Introduction to MIMO-FMCW Radar with MATLAB Examples

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CHAPTER 1

INTRODUCTION

This document serves as an introduction to concepts pertaining to frequency modulated continuous wave (FMCW) radar signal processing. It is highly recommended to follow the derivations step-by-step by writing each equation, understanding its importance/relevance, and showing the full work for each step of the derivation to develop a rigorous intuition of the signal model. MATLAB examples are also provided for further study.

CHAPTER 2

PRELIMINARIES OF FMCW SIGNALING

2.1 Monostatic FMCW Signal Model

2.1.1 FMCW Chirp

In this section we derive the simple signal model for a monostatic (Tx and Rx are assumed to be at the same position in space) scenario with a single, ideal point scatterer (reflector). By definition, an FMCW signal, called an FMCW chirp, has a frequency linearly increasing with time. We can express this relationship as

$$f(t) \triangleq f_0 + Kt, \quad 0 \leq t \leq T, \tag{2.1}$$

where f is the instantaneous frequency as a function of t, f_0 is the start frequency (frequency at time $t = 0$), K is the slope of the chirp, the bandwidth covered by the chirp is given by $B = KT$, and t is called the "fast time" variable.

The definition of instantaneous frequency is given by

$$f(t) \triangleq \frac{1}{2\pi} \frac{\partial}{\partial t} \phi(t), \tag{2.2}$$

where $\phi(t)$ is the phase of the signal, containing the frequency content, e.g.,

$$m(t) \triangleq e^{j\phi(t)}. \tag{2.3}$$

Hence, the phase term $\phi(t)$ can be expressed as

$$\phi(t) = 2\pi \int_0^t f(t')dt'. \tag{2.4}$$

Substituting (2.1) into (2.4) yields

$$\begin{aligned} \phi(t) &= 2\pi \int_0^t (f_0 + Kt')dt', \\ &= 2\pi \left[f_0 t' + 0.5K(t')^2 \right]_0^t, \\ &= 2\pi (f_0 t + 0.5Kt^2). \end{aligned} \tag{2.5}$$

2.1.2 FMCW Beat Signal

Thus, by the definition of the FMCW chirp as a sinusoidal signal whose frequency linearly increases with time, we can derive the phase of the signal and express the transmitted FMCW pulse by substituting (2.5) into (2.3) as

$$m(t) = e^{j\phi(t)} = e^{j2\pi(f_0 t + 0.5Kt^2)}, \quad 0 \leq t \leq T, \tag{2.6}$$

where T is the duration of the FMCW pulse.

Assuming the monostatic radar antenna element is located at (x', y', z') and the point scatterer is located at (x_0, y_0, z_0), the distance between the radar and point target can be expressed as

$$R_0 = \sqrt{(x_0 - x')^2 + (y_0 - y')^2 + (z_0 - z')^2}. \tag{2.7}$$

The FMCW pulse, expressed in (2.6) is transmitted from the antenna, propagates through space traveling a distance of R_0 to the point scatter, reflects from the point scatterer, travels another R_0 back to the radar. The round-trip time delay required for this propagation is given by

$$\tau_0 = \frac{2R_0}{c}, \tag{2.8}$$

where c is the speed of light.

As a result, the signal received at the radar is a time-delayed and scaled version of the transmitted signal as

$$\begin{aligned}
\hat{s}(t) &= \frac{\sigma}{R_0^2} m(t - \tau_0), \\
&= \frac{\sigma}{R_0^2} e^{j2\pi(f_0(t-\tau_0) + 0.5K(t-\tau_0)^2)}, \\
&= \frac{\sigma}{R_0^2} e^{j2\pi(f_0 t - f_0\tau_0 + 0.5Kt^2 - K\tau_0 t + 0.5K\tau_0^2)}, \\
&= \frac{\sigma}{R_0^2} e^{j2\pi(f_0 t + 0.5Kt^2 - f_0\tau_0 - K\tau_0 t + 0.5K\tau_0^2}, \\
&= \frac{\sigma}{R_0^2} \underbrace{e^{j2\pi(f_0 t + 0.5Kt^2)}}_{m(t)} e^{-j2\pi(f_0\tau_0 + K\tau_0 t - 0.5K\tau_0^2)},
\end{aligned} \tag{2.9}$$

where σ is known as the "reflectivity" of the scatterer (how reflective the point target is) and the $1/R_0^2$ term is the round-trip path loss or amplitude decay (the intuition here is that farther targets give weaker reflections). Note that the received signal $\hat{s}(t)$ contains a factor which is the transmitted signal $m(t)$.

The next step in the signal chain is known as "dechirping" and removes this factor of $m(t)$ by multiplying the conjugate of the received signal by the transmitted signal. After dechirping, the signal is known as the IF signal or beat signal. The beat signal can be expressed as

$$\begin{aligned} s(t) &= m(t)\hat{s}^*(t), \\ &= e^{j2\pi(f_0 t+0.5Kt^2)}\frac{\sigma}{R_0^2}e^{-j2\pi(f_0 t+0.5Kt^2)}e^{j2\pi(f_0\tau_0+K\tau_0 t-0.5K\tau_0^2)}, \\ &= \frac{\sigma}{R_0^2}e^{j2\pi(f_0 t+0.5Kt^2)-j2\pi(f_0 t+0.5Kt^2)}e^{j2\pi(f_0\tau_0+K\tau_0 t-0.5K\tau_0^2)}, \\ &= \frac{\sigma}{R_0^2}e^{j2\pi(f_0\tau_0+K\tau_0 t-0.5K\tau_0^2)}. \end{aligned} \tag{2.10}$$

In radar literature, it is common practice to express the beat signal as a function of k rather than t, where $k(t)$ is the instantaneous wavenumber corresponding to the instantaneous frequency $f(t)$ given by

$$k(t) \triangleq \frac{2\pi}{c}f(t). \tag{2.11}$$

Substituting (2.1) into (2.11) yields

$$k(t) = \frac{2\pi}{c}(f_0 + Kt), \quad 0 \le t \le T. \tag{2.12}$$

Hence, let us express $s(t)$ as a function of k by rewriting (2.10) in terms of (2.12). Note that for short distances τ_0^2 is negligible and the last term in (2.10) can be ignored.

