Method of Stationary Phase for SAR/ISAR Image Reconstruction

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***Abstract*—Using the method of stationary phase for spherical wave decomposition.**

I. Basic Method of Stationary Phase

In its simplest form, neglecting amplitude terms, the method of stationary phase (MSP) can be written as:

$$\iint e^{j\phi(x,y)} dxdy \approx e^{j\phi(x_0,y_0)} \tag{1}$$

Where x_0 and y_0 are obtained by the system:

$$\left.\frac{\partial\phi(x,y)}{\partial x}\right|_{(x=x_0,y=y_0)} = 0 \tag{2}$$

$$\left.\frac{\partial\phi(x,y)}{\partial y}\right|_{(x=x_0,y=y_0)} = 0 \tag{3}$$

A. Application to FMCW Wavefront Decomposition

Given that a FMCW signal is transmitted from an ideal, monostatic, full-duplex transceiver scanned across a rectilinear scanner in the x', y' plane, the return signal can be modeled as:

$$s(x',y',k) = \iiint \frac{p(x,y,z)}{R^2} e^{j2kR} dxdydz \tag{4}$$

$$R = \sqrt{(x'-x)^2 + (y'-y)^2 + (Z_1-z)^2} \tag{5}$$

where the transceiver is located at the point (x', y', Z_1) and $p(x,y,z)$ is the complex reflectivity function of the target scene.

This expression can be simplified by ignoring the amplitude terms and reduces to:

$$s(x',y',k) = \iiint p(x,y,z) e^{j2kR} dxdydz \tag{6}$$

MSP allows a decomposition of the spherical wavefront contained in the exponential term of equation (4). First, we take a 2D Fourier Transform across the x' and y' domains.

$$FT_{2D}^{(x',y')}[e^{j2kR}] = e^{-j(k_{x'}x + k_{y'}y)} \times \iint e^{j(2k\sqrt{x'^2+y'^2+(Z_1-z)^2} - k_{x'}x' - k_{y'}y')} dx'dy' \tag{7}$$

Note that the exponential term outside of the integrand is from the shift in the spatial domain.

Now, applying MSP to the double integral above:

$$\phi(x',y') = 2k\sqrt{x'^2+y'^2+(Z_1-z)^2} - k_{x'}x' - k_{y'}y' \tag{8}$$

Solving the simultaneous system described in equations (2) and (3) yields the stationary points:

$$x_0 = \frac{k_{x'}(Z_1-z)}{\sqrt{4k^2 - k_{x'}^2 - k_{y'}^2}} \tag{9}$$

$$y_0 = \frac{k_{y'}(Z_1-z)}{\sqrt{4k^2 - k_{x'}^2 - k_{y'}^2}} \tag{10}$$

Substituting these into equation (1) completes the approximation:

$$\iint e^{j(2k\sqrt{x'^2+y'^2+(Z_1-z)^2} - k_{x'}x' - k_{y'}y')} dx'dy' \approx e^{jk_{z'}(Z_1-z)} \tag{11}$$

$$k_{z'} = \sqrt{4k^2 - k_{x'}^2 - k_{y'}^2} \tag{12}$$

which can then be applied to equation (7) as:

$$FT_{2D}^{(x',y')}[e^{j2kR}] \approx e^{j(-k_{x'}x - k_{y'}y + k_{z'}(Z_1-z))} \tag{13}$$

Taking an 2D inverse Fourier transform across the $k_{x'}$ and $k_{y'}$ domains yields:

$$e^{j2kR} \approx \iint \left[e^{j(-k_{x'}x - k_{y'}y + k_{z'}(Z_1-z))} \right] e^{j(k_{x'}x' + k_{y'}y')} dk_{x'}dk_{y'} \tag{14}$$

Simplified:

$$e^{j2kR} \approx \iint e^{j(k_{x'}(x'-x) + k_{y'}(y'-y) + k_{z'}(Z_1-z))} dk_{x'}dk_{y'} \tag{15}$$

Substitute into equation (6):

$$s(x',y',k) \approx \iiint p(x,y,z) \iint e^{j(k_{x'}(x'-x) + k_{y'}(y'-y) + k_{z'}(Z_1-z))} \times dk_{x'}dk_{y'}dxdydz \tag{16}$$

Rearranging the integrands of equation (16) to produce:

$$s(x',y',k) \approx \iint \left[\iiint p(x,y,z) e^{-j(k_{x'}x + k_{y'}y + k_{z'}z)} dxdydz \right] \times e^{j(k_{x'}x' + k_{y'}y' + k_{z'}Z_1)} dk_{x'}dk_{y'} \tag{17}$$

where the triple integral inside of the [•] brackets represents the 3D Fourier transform of the reflectivity function.

$$P(k_{x'}, k_{y'}, k_{z'}) = FT_{3D}^{(x,y,z)}[p(x,y,z)]$$
$$= \iiint p(x,y,z) e^{-j(k_{x'}x + k_{y'}y + k_{z'}z)} dxdydz \quad (18)$$

Now, (17) can be updated as:

$$s(x', y', k) \approx$$
$$\iint P(k_{x'}, k_{y'}, k_{z'}) e^{j(k_{x'}x' + k_{y'}y' + k_{z'}Z_1)} dk_{x'} dk_{y'} \quad (19)$$

Rearranged to expose inverse Fourier identity, assuming close approximation to equality:

$$s(x', y', k) =$$
$$\iint \left[P(k_{x'}, k_{y'}, k_{z'}) e^{jk_{z'}Z_1}\right] e^{j(k_{x'}x' + k_{y'}y')} dk_{x'} dk_{y'} \quad (20)$$

$$s(x', y', k) \; = \; IFT_{2D}^{(k_{x'}, k_{y'})} \left[P(k_{x'}, k_{y'}, k_{z'}) e^{jk_{z'}Z_1}\right] \quad (21)$$

Take a 2D Fourier transform across the x' and y' domains on both sides, dropping distinction between primed and unprimed coordinate systems due to coincidence, yields:

$$FT_{2D}^{(x,y)}[s(x,y,k)] = S(k_x, k_y, k) = P(k_x, k_y, k_z) e^{jk_z Z_1} \quad (22)$$

Inverting equation (22):

$$p(x,y,z) = IFT_{3D}^{(k_x, k_y, k_z)}[P(k_x, k_y, k_z)] \quad (23)$$
$$P(k_x, k_y, k_z) = S(k_x, k_y, k) e^{-jk_z Z_1} \quad (24)$$

However, Stolt interpolation must be performed to map (k_x, k_y, k) to (k_x, k_y, k_z) by the dispersion relation for plane waves in free-space or a uniform dielectric, which can also be obtained by solving equation (12) for k as:

$$k = \frac{1}{2}\sqrt{k_x^2 + k_y^2 + k_z^2} \quad (25)$$

where $k_z \in [0, 2k]$ for the visible portion of the spectrum.

Combining the above relations, the resulting image reconstruction algorithm follows:

$$p(x,y,z) = IFT_{3D}\left[FT_{2D}^{(x,y)}[s(x,y,k)] e^{-jk_z Z_1}\right] \quad (26)$$