Introduction to MIMO-FMCW Radar with MATLAB Examples

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1 Introduction

This document serves as an introduction to concepts pertaining to frequency modulated continuous wave (FMCW) radar signal processing. It is highly recommended to follow the derivations step-by-step by writing each equation, understanding its importance/relevance, and showing the full work for each step of the derivation to develop a rigorous intuition of the signal model. MATLAB examples are also provided for further study.

2 Monostatic FMCW Signal Model

2.1 FMCW Chirp

In this section we derive the simple signal model for a monostatic (Tx and Rx are assumed to be at the same position in space) scenario with a single, ideal point scatterer (reflector). By definition, an FMCW signal, called an FMCW chirp, has a frequency linearly increasing with time. We can express this relationship as

$$f(t) \triangleq f_0 + Kt, \quad 0 \leq t \leq T, \tag{1}$$

where f is the instantaneous frequency as a function of t, f_0 is the start frequency (frequency at time $t = 0$), K is the slope of the chirp, the bandwidth covered by the chirp is given by $B = KT$, and t is called the "fast time" variable.

The definition of instantaneous frequency is given by

$$f(t) \triangleq \frac{1}{2\pi}\frac{\partial}{\partial t}\phi(t), \tag{2}$$

where $\phi(t)$ is the phase of the signal, containing the frequency content, e.g.,

$$m(t) \triangleq e^{j\phi(t)}. \tag{3}$$

Hence, the phase term $\phi(t)$ can be expressed as

$$\phi(t) = 2\pi \int_0^t f(t')dt'. \tag{4}$$

Substituting (1) into (4) yields

$$\begin{aligned}\phi(t) &= 2\pi \int_0^t (f_0 + Kt')dt', \\ &= 2\pi \left[f_0 t' + 0.5K(t')^2\right]_0^t, \\ &= 2\pi(f_0 t + 0.5Kt^2).\end{aligned} \tag{5}$$

2.2 FMCW Beat Signal

Thus, by the definition of the FMCW chirp as a sinusoidal signal whose frequency linearly increases with time, we can derive the phase of the signal and express the transmitted FMCW pulse by substituting (5) into (3) as

$$m(t) = e^{j\phi(t)} = e^{j2\pi(f_0 t + 0.5Kt^2)}, \quad 0 \le t \le T, \tag{6}$$

where T is the duration of the FMCW pulse.

Assuming the monostatic radar antenna element is located at (x', y', z') and the point scatterer is located at (x_0, y_0, z_0), the distance between the radar and point target can be expressed as

$$R_0 = \sqrt{(x_0 - x')^2 + (y_0 - y')^2 + (z_0 - z')^2}. \tag{7}$$

The FMCW pulse, expressed in (6) is transmitted from the antenna, propagates through space traveling a distance of R_0 to the point scatter, reflects from the point scatterer, travels another R_0 back to the radar. The round-trip time delay required for this propagation is given by

$$\tau_0 = \frac{2R_0}{c}, \tag{8}$$

where c is the speed of light.

As a result, the signal received at the radar is a time-delayed and scaled version of the transmitted signal as

$$\begin{aligned}\hat{s}(t) &= \frac{\sigma}{R_0^2} m(t - \tau_0), \\ &= \frac{\sigma}{R_0^2} e^{j2\pi(f_0(t-\tau_0) + 0.5K(t-\tau_0)^2)}, \\ &= \frac{\sigma}{R_0^2} e^{j2\pi(f_0 t - f_0\tau_0 + 0.5Kt^2 - K\tau_0 t + 0.5K\tau_0^2)}, \\ &= \frac{\sigma}{R_0^2} e^{j2\pi(f_0 t + 0.5Kt^2 - f_0\tau_0 - K\tau_0 t + 0.5K\tau_0^2}, \\ &= \frac{\sigma}{R_0^2} \underbrace{e^{j2\pi(f_0 t + 0.5Kt^2)}}_{m(t)} e^{-j2\pi(f_0\tau_0 + K\tau_0 t - 0.5K\tau_0^2)},\end{aligned} \tag{9}$$

where σ is known as the "reflectivity" of the scatterer (how reflective the point target is) and the $1/R_0^2$ term is the round-trip path loss or amplitude decay (the intuition here is that farther targets give weaker reflections). Note that the received signal $\hat{s}(t)$ contains a factor which is the transmitted signal $m(t)$.

