6 Continuous Time Markov Chains

6.1 Pure Birth Processes

In this chapter, we present several important examples of continuous time, discrete state, and Markov processes. Specifically, we deal here with a family of random variables $\{X(t); 0 \leq t < \infty\}$ where the possible values of $X(t)$ are the nonnegative integers. We shall restrict attention to the case where $\{X(t)\}$ is a Markov process with stationary transition probabilities. Thus, the transition probability function for $t > 0$,

$$P_{ij}(t) = \Pr\{X(t+u) = j | X(u) = i\}, \qquad i, j = 0, 1, 2, \ldots,$$

is independent of $u \geq 0$.

It is usually more natural in investigating particular stochastic models based on physical phenomena to prescribe the so-called infinitesimal probabilities relating to the process and then derive from them an explicit expression for the transition probability function. For the case at hand, we will postulate the form of $P_{ij}(h)$ for h small, and, using the Markov property, we will derive a system of differential equations satisfied by $P_{ij}(t)$ for all $t > 0$. The solution of these equations under suitable boundary conditions gives $P_{ij}(t)$.

By way of introduction to the general pure birth process, we review briefly the axioms characterizing the Poisson process.

6.1.1 Postulates for the Poisson Process

The Poisson process is the prototypical pure birth process. Let us point out the relevant properties. The Poisson process is a Markov process on the nonnegative integers for which

(i) $\Pr\{X(t+h) - X(t) = 1 | X(t) = x\} = \lambda h + o(h)$ as $h \downarrow 0$
$(x = 0, 1, 2, \ldots)$.

(ii) $\Pr\{X(t+h) - X(t) = 0 | X(t) = x\} = 1 - \lambda h + o(h)$ as $h \downarrow 0$.

(iii) $X(0) = 0$.

The precise interpretation of (i) is the relationship

$$\lim_{h \to 0+} \frac{\Pr\{X(t+h) - X(t) = 1 | X(t) = x\}}{h} = \lambda.$$

The $o(h)$ symbol represents a negligible remainder term in the sense that if we divide the term by h, then the resulting value tends to zero as h tends to zero. Notice that the right side of (i) is independent of x.

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These properties are easily verified by direct computation, since the explicit formulas for all the relevant properties are available. Problem 6.1.13 calls for showing that these properties, in fact, define the Poisson process.

6.1.2 Pure Birth Process

A natural generalization of the Poisson process is to permit the chance of an event occurring at a given instant of time to depend upon the number of events that have already occurred. An example of this phenomenon is the reproduction of living organisms (and hence the name of the process), in which under certain conditions—e.g., sufficient food, no mortality, no migration—the infinitesimal probability of a birth at a given instant is proportional (directly) to the population size at that time. This example is known as the *Yule process* and will be considered in detail later.

Consider a sequence of positive numbers, $\{\lambda_k\}$. We define a pure birth process as a Markov process satisfying the following postulates:

1. $\Pr\{X(t+h) - X(t) = 1 | X(t) = k\} = \lambda_k h + o_{1,k}(h) (h \to 0+)$.
2. $\Pr\{X(t+h) - X(t) = 0 | X(t) = k\} = 1 - \lambda_k h + o_{2,k}(h)$. (6.1)
3. $\Pr\{X(t+h) - X(t) < 0 | X(t) = k\} = 0 \ (k \geq 0)$.
As a matter of convenience, we often add the postulate
4. $X(0) = 0$.

With this postulate, $X(t)$ does not denote the population size but, rather, the number of births in the time interval $(0, t]$.

Note that the left sides of Postulates (1) and (2) are just $P_{k,k+1}(h)$ and $P_{k,k}(h)$, respectively (owing to stationarity), so that $o_{1,k}(h)$ and $o_{2,k}(h)$ do not depend upon t.

We define $P_n(t) = \Pr\{X(t) = n\}$, assuming $X(0) = 0$.

By analyzing the possibilities at time t just prior to time $t+h$ (h small), we will derive a system of differential equations satisfied by $P_n(t)$ for $t \geq 0$, namely

$$
\begin{aligned}
P_0'(t) &= -\lambda_0 P_0(t), \\
P_n'(t) &= -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \qquad \text{for } n \geq 1,
\end{aligned}
\tag{6.2}
$$

with initial conditions

$$
P_0(0) = 1, \qquad P_n(0) = 0, \qquad n > 0.
$$

Indeed, if $h > 0, n \geq 1$, then by invoking the law of total probability, the Markov property, and Postulate (3), we obtain

$$
\begin{aligned}
P_n(t+h) &= \sum_{k=0}^{\infty} P_k(t) \Pr\{X(t+h) = n | X(t) = k\} \\
&= \sum_{k=0}^{\infty} P_k(t) \Pr\{X(t+h) - X(t) = n - k | X(t) = k\} \\
&= \sum_{k=0}^{n} P_k(t) \Pr\{X(t+h) - X(t) = n - k | X(t) = k\}.
\end{aligned}
$$

Now for $k = 0, 1, \ldots, n-2$, we have

$$\begin{aligned}
&\Pr\{X(t+h) - X(t) = n-k|X(t) = k\} \\
&\quad \le \Pr\{X(t+h) - X(t) \ge 2|X(t) = k\} \\
&\quad = o_{1,k}(h) + o_{2,k}(h),
\end{aligned}$$

or

$$\Pr\{X(t+h) - X(t) = n-k|X(t) = k\} = o_{3,n,k}(h), \qquad k = 0, \ldots, n-2.$$

Thus,

$$\begin{aligned}
P_n(t+h) = P_n(t)\left[1 - \lambda_n h + o_{2,n}(h)\right] + P_{n-1}(t)\left[\lambda_{n-1}h + o_{1,n-1}(h)\right] \\
+ \sum_{k=0}^{n-2} P_k(t) o_{3,n,k}(h)k,
\end{aligned}$$

or

$$\begin{aligned}
&P_n(t+h) - P_n(t) \\
&\quad = P_n(t)\left[-\lambda_n h + o_{2,n}(h)\right] + P_{n-1}(t)\left[\lambda_{n-1}h + o_{1,n-1}(h)\right] + o_n(h),
\end{aligned} \tag{6.3}$$

where, clearly, $\lim_{h\downarrow 0} o_n(h)/h = 0$ uniformly in $t \ge 0$, since $o_n(h)$ is bounded by the finite sum $\Sigma_{k=0}^{n-2} o_{3,n,k}(h)$, which does not depend on t.

Dividing by h and passing to the limit $h \downarrow 0$, we validate the relations (6.2), where on the left side we should, to be precise, write the derivative from the right. With a little more care, however, we can derive the same relation involving the derivative from the left. In fact, from (6.3), we see at once that the $P_n(t)$ are continuous functions of t. Replacing t by $t - h$ in (6.3), dividing by h, and passing to the limit $h \downarrow 0$, we find that each $P_n(t)$ has a left derivative that also satisfies equation (6.2).

The first equation of (6.2) can be solved immediately and yields

$$P_0(t) = \exp\{-\lambda_0 t\} \qquad \text{for } t > 0. \tag{6.4}$$

Define S_k as the time between the kth and the $(k+1)$st birth, so that

$$P_n(t) = \Pr\left\{\sum_{i=0}^{n-1} S_i \le t < \sum_{i=0}^{n} S_i\right\}.$$

The random variables S_k are called the "sojourn times" between births, and

$$W_k = \sum_{i=0}^{k-1} S_i = \text{the time at which the } k\text{th birth occurs.}$$

We have already seen that $P_0(t) = \exp\{-\lambda_0 t\}$. Therefore,

$$\Pr\{S_0 \leq t\} = 1 - \Pr\{X(t) = 0\} = 1 - \exp\{-\lambda_0 t\};$$

that is S_0 has an exponential distribution with parameter λ_0. It may be deduced from Postulates (1) through (4) that $S_k, k > 0$, also has an exponential distribution with parameter λ_k and that the S_i's are mutually independent.

This description characterizes the pure birth process in terms of its sojourn times, in contrast to the infinitesimal description corresponding to (6.1).