Substituting (2.8) into (2.10) and recalling the definitions in (2.1) and (2.11) yields

$$\begin{aligned} s(t) &= \frac{\sigma}{R_0^2} e^{j2\pi(f_0\tau_0 + K\tau_0 t)}, \\ &= \frac{\sigma}{R_0^2} e^{j2\pi\tau_0(f_0 + Kt)}, \\ &= \frac{\sigma}{R_0^2} e^{j2\pi\frac{2R_0}{c}(f_0 + Kt)}, \\ &= \frac{\sigma}{R_0^2} e^{j2\pi\frac{2R_0}{c}f(t)}, \\ &= \frac{\sigma}{R_0^2} e^{j2R_0\frac{2\pi}{c}f(t)}, \\ s(k) &= \frac{\sigma}{R_0^2} e^{j2R_0 k}, \quad 0 \le t \le T. \end{aligned} \tag{2.13}$$

More commonly, the beat signal is expressed as

$$s(k) = \frac{\sigma}{R_0^2} e^{j2kR_0} \tag{2.14}$$

We have derived a compact representation of the FMCW beat signal, $s(k)$. It is clear from (2.14) that the frequency of the beat signal corresponds directly with the radial distance, known as the "range", R_0. For this single point scatter case, the radar beat signal $s(k)$ is a single tone sinusoid whose frequency corresponds with R_0. Hence, the Fourier transform of $s(k)$ would have a single peak at a position corresponding to the range R_0.

This section detailed the derivation of the radar beat signal. However, now that the beat signal has been derived, it can be applied simply by the definition of $s(k)$ in (2.14) without needing to go through all steps every time.

2.1.3 Multiple Targets

Suppose N point scatterers are in the radar FOV such that the n-th point scatterer is located at (x_n, y_n, z_n) and has reflectivity σ_n. In this case, the transmit signal is the same, but the received signal is now a sum (by superposition) of the received signals from each of the point scatterers. As a result, the radar beat signal can be written as

$$s(k) = \sum_{n=1}^{N} \frac{\sigma_n}{R_n^2} e^{j2kR_n}, \quad 0 \leq t \leq T, \tag{2.15}$$

which is clearly a sum of sinusoidal signals whose frequencies depend on the distances R_n. Hence, the Fourier transform of (2.15) would result in multiple peaks such that the location of the n-th peak corresponds with the distance R_n.

Alternatively, if we model the target as a continuous set of point targets, rather than a discrete set as in (2.15), where the target is located in a volume V inside (x, y, z) space, we express the reflectivity as a continuous function $p(x, y, z)$ and the summation expressed in (2.15) becomes an integral as

$$s(k) = \iiint_V \frac{p(x, y, z)}{R^2} e^{j2kR} dxdydz, \tag{2.16}$$

where

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \tag{2.17}$$

recall the position of the antenna is (x', y', z').

CHAPTER 3

METHOD OF STATIONARY PHASE

In this chapter, we discuss the important topic of the Method of Stationary Phase (MSP) required for many of the subsequent imaging algorithms detailed in following chapters. A highly recommended exercise to the reader is to follow the example derivation in Section 3.3.1 closely and derive the helpful approximations given in (3.22)–(3.27) showing every step. Additionally, a careful review of the spatial Fourier relationships in Appendix A is recommended.

As discussed in [1, 2, 3], the general form of the n-dimensional Method of Stationary Phase (MSP) can be expressed as the following. A rigorous mathematical perspective is offered in [3], whereas our discussion does not comprehensively address the underlying assumptions and constraints. Rather, this section is meant to serve as a resource to researchers and engineers to apply the results of the MSP approximation to near-field spherical wave decomposition problems.

Given an oscillatory integral with a wide phase variation of the form

$$I(\mathbf{x}) = \int g(\mathbf{x}) e^{jf(\mathbf{x})} d\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{3.1}$$

where $f(\mathbf{x})$ is assumed to be twice-continuously differentiable, the major contribution to the quantity $I(\mathbf{x})$ is from the stationary points, $\mathbf{x}_0$, which are calculated by

$$\nabla f(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0} = 0 \tag{3.2}$$

Thus, the integral can be approximated by

$$I(\mathbf{x}) \approx \frac{g(\mathbf{x}_0)}{\sqrt{\det \mathbf{A}}} e^{jf(\mathbf{x}_0)}, \tag{3.3}$$

where $\mathbf{x}_0$ is the set of stationary points and $\mathbf{A}$ is the Hessian matrix of $f(\mathbf{x})$ evaluated at $\mathbf{x}_0$ and defined as

$$\mathbf{A} = \left(\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \right) \Bigg|_{\mathbf{x}=\mathbf{x}_0}. \tag{3.4}$$

For the derivations required in this article, we can limit n to 1 or 2 dimensions.

3.1 1-D Method of Stationary Phase

The 1-D MSP can be written as the following. The following integral with the same assumptions as the MSP,

$$I(u) = \int g(u)e^{jf(u)}du, \tag{3.5}$$

can be approximated as

$$I(u) \approx \frac{g(u_0)}{\sqrt{f''(u_0)}}e^{jf(u_0)}, \tag{3.6}$$

where u_0 is the stationary point calculated by

$$\left.\frac{\partial f(u)}{\partial u}\right|_{u=u_0}, \tag{3.7}$$

and $f''(u_0)$ is the second derivative of $f(u)$ evaluated at the stationary point u_0.

