Helpful Proofs

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$\cancel{*}$ to ignore amplitude terms, ignore on both sides

① $e^{jr\sqrt{(x-u)^2+s^2}} \approx \int e^{jk_u(u\pm x)\pm jk_s s}\,dk_u \quad , \quad k_s^2 = r^2 - k_u^2$

② $\dfrac{e^{jr\sqrt{(x-u)^2+(y-v)^2+s^2}}}{\sqrt{(x-u)^2+(y-v)^2+s^2}} \approx \iint \dfrac{e^{jk_u(u\pm x)+jk_v(v\pm y)\pm jk_s s}}{k_s}\,dk_u\,dk_v \quad , \quad k_s^2 = r^2 - k_u^2 - k_v^2$

③ $e^{jr\sqrt{(x-u)^2+(z-s)^2}} \approx \iint e^{jk_u(u\pm x)+jk_s(s\pm z)}\,dk_u\,dk_s$

④ $e^{jr\sqrt{(x-u)^2+(y-v)^2+(z-s)^2}} \approx \iiint e^{jk_u(u\pm x)+jk_v(v\pm y)+jk_s(s\pm z)}\,dk_u\,dk_v\,dk_s$

① $I = e^{jr\sqrt{(x-u)^2+s^2}}$

$$FT_{1D}^{(u)}\left[e^{jr\sqrt{(x-u)^2+s^2}}\right] = e^{\pm jk_u x}\int e^{jr\sqrt{u^2+s^2}-jk_u u}\,du$$

$$f(u) = r\sqrt{u^2+s^2} - k_u u$$

$$\frac{\partial f(u)}{\partial u}\bigg|_{u=u_0} = \frac{r u_0}{\sqrt{u_0^2+s^2}} - k_u = 0$$

$$r^2 u_0^2 = k_u^2(u_0^2+s^2)$$

$$(r^2-k_u^2)\,u_0^2 = k_u^2 s^2$$

$$\boxed{u_0^2 = \frac{k_u^2 s^2}{r^2-k_u^2}}$$

$$f(u_0) = r\sqrt{\frac{k_u^2 s^2}{r^2-k_u^2} + \frac{(r^2-k_u^2)s^2}{r^2-k_u^2}} - \frac{k_u^2 s}{\sqrt{r^2-k_u^2}}$$

$$= \frac{r(rs) - k_u^2 s}{\sqrt{r^2-k_u}}$$

$$= \pm s\sqrt{r^2-k_u^2} = \pm k_s s \quad , \quad k_s = \sqrt{r^2-k_u^2}$$

$$\therefore \quad FT_{1D}^{(u)}\left[e^{jr\sqrt{(x-u)^2+s^2}}\right] \approx e^{\pm jk_u x \pm jk_s}$$

$$e^{jr\sqrt{(x-u)^2+s^2}} = \int e^{jk_u(u\pm x)\pm jk_s s}\,dk_u$$

② $$I = \frac{e^{jr\sqrt{(x-u)^2+(y-v)^2+s^2}}}{\sqrt{(x-u)^2+(y-v)^2+s^2}}$$

$$FT_{2D}^{(u,v)}\left[I \right] = e^{\pm jk_u x \pm jk_v y} \iint \frac{e^{jr\sqrt{u^2+v^2+s^2} - k_u u - k_v v}}{\sqrt{u^2+v^2+s^2}} \, du \, dv$$

$$f(u,v) = r\sqrt{u^2+v^2+s^2} - k_u u - k_v v \quad , \quad g(u,v) = \frac{1}{\sqrt{u^2+v^2+s^2}}$$

$$\left.\frac{\partial f(u,v)}{\partial u}\right|_{u=u_0,\, v=v_0} = \frac{r u_0}{\sqrt{u_0^2+u_0^2+s^2}} - k_u = 0$$

$$u_0^2 = \frac{k_u^2 (v_0^2+s^2)}{r^2 - k_u^2}$$

$$\left.\frac{\partial f(u,v)}{\partial v}\right|_{u=u_0,\, v=v_0} = \frac{r v_0}{\sqrt{u_0^2+v_0^2+s^2}} - k_v = 0$$

$$r^2 v_0^2 = k_v^2 (u_0^2 + v_0^2 + s^2)$$

$$r^2 v_0^2 = k_v^2 \left(\frac{k_u^2 (v_0^2+s^2)}{r^2-k_u^2} + \frac{(r^2-k_u^2)(v_0^2+s^2)}{r^2-k_u^2} \right)$$