To solve the differential equations of (6.2) recursively, introduce $Q_n(t) = \mathrm{e}^{\lambda_n t} P_n(t)$ for $n = 0, 1, \ldots$. Then,

$$\begin{aligned} Q_n'(t) &= \lambda_n \mathrm{e}^{\lambda_n t} P_n(t) + \mathrm{e}^{\lambda_n t} P_n'(t) \\ &= \mathrm{e}^{\lambda_n t}\left[\lambda_n P_n(t) + P_n'(t)\right] \\ &= \mathrm{e}^{\lambda_n t} \lambda_{n-1} P_{n-1}(t) \qquad [\text{using (6.2)}]. \end{aligned}$$

Integrating both sides of these equations and using the boundary condition $Q_n(0) = 0$ for $n \geq 1$ gives

$$Q_n(t) = \int_0^t \mathrm{e}^{\lambda_n x} \lambda_{n-1} P_{n-1}(x) \mathrm{d}x,$$

or

$$P_n(t) = \lambda_{n-1} \mathrm{e}^{-\lambda_n t} \int_0^t \mathrm{e}^{\lambda_n x} P_{n-1}(x) \mathrm{d}x, \qquad n = 1, 2, \ldots. \tag{6.5}$$

It is now clear that all $P_k(t) \geq 0$, but there is still a possibility that

$$\sum_{n=0}^{\infty} P_n(t) < 1.$$

To secure the validity of the process, i.e., to assure that $\Sigma_{n=0}^{\infty} P_n(t) = 1$ for all t, we must restrict the λ_k according to the following:

$$\sum_{n=0}^{\infty} P_n(t) = 1 \quad \text{if and only if} \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty. \tag{6.6}$$

The intuitive argument for this result is as follows: The time S_k between consecutive births is exponentially distributed with a corresponding parameter λ_k. Therefore, the quantity $\Sigma_n 1/\lambda_n$ equals the expected time before the population becomes infinite. By comparison, $1 - \Sigma_{n=0}^{\infty} P_n(t)$ is the probability that $X(t) = \infty$.

If $\Sigma_n \lambda_n^{-1} < \infty$, then the expected time for the population to become infinite is finite. It is then plausible that for all $t > 0$, the probability that $X(t) = \infty$ is positive.

When no two of the birth parameters $\lambda_0, \lambda_1, \ldots$ are equal, the integral equation (6.5) may be solved to give the explicit formula

$$\begin{aligned} P_0(t) &= \mathrm{e}^{-\lambda_0 t}, \\ P_1(t) &= \lambda_0 \left(\frac{1}{\lambda_1 - \lambda_0} \mathrm{e}^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} \mathrm{e}^{-\lambda_1 t} \right) \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} P_n(t) &= \Pr\{X(t) = n | X(0) = 0\} \\ &= \lambda_0 \cdots \lambda_{n-1} \left[B_{0,n} \mathrm{e}^{-\lambda_0 t} + \cdots + B_{n,n} \mathrm{e}^{-\lambda_n t} \right] \qquad \text{for } n > 1, \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} B_{0,n} &= \frac{1}{(\lambda_1 - \lambda_0) \cdots (\lambda_n - \lambda_0)}, \\ B_{k,n} &= \frac{1}{(\lambda_0 - \lambda_k) \cdots (\lambda_{k-1} - \lambda_k)(\lambda_{k+1} - \lambda_k) \cdots (\lambda_n - \lambda_k)} \\ & \qquad \qquad \text{for } 0 < k < n \end{aligned} \tag{6.9}$$

and

$$B_{n,n} = \frac{1}{(\lambda_0 - \lambda_n) \cdots (\lambda_{n-1} - \lambda_n)}.$$

Because $\lambda_j \neq \lambda_k$ when $j \neq k$ by assumption, the denominator in (6.9) does not vanish, and $B_{k,n}$ is well defined.

We will verify that $P_1(t)$, as given by (6.7), satisfies (6.5). Equation (6.4) gives $P_0(t) = \mathrm{e}^{-\lambda_0 t}$. We next substitute this in (6.5) when $n = 1$, thereby obtaining

$$\begin{aligned} P_1(t) &= \lambda_0 \mathrm{e}^{-\lambda_1 t} \int_0^t \mathrm{e}^{\lambda_1 x} \mathrm{e}^{-\lambda_0 x} \mathrm{d}x \\ &= \lambda_0 \mathrm{e}^{-\lambda_1 t} (\lambda_0 - \lambda_1)^{-1} \left[1 - \mathrm{e}^{-(\lambda_0 - \lambda_1) t} \right] \\ &= \lambda_0 \left(\frac{1}{\lambda_1 - \lambda_0} \mathrm{e}^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} \mathrm{e}^{-\lambda_1 t} \right), \end{aligned}$$

in agreement with (6.7).

The induction proof for general n involves tedious and difficult algebra. The case $n = 2$ is suggested as a problem.

6.1.3 The Yule Process

The Yule process arises in physics and biology and describes the growth of a population in which each member has a probability $\beta h + o(h)$ of giving birth to a new member during an interval of time of length $h(\beta > 0)$. Assuming independence and no interaction among members of the population, the binomial theorem gives

$$\Pr\{X(t+h) - X(t) = 1 | X(t) = n\} = \binom{n}{1} [\beta h + o(h)] [1 - \beta h + o(h)]^{n-1}$$
$$= n\beta h + o_n(h);$$

for the Yule process the infinitesimal parameters are $\lambda_n = n\beta$. In words, the total population birth rate is directly proportional to the population size, the proportionality constant being the individual birth rate β. As such, the Yule process forms a stochastic analog of the deterministic population growth model represented by the differential equation $\mathrm{d}y/\mathrm{d}t = \alpha y$. In the deterministic model, the rate $\mathrm{d}y/\mathrm{d}t$ of population growth is directly proportional to population size y. In the stochastic model, the infinitesimal deterministic increase $\mathrm{d}y$ is replaced by the probability of a unit increase during the infinitesimal time interval $\mathrm{d}t$. Similar connections between deterministic rates and birth (and death) parameters arise frequently in stochastic modeling. Examples abound in this chapter.

The system of equations (6.2) in the case that $X(0) = 1$ becomes

$$P_n'(t) = -\beta [nP_n(t) - (n-1)P_{n-1}(t)], \qquad n = 1, 2, \ldots,$$

under the initial conditions

$$P_1(0) = 1, \qquad P_n(0) = 0, \qquad n = 2, 3, \ldots.$$

Its solution is

$$P_n(t) = \mathrm{e}^{-\beta t}\left(1 - \mathrm{e}^{-\beta t}\right)^{n-1}, \qquad n \geq 1, \tag{6.10}$$

as may be verified directly. We recognize (6.10) as the geometric distribution in Chapter 1, (1.26) with $p = \mathrm{e}^{-\beta t}$.

The general solution analogous to (6.8) but for pure birth processes starting from $X(0) = 1$ is

$$P_n(t) = \lambda_1 \cdots \lambda_{n-1}\left[B_{1,n}\mathrm{e}^{-\lambda_1 t} + \cdots + B_{n,n}\mathrm{e}^{-\lambda_n t}\right], \quad n > 1. \tag{6.11}$$

When $\lambda_n = \beta n$, we will show that (6.11) reduces to the solution given in (6.10) for a Yule process with parameter β. Then,

$$\begin{aligned} B_{1,n} &= \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)\cdots(\lambda_n - \lambda_1)} \\ &= \frac{1}{\beta^{n-1}(1)(2)\cdots(n-1)} \\ &= \frac{1}{\beta^{n-1}(n-1)!}, \end{aligned}$$

$$\begin{aligned}B_{2,n} &= \frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\cdots(\lambda_n-\lambda_2)}\\ &= \frac{1}{\beta^{n-1}(-1)(1)(2)\cdots(n-2)}\\ &= \frac{-1}{\beta^{n-1}(n-2)!},\end{aligned}$$

and

$$\begin{aligned}\beta_{k,n} &= \frac{1}{(\lambda_1-\lambda_k)\cdots(\lambda_{k-1}-\lambda_k)(\lambda_{k+1}-\lambda_k)\cdots(\lambda_n-\lambda_k)}\\ &= \frac{(-1)^{k-1}}{\beta^{n-1}(k-1)!\,(n-k)!}.\end{aligned}$$

Thus, according to (6.11),

$$\begin{aligned}P_n(t) &= \beta^{n-1}(n-1)!\left(B_{1,n}\mathrm{e}^{-\beta t}+\cdots+B_{n,n}\mathrm{e}^{-n\beta t}\right)\\ &= \sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!\,(n-k)!}(-1)^{k-1}\mathrm{e}^{-k\beta t}\\ &= \mathrm{e}^{-\beta t}\sum_{j=0}^{n-1}\frac{(n-1)!}{j!\,(n-1-j)!}\left(-\mathrm{e}^{-\beta t}\right)^j\\ &= \mathrm{e}^{-\beta t}\left(1-\mathrm{e}^{-\beta t}\right)^{n-1} \qquad \text{[see Chapter 1, (1.67)]},\end{aligned}$$

which establishes (6.10).