3.2 2-D Method of Stationary Phase

Similarly, for the 2-D case, the integral,

$$I(u,v) = \iint g(u,v)e^{jf(u,v)}dudv, \tag{3.8}$$

can be approximated by

$$I(u,v) \approx \frac{g(u_0,v_0)}{\sqrt{f_{uu}f_{vv} - f_{uv}^2}}e^{jf(u_0,v_0)}, \tag{3.9}$$

where the stationary points u_0, v_0 are calculated by

$$\left.\frac{\partial f(u,v)}{\partial u}\right|_{(u=u_0,v=v_0)} = 0, \tag{3.10}$$

$$\left.\frac{\partial f(u,v)}{\partial v}\right|_{(u=u_0,v=v_0)} = 0, \tag{3.11}$$

and f_{uu}, f_{vv}, f_{uv} are the second partial derivatives of $f(u,v)$ evaluated at the stationary points.

3.3 Useful MSP Identities

Using the aforementioned method for the 1-D and 2-D cases, the MSP is applied to several integrals and the corresponding approximations are provided for reference in this section.

We will demonstrate the steps for the approximation below which have been applied to the other spherical wavefronts to yield the approximations in (3.22)–(3.27).

3.3.1 Example MSP Derivation

We consider the linear array case with a monostatic single antenna array being scanned along the x-axis at the positions labeled x'. Further, we consider a 1-D target at some line z_0 in the x-z plane, where the x and x' coordinate systems are coincident. Thus, the radar beat signal can be modeled, neglecting path loss, as

$$s(x', k) = \int p(x) e^{j2kR} dx, \tag{3.12}$$

where R is the radial distance from each of the antenna locations $(x', 0)$ to the target locations (x, z) and is expressed as

$$R = \sqrt{(x - x')^2 + z_0^2}. \tag{3.13}$$

It is desired to approximate the spherical wavefront term in (3.12), e^{j2kR}, as a more tractable expression. Thus, the MSP is exploited. For generality, the following substitutions are made $v = x'$, $r = 2k$, $w = z_0$. The 1-D spatial Fourier transform (A.1) is performed over the u dimension of the spherical wave term and the spatial translation property (A.7) is applied as

$$\mathrm{FT}_{\mathrm{1D}}^{(u)} \left[e^{jr\sqrt{(x-u)^2+w^2}} \right] = e^{-jk_u x} \int e^{jr\sqrt{u^2+w^2} - jk_u u} du. \tag{3.14}$$

The MSP will be applied to the Fourier integral in (3.14), implying for this example

$$g(u) = 1, \tag{3.15}$$

$$f(u) = r\sqrt{u^2 + w^2} - k_u u. \tag{3.16}$$

Using (3.7), the stationary point u_0 can be computed as

$$\left.\frac{\partial f(u)}{\partial u}\right|_{u=u_0} = \frac{ru_0}{\sqrt{u_0^2+w^2}} - k_u = 0, \tag{3.17}$$

$$u_0 = \frac{k_u w}{\sqrt{r^2-k_u^2}}, \tag{3.18}$$

$$f(u_0) = w\sqrt{r^2-k_u^2} \tag{3.19}$$

Finally, u_0 can be substituted into (3.6) ignoring the factor of $1/f''(u_0)$ as

$$\int e^{jr\sqrt{u^2+w^2}-jk_u u}du \approx e^{jw\sqrt{r^2-k_u^2}}. \tag{3.20}$$

Substituting (3.20) into (3.14) yields

$$\mathrm{FT}_{\mathrm{1D}}^{(u)}\left[e^{jr\sqrt{(x-u)^2+w^2}}\right] = e^{-jk_u x+jw\sqrt{r^2-k_u^2}}. \tag{3.21}$$

Taking the 1-D inverse spatial Fourier transform of (3.21) results in (3.22), labeled Approximation 1 below. This example illustrates the key steps of the spherical wave decomposition using the method of stationary phase. Similar analysis has been employed on the other examples below yielding the corresponding approximations using the MSP.

3.3.2 Useful MSP Approximations

Approximation 1:

$$e^{jr\sqrt{(x-u)^2+w^2}} \approx \int e^{jk_u(u-x)+jk_w w}dk_u, \tag{3.22}$$

where

$$k_w^2 = r^2 - k_u^2. \tag{3.23}$$

Approximation 2:

$$\frac{e^{jr\sqrt{(x-u)^2+(y-v)^2+w^2}}}{\sqrt{(x-u)^2+(y-v)^2+w^2}} \approx \iint \frac{1}{k_w} e^{jk_u(u-x)+jk_v(v-y)+k_w w}dk_u dk_v, \tag{3.24}$$

where

$$k_w^2 = r^2 - k_u^2 - k_v^2. \tag{3.25}$$

Approximation 3:

$$e^{jr\sqrt{(x-u)^2+(z-w)^2}} \approx \iint e^{jk_u(u-x)+jk_w(w-z)} dk_u dk_w. \tag{3.26}$$

Approximation 4:

$$e^{jr\sqrt{(x-u)^2+(y-v)^2+(z-w)^2}} \approx \iiint e^{jk_u(u-x)+jk_v(v-y)+jk_w(w-z)} dk_u dk_v dk_w. \tag{3.27}$$