$$\cancel{r^2} v_0^2 = \frac{k_v^2 \cancel{r^2} (v_0^2+s^2)}{r^2-k_u^2}$$

$$(r^2-k_u^2) v_0^2 = k_v^2 (v_0^2+s^2)$$

$$\begin{cases} v_0^2 = \dfrac{k_v^2 s^2}{r^2-k_u^2-k_v^2} \\[2ex] u_0^2 = \dfrac{k_v^2 s^2}{r^2-k_u^2-k_v^2} \end{cases}$$

$$f(u_0, v_0) = r\sqrt{\frac{k_u^2 s^2 + k_v^2 s^2 + (r^2-k_u^2-k_v^2)s^2}{r^2-k_u^2-k_v^2}} - \frac{k_u^2 s}{\sqrt{r^2-k_u^2-k_v^2}} - \frac{k_v^2 s}{\sqrt{r^2-k_u^2-k_v^2}}$$

$$f(u_0,v_0) = \pm k_s s \quad , \quad k_s = \sqrt{r^2-k_u^2-k_v^2}$$

note: $$u_0^2 + v_0^2 + s^2 = \frac{r^2 s^2}{r^2-k_u^2-k_v^2}$$

$$f_{uu} = \frac{r}{\sqrt{u_0^2+v_0^2+s^2}} - \frac{r u_0^2}{(u_0^2+v_0^2+s^2)^{3/2}}$$

$$= \frac{r(v_0^2+s^2)}{(u_0^2+v_0^2+s^2)^{3/2}}$$

$$= r\left(\frac{k_v^2 s^2}{r^2-k_u^2-k_v^2} + s^2 \right) \frac{(r^2-k_u^2-k_v^2)^{3/2}}{r^3 s^3}$$

$$= \frac{(r^2 - k_u^2)\sqrt{r^2 - k_u^2 - k_v^2}}{r^2 s}$$

$$f_{uu} = \frac{k_s(r^2 - k_u^2)}{r^2 s}$$

$$f_{vv} = \frac{k_s(r^2 - k_v)}{r^2 s}$$

$$f_{uv} = \frac{-r\, u_0 v_0}{(u_0^2 + v_0^2 + s^2)^{3/2}} = \frac{-r\, k_u k_v s^2}{r^2 - k_u^2 - k_v^2} \cdot \frac{(r^2 - k_u^2 - k_v^2)^{3/2}}{r^3 s^3}$$

$$f_{uv} = \frac{-k_u k_v k_s}{r^2 s}$$

$$g(u_0, v_0) = \frac{k_s}{r s}$$

$$\frac{g(u_0, v_0)}{\sqrt{f_{uu} f_{vv} - f_{uv}^2}} = \frac{k_s}{r s}\left(\frac{k_s^2 (r^2 - k_u^2)(r^2 - k_v^2)}{r^4 s^2} - \frac{k_s^2 k_u^2 k_v^2}{r^4 s^2} \right)^{-1/2}$$

$$= \frac{\cancel{k_s}}{\cancel{r s}} \cdot \frac{r^{\cancel{2}} \cancel{s}}{\cancel{k_s}\sqrt{r^4 - r^2(k_u^2 + k_v^2) + k_u^2 k_v^2 - k_u^2 k_v^2}}$$

recall: $r^2 - k_s^2 = k_u^2 + k_v^2$

$$= \frac{r}{\sqrt{r^2 - r^4 + r^2 k_s^2}} = \frac{1}{k_s}$$

$$\therefore \quad FT_{2D}^{(u,v)}[I] \approx \frac{e^{\pm j k_u x \pm j k_v y \pm j k_s s}}{k_s}$$

$$I \approx \iint \frac{e^{j k_u (u \pm x) + j k_v (v \pm y) \pm k_s s}}{k_s}\, dk_u\, dk_v$$