Exercises

6.1.1 A pure birth process starting from $X(0)=0$ has birth parameters $\lambda_0=1$, $\lambda_1=3, \lambda_2=2$, and $\lambda_3=5$. Determine $P_n(t)$ for $n=0,1,2,3$.

6.1.2 A pure birth process starting from $X(0)=0$ has birth parameters $\lambda_0=1$, $\lambda_1=3, \lambda_2=2$, and $\lambda_3=5$. Let W_3 be the random time that it takes the process to reach state 3.

- **(a)** Write W_3 as a sum of sojourn times and thereby deduce that the mean time is $E[W_3]=\frac{11}{6}$.
- **(b)** Determine the mean of $W_1+W_2+W_3$.
- **(c)** What is the variance of W_3?

6.1.3 A population of organisms evolves as follows. Each organism exists, independent of the other organisms, for an exponentially distributed length of time with parameter θ, and then splits into two new organisms, each of which exists, independent of the other organisms, for an exponentially distributed length of time with parameter θ, and then splits into two new organisms, and so on. Let $X(t)$ denote the number of organisms existing at time t. Show that $X(t)$ is a Yule process.

6.1.4 Consider an experiment in which a certain event will occur with probability αh and will not occur with probability $1-\alpha h$, where α is a fixed positive parameter and h is a small ($h < 1/\alpha$) positive variable. Suppose that n independent trials of the experiment are carried out, and the total number of times that the event occurs is noted. Show that

(a) The probability that the event never occurs during the n trials is $1-n\alpha h+o(h)$;

(b) The probability that the event occurs exactly once is $n\alpha h+o(h)$;

(c) The probability that the event occurs twice or more is $o(h)$.

Hint: Use the binomial expansion

$$(1-\alpha h)^n = 1-n\alpha h+\frac{n(n-1)}{2}(\alpha h)^2 - \cdots .$$

6.1.5 Using equation (6.10), calculate the mean and variance for the Yule process where $X(0)=1$.

6.1.6 Operations 1, 2, and 3 are to be performed in succession on a major piece of equipment. Operation k takes a random duration S_k that is exponentially distributed with parameter λ_k for $k=1,2,3$, and all operation times are independent. Let $X(t)$ denote the operation being performed at time t, with time $t=0$ marking the start of the first operation. Suppose that $\lambda_1=5, \lambda_2=3$, and $\lambda_3=13$. Determine

(a) $P_1(t)=\Pr\{X(t)=1\}$.

(b) $P_2(t)=\Pr\{X(t)=2\}$.

(c) $P_3(t)=\Pr\{X(t)=3\}$.

Problems

6.1.1 Let $X(t)$ be a Yule process that is observed at a random time U, where U is uniformly distributed over $[0,1)$. Show that $\Pr\{X(U)=k\}=p^k/(\beta k)$ for $k=1,2,\ldots$, with $p=1-e^{-\beta}$.

Hint: Integrate (6.10) over t between 0 and 1.

6.1.2 A Yule process with immigration has birth parameters $\lambda_k=\alpha+k\beta$ for $k=0,1,2,\ldots$. Here, α represents the rate of immigration into the population, and β represents the individual birth rate. Supposing that $X(0)=0$, determine $P_n(t)$ for $n=0,1,2,\ldots$.

6.1.3 Consider a population comprising a fixed number N of individuals. Suppose that at time $t=0$, there is exactly one *infected* individual and $N-1$ *susceptible* individuals in the population. Once infected, an individual remains in that state forever. In any short time interval of length h, *any given infected person* will transmit the disease to *any given susceptible person* with probability $\alpha h+o(h)$. (The parameter α is the *individual infection rate*.) Let $X(t)$ denote the number of infected individuals in the population at time $t\geq 0$. Then, $X(t)$ is a pure birth process on the states $0,1,\ldots,N$. Specify the birth parameters.

6.1.4 A new product (a "Home Helicopter" to solve the commuting problem) is being introduced. The sales are expected to be determined by both media (newspaper and television) advertising and word-of-mouth advertising, wherein satisfied customers tell others about the product. Assume that media advertising creates new customers according to a Poisson process of rate $\alpha = 1$ customer per month. For the word-of-mouth advertising, assume that each purchaser of a Home Helicopter will generate sales to new customers at a rate of $\theta = 2$ customers per month. Let $X(t)$ be the total number of Home Helicopter customers up to time t.

(a) Model $X(t)$ as a pure birth process by specifying the birth parameters λ_k, for $k = 0, 1, \ldots$.

(b) What is the probability that exactly two Home Helicopters are sold during the first month?

6.1.5 Let W_k be the time to the kth birth in a pure birth process starting from $X(0) = 0$. Establish the equivalence

$$\Pr\{W_1 > t, W_2 > t + s\} = P_0(t)[P_0(s) + P_1(s)].$$

From this relation together with equation (6.7), determine the joint density for W_1 and W_2, and then the joint density of $S_0 = W_1$ and $S_1 = W_2 - W_1$.

6.1.6 A fatigue model for the growth of a crack in a discrete lattice proposes that the size of the crack evolves as a pure birth process with parameters

$$\lambda_k = (1 + k)^{\rho} \qquad \text{for } k = 1, 2, \ldots.$$

The theory behind the model postulates that the growth rate of the crack is proportional to some power of the stress concentration at its ends and that this stress concentration is itself proportional to some power of $1 + k$, where k is the crack length. Use the sojourn time description to deduce that the mean time for the crack to grow to infinite length is finite when $\rho > 1$ and that, therefore, the failure time of the system is a well-defined and finite-valued random variable.

6.1.7 Let λ_0, λ_1, and λ_2 be the parameters of the independent exponentially distributed random variables S_0, S_1, and S_2. Assume that no two of the parameters are equal.

(a) Verify that

$$\Pr\{S_0 > t\} = e^{-\lambda_0 t},$$

$$\Pr\{S_0 + S_1 > t\} = \frac{\lambda_1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 t},$$

and evaluate in similar terms

$$\Pr\{S_0 + S_1 + S_2 > t\}.$$

(b) Verify equation (6.8) in the case that $n = 2$ by evaluating

$$P_2(t) = \Pr\{X(t) = 2\} = \Pr\{S_0 + S_1 + S_2 > t\} - \Pr\{S_0 + S_1 > t\}.$$

6.1.8 Let $N(t)$ be a pure birth process for which

$$\Pr\{\text{an event happens in}(t, t+h)|N(t) \text{ is odd}\} = \alpha h + o(h),$$
$$\Pr\{\text{an event happens in}(t, t+h)|N(t) \text{ is even}\} = \beta h + o(h),$$

where $o(h)/h \to 0$ as $h \downarrow 0$. Take $N(0) = 0$. Find the following probabilities:

$$P_0(t) = \Pr\{N(t) \text{ is even}\}; \qquad P_1(t) = \Pr\{N(t) \text{ is odd}\}.$$

Hint: Derive the differential equations

$$P_0'(t) = \alpha P_1(t) - \beta P_0(t) \quad \text{and} \quad P_1'(t) = -\alpha P_1(t) + \beta P_0(t)$$

and solve them by using $P_0(t) + P_1(t) = 1$.