CHAPTER 4

EFFICIENT NEAR-FIELD SAR IMAGE RECONSTRUCTION ALGORITHMS FOR VARIOUS GEOMETRIES

In this chapter, we detail the efficient image reconstruction algorithms for

4.1 1-D Linear Synthetic Array 1-D Imaging - Fourier-based

In this section, we derive the image reconstruction algorithm for recovering a 1-D reflectivity function from a 1-D linear SAR scenario in the near-field. Given a 1-D linear SISO synthetic array whose elements are located at the points (y', Z_0) in the y-z plane and a 1-D target with reflectivity function $p(y)$ located at the points (y, z_0), the isotropic beat signal can be written as

$$s(y', k) = \int \frac{p(y)}{R^2} e^{j2kR} dy, \tag{4.1}$$

where

$$R = \sqrt{(y - y')^2 + (z_0 - Z_0)^2}. \tag{4.2}$$

Ignoring amplitude terms, applying the MSP derived in (3.22), the spherical phase term in (4.1) can be substituted yielding

$$s(y', k) = \iint p(y) e^{j(k'_y(y'-y) + k_z(z_0 - Z_0))} dy dk'_y, \tag{4.3}$$

where

$$k_z = \sqrt{4k^2 - k_y^2}. \tag{4.4}$$

Rearranging the phase terms in (4.3), a forward spatial Fourier transform on y and inverse spatial Fourier transform on y' become evident as

$$s(y', k) = \int \left[\int p(y) e^{-jk'_y y} dy \right] e^{j(k'_y y' + k_z(z_0 - Z_0))} dk'_y. \tag{4.5}$$

The term inside the brackets can be rewritten as the spatial-spectral representation of the target reflectivity function, $P(k_y)$. Then, performing a forward Fourier transform along y' on both sides simplifies the expression as the following. Note that the distinction between the primed and unprimed domains can be dropped in the spatial Fourier domain as they coincide.

$$s(y', k) = \int \left[P(k_y)e^{jk_z(z_0-Z_0)}\right] e^{jk'_y y'} dk'_y, \tag{4.6}$$

$$S(k_y, k) = P(k_y)e^{jk_z(z_0-Z_0)}, \tag{4.7}$$

$$P(k_y) = S(k_y, k)e^{-jk_z(z_0-Z_0)}. \tag{4.8}$$

For wideband waveforms, (4.8) is evaluated at multiple wavenumbers thus coherent summation is performed over k. Hence, the complete expression for the Fourier-based 1-D image reconstruction algorithm for a 1-D linear SISO synthetic array is

$$p(y) = \int \text{IFT}_{\text{1D}}^{(k_y)} \left[\text{FT}_{\text{1D}}^{(y')}[s(y', k)]e^{-jk_z(z_0-Z_0)}\right] dk. \tag{4.9}$$

4.2 1-D Linear Synthetic Array 2-D Imaging - Range Migration Algorithm

In this section we derive the image reconstruction algorithm for recovering a 2-D reflectivity function from a 1-D linear SAR scenario in the near-field. Given a 1-D linear SISO synthetic array whose elements are located at the points (y', Z_0) in the y-z plane and a 2-D target with reflectivity function $p(y, z)$ located at the points (y, z), the isotropic beat signal can be written as

$$s(y', k) = \iint \frac{p(y, z)}{R^2} e^{j2kR} dydz, \tag{4.10}$$

where

$$R = \sqrt{(y - y')^2 + (z - Z_0)^2}. \tag{4.11}$$

Ignoring amplitude terms, applying the MSP derived in (3.22), the spherical phase term in (4.10) can be substituted yielding

$$s(y', k) = \iiint p(y, z) e^{j(k_y'(y'-y)+k_z(z-Z_0))} dy dz dk_y', \tag{4.12}$$

where

$$k_z = \sqrt{4k^2 - k_y^2}. \tag{4.13}$$

Leveraging conjugate symmetry of the spherical wavefront, (4.12) can be rewritten in the following form to exploit the spatial Fourier transform on z

$$s^*(y', k) = \iiint p(y, z) e^{j(k_y'(y'-y)-k_z(z-Z_0))} dy dz dk_y', \tag{4.14}$$

where $(\bullet)^*$ is the complex conjugate operation.

Rearranging the phase terms in (4.14), a forward spatial Fourier transform on y-z and inverse spatial Fourier transform on y' become evident as

$$s^*(y', k) = \int \left[\iint p(y, z) e^{-j(k_y' y + k_z z)} dy dz \right] e^{j(k_y' y' + k_z Z_0)} dk_y'. \tag{4.15}$$

The term inside the brackets can be rewritten as the spatial-spectral representation of the target reflectivity function, $P(k_y, k_z)$. Then, performing a forward Fourier transform along y' on both sides simplifies the expression as the following. Note that the distinction between the primed and unprimed domains can be dropped in the spatial Fourier domain as they coincide.

$$s^*(y', k) = \int \left[P(k_y, k_z) e^{jk_z Z_0} \right] e^{jk_y' y'} dk_y', \tag{4.16}$$

$$\tilde{S}(k_y, k) = \mathrm{FT}_{\mathrm{1D}}^{(y')}[s^*(y', k)], \tag{4.17}$$

$$\tilde{S}(k_y, k) = P(k_y, k_z) e^{jk_z Z_0}, \tag{4.18}$$

$$P(k_y, k_z) = \tilde{S}(k_y, k) e^{-jk_z Z_0}. \tag{4.19}$$

The direct relationship between $P(k_y, k_z)$ and $\tilde{S}(k_y, k)$ is now obvious in (4.19); however, $P(k_y, k_z)$ is sampled on a uniform k_y-k_z grid and $\tilde{S}(k_y, k)$ is sampled on a uniform k_y-k grid. Before the reflectivity function can be recovered using in inverse Fourier transform, $\tilde{S}(k_y, k)e^{-jk_z Z_0}$ must be interpolated to a uniform k_y-k_z grid using Stolt interpolation, represented by the $\mathcal{S}[\bullet]$ operator, to account for the curvature of the wavefront [4].