③ $I = e^{j r \sqrt{(x-u)^2 + (z-s)^2}}$

$$FT_{2D}^{(u,s)}[I] = e^{\pm j k_u x \pm j k_s s} \iint e^{j r \sqrt{u^2 + s^2} - k_u u - k_s s}\, dk_u\, dk_s$$

$$f(u,s) = r\sqrt{u^2 + s^2} - k_u u - k_s s$$

$$\left.\frac{\partial f(u,s)}{\partial u}\right|_{u=u_0,\, s=s_0} = \frac{r u_0}{\sqrt{u_0^2 + s_0^2}} - k_u = 0$$

$$u_0^2 = \frac{k_u^2 s_0^2}{r^2 - k_u^2}$$

$$r^2 s_o^2 = k_s^2 (u_o^2 + s_o^2)$$

$$r^2 s_o^2 = k_s^2 \left(\frac{k_u^2 s_o^2 + (r^2 - k_u^2) s_o^2}{r^2 - k_u^2} \right)$$

$$\cancel{r^2} s_o^2 = \frac{\cancel{r^2} k_s^2 s_o^2}{r^2 - k_u^2}$$

$$(r^2 - k_u^2) s_o^2 = k_s^2 s_o^2$$

$$(r^2 - k_u^2 - k_s^2) s_o^2 = 0$$

$$\begin{cases} s_o = 0 \\ u_o = 0 \end{cases}$$

$$f(u_o, s_o) = 0$$

$$\therefore \ FT_{2D}^{(u,s)}[I] \approx e^{\pm j k_u x \pm j k_s z}$$

$$I \approx \iint e^{j k_u (u \pm x) + j k_s (s \pm z)} \, dk_u \, dk_s$$

4 $$I = e^{jr\sqrt{(x-u)^2 + (y-v)^2 + (z-s)^2}}$$

$$FT_{3D}^{(u,v,s)}[I] = e^{\pm j k_u x \pm j k_v y \pm k_s z} \iiint e^{jr\sqrt{u^2+v^2+s^2} - j k_u u - j k_v v - j k_s s} \, du \, dv \, ds$$

$$f_u = \frac{r u_o}{\sqrt{u_o^2 + v_o^2 + s_o^2}} - k_u = 0 \ , \qquad f(u,v,s) = r\sqrt{u^2 + v^2 + s^2} - k_u u - k_v v - k_s s$$

$$\begin{cases} u_o^2 = \dfrac{k_u^2 (v_o^2 + s_o^2)}{r^2 - k_u^2} \\[2ex] v_o^2 = \dfrac{k_v^2 (u_o^2 + s_o^2)}{r^2 - k_v^2} \\[2ex] s_o^2 = \dfrac{k_s^2 (u_o^2 + v_o^2)}{r^2 - k_s^2} \end{cases}$$

$$u_o \to v_o : \quad r^2 v_o^2 = k_v^2 \left(\frac{k_u^2 (v_o^2 + s_o^2) + (r^2 - k_u^2)(v_o^2 + s_o^2)}{r^2 - k_u^2} \right)$$

$$(r^2 - k_v^2) v_o^2 = k_v^2 (v_o^2 + s_o^2)$$

$$(r^2 - k_u^2 - k_v^2) v_o^2 = k_v^2 s_o^2$$

$$v_o^2 = \frac{k_v^2 s_o^2}{} \qquad u_o^2 = \frac{k_u^2 s_o^2}{}$$

$$v_0^2 = \frac{k_v^2 s_0^2}{r^2 - k_u^2 - k^2} \quad , \qquad u_0^2 = \frac{k_u^2 s_0^2}{r^2 - k_u^2 - k_v^2}$$

$v_0 \rightarrow s_0$

$$r^2 s_0^2 = k_s^2 (u_0^2 + v_0^2 + s_0^2)$$

$$r^2 s_0^2 = k_s^2 \left(\frac{k_u^2 s_0^2 + k_v^2 s_0^2 + (r^2 - k_u^2 - k_v^2) s_0^2}{r^2 - k_u^2 - k_v^2} \right)$$

$$(r^2 - k_u^2 - k_v^2) s_0^2 = k_s^2 s_0^2$$

$$(r^2 - k_u^2 - k_v^2 - k_s^2) s_0^2 = 0$$

$$s_0 = 0$$

$$\left\{ \begin{array}{l} u_0 = 0 \\ v_0 = 0 \\ s_0 = 0 \end{array} \right.$$

$$f(u_0, v_0, s_0) = 0$$

$$\therefore \quad FT_{3D}^{(u,v,s)}[I] \approx e^{\pm j k_u x \pm j k_v y \pm j k_s z}$$

$$I \approx \iiint e^{j k_u (u \pm x) + j k_v (v \pm y) + j k_s (s \pm z)} \, dk_u \, dk_v \, dk_z$$