6.1.9 Under the conditions of Problem 6.8, determine $E[N(t)]$.

6.1.10 Consider a pure birth process on the states $0, 1, \ldots, N$ for which $\lambda_k = (N-k)\lambda$ for $k = 0, 1, \ldots, N$. Suppose that $X(0) = 0$. Determine $P_n(t) = \Pr\{X(t) = n\}$ for $n = 0, 1$, and 2.

6.1.11 Beginning with $P_0(t) = \mathrm{e}^{-\lambda_0 t}$ and using equation (6.5), calculate $P_1(t), P_2(t)$, and $P_3(t)$ and verify that these probabilities conform with equation (6.7), assuming distinct birth parameters.

6.1.12 Verify that $P_2(t)$, as given by (6.8), satisfies (6.5) by following the calculations in the text that showed that $P_1(t)$ satisfies (6.5).

6.1.13 Using (6.5), derive $P_n(t)$ when all birth parameters are the same constant λ and show that

$$P_n(t) = \frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!}, \qquad n = 0, 1, \ldots.$$

Thus, the postulates of Section 6.1.1 serve to define the Poisson processes.

6.2 Pure Death Processes

Complementing the increasing pure birth process is the decreasing pure death process. It moves successively through states $N, N-1, \ldots, 2, 1$ and ultimately is absorbed in state 0 (extinction). The process is specified by the death parameters $\mu_k > 0$ for $k = 1, 2, \ldots, N$, where the sojourn time in state k is exponentially distributed with parameter μ_k, all sojourn times being independent. A typical sample path is depicted in Figure 6.1.

Alternatively, we have the infinitesimal description of a pure death process as a Markov process $X(t)$ whose state space is $0, 1, \ldots, N$ and for which

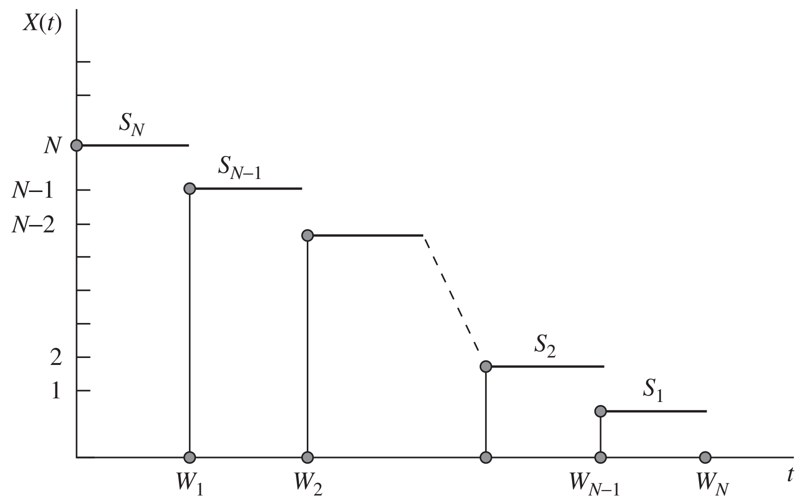

Figure 6.1 A typical sample path of a pure death process, showing the sojourn times $S_N, \ldots, S_1$ and the waiting times $W_1, W_2, \ldots, W_N$.

$$\begin{aligned}
&\textbf{(i)}\ \Pr\{X(t+h) = k-1|X(t) = k\} = \mu_k h + o(h), k = 1, \ldots, N;\\
&\textbf{(ii)}\ \Pr\{X(t+h) = k|X(t) = k\} = 1 - \mu_k h + o(h), k = 1, \ldots, N;\\
&\textbf{(iii)}\ \Pr\{X(t+h) > k|X(t) = k\} = 0, k = 0, 1, \ldots, N.
\end{aligned} \tag{6.12}$$

The parameter μ_k is the "death rate" operating or in effect while the process sojourns in state k. It is a common and useful convention to assign $\mu_0 = 0$.

When the death parameters $\mu_1, \mu_2, \ldots, \mu_N$ are distinct, i.e., $\mu_j \neq \mu_k$ if $j \neq k$, then we have the explicit transition probabilities

$$P_N(t) = \mathrm{e}^{-\mu_N t};$$

and for $n < N$,

$$\begin{aligned}
P_n(t) &= \Pr\{X(t) = n|X(0) = N\}\\
&= \mu_{n+1}\mu_{n+2}\cdots\mu_N\left[A_{n,n}\mathrm{e}^{-\mu_n t} + \cdots + A_{N,n}\mathrm{e}^{-\mu_N t}\right],
\end{aligned} \tag{6.13}$$

where

$$A_{k,n} = \frac{1}{(\mu_N - \mu_k)\cdots(\mu_{k+1} - \mu_k)(\mu_{k-1} - \mu_k)\cdots(\mu_n - \mu_k)}.$$

6.2.1 The Linear Death Process

As an example, consider a pure death process in which the death rates are proportional to population size. This process, which we will call the *linear death process*, complements the Yule, or linear birth, process. The parameters are $\mu_k = k\alpha$, where α is the individual death rate in the population. Then,

$$
\begin{aligned}
A_{n,n} &= \frac{1}{(\mu_N - \mu_n)(\mu_{N-1} - \mu_n)\cdots(\mu_{n+1} - \mu_n)} \\
&= \frac{1}{\alpha^{N-n-1}(N-n)(N-n-1)\cdots(2)(1)}, \\
A_{n+1,n} &= \frac{1}{(\mu_N - \mu_{n+1})\cdots(\mu_{n+2} - \mu_{n+1})(\mu_n - \mu_{n+1})} \\
&= \frac{1}{\alpha^{N-n-1}(N-n-1)\cdots(1)(-1)}, \\
A_{k,n} &= \frac{1}{(\mu_N - \mu_k)\cdots(\mu_{k+1} - \mu_k)(\mu_{k-1} - \mu_k)\cdots(\mu_n - \mu_k)} \\
&= \frac{1}{\alpha^{N-n-1}(N-k)\cdots(1)(-1)(-2)\cdots(n-k)} \\
&= \frac{1}{\alpha^{N-n-1}(-1)^{k-n}(N-k)!\,(k-n)!}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
P_n(t) &= \mu_{n+1}\mu_{n+2}\cdots\mu_N \sum_{k=n}^{N} A_{k,n}\mathrm{e}^{-\mu_k t} \\
&= \alpha^{N-n-1}\frac{N!}{n!}\sum_{k=n}^{N} \frac{\mathrm{e}^{-k\alpha t}}{\alpha^{N-n-1}(-1)^{k-n}(N-k)!\,(k-n)!} \\
&= \frac{N!}{n!}\mathrm{e}^{-n\alpha t}\sum_{j=0}^{N-n} \frac{(-1)^j \mathrm{e}^{-j\alpha t}}{(N-n-j)!\,j!} \\
&= \frac{N!}{n!\,(N-n)!}\mathrm{e}^{-n\alpha t}\left(1 - \mathrm{e}^{-\alpha t}\right)^{N-n}, \quad n = 0, \ldots, N.
\end{aligned}
\tag{6.14}
$$

Let T be the time of population extinction. Formally, $T = \min\{t \geq 0; X(t) = 0\}$. Then, $T \leq t$ if and only if $X(t) = 0$, which leads to the cumulative distribution function of T via

$$
\begin{aligned}
F_T(t) &= \Pr\{T \leq t\} = \Pr\{X(t) = 0\} \\
&= P_0(t) = \left(1 - \mathrm{e}^{-\alpha t}\right)^N, \quad t \geq 0.
\end{aligned}
\tag{6.15}
$$

The linear death process can be viewed in yet another way, a way that again confirms the intimate connection between the exponential distribution and a continuous time parameter Markov chain. Consider a population consisting of N individuals, each of whose lifetimes is an independent exponentially distributed random variable with

