Introduction to Model Predictive Control

Model Predictive Control (MPC)

MPC minimizes the cost function *explicitly* using numerical methods.

Disadvantages:
- Computationally intensive
- Only applicable to discrete-time systems

Advantages:
- Simple*
- Ability to handle constraints optimally

What types of constraints exist in real-world scenarios?

MPC constraint for self-driving car: minimize a cost function of performance plus braking and engine actuators ***but avoid hitting a pedestrian at all costs.***

Convert cost function to the "standard form" of a quadratic program

Control system formulation

$$J_k = \sum_{i=1}^{N} x_{k+i}^T Q x_{k+i} + u_{k+i-1}^T R u_{k+i-1}$$

Subject to:

- Dynamics constraint

$$x_{i+1} = Ax_i + Bu_i$$

- Input constraints
- Output constraints

LQR is a special case of this with N = ∞ and no input or output constraints

Standard form for quadratic programs

$$J_k(z) = z^T H z + f^T z$$

Subject to:

$$A_{eq} z = b_{eq}$$

$$A_{ineq} z \leq b_{ineq}$$

Cost function is an important design choice

$$J = \sum_{i=1}^{N} \left(x_i^T Q x_i + u_i^T R u_i \right)$$

"Basic" form

$$J = \sum_{i=1}^{N} \left(x_i^T Q x_i + u_i^T R u_i \right) + x_F^T Q_F x_F$$

Terminal cost – can improve stability, performance

$$J = \sum_{i=1}^{N} \left((r_i - y_i)^T Q (r_i - y_i) + u_i^T R u_i \right)$$

Reference-tracking

$$J = \sum_{i=1}^{N} \left((r_i - y_i)^T Q (r_i - y_i) + \Delta u_i^T R \Delta u_i \right)$$

Velocity form – similar to integral control, can remove steady-state error.

How is a quadratic program solved?

Lots of computational tools available to solve quadratic programs (**quadprog** in MATLAB)

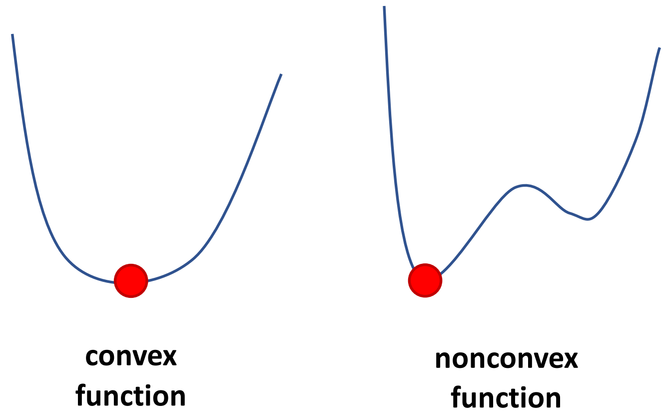
Efficient, due to the fact that quadratic programs are convex. Rely on methods from *convex optimization*.

$$J_k(z) = z^T H z + f^T z$$

Subject to:

$A_{eq} z = b_{eq}$

$A_{ineq} z \leq b_{ineq}$



Property of convex functions: any local minimum is also the global minimum

How to solve:

1. If there are no constraints, set the derivative of J to zero

$$0 = \frac{\partial J}{\partial z} = Hz + f \implies z = -H^{-1} f$$

2. If there are only equality constraints, use method of Lagrange multipliers
3. If there are inequality constraints, the solution must satisfy the KKT conditions

Convert cost function to the standard form of a quadratic program

Predict the model forward:

$$x_{k+1} = Ax_k + Bu_k$$

$$\begin{aligned} x_{k+2} &= Ax_{k+1} + Bu_{k+1} \\ &= A[Ax_k + Bu_k] + Bu_{k+1} \\ &= A^2 x_k + \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u_k \\ u_{k+1} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} x_{k+3} &= Ax_{k+2} + Bu_{k+2} \\ &= A^3 x_k + \begin{bmatrix} A^2 B & AB & B \end{bmatrix} \begin{bmatrix} u_k \\ u_{k+1} \\ u_{k+2} \end{bmatrix} \end{aligned}$$

$$\vdots$$

$$\underbrace{\begin{bmatrix} x_{k+1} \\ x_{k+2} \\ \vdots \\ x_{k+N} \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{E} x_k + \underbrace{\begin{bmatrix} B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+N-1} \end{bmatrix}}_{\hat{u}}$$

$$\hat{x} = Ex_k + F\hat{u}$$

Convert cost function to the standard form of a quadratic program

$$J_k = \sum_{i=1}^{N} x_{k+i}^T Q x_{k+i} + u_{k+i-1}^T R u_{k+i-1}$$

$$= \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ \vdots \\ x_{k+N} \end{bmatrix}^T \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ \vdots \\ x_{k+N} \end{bmatrix} + \begin{bmatrix} u_{k} \\ u_{k+1} \\ \vdots \\ u_{k+N-1} \end{bmatrix}^T \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix} \begin{bmatrix} u_{k+1} \\ u_{k+2} \\ \vdots \\ u_{k+N} \end{bmatrix}$$

$$= \hat{x}^T \hat{Q} + \hat{x} + \hat{u}^T + \hat{R} + \hat{u}$$

$$J_k = \hat{u}^T \underbrace{\left(F^T \hat{Q} F + \hat{R}\right)}_{H} \hat{u} + 2 \underbrace{\hat{x}_k E^T \hat{Q} F}_{f} \hat{u}$$

Connection between MPC and LQR (constrained LQR)

$$J = \int_{t_0}^{t_1} \left[x^T Q x + u^T R u\right] dt + \underbrace{\int_{t_1}^{\infty} \left[x^T Q x + u^T R u\right] dt}$$

$J_1 = x(t_1)^T P x(t_1)$ where P is the solution to the CARE $A^T P + PA - PBR^{-1}B^T P + Q = 0$

$$J = \sum_{i=1}^{N-1} x_i^T Q x_i + u_{i-1}^T R u_{i-1} + \underbrace{\sum_{i=N}^{\infty} x_i^T Q x_i + u_{i-1}^T R u_{i-1}}$$

$J_1 = x_N^T P x_N$ where P is the solution to the DARE $P = Q + A^T P A - (A^T P B)(R + B^T P B)^{-1}(B^T P A)$

Constrained LQR: $J = \sum_{i=1}^{N-1} \left[x_i^T Q x_i + u_{i-1}^T R u_{i-1}\right] + x_N^T P x_N$