$$S(k_y, k_z) = \mathcal{S}\left[\tilde{S}(k_y, k)e^{-jk_z Z_0}\right]. \tag{4.20}$$

Finally, the complete expression for the Fourier-based 2-D image reconstruction algorithm for a 1-D linear SISO synthetic array can be written as

$$p(y, z) = \mathrm{IFT}_{\mathrm{2D}}^{(k_y, k_z)}\left[\mathcal{S}\left[\mathrm{FT}_{\mathrm{1D}}^{(y')}[s^*(y', k)]e^{-jk_z Z_0}\right]\right]. \tag{4.21}$$

4.3 2-D Rectilinear Array 2-D Imaging - Fourier-based

In this section, we derive the image reconstruction algorithm for recovering a 2-D reflectivity function from a 2-D rectilinear SAR scenario in the near-field [5, 6, 7]. Given a 2-D rectilinear SISO synthetic array whose elements are located at the points (x', y', Z_0) in x-y-z space and a 2-D target with reflectivity function $p(x, y)$ located at the points (x, y, z_0), the isotropic beat signal can be written as

$$s(x', y', k) = \iint \frac{p(x, y)}{R^2} e^{j2kR} dxdy, \tag{4.22}$$

where

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z_0 - Z_0)^2}. \tag{4.23}$$

Assuming the points of the target scene are closely located, the R^{-2} factor in (4.22) can be approximated as R^{-1} [6]. Applying the MSP derived in (3.24), the spherical phase term in (4.22) can be substituted yielding

$$s(x', y', k) = \iiiint \frac{p(x, y)}{k_z} e^{j(k'_x(x'-x)+k'_y(y'-y))} e^{jk_z(z_0 - Z_0)} dxdydk'_x dk'_y, \tag{4.24}$$

where

$$k_z = \sqrt{4k^2 - k_x^2 - k_y^2}. \tag{4.25}$$

Rearranging the phase terms in (4.24), a forward spatial Fourier transform on x-y and inverse spatial Fourier transform on x'-y' become evident as

$$s(x'y',k) = \iint \left[\iint \frac{p(x,y)}{k_z} e^{-j(k_x' x + k_y' y)} dxdy \right] e^{j(k_x' x' + k_y' y') + jk_z(z_0 - Z_0)} dk_x' dk_y'. \tag{4.26}$$

The term inside the brackets can be rewritten as the spatial-spectral representation of the target reflectivity function. Then, performing a forward Fourier transform along x'-y' on both sides simplifies the expression as the following. Note that the distinction between the primed and unprimed domains can be dropped in the spatial Fourier domain as they coincide.

$$s(x',y',k) = \int \left[\frac{P(k_x,k_y)}{k_z} e^{jk_z(z_0 - Z_0)} \right] e^{j(k_x' x' + k_y' y')} dk_x' dk_y', \tag{4.27}$$

$$S(k_x,k_y,k) = \frac{P(k_x,k_y)}{k_z} e^{jk_z(z_0 - Z_0)}, \tag{4.28}$$

$$P(k_x,k_y) = S(k_y,k) k_z e^{-jk_z(z_0 - Z_0)}. \tag{4.29}$$

For wideband waveforms, (4.29) is evaluated at multiple wavenumbers thus coherent summation is performed over k. Hence, the complete expression for the Fourier-based 2-D image reconstruction algorithm for a 2-D rectilinear SISO synthetic array is

$$p(x,y) = \int \text{IFT}_{\text{2D}}^{(k_x,k_y)} \left[\text{FT}_{\text{2D}}^{(x',y')} [s(x',y',k)] k_z e^{-jk_z(z_0 - Z_0)} \right] dk. \tag{4.30}$$

4.4 2-D Rectilinear Array 3-D Imaging - Range Migration Algorithm

In this section we derive the image reconstruction algorithm for recovering a 3-D reflectivity function from a 2-D rectilinear SAR scenario in the near-field. Given a 2-D rectilinear SISO synthetic array whose elements are located at the points (x',y',Z_0) in x-y-z space and a 3-D target with reflectivity function $p(x, y, z)$ located at the points (x, y, z), the isotropic beat signal can be written as

$$s(x', y', k) = \iiint \frac{p(x, y, z)}{R^2} e^{j2kR} dxdydz, \tag{4.31}$$

where

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - Z_0)^2}. \tag{4.32}$$

Assuming the points of the target scene are closely located, the R^{-2} factor in (4.31) can be approximated as R^{-1} [6]. Applying the MSP derived in (3.24), the spherical phase term in (4.31) can be substituted yielding

$$s(x', y', k) = \iint \left[\iiint p(x, y, z) e^{j(k'_x(x'-x)+k'_y(y'-y))} e^{jk_z(z-Z_0)} dxdydz \right] dk'_x dk'_y, \tag{4.33}$$

where

$$k_z = \sqrt{4k^2 - k_x^2 - k_y^2}. \tag{4.34}$$

Leveraging conjugate symmetry of the spherical wavefront, (4.33) can be rewritten in the following form to exploit the spatial Fourier transform on z

$$s^*(x', y', k) = \iint \left[\iiint p(x, y, z) e^{j(k'_x(x'-x)+k'_y(y'-y))} e^{-jk_z(z-Z_0)} dxdydz \right] dk'_x dk'_y, \tag{4.35}$$

where $(\bullet)^*$ is the complex conjugate operation.