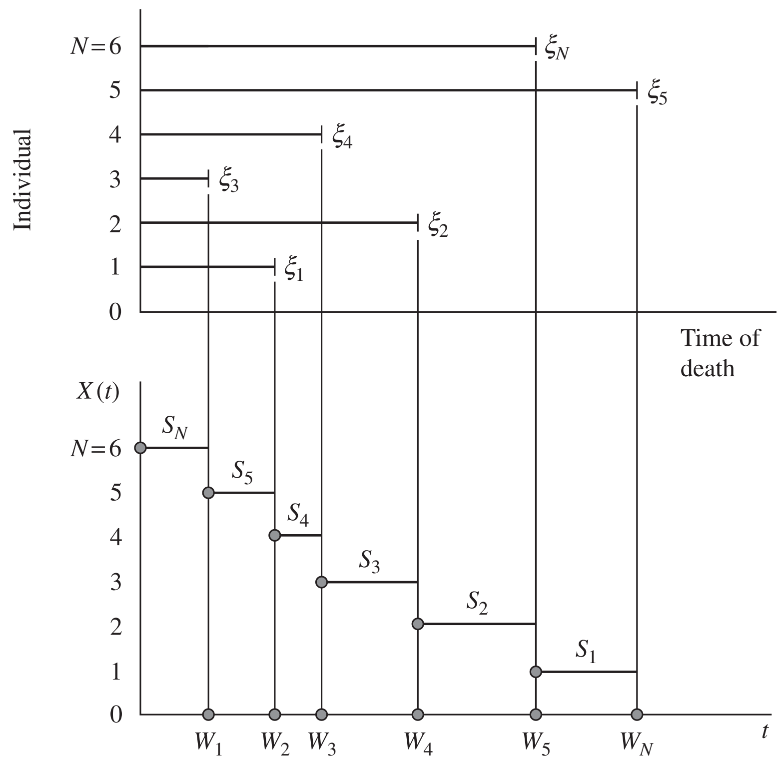

Figure 6.2 The linear death process. As depicted here, the third individual is the first to die, the first individual is the second to die, etc.

parameter α. Let $X(t)$ be the number of survivors in this population at time t. Then, $X(t)$ is the linear pure death process whose parameters are $\mu_k = k\alpha$ for $k = 0, 1, \ldots, N$. To help understand this connection, let $\xi_1, \xi_2, \ldots, \xi_N$ denote the times of death of the individuals labeled $1, 2, \ldots, N$, respectively. Figure 6.2 shows the relation between the individual lifetimes $\xi_1, \xi_2, \ldots, \xi_N$ and the death process $X(t)$.

The sojourn time in state N, denoted by S_N, equals the time of the earliest death, or $S_N = \min\{\xi_1, \ldots, \xi_N\}$. Since the lifetimes are independent and have the same exponential distribution,

$$\begin{aligned}
\Pr\{S_N > t\} &= \Pr\{\min\{\xi_1, \ldots, \xi_N\} > t\} \\
&= \Pr\{\xi_1 > t, \ldots, \xi_N > t\} \\
&= [\Pr\{\xi_1 > t\}]^N \\
&= e^{-N\alpha t}.
\end{aligned}$$

That is, S_N has an exponential distribution with parameter $N\alpha$. Similar reasoning applies when there are k members alive in the population. The memoryless property of the exponential distribution implies that the remaining lifetime of each of these k individuals is exponentially distributed with parameter α. Then, the sojourn time S_k is the minimum of these k remaining lifetimes and hence is exponentially distributed with parameter $k\alpha$. To give one more approach in terms of transition rates, each individual in the population has a constant death rate of α in the sense that

$$\begin{aligned}\Pr\{t < \xi_1 < t+h | t < \xi_1\} &= \frac{\Pr\{t < \xi_1 < t+h\}}{\Pr\{t < \xi_1\}} \\ &= \frac{e^{-\alpha t} - e^{-\alpha(t+h)}}{e^{-\alpha t}} \\ &= 1 - e^{-\alpha h} \\ &= \alpha h + o(h) \quad \text{as } h \downarrow 0.\end{aligned}$$

If each of k individuals alive in the population at time t has a constant death rate of α, then the total population death rate should be $k\alpha$, directly proportional to the population size. This shortcut approach to specifying appropriate death parameters is a powerful and often-used tool of stochastic modeling. The next example furnishes another illustration of its use.

6.2.2 Cable Failure Under Static Fatigue

A cable composed of parallel fibers under tension is being designed to support a high-altitude weather balloon. With a design load of 1000 kg and a design lifetime of 100 years, how many fibers should be used in the cable?

The low-weight, high-strength fibers to be used are subject to *static fatigue*, or eventual failure when subjected to a constant load. The higher the constant load, the shorter the life, and experiments have established a linear plot on log–log axes between average failure time and load that is shown in Figure 6.3.

The relation between mean life μ_T and load l that is illustrated in Figure 6.3 takes the analytic form

$$\log_{10} \mu_T = 2 - 40 \log_{10} l.$$

Were the cable to be designed on the basis of average life, to achieve the 100 year design target each fiber should carry 1 kg. Since the total load is 1000 kg, $N = \frac{1000}{1} = 1000$ fibers should be used in the cable.

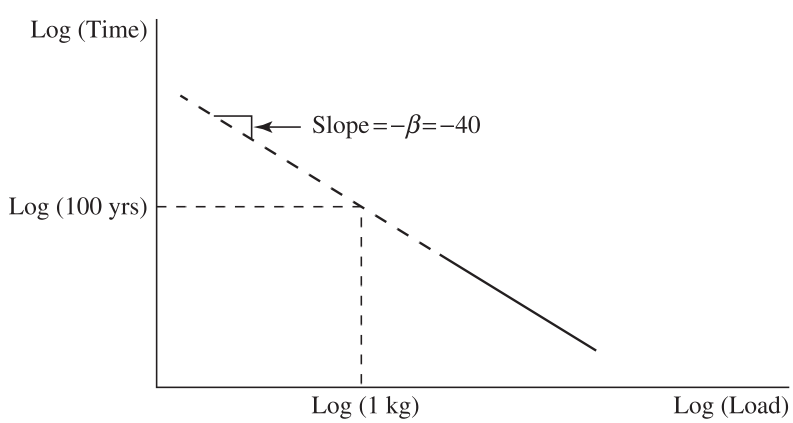

Figure 6.3 A linear relation between log mean failure time and log load.

One might suppose that this large number ($N = 1000$) of fibers would justify designing the cable based on average fiber properties. We shall see that such reasoning is dangerously wrong.

Let us suppose, however, as is the case with many modern high-performance structural materials, that there is a large amount of random scatter of individual fiber lifetimes about the mean. How does this randomness affect the design problem?

Some assumption must be made concerning the probability distribution governing individual fiber lifetimes. In practice, it is extremely difficult to gather sufficient data to determine this distribution with any degree of certainty. Most data do show, however, a significant degree of skewness, or asymmetry. Because it qualitatively matches observed data and because it leads to a pure death process model that is accessible to exhaustive analysis, we will assume that the probability distribution for the failure time T of a single fiber subjected to the time-varying tensile load $l(t)$ is given by

$$\Pr\{T \le t\} = 1 - \exp\left\{-\int_0^t K[l(s)]\mathrm{d}s\right\}, \quad t \ge 0.$$

This distribution corresponds to a *failure rate*, or *hazard rate*, of $r(t) = K[l(t)]$ wherein a single fiber, having not failed prior to time t and carrying the load $l(t)$, will fail during the interval $(t, t + \Delta t]$ with probability

$$\Pr\{t < T \le t + \Delta t | T > t\} = K[l(t)]\Delta t + o(\Delta t).$$

The function $K[l]$, called the *breakdown rule*, expresses how changes in load affect the failure probability. We are concerned with the *power law breakdown rule* in which $K[l] = l^{\beta}/A$ for some positive constants A and β. Assuming power law breakdown, under a constant load $l(t) = l$, the single fiber failure time is exponentially distributed with mean $\mu_T = E[T|l] = 1/K[l] = Al^{-\beta}$. A plot of mean failure time versus load is linear on log–log axes, matching the observed properties of our fiber type. For the design problem, we have $\beta = 40$ and $A = 100$.

Now, place N of these fibers in parallel and subject the resulting bundle or cable to a total load, constant in time, of NL, where L is the nominal load per fiber. What is the probability distribution of the time at which the cable fails? Since the fibers are in parallel, this system failure time equals the failure time of the last fiber.

