5 Poisson Processes

5.1 The Poisson Distribution and the Poisson Process

Poisson behavior is so pervasive in natural phenomena and the Poisson distribution is so amenable to extensive and elaborate analysis as to make the Poisson process a cornerstone of stochastic modeling.

5.1.1 The Poisson Distribution

The Poisson distribution with parameter $\mu > 0$ is given by

$$p_k = \frac{e^{-\mu}\mu^k}{k!} \quad \text{for } k = 0, 1, \ldots. \tag{5.1}$$

Let X be a random variable having the Poisson distribution in (5.1). We evaluate the mean, or first moment, via

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} k p_k = \sum_{k=1}^{\infty} \frac{k e^{-\mu}\mu^k}{k!} \\ &= \mu e^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^{(k-1)}}{(k-1)!} \\ &= \mu. \end{aligned}$$

To evaluate the variance, it is easier first to determine

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{\infty} k(k-1) p_k \\ &= \mu^2 e^{-\mu} \sum_{k=2}^{\infty} \frac{\mu^{(k-2)}}{(k-2)!} \\ &= \mu^2. \end{aligned}$$

Then

$$\begin{aligned} E\left[X^2\right] &= E[X(X-1)] + E[X] \\ &= \mu^2 + \mu, \end{aligned}$$

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while

$$\sigma_X^2 = \mathrm{Var}[X] = E\left[X^2\right] - \{E[X]\}^2$$
$$= \mu^2 + \mu - \mu^2 = \mu.$$

Thus, the Poisson distribution has the unusual characteristic that both the mean and the variance are given by the same value μ.

Two fundamental properties of the Poisson distribution, which will arise later in a variety of forms, concern the sum of independent Poisson random variables and certain random decompositions of Poisson phenomena. We state these properties formally as Theorems 5.1 and 5.2.

Theorem 5.1. *Let X and Y be independent random variables having Poisson distributions with parameters μ and ν, respectively. Then the sum $X+Y$ has a Poisson distribution with parameter $\mu+\nu$.*

Proof. By the law of total probability,

$$\begin{aligned}
\Pr\{X+Y=n\} &= \sum_{k=0}^{n} \Pr\{X=k, Y=n-k\} \\
&= \sum_{k=0}^{n} \Pr\{X=k\}\ \Pr\{Y=n-k\} \\
&\qquad\qquad\qquad (X \text{ and } Y \text{ are independent}) \\
&= \sum_{k=0}^{n} \left\{\frac{\mu^k e^{-\mu}}{k!}\right\}\left\{\frac{\nu^{n-k} e^{-\nu}}{(n-k)!}\right\} \\
&= \frac{e^{-(\mu+\nu)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\mu^k \nu^{n-k}.
\end{aligned} \tag{5.2}$$

The binomial expansion of $(\mu+\nu)^n$ is, of course,

$$(\mu+\nu)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\mu^k \nu^{n-k},$$

and so (5.2) simplifies to

$$\Pr\{X+Y=n\} = \frac{e^{-(\mu+\nu)}(\mu+\nu)^n}{n!}, \quad n = 0, 1, \ldots,$$

the desired Poisson distribution.

To describe the second result, we consider first a Poisson random variable N where the parameter is $\mu > 0$. Write N as a sum of ones in the form

$$N = \underbrace{1 + 1 + \cdots + 1}_{N \text{ ones}},$$

and next, considering each one separately and independently, erase it with probability $1 - p$ and keep it with probability p. What is the distribution of the resulting sum M, of the form $M = 1 + 0 + 0 + 1 + \cdots + 1$?

The next theorem states and answers the question in a more precise wording. ■

Theorem 5.2. *Let N be a Poisson random variable with parameter μ, and conditional on N, let M have a binomial distribution with parameters N and p. Then the unconditional distribution of M is Poisson with parameter μp.*

Proof. The verification proceeds via a direct application of the law of total probability. Then

$$\begin{aligned}
\Pr\{M = k\} &= \sum_{n=0}^{\infty} \Pr\{M = k | N = n\} \Pr\{N = n\} \\
&= \sum_{n=k}^{\infty} \left\{ \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \right\} \left\{ \frac{\mu^n \mathrm{e}^{-\mu}}{n!} \right\} \\
&= \frac{\mathrm{e}^{-\mu} (\mu p)^k}{k!} \sum_{n=k}^{\infty} \frac{[\mu(1-p)]^{n-k}}{(n-k)!} \\
&= \frac{\mathrm{e}^{-\mu} (\mu p)^k}{k!} \mathrm{e}^{\mu(1-p)} \\
&= \frac{\mathrm{e}^{-\mu p} (\mu p)^k}{k!} \quad \text{for } k = 0, 1, \ldots,
\end{aligned}$$

which is the claimed Poisson distribution. ■

5.1.2 The Poisson Process

The Poisson process entails notions of both independence and the Poisson distribution.

Definition A Poisson process of intensity, or rate, $\lambda > 0$ is an integer-valued stochastic process $\{X(t);\ t \geq 0\}$ for which

1. for any time points $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, the process increments

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent random variables;

2. for $s \geq 0$ and $t > 0$, the random variable $X(s+t) - X(s)$ has the Poisson distribution

$$\Pr\{X(s+t) - X(s) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad \text{for } k = 0, 1, \ldots;$$

3. $X(0) = 0$.

In particular, observe that if $X(t)$ is a Poisson process of rate $\lambda > 0$, then the moments are

$$E[X(t)] = \lambda t \quad \text{and} \quad \mathrm{Var}[X(t)] = \sigma^2_{X(t)} = \lambda t.$$

Example Defects occur along an undersea cable according to a Poisson process of rate $\lambda = 0.1$ per mile. (a) What is the probability that no defects appear in the first two miles of cable? (b) Given that there are no defects in the first two miles of cable, what is the conditional probability of no defects between mile points two and three? To answer (a) we observe that $X(2)$ has a Poisson distribution whose parameter is $(0.1)(2) = 0.2$. Thus, $\Pr\{X(2) = 0\} = e^{-0.2} = 0.8187$. In part (b), we use the independence of $X(3) - X(2)$ and $X(2) - X(0) = X(2)$. Thus, the conditional probability is the same as the unconditional probability, and

$$\Pr\{X(3) - X(2) = 0\} = \Pr\{X(1) = 0\} = e^{-0.1} = 0.9048.$$

Example Customers arrive in a certain store according to a Poisson process of rate $\lambda = 4$ per hour. Given that the store opens at 9:00 A.M., what is the probability that exactly one customer has arrived by 9:30 and a total of five have arrived by 11:30 A.M.?

Measuring time t in hours from 9:00 A.M., we are asked to determine $\Pr\left\{X\left(\frac{1}{2}\right) = 1, X\left(\frac{5}{2}\right) = 5\right\}$. We use the independence of $X\left(\frac{5}{2}\right) - X\left(\frac{1}{2}\right)$ and $X\left(\frac{1}{2}\right)$ to reformulate the question thus:

$$\begin{aligned}
\Pr\left\{X\left(\frac{1}{2}\right) = 1, X\left(\frac{5}{2}\right) = 5\right\} &= \Pr\left\{X\left(\frac{1}{2}\right) = 1, X\left(\frac{5}{2}\right) - X\left(\frac{1}{2}\right) = 4\right\} \\
&= \left\{\frac{e^{-4(1/2)} 4\left(\frac{1}{2}\right)}{1!}\right\}\left\{\frac{e^{-4(2)}[4(2)]^4}{4!}\right\} \\
&= \left(2e^{-2}\right)\left(\frac{512}{3}e^{-8}\right) = 0.0154