Rearranging the phase terms in (4.35), a forward spatial Fourier transform on x,y,z and inverse spatial Fourier transform on x',y' become evident as

$$s^*(x', y', k) = \iint \left[\iiint p(x, y, z) e^{-(jk'_x x + jk'_y y + jk_z z)} dxdydz \right] e^{j(k'_x x' + k'_y y') + jk_z Z_0} dk'_x dk'_y. \tag{4.36}$$

The term inside the brackets can be rewritten as the spatial-spectral representation of the target reflectivity function. Then, performing a forward Fourier transform along x', y' on both sides simplifies the expression as the following. Note that the distinction between the primed and unprimed domains can be dropped in the spatial Fourier domain as they coincide.

$$s^*(x',y',k) = \int \left[P(k_x,k_y,k_z)e^{jk_zZ_0}\right] e^{j(k'_x x' + k'_y y')} dk'_x dk'_y, \tag{4.37}$$

$$\tilde{S}(k_x,k_y,k) = \mathrm{FT}_{\mathrm{2D}}^{(x',y')}[s^*(x',y',k)], \tag{4.38}$$

$$\tilde{S}(k_x,k_y,k) = P(k_x,k_y,k_z)e^{jk_zZ_0}, \tag{4.39}$$

$$P(k_x,k_y,k_z) = \tilde{S}(k_x,k_y,k)e^{-jk_zZ_0}. \tag{4.40}$$

The direct relationship between $P(k_x,k_y,k_z)$ and $\tilde{S}(k_x,k_y,k)$ is now obvious in (4.40); however, $P(k_x,k_y,k_z)$ is sampled on a uniform k_x-k_y-k_z grid and $\tilde{S}(k_x,k_y,k)$ is sampled on a uniform k_x-k_y-k grid. Before the reflectivity function can be recovered using an inverse Fourier transform, $\tilde{S}(k_x,k_y,k)e^{-jk_zZ_0}$ must be interpolated to a uniform k_x, k_y-k_z grid using the Stolt interpolation, represented by the $\mathcal{S}[\bullet]$ operator, to account for the curvature of the wavefront [4].

$$S(k_x,k_y,k_z) = \mathcal{S}\left[\tilde{S}(k_x,k_y,k)e^{-jk_zZ_0}\right]. \tag{4.41}$$

Finally, the complete expression for the Fourier-based 3-D image reconstruction algorithm for a 2-D rectilinear SISO synthetic array can be written as

$$p(x,y,z) = \mathrm{IFT}_{\mathrm{3D}}^{(k_x,k_y,k_z)}\left[\mathcal{S}\left[\mathrm{FT}_{\mathrm{2D}}^{(x',y')}[s^*(x',y',k)]e^{-jk_zZ_0}\right]\right]. \tag{4.42}$$

4.5 1-D Circular Synthetic Array 2-D Imaging - Polar Formatting Algorithm

In this section, we derive the image reconstruction algorithm for recovering a 2-D reflectivity function from a 1-D circular SAR scenario in the near-field. Given a 1-D circular SISO synthetic array whose elements are located at the points $(R_0\cos\theta, R_0\sin\theta)$ in the x-z plane at $y = 0$, where R_0 and θ are the constant radial distance from the antenna elements to the origin and the angular dimension, respectively, and a 2-D target with reflectivity function $p(x, z)$ located at the points (x, z), the isotropic beat signal can be written as

$$s(\theta, k) = \iint \frac{p(x, z)}{R^2} e^{j2kR} dx dz, \tag{4.43}$$

where

$$R = \sqrt{(x - R_0 \cos\theta)^2 + (z - R_0 \sin\theta)^2}. \tag{4.44}$$

The MSP derived in (3.26) can be applied to the spherical phase term in (4.43) after the following substitutions

$$x' = R_0 \cos\theta, \tag{4.45}$$

$$z' = R_0 \cos\theta, \tag{4.46}$$

$$k'_x = k_r \cos\alpha, \tag{4.47}$$

$$k'_z = k_r \cos\alpha, \tag{4.48}$$

$$k_r^2 = k_x'^2 + k_z'^2, \tag{4.49}$$

yielding

$$e^{j2kR} \approx \iint e^{j(k'_x(x'-x)+k'_z(z'-z))} dk'_x dk'_z. \tag{4.50}$$

Neglecting path loss, (4.43) and (4.50) can be combined as

$$s(\theta, k) = \iiiint p(x, z) e^{j(k'_x(x'-x)+k'_z(z'-z))} dx dz dk'_x dk'_z. \tag{4.51}$$

Rearranging the phase terms in (4.51), a forward spatial Fourier transform on x-z and inverse spatial Fourier transform on x'-z' become evident as

$$s(\theta, k) = \iint \left[\iint p(x, z) e^{-j(k'_x x + k'_z z)} dx dz \right] e^{j(k'_x x' + k'_z z')} dk'_x dk'_z. \tag{4.52}$$

The term inside the brackets can be rewritten as the spatial spectral representation of the target reflectivity function, $P(k_x, k_z)$. Then using the relations (4.45)-(4.49), the expression in (4.52) can be rewritten as

$$s(\theta, k) = \iint P(k_x, k_z) e^{j(k_r \cos\theta R_0 \cos\alpha + k_r \sin\theta R_0 \sin\alpha)} k_r dk_r d\alpha. \tag{4.53}$$

Rewriting the spectral $P(k_x, k_z)$ as its equivalent spectral polar form $P(\alpha, k_r)$ and simplifying the phase term

$$s(\theta, k) = \int \left[\int P(\alpha, k_r) e^{jk_r R_0 cos(\theta-\alpha)} d\alpha \right] k_r dk_r. \tag{4.54}$$

