Qualifying Exam

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Abstract

Equations used on my qualifying exam spring 2020.

1 2D Rectilinear SAR - BPA

Echo signal from a 2D rectilinear scanner can be modeled as:

$$s(x', y', k) = \iiint \frac{p(x, y, z)}{R^2} e^{j2kR} dxdydz \tag{1}$$

$$R = \sqrt{(x - x')^2 + (y - y')^2 + z^2} \tag{2}$$

The back-projection algorithm:

$$p(x, y, z) = \iiint s(x', y', k) R^2 e^{-j2kR} dx'dy'dk \tag{3}$$

Neglecting amplitude terms:

$$p(x, y, z) = \iiint s(x', y', k) e^{-j2kR} dx'dy'dk \tag{4}$$

$$f(x, y) \circledast_{(x,y)} h(x, y) = IFT_{2D}^{(k_x, k_y)} \left[F(k_x, k_y) H(k_x, k_y) \right] \tag{5}$$

2 2D Rectilinear SAR - MF

The back-projection algorithm:

$$p(x, y, z) = \iiint s(x', y', k) R^2 e^{-j2kR} dx' dy' dk \tag{6}$$

Define the filter with or without considering the amplitude term

$$h(x, y, k) = (x^2 + y^2 + z^2) e^{-j2k\sqrt{x^2+y^2+z^2}} \tag{7}$$

$$h(x, y, k) = e^{-j2k\sqrt{x^2+y^2+z^2}} \tag{8}$$

Now:

$$p(x, y, z) = \iiint s(x', y', k) h(x - x', y - y', k) dx' dy' dk \tag{9}$$

$$p(x, y, z) = \int s(x, y, k) \circledast_{(x', y')} h(x, y, k) dk \tag{10}$$

$$P(k_x, k_y, z) = \int S(k_x, k_y, k) H(k_x, k_y, k) dk \tag{11}$$

Finally, the BPA matched filter algorithm:

$$p(x, y, z) = \int IFT_{2D}^{(k_x, k_y)} [S(k_x, k_y, k) H(k_x, k_y, k)] dk \tag{12}$$

3 2D Rectilinear SAR - RMA

Echo signal from a 2D rectilinear scanner can be modeled as:

$$s(x', y', k) = \iiint \frac{p(x, y, z)}{R^2} e^{j2kR} dx dy dz \tag{13}$$

$$R = \sqrt{(x - x')^2 + (y - y')^2 + z^2} \tag{14}$$

Using the method of stationary phase (MSP), the phase term of the echo signal can be decomposed into plane wave components.

$$e^{j2kR} \approx \iint e^{j(k_{x'}(x'-x) + k_{y'}(y'-y) - k_{z'}(z))} dk_{x'} dk_{y'} \tag{15}$$

Substituting into echo signal and rearranging the integrals:

$$s(x', y', k) \approx \iint \left[\iiint p(x, y, z) e^{-j(k_{x'}x + k_{y'}y + k_{z'}z)} dx dy dz \right] e^{j(k_{x'}x' + k_{y'}y')} dk_{x'} dk_{y'} \tag{16}$$

Inner integral can be viewed as a 3D Fourier transform:

$$s(x', y', k) = \iint P(k_x, k_y, k_z) e^{j(k_{x'}x' + k_{y'}y')} dk_{x'} dk_{y'} \tag{17}$$

Noting the 2D inverse Fourier transform:

$$S(k_x, k_y, k) = P(k_x, k_y, k_z) \tag{18}$$

$$p(x, y, z) = IFT_{3D}^{(k_x,k_y,k_z)}[\mathcal{RMA}[S(k_x, k_y, k)]] \tag{19}$$

$\mathcal{RMA}[\bullet]$ is the range migration algorithm (RMA) Stolt interpolation operation:

$$P(k_x, k_y, k_z) = \mathcal{RMA}[S(k_x, k_y, k)] = S(k_x, k_y, k)\Big|^{k=\frac{1}{2}\sqrt{k_y^2+k_x^2+k_z^2}} \tag{20}$$

The RMA can be summarized by:

$$p(x, y, z) = IFT_{3D}^{(k_x,k_y,k_z)}\left[\mathcal{RMA}\left[FT_{2D}^{(x,y)}[s(x, y, k)]\right]\right] \tag{21}$$

Stolt Interpolation Step $(k_x, k_y, k) \rightarrow (k_x, k_y, k_z)$

4 2D Cylindrical SAR - BPA

Echo signal from a 2D cylindrical scanner can be modeled as:

$$s(\theta, k, y') = \iiint \frac{p(x, y, z)}{R^2} e^{j2kR} dxdydz \tag{22}$$

$$R = \sqrt{(x - R_0 cos\theta))^2 + (z - R_0 sin\theta)^2 + (y - y')^2} \tag{23}$$

The back-projection algorithm:

$$p(x, y, z) = \iiint s(\theta, k, y') R^2 e^{-j2kR} d\theta dk dy' \tag{24}$$

Neglecting amplitude terms:

$$p(x, y, z) = \iiint s(\theta, k, y') e^{-j2kR} d\theta dk dy' \tag{25}$$

5 2D Cylindrical SAR - PFA

Echo signal from a 2D cylindrical scanner can be modeled as:

$$s(\theta, k, y') = \iiint \frac{p(x, y, z)}{R^2} e^{j2kR} dxdydz \tag{26}$$

$$R = \sqrt{(x - R_0 cos\theta))^2 + (z - R_0 sin\theta)^2 + (y - y')^2} \tag{27}$$

Using the method of stationary phase (MSP), the phase term of the echo signal can be decomposed into plane wave components.

$$\begin{aligned} &e^{j2k\sqrt{(R_0 cos\theta - x)^2 + (R_0 sin\theta - z)^2 + (y-y')^2}} = \\ &\iint e^{jk_r cos\phi (R_0 cos\theta - x) + jk_r sin\phi (R_0 sin\theta - z) + jk_{y'}(y - y'))} d\phi dk_{y'} \end{aligned} \tag{28}$$

By some Fourier analysis, $(\bullet)^*$ is the complex conjugate operation,

$$\begin{aligned} G(k_\theta, k, k_y) &= FT_{1D}^{(\theta)} \left[e^{j\sqrt{4k^2 - k_y^2} R_0 cos\theta} \right] \\ P(k_\theta, k, k_y) &= S(k_\theta, k, k_y) G^*(k_\theta, k, k_y) \end{aligned} \tag{29}$$

$\mathcal{PFA}[\bullet]$ is the polar formatting algorithm (PFA) Stolt interpolation operation:

$$\begin{aligned} P(k_x, k_y, k_z) &= \mathcal{PFA}\left[P(k_\theta, k, k_y)\right] \\ k_\theta &= tan^{-1}\left(\frac{k_z}{k_y}\right) \\ k &= \frac{1}{2}\sqrt{k_x^2 + k_y^2 + k_z^2} \end{aligned} \tag{30}$$

Summary:

$$p(x, y, z) = FT_{3D}^{(k_x, k_y, k_z)} \left[\mathcal{PFA} \left[IFT_{1D}^{(\Theta)} \left[FT_{2D}^{(\theta, y)} \left[s(\theta, k, y) \right] G^*(k_\theta, k, k_y) \right] \right] \right] \tag{31}$$