^{(k-1)} & \cdots & a_{n,n}^{(k-1)}
\end{array}\right]
$$

(Later on we will discuss *complete pivoting*, in which both rows and columns are interchanged.) Notice that the pivots end up on the main diagonal of the matrix U of the LU decomposition. Also, the choice of pivots implies that all of the multipliers will satisfy $|m_{ij}| \leq 1$. Thus in the matrix L of the LU decomposition, all entries will have absolute values less than or equal to 1.

After $n-1$ steps, the decomposition is complete. One final check must be made: If $a_{nn}^{(n-1)} = 0$, A is singular; set a flag. This is the last pivot. It is not used to create zeros in other rows, but, being the (n, n) entry of U, it is used as a divisor in the back-substitution process.

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Solving the system $Ax = b$ is the same as solving a system $\hat{A}x = \hat{b}$, obtained by interchanging the equations. Since we have saved a record of the row interchanges, it is easy to permute the entries of b to obtain $\hat{b}$. We then solve $Ly = \hat{b}$ by forward substitution and $Ux = y$ by back substitution to get the solution vector x.

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Example 1 We will solve the system

$$\begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ -1 \end{bmatrix}$$

In step 1 the pivotal position is $(1,1)$. Since there is a zero there, a row interchange is absolutely necessary. Since the largest potential pivot is the 2 in the $(3,1)$ position, we interchange rows 1 and 3 to get

$$\begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{bmatrix}.$$

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

The multipliers for the first step are $1/2$ and 0. Subtracting $1/2$ the first row from the second row and storing the multipliers, we have

$$\left[\begin{array}{c|cc} 2 & -2 & 1 \\ \hline 1/2 & 2 & 5/2 \\ 0 & 4 & 1 \end{array}\right].$$

The pivotal position for the second step is $(2,2)$. Since the 4 in the $(3,2)$ position is larger than the 2 in the $(2,2)$ position, we interchange rows 2 and 3 (including the multipliers) to get the 4 into the pivotal position:

$$\left[\begin{array}{c|cc} 2 & -2 & 1 \\ \hline 0 & 4 & 1 \\ 1/2 & 2 & 5/2 \end{array}\right].$$

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

The multiplier for the third step is $1/2$. Subtracting $1/2$ the second row from the third row and storing the multiplier, we have

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 1/2 & 1/2 & 2 \end{bmatrix}. \tag{1}$$

This completes the Gaussian elimination process. Noting that the pivot in the $(3,3)$ position is nonzero, we conclude that A is nonsingular, and the system has a unique solution. Encoded in 1 is the LU decomposition of

$$\hat{A} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix},$$

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

$$\hat{A} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix},$$

which was obtained from A by making appropriate row interchanges. You can check that $\hat{A} = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

To solve $Ax = b$, we first transform b to $\hat{b}$. Since we interchanged rows 1 and 3 at step 1, we must first interchange components 1 and 3 of b to obtain $[-1,\ 6,\ 9]^T$. At step 2 we interchanged rows 2 and 3, so we must interchange components 2 and 3 to get $\hat{b} = [-1,\ 9,\ 6]^T$. We now solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 6 \end{bmatrix}$$

to get $y = [-1,\ 9,\ 2]^T$. Finally we solve

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 2 \end{bmatrix}$$

to get $x = [1,\ 2,\ 1]^T$. You can easily check that this is correct by substituting it back into the original equation. $\square$

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Exercise 1 Let

$$A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}.$$

Use Gaussian elimination with partial pivoting (by hand) to find matrices L and U such that U is upper triangular, L is unit lower triangular with $|l_{ij}| \leq 1$ for all $i > j$, and $LU = \hat{A}$, where $\hat{A}$ can be obtained from A by making row interchanges. Use your LU decomposition to solve the system $Ax = b$. □

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Permuting the rows of a matrix is equivalent to multiplying the matrix by a permutation matrix.

Example 2 Let A and $\hat{A}$ of example 1

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \qquad \hat{A} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix},$$

We can easily check that $\hat{A}$ = PA, where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Exercise 2:

Show that if P is a permutation matrix, then $P^T P = PP^T = I$. Thus P is nonsingular, and $P^{-1} = P^T$. $\square$

Exercise 3:

Let A be an $n \times m$ matrix, and let $\hat{A}$ be a matrix obtained from A by scrambling the rows. Show that there is a unique $n \times n$ permutation matrix P such that $\hat{A} = PA$.

Theorem *Gaussian elimination with partial pivoting, applied to an $n \times n$ matrix A produces a unit lower-triangular matrix L such that $|l_{ij}| \leq 1$, an upper triangular matrix U, and a permutation matrix P such that*

$$A = P^T LU.$$

Proof. We have seen that Gaussian elimination with partial pivoting produces L and U of the required form such that $\hat{A} = LU$, where $\hat{A}$ was obtained from A by permuting the rows. By Exercise 1 , there is a permutation matrix P such that $\hat{A} = PA$. Thus $PA = LU$ or $A = P^{-1}LU = P^T LU$. $\square$

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Given a decomposition $A = P^T LU$, we can use it to solve a system $Ax = b$, by writing $P^T LUx = b$, then solving, successively, $P^T \hat{b} = b$, $Ly = \hat{b}$, and $Ux = y$.

Earlier in this course there were quite a few exercises that used Julia to solve system of linear equations.

Whenever the command x=A\b is invoked to solve the Ax = b (where A is $n \times n$ and nonsingular), Julia uses Gaussian elimination with partial pivoting to solve the system.

If you want to see the LU decomposition, use Julia command lu.

SYSTEM OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION WITH PIVOTING

Exercice Use Julia to check the LU decomposition of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}$$

Enter the matrix A, then type F = lu(A). The factors are obtained as F.L F.U and permutation matrix as F.P. You can check that A == F.P'*F.L*F.U is true or A \approx +Tab F.P'*F.L*F.U is also true. (In Julia the prime symbol, applied to a real matrix, mean "transpose". You can also use transpose(P) for transpose.

For a description of the lu command, type ?lu (? Stand for help)