The term inside the brackets in (4.54) is a convolution operation in the θ domain, where the θ and α domains are coincident and can be exploited using Fourier relations by taking a Fourier transform across θ on both sides of the equation as

$$S(k_\theta, k) = \int P(k_\theta, k_r) \times \mathrm{FT}_{\mathrm{1D}}^{(\theta)} \left[e^{jk_r R_0 \cos\theta} \right] k_r dk_r. \tag{4.55}$$

Considering only the values lying on the Ewald sphere, $k_r^2 = 4k^2$ imposes a δ-function behavior of the integrand in (4.55) with respect to k_r [8]. As such, (4.55) can be simplified as such, substituting $k_r = 2k$,

$$P(k_\theta, k_r) = S(k_\theta, k) G^*(k_\theta, k), \tag{4.56}$$

where

$$G(k_\theta, k) = \mathrm{FT}_{\mathrm{1D}}^{(\theta)} \left[e^{j2kR_0 \cos\theta} \right]. \tag{4.57}$$

The spatial spectral reflectivity function in polar coordinates can be recovered from (4.56) as

$$P(\theta, k_r) = \mathrm{IFT}_{\mathrm{1D}}^{(k_\theta)} \left[S(k_\theta, k) G^*(k_\theta, k) \right]. \tag{4.58}$$

Finally, the reflectivity function $p(x, z)$ can be recovered using a nonuniform FFT (NUFFT) [9] or via interpolation to the rectangular spatial Fourier domain k_x-k_z followed by a uniform

IFFT. This interpolation operation, known as the polar formatting algorithm (PFA), is denoted by $\mathcal{P}[\bullet]$. Thus, the final step in the image recovery process is (prime notation will be ignored for the remainder of this derivation as the primed and unprimed coordinate systems are coincident)

$$p(x,z) = \text{IFT}_{\text{2D}}^{(k_x,k_z)} \left[\mathcal{P}[P(\theta,k_r)]\right]. \tag{4.59}$$

Finally, the complete expression for the Fourier-based 2-D image reconstruction algorithm for a 1-D circular SISO synthetic array can be written as

$$p(x,z) = \text{IFT}_{\text{2D}}^{(k_x,k_z)} \left[\mathcal{P}\left[\text{IFT}_{\text{1D}}^{(k_\theta)}\left[S(k_\theta,k)\text{FT}_{\text{1D}}^{(\theta)}\left[e^{j2kR_0\cos\theta}\right]^*\right]\right]\right]. \tag{4.60}$$

4.6 2-D Cylindrical Synthetic Array 3-D Imaging - Polar Formatting Algorithm

In this section, we derive the image reconstruction algorithm for recovering a 3-D reflectivity function from a 2-D cylindrical SAR (also known as ECSAR) scenario in the near-field. Given a 2-D cylindrical SISO synthetic array whose elements are located at the points $(R_0\cos\theta, y', R_0\sin\theta)$ in x-y-z space, where R_0 and θ are the constant radial distance from the antenna elements to the origin and the angular dimension, respectively, and a 3-D target with reflectivity function $p(x,y,z)$ located at the points (x,y,z), the isotropic beat signal can be written as

$$s(\theta,y',k) = \iiint \frac{p(x,y,z)}{R^2} e^{j2kR} dx dy dz, \tag{4.61}$$

where

$$R = \sqrt{(x-R_0\cos\theta)^2 + (y-y')^2 + (z-R_0\sin\theta)^2}. \tag{4.62}$$

The MSP derived in (3.27) can be applied to the spherical phase term in (4.61) after the following substitutions

$$x' = R_0 \cos\theta, \tag{4.63}$$

$$z' = R_0 \cos\theta, \tag{4.64}$$

$$k'_x = k_r \cos\alpha, \tag{4.65}$$

$$k'_z = k_r \cos\alpha, \tag{4.66}$$

$$k_r^2 = k_x'^2 + k_z'^2 = 4k^2 - k_y'^2, \tag{4.67}$$

yielding

$$e^{j2kR} \approx \iint e^{j(k'_x(x'-x)+k'_y(y'-y)+k'_z(z'-z))} dk'_x dk'_y dk'_z. \tag{4.68}$$

Neglecting path loss, (4.61) and (4.68) can be combined as

$$\begin{aligned} s(\theta, y', k) = \iiint \Big[\iiint & p(x, y, z) \\ & \times e^{j(k'_x(x'-x)+k'_y(y'-y)+k'_z(z'-z))} \\ & \times dxdydz \Big] dk'_x dk'_y dk'_z. \end{aligned} \tag{4.69}$$

Rearranging the phase terms in (4.69), a forward spatial Fourier transform on x-y-z and inverse spatial Fourier transform on x'-y'-z' become evident as

$$s(\theta, y', k) = \iiint \left[\iiint p(x, y, z) e^{-j(k'_x x + k'_y y + k'_z z)} dxdydz \right] e^{j(k'_x x' + k'_y y' + k'_z z')} dk'_x dk'_y dk'_z. \tag{4.70}$$

The term inside the brackets can be rewritten as the spatial spectral representation of the target reflectivity function, $P(k_x, k_y, k_z)$. Then using the relations (4.63)-(4.67), the expression in (4.70) can be rewritten as

$$s(\theta, y', k) = \iiint P(k_x, k_y, k_z) e^{j(k_r \cos\theta R_0 \cos\alpha} e^{j(k_r \sin\theta R_0 \sin\alpha) + k'_y y')} k_r dk'_y dk_r d\alpha. \tag{4.71}$$

Taking a Fourier transform on both side with respect to y', rewriting the spectral $P(k_x, k_y, k_z)$ as its equivalent spectral polar form $P(\alpha, k_y, k_r)$, and simplifying the phase term yields (prime notation will be dropped for the remainder of this derivation as the primed and unprimed coordinate systems are coincident)

$$s(\theta, k_y, k) = \int \left[\int P(\alpha, k_y, k_r) e^{jk_r R_0 cos(\theta - \alpha)} d\alpha \right] k_r dk_r. \tag{4.72}$$