Under the stated assumptions governing single-fiber behavior, $X(t)$, the number of unfailed fibers in the cable at time t, evolves as a pure death process with parameters $\mu_k = kK[NL/k]$ for $k = 1, 2, \ldots, N$. Given $X(t) = k$ surviving fibers at time t and assuming that the total bundle load NL is shared equally among them, then each carries load NL/k and has a corresponding failure rate of $K[NL/k]$. As there are k such survivors in the bundle, the bundle, or system, failure rate is $\mu_k = kK[NL/k]$ as claimed.

It was mentioned earlier that the system failure time was W_N, the waiting time to the Nth fiber failure. Then, $\Pr\{W_N \le t\} = \Pr\{X(t) = 0\} = P_0(t)$, where $P_n(t)$ is given explicitly by (2.13) in terms of $\mu_1, \ldots, \mu_N$. Alternatively, we may bring to bear the sojourn time description of the pure death process and, following Figure 6.1, write

$$W_N = S_N + S_{N-1} + \cdots + S_1,$$

where $S_N, S_{N-1}, \ldots, S_1$ are independent exponentially distributed random variables and S_k has parameter $\mu_k = kK[NL/k] = k(NL/k)^\beta/A$. The mean system failure time is readily computed to be

$$\begin{aligned} E[W_N] &= E[S_N] + \cdots + E[S_1] \\ &= AL^{-\beta}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{k}{N}\right)^{\beta} \\ &= AL^{-\beta}\sum_{k=1}^{N}\left(\frac{k}{N}\right)^{\beta-1}\left(\frac{1}{N}\right). \end{aligned} \tag{6.16}$$

The sum in the expression for $E[W_N]$ seems formidable at first glance, but a very close approximation is readily available when N is large. Figure 6.4 compares the sum to an integral.

From Figure 6.4, we see that

$$\sum_{k=1}^{N}\left(\frac{k}{N}\right)^{\beta-1}\left(\frac{1}{N}\right) \approx \int_0^1 x^{\beta-1}\mathrm{d}x = \frac{1}{\beta}.$$

Indeed, we readily obtain

$$\begin{aligned} \frac{1}{\beta} = \int_0^1 x^{\beta-1}\mathrm{d}x \le \sum_{k=1}^{N}\left(\frac{k}{N}\right)^{\beta-1}\left(\frac{1}{N}\right) &\le \int_{1/N}^{1+1/N} x^{\beta-1}\mathrm{d}x \\ &= \left(\frac{1}{\beta}\right)\left[\left(1+\frac{1}{N}\right)^{\beta} - \left(\frac{1}{N}\right)^{\beta}\right]. \end{aligned}$$

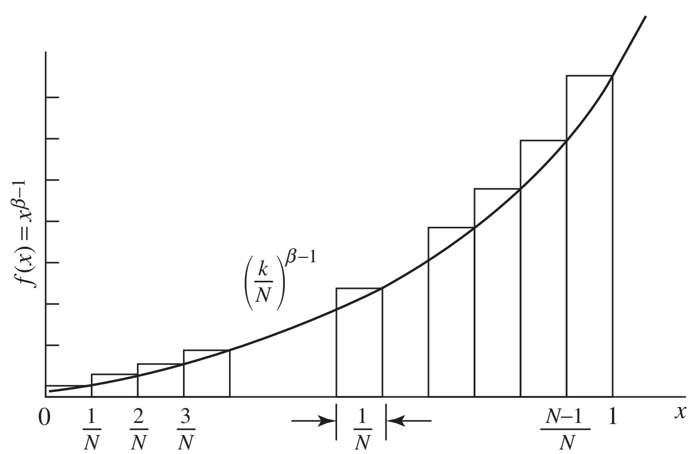

Figure 6.4 The sum $\Sigma_{k=1}^{N}(k/N)^{\beta-1}(1/N)$ is a Riemann approximation to $\int_0^1 x^{\beta-1}\mathrm{d}x = 1/\beta$.

When $N = 1000$ and $\beta = 40$, the numerical bounds are

$$\left(\frac{1}{40}\right) \leq \sum_{k=1}^{N} \left(\frac{k}{N}\right)^{\beta-1} \left(\frac{1}{N}\right) \leq \left(\frac{1}{40}\right)(1.0408),$$

which shows that the integral determines the sum to within about 4%.

Substituting $1/\beta$ for the sum in (6.16) gives the average cable life

$$E[W_N] \approx \frac{A}{\beta L^\beta}$$

to be compared with the average fiber life of

$$\mu_T = \frac{A}{L^\beta}.$$

That is, a cable lasts only about $1/\beta$ as long as an average fiber under an equivalent load. With $A = 100$, $\beta = 40$, and $N = 1000$, the designed cable would last, on the average, $100\big/\big[40(1)^{40}\big] = 2.5$ years, far short of the desired life of 100 years. The cure is to increase the number of fibers in the cable, thereby decreasing the per fiber load. Increasing the number of fibers from N to N' decreases the nominal load per fiber from L to $L' = NL/N'$. To achieve parity in fiber-cable lifetimes, we equate

$$\frac{A}{L^\beta} = \frac{A}{\beta(NL/N')^\beta},$$

or

$$N' = N\beta^{1/\beta}.$$

For the given data, this calls for $N' = 1000(40)^{1/40} = 1097$ fibers. That is, the design lifetime can be restored by increasing the number of fibers in the cable by about 10%.

Exercises

6.2.1 A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0$, $\mu_1 = 3$, $\mu_2 = 2$, and $\mu_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

6.2.2 A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0$, $\mu_1 = 3$, $\mu_2 = 2$, and $\mu_3 = 5$. Let W_3 be the random time that it takes the process to reach state 0.

(a) Write W_3 as a sum of sojourn times and thereby deduce that the mean time is $E[W_3] = \frac{31}{30}$.

(b) Determine the mean of $W_1 + W_2 + W_3$.

(c) What is the variance of W_3?

6.2.3 Give the transition probabilities for the pure death process described by $X(0)=3$, $\mu_3=1$, $\mu_2=2$, and $\mu_1=3$.

6.2.4 Consider the linear death process (Section 6.2.1) in which $X(0)=N=5$ and $\alpha=2$. Determine $\Pr\{X(t)=2\}$.

Hint: Use equation (6.14).

Problems

6.2.1 Let $X(t)$ be a pure death process starting from $X(0)=N$. Assume that the death parameters are $\mu_1, \mu_2, \ldots, \mu_N$. Let T be an independent exponentially distributed random variable with parameter θ. Show that

$$\Pr\{X(T)=0\}=\prod_{i=1}^{N}\frac{\mu_i}{\mu_i+\theta}.$$

6.2.2 Let $X(t)$ be a pure death process with constant death rates $\mu_k=\theta$ for $k=1,2,\ldots,N$. If $X(0)=N$, determine $P_n(t)=\Pr\{X(t)=n\}$ for $n=0,1,\ldots,N$.

6.2.3 A pure death process $X(t)$ with parameters $\mu_1, \mu_2, \ldots$ starts at $X(0)=N$ and evolves until it reaches the absorbing state 0. Determine the mean area under the $X(t)$ trajectory.

Hint: This is $E[W_1+W_2+\cdots+W_N]$.

6.2.4 A chemical solution contains N molecules of type A and M molecules of type B. An irreversible reaction occurs between type A and B molecules in which they bond to form a new compound AB. Suppose that in any small time interval of length h, any particular unbonded A molecule will react with any particular unbonded B molecule with probability $\theta h+o(h)$, where θ is a reaction rate. Let $X(t)$ denote the number of unbonded A molecules at time t.

(a) Model $X(t)$ as a pure death process by specifying the parameters.

(b) Assume that $N<M$ so that eventually all of the A molecules become bonded. Determine the mean time until this happens.

6.2.5 Consider a cable composed of fibers following the breakdown rule $K[l]=\sinh(l)=\frac{1}{2}\left(e^{l}-e^{-l}\right)$ for $l\ge 0$. Show that the mean cable life is given by

$$\begin{aligned}E[W_N]=\sum_{k=1}^{N}\{k\sinh(NL/k)\}^{-1}&=\sum_{k=1}^{N}\left\{\frac{k}{N}\sinh\left(\frac{L}{k/N}\right)\right\}^{-1}\left(\frac{1}{N}\right)\\&\approx\int_0^1\{x\sinh(L/x)\}^{-1}\mathrm{d}x.\end{aligned}$$

6.2.6 Let T be the time to extinction in the linear death process with parameters $X(0)=N$ and α (see Section 6.2.1).

(a) Using the sojourn time viewpoint, show that

$$E[T] = \frac{1}{\alpha}\left[\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}\right].$$

(b) Verify the result of (a) by using equation (6.15) in

$$E[T] = \int_0^\infty \Pr\{T > t\}\mathrm{d}t = \int_0^\infty [1 - F_T(t)]\mathrm{d}t.$$

Hint: Let $y = 1 - \mathrm{e}^{-\alpha t}$.