The next step in the signal chain is known as "dechirping" and removes this factor of $m(t)$ by multiplying the conjugate of the received signal by the transmitted signal. After dechirping, the signal is known as the IF signal or beat signal. The beat signal can be expressed as

$$\begin{aligned} s(t) &= m(t)\hat{s}^*(t), \\ &= e^{j2\pi(f_0 t+0.5Kt^2)}\frac{\sigma}{R_0^2}e^{-j2\pi(f_0 t+0.5Kt^2)}e^{j2\pi(f_0\tau_0+K\tau_0 t-0.5K\tau_0^2)}, \\ &= \frac{\sigma}{R_0^2}e^{j2\pi(f_0 t+0.5Kt^2)-j2\pi(f_0 t+0.5Kt^2)}e^{j2\pi(f_0\tau_0+K\tau_0 t-0.5K\tau_0^2)}, \\ &= \frac{\sigma}{R_0^2}e^{j2\pi(f_0\tau_0+K\tau_0 t-0.5K\tau_0^2)}. \end{aligned} \tag{10}$$

In radar literature, it is common practice to express the beat signal as a function of k rather than t, where $k(t)$ is the instantaneous wavenumber corresponding to the instantaneous frequency $f(t)$ given by

$$k(t) \triangleq \frac{2\pi}{c}f(t). \tag{11}$$

Substituting (1) into (11) yields

$$k(t) = \frac{2\pi}{c}(f_0+Kt), \quad 0 \le t \le T. \tag{12}$$

Hence, let us express $s(t)$ as a function of k by rewriting (10) in terms of (12). Note that for short distances τ_0^2 is negligible and the last term in (10) can be ignored. Substituting (8) into (10) and recalling the definitions in (1) and (11) yields

$$\begin{aligned} s(t) &= \frac{\sigma}{R_0^2}e^{j2\pi(f_0\tau_0+K\tau_0 t)}, \\ &= \frac{\sigma}{R_0^2}e^{j2\pi\tau_0(f_0+Kt)}, \\ &= \frac{\sigma}{R_0^2}e^{j2\pi\frac{2R_0}{c}(f_0+Kt)}, \\ &= \frac{\sigma}{R_0^2}e^{j2\pi\frac{2R_0}{c}f(t)}, \\ &= \frac{\sigma}{R_0^2}e^{j2R_0\frac{2\pi}{c}f(t)}, \\ s(k) &= \frac{\sigma}{R_0^2}e^{j2R_0 k}, \quad 0 \le t \le T. \end{aligned} \tag{13}$$

More commonly, the beat signal is expressed as

$$s(k) = \frac{\sigma}{R_0^2}e^{j2kR_0} \tag{14}$$

We have derived a compact representation of the FMCW beat signal, $s(k)$. It is clear from (14) that the frequency of the beat signal corresponds directly with the radial distance, known as the "range", R_0. For this single point scatter case, the radar beat signal $s(k)$ is a single tone sinusoid whose frequency corresponds with R_0. Hence, the Fourier transform of $s(k)$ would have a single peak at a position corresponding to the range R_0.

This section detailed the derivation of the radar beat signal. However, now that the beat signal has been derived, it can be applied simply by the definition of $s(k)$ in (14) without needing to go through all steps every time.

2.3 Multiple Targets

Suppose N point scatterers are in the radar FOV such that the n-th point scatterer is located at (x_n, y_n, z_n) and has reflectivity σ_n. In this case, the transmit signal is the same, but the received signal is now a sum (by superposition) of the received signals from each of the point scatterers. As a result, the radar beat signal can be written as

$$s(k) = \sum_{n=1}^{N} \frac{\sigma_n}{R_n^2} e^{j2kR_n}, \quad 0 \leq t \leq T, \tag{15}$$

which is clearly a sum of sinusoidal signals whose frequencies depend on the distances R_n. Hence, the Fourier transform of (15) would result in multiple peaks such that the location of the n-th peak corresponds with the distance R_n.

Alternatively, if we model the target as a continuous set of point targets, rather than a discrete set as in (15), where the target is located in a volume V inside (x, y, z) space, we express the reflectivity as a continuous function $p(x, y, z)$ and the summation expressed in (15) becomes an integral as

$$s(k) = \iiint_V \frac{p(x, y, z)}{R^2} e^{j2kR} dxdydz, \tag{16}$$

where

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \tag{17}$$

recall the position of the antenna is (x', y', z').

A Fourier Transform Applications to FMCW Radar