The term inside the brackets in (4.72) is a convolution operation in the θ domain, where the θ and α domains are coincident and can be exploited using Fourier relations by taking a Fourier transform across θ on both sides of the equation as

$$S(k_\theta, k_y, k) = \int P(k_\theta, k_y, k_r) \mathrm{FT}_{\mathrm{1D}}^{(\theta)} \left[e^{jk_r R_0 \cos\theta} \right] k_r dk_r. \tag{4.73}$$

Considering only the values lying on the Ewald sphere, $k_r^2 = 4k^2 - k_y^2$ imposes a δ-function behavior of the integrand in (4.73) with respect to k_r [8]. As such, (4.73) can be simplified as such, substituting $k_r = \sqrt{4k^2 - k_y^2}$,

$$P(k_\theta, k_y, k_r) = S(k_\theta, k_y, k) G^*(k_\theta, k_y, k), \tag{4.74}$$

where

$$G(k_\theta, k_y, k) = \mathrm{FT}_{\mathrm{1D}}^{(\theta)} \left[e^{j\sqrt{4k^2 - k_y^2} R_0 \cos\theta} \right]. \tag{4.75}$$

The spatial spectral reflectivity function in polar coordinates can be recovered from (4.74) as

$$P(\theta, k_y, k_r) = \mathrm{IFT}_{\mathrm{1D}}^{(k_\theta)} \left[S(k_\theta, k_y, k) G^*(k_\theta, k_y, k) \right]. \tag{4.76}$$

Finally, the reflectivity function $p(x, y, z)$ can be recovered using a nonuniform FFT (NUFFT) [9] or via interpolation to the rectangular spatial Fourier domain k_x-k_y-k_z followed by a uniform IFFT. This interpolation operation, known as the polar formatting algorithm (PFA), is denoted by $\mathcal{P}[\bullet]$. Thus, the final step in the image recovery process is

$$p(x, y, z) = \mathrm{IFT}_{\mathrm{3D}}^{(k_x, k_y, k_z)} \left[\mathcal{P}[P(\theta, k_y, k_r)] \right]. \tag{4.77}$$

Finally, the complete expression for the Fourier-based 3-D image reconstruction algorithm for a 2-D cylindrical SISO synthetic array can be written as

$$p(x,y,z) = \mathrm{IFT}_{\mathrm{3D}}^{(k_x,k_y,k_z)}\left[\mathcal{P}\left[\mathrm{IFT}_{\mathrm{1D}}^{(k_\theta)}\left[S(k_\theta,k_y,k)\times \mathrm{FT}_{\mathrm{1D}}^{(\theta)}\left[e^{j\sqrt{4k^2-k_y^2}R_0\cos\theta}\right]^*\right]\right]\right]. \qquad (4.78)$$

APPENDIX A

SPATIAL FOURIER TRANSFORM AND RELATIONS

Neglecting amplitude terms, the 1-D, 2-D and 3-D spatial Fourier transforms can be defined as [6]

$$\mathrm{FT}_{\mathrm{1D}}^{(u)}\left[s(u)\right] = S(k_u) = \int s(u) e^{-jk_u u} du, \tag{A.1}$$

$$\mathrm{FT}_{\mathrm{2D}}^{(u,v)}\left[s(u,v)\right] = S(k_u,k_v) = \iint s(u,v) e^{-j(k_u u + k_v v)} dudv, \tag{A.2}$$

$$\mathrm{FT}_{\mathrm{3D}}^{(u,v,w)}\left[s(u,v,w)\right] = S(k_u,k_v,k_w) = \iiint s(u,v,w) e^{-j(k_u u + k_v v + k_w w)} dudvdw. \tag{A.3}$$

Similarly, the 1-D, 2-D and 3-D inverse spatial Fourier transforms can be expressed as

$$\mathrm{IFT}_{\mathrm{1D}}^{(k_u)}\left[S(k_u)\right] = s(u) = \int S(k_u) e^{jk_u u} du, \tag{A.4}$$

$$\mathrm{IFT}_{\mathrm{2D}}^{(k_u,k_v)}\left[S(k_u,k_v)\right] = s(u,v) = \iint S(k_u,k_v) e^{j(k_u u + k_v v)} dudv, \tag{A.5}$$

$$\mathrm{IFT}_{\mathrm{3D}}^{(k_u,k_v,k_w)}\left[S(k_u,k_v,k_w)\right] = s(u,v,w) = \iiint S(k_u,k_v,k_w) e^{j(k_u u + k_v v + k_w w)} dudvdw. \tag{A.6}$$

A shift in the spatial domain results in a corresponding phase shift in the spatial spectral domain. The example given here is in the 3-D spatial domain but holds true for the 2-D and 1-D cases also:

$$\mathrm{FT}_{\mathrm{3D}}^{(u,v,w)}\left[s(u-u_0, v-v_0, w-w_0)\right] = e^{-j(k_u u_0 + k_v v_0 + k_w w_0)} S(k_u,k_v,k_w). \tag{A.7}$$

Similarly, a shift in the spatial spectral domain results in a phase shift in the spatial domain:

$$\mathrm{IFT}_{\mathrm{3D}}^{(k_u,k_v,k_w)}\left[S(k_u - k_0^u, k_v - k_0^v, k_w - k_0^w)\right] = e^{j(k_u u_0 + k_v v_0 + k_w w_0)} s(u,v,w). \tag{A.8}$$

These spatial Fourier transform definitions and relations are useful in deriving the reconstruction algorithms discussed in the subsequent appendices.

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