6.3 Birth and Death Processes

An obvious generalization of the pure birth and pure death processes discussed in Sections 6.1 and 6.2 is to permit $X(t)$ both to increase and to decrease. Thus, if at time t the process is in state n, it may, after a random sojourn time, move to either of the neighboring states $n+1$ or $n-1$. The resulting *birth and death process* can then be regarded as the continuous-time analog of a random walk (Chapter 3, Section 3.5.3).

Birth and death processes form a powerful tool in the kit of the stochastic modeler. The richness of the birth and death parameters facilitates modeling a variety of phenomena. At the same time, standard methods of analysis are available for determining numerous important quantities such as stationary distributions and mean first passage times. This section and later sections contain several examples of birth and death processes and illustrate how they are used to draw conclusions about phenomena in a variety of disciplines.

6.3.1 Postulates

As in the case of the pure birth processes, we assume that $X(t)$ is a Markov process on the states $0, 1, 2, \ldots$ and that its transition probabilities $P_{ij}(t)$ are stationary; that is

$$P_{ij}(t) = \Pr\{X(t+s) = j | X(s) = i\} \quad \text{for all} \quad s \geq 0.$$

In addition, we assume that the $P_{ij}(t)$ satisfy

1. $P_{i,i+1}(h) = \lambda_i h + o(h)$ as $h \downarrow 0, i \geq 0$;
2. $P_{i,i-1}(h) = \mu_i h + o(h)$ as $h \downarrow 0, i \geq 1$;
3. $P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$ as $h \downarrow 0, i \geq 0$;
4. $P_{ij}(0) = \delta_{ij}$;
5. $\mu_0 = 0, \lambda_0 > 0, \mu_i, \lambda_i > 0, i = 1, 2, \ldots$.

The $o(h)$ in each case may depend on i. The matrix

$$\mathbf{A} = \begin{Vmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{Vmatrix} \tag{6.17}$$

is called the *infinitesimal generator* of the process. The parameters λ_i and μ_i are called, respectively, the infinitesimal birth and death rates. In Postulates (1) and (2), we are assuming that if the process starts in state i, then in a small interval of time the probabilities of the population increasing or decreasing by 1 are essentially proportional to the length of the interval.

Since the $P_{ij}(t)$ are probabilities, we have $P_{ij}(t) \geq 0$ and

$$\sum_{j=0}^{\infty} P_{ij}(t) \leq 1. \tag{6.18}$$

Using the Markov property of the process, we may also derive the so-called *Chapman–Kolmogorov equation*

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s). \tag{6.19}$$

This equation states that in order to move from state i to state j in time $t+s, X(t)$ moves to some state k in time t and then from k to j in the remaining time s. This is the continuous-time analog of formula (3.11) in Chapter 3.

Thus far, we have mentioned only the transition probabilities $P_{ij}(t)$. In order to obtain the probability that $X(t) = n$, we must specify where the process starts or more generally the probability distribution for the initial state. We then have

$$\Pr\{X(t) = n\} = \sum_{i=0}^{x} q_i P_{in}(t),$$

where

$$q_i = \Pr\{X(0) = i\}.$$

6.3.2 Sojourn Times

With the aid of the preceding assumptions, we may calculate the distribution of the random variable S_i, which is the sojourn time of $X(t)$ in state i; that is, given that the process is in state i, what is the distribution of the time S_i until it first leaves state i? If we let

$$\Pr\{S_i \geq t\} = G_i(t),$$

it follows easily by the Markov property that as $h \downarrow 0$,

$$\begin{aligned} G_i(t+h) &= G_i(t)G_i(h) = G_i(t)[P_{ii}(h) + o(h)] \\ &= G_i(t)[1 - (\lambda_i + \mu_i)h] + o(h), \end{aligned}$$

or

$$\frac{G_i(t+h) - G_i(t)}{h} = -(\lambda_i + \mu_i)G_i(t) + o(1)$$

so that

$$G_i'(t) = -(\lambda_i + \mu_i)G_i(t). \tag{6.20}$$

If we use the conditions $G_i(0) = 1$, the solution of this equation is

$$G_i(t) = \exp[-(\lambda_i + \mu_i)t];$$

that is, S_i follows an exponential distribution with mean $(\lambda_i + \mu_i)^{-1}$. The proof presented here is not quite complete, since we have used the intuitive relationship

$$G_i(h) = P_{ii}(h) + o(h)$$

without a formal proof.

According to Postulates (1) and (2), during a time duration of length h, a transition occurs from state i to $i+1$ with probability $\lambda_i h + o(h)$ and from state i to $i-1$ with probability $\mu_i h + o(h)$. It follows intuitively that, given that a transition occurs at time t, the probability that this transition is to state $i+1$ is $\lambda_i/(\lambda_i + \mu_i)$ and to state $i-1$ is $\mu_i/(\lambda_i + \mu_i)$. The rigorous demonstration of this result is beyond the scope of this book.

It leads to an important characterization of a birth and death process, however, wherein the description of the motion of $X(t)$ is as follows: The process sojourns in a given state i for a random length of time whose distribution function is an exponential distribution with parameter $(\lambda_i + \mu_i)$. When leaving state i the process enters either state $i+1$ or state $i-1$ with probabilities $\lambda_i/(\lambda_i + \mu_i)$ and $\mu_i/(\lambda_i + \mu_i)$, respectively. The motion is analogous to that of a random walk except that transitions occur at random times rather than at fixed time periods.

The traditional procedure for constructing birth and death processes is to prescribe the birth and death parameters $\{\lambda_i, \mu_i\}_{i=0}^{\infty}$ and build the path structure by utilizing the preceding description concerning the waiting times and the conditional transition probabilities of the various states. We determine realizations of the process as follows.

Suppose $X(0) = i$; the particle spends a random length of time, exponentially distributed with parameter $(\lambda_i + \mu_i)$, in state i and subsequently moves with probability $\lambda_i/(\lambda_i + \mu_i)$ to state $i+1$ and with probability $\mu_i/(\lambda_i + \mu_i)$ to state $i-1$. Next, the particle sojourns a random length of time in the new state and then moves to one of its neighboring states and so on. More specifically, we observe a value t_1 from the exponential distribution with parameter $(\lambda_i + \mu_i)$ that fixes the initial sojourn time in state i. Then, we toss a coin with probability of heads $p_i = \lambda_i/(\lambda_i + \mu_i)$. If heads (tails) appear, we move the particle to state $i+1(i-1)$. In state $i+1$, we observe a value t_2 from the exponential distribution with parameter $(\lambda_{i+1} + \mu_{i+1})$ that fixes the sojourn time in the seconds state visited. If the particle at the first transition enters state $i-1$, the subsequent sojourn time t_2' is an observation from the exponential distribution with parameter $(\lambda_{i-1} + \mu_{i-1})$. After the second wait is completed, a Bernoulli trial is performed that chooses the next state to be visited, and the process continues in the same way.

A typical outcome of these sampling procedures determines a realization of the process. Its form might be, e.g.,

$$X(t) = \begin{cases} i, & \text{for } 0 < t < t_1, \\ i+1, & \text{for } t_1 < t < t_1 + t_2, \\ i, & \text{for } t_1 + t_2 < t < t_1 + t_2 + t_3, \\ \vdots & \qquad\qquad \vdots \end{cases}$$

Thus, by sampling from exponential and Bernoulli distributions appropriately, we construct typical sample paths of the process. Now, it is possible to assign to this set of paths (realizations of the process) a probability measure in a consistent way so that $P_{ij}(t)$ is determined satisfying (6.18) and (6.19). This result is rather deep, and its rigorous discussion is beyond the level of this book. The process obtained in this manner is called the minimal process associated with the infinitesimal matrix **A** defined in (6.17).

The preceding construction of the minimal process is fundamental, since the infinitesimal parameters need not determine a unique stochastic process obeying (6.18), (6.19), and Postulates 1 through 5 of Section 6.3.1. In fact, there could be several Markov processes that possess the same infinitesimal generator. Fortunately, such complications do not arise in the modeling of common phenomena. In the special case of birth and death processes for which $\lambda_0 > 0$, a sufficient condition that there exists a unique Markov process with transition probability function $P_{ij}(t)$ for which the infinitesimal relations (6.18) and (6.19) hold is that

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n \theta_n} \sum_{k=0}^{n} \theta_k = \infty, \tag{6.21}$$

where

$$\theta_0 = 1, \quad \theta_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \quad n = 1, 2, \ldots.$$

In most practical examples of birth and death processes, the condition (6.21) is met, and the birth and death process associated with the prescribed parameters is uniquely determined.

6.3.3 Differential Equations of Birth and Death Processes

As in the case of the pure birth and pure death processes, the transition probabilities $P_{ij}(t)$ satisfy a system of differential equations known as the backward Kolmogorov differential equations. These are given by

$$\begin{aligned} P'_{0j}(t) &= -\lambda_0 P_{0j}(t) + \lambda_0 P_{1j}(t), \\ P'_{ij}(t) &= \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t), \quad i \geq 1, \end{aligned} \tag{6.22}$$

and the boundary condition $P_{ij}(0) = \delta_{ij}$.

To derive these, we have, from equation (6.19),

$$\begin{aligned} P_{ij}(t+h) &= \sum_{k=0}^{\infty} P_{ik}(h) P_{kj}(t) \\ &= P_{i,i-1}(h) P_{i-1,j}(t) + P_{i,i}(h) P_{ij}(t) + P_{i,i+1}(h) P_{i+1,j}(t) \\ &\quad + {\sum_k}' P_{ik}(h) P_{kj}(t), \end{aligned} \tag{6.23}$$

where the last summation is over all $k \neq i-1, i, i+1$. Using Postulates (1), (2), and (3) of Section 6.3.1, we obtain

$$\begin{aligned} {\sum_k}' P_{ik}(h) P_{kj}(t) &\leq {\sum_k}' P_{ik}(h) \\ &= 1 - [P_{i,i}(h) + P_{i,i-1}(h) + P_{i,i+1}(h)] \\ &= 1 - [1 - (\lambda_i + \mu_i)h + o(h) + \mu_i h + o(h) + \lambda_i h + o(h)] \\ &= o(h) \end{aligned}$$

so that

$$P_{ij}(t+h) = \mu_i h P_{i-1,j}(t) + [1 - (\lambda_i + \mu_i)h] P_{ij}(t) + \lambda_i h P_{i+1,j}(t) + o(h).$$

Transposing the term $P_{ij}(t)$ to the left-hand side and dividing the equation by h, we obtain, after letting $h \downarrow 0$,

$$P'_{ij}(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t).$$

The backward equations are deduced by decomposing the time interval $(0, t+h)$, where h is positive and small, into the two periods

$$(0, h), \quad (h, t+h)$$

and examining the transition in each period separately. In this sense, the backward equations result from a "first step analysis," the first step being over the short time interval of duration h.

A different result arises from a "last step analysis," which proceeds by splitting the time interval $(0, t+h)$ into the two periods

$$(0, t), \quad (t, t+h)$$

and adapting the preceding reasoning. From this viewpoint, under more stringent conditions, we can derive a further system of differential equations

$$\begin{aligned} P'_{i0}(t) &= -\lambda_0 P_{i,0}(t) + \mu_1 P_{i,1}(t), \\ P'_{ij}(t) &= \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t), \quad j \geq 1, \end{aligned} \tag{6.24}$$

with the same initial condition $P_{ij}(0) = \delta_{ij}$. These are known as the forward Kolmogorov differential equations. To derive these equations, we interchange t and h in equation (6.23), and under stronger assumptions in addition to Postulates (1), (2), and (3), it can be shown that the last term is again $o(h)$. The remainder of the argument is the same as before. The usefulness of the differential equations will become apparent in the examples that we study in this and the next section.

A sufficient condition that (6.24) hold is that $[P_{kj}(h)]/h = o(1)$ for $k \neq j, j-1, j+1$, where the $o(1)$ term apart from tending to zero is uniformly bounded with respect to k for fixed j as $h \to 0$. In this case, it can be proved that $\sum'_k P_{ik}(t) P_{kj}(h) = o(h)$.

Example *Linear Growth with Immigration* A birth and death process is called a linear growth process if $\lambda_n = \lambda n + a$ and $\mu_n = \mu n$ with $\lambda > 0$, $\mu > 0$, and $a > 0$. Such processes occur naturally in the study of biological reproduction and population growth. If the state n describes the current population size, then the average instantaneous rate of growth is $\lambda n + a$. Similarly, the probability of the state of the process decreasing by one after the elapse of a small duration of time h is $\mu n h + o(h)$. The factor λn represents the natural growth of the population owing to its current size, while the second factor a may be interpreted as the infinitesimal rate of increase of the population due to an external source such as immigration. The component μn, which gives the mean infinitesimal death rate of the present population, possesses the obvious interpretation.

If we substitute the above values of λ_n and μ_n in (6.24), we obtain

$$\begin{aligned} P'_{i0}(t) &= -a P_{i0}(t) + \mu P_{i1}(t), \\ P'_{ij}(t) &= [\lambda(j-1) + a] P_{i,j-1}(t) - [(\lambda + \mu) j + a] P_{ij}(t) \\ &\quad + \mu(j+1) P_{i,j+1}(t), \quad j \geq 1. \end{aligned}$$

Now, if we multiply the jth equation by j and sum, it follows that the expected value

$$E[X(t)] = M(t) = \sum_{j=1}^{\infty} j P_{ij}(t)$$

satisfies the differential equation

$$M'(t) = a + (\lambda - \mu)M(t),$$

with initial condition $M(0) = i$, if $X(0) = i$. The solution of this equation is

$$M(t) = at + i \quad \text{if } \lambda = \mu$$

and

$$M(t) = \frac{a}{\lambda - \mu}\left\{e^{(\lambda-\mu)t} - 1\right\} + ie^{(\lambda-\mu)t} \quad \text{if } \lambda \neq \mu. \tag{6.25}$$

The second moment, or variance, may be calculated in a similar way. It is interesting to note that $M(t) \to \infty$ as $t \to \infty$ if $\lambda \geq \mu$, while if $\lambda < \mu$, the mean population size for large t is approximately

$$\frac{a}{\mu - \lambda}.$$

These results suggest that in the second case, wherein $\lambda < \mu$, the population stabilizes in the long run in some form of statistical equilibrium. Indeed, it can be shown that a limiting probability distribution $\{\pi_j\}$ exists for which $\lim_{t\to\infty} P_{ij}(t) = \pi_j, j = 0, 1, \ldots$. Such limiting distributions for general birth and death processes are the subject of Section 6.4.

Example *The Two-State Markov Chain* Consider a Markov chain $\{X(t)\}$ with state $\{0, 1\}$ whose infinitesimal matrix is

$$\mathbf{A} = \begin{array}{c|cc|} & 0 & 1 \\ 0 & -\alpha & \alpha \\ 1 & \beta & -\beta \end{array}. \tag{6.26}$$

The process alternates between states 0 and 1. The sojourn times in state 0 are independent and exponentially distributed with parameter α. Those in state 1 are independent and exponentially distributed with parameter β. This is a finite-state birth and death process for which $\lambda_0 = \alpha, \lambda_1 = 0, \mu_0 = 0$, and $\mu_1 = \beta$. The first Kolmogorov forward equation in (6.24) becomes

$$P'_{00}(t) = -\alpha P_{00}(t) + \beta P_{01}(t). \tag{6.27}$$

Now, $P_{01}(t) = 1 - P_{00}(t)$, which placed in (6.27) gives

$$P'_{00}(t) = \beta - (\alpha + \beta)P_{00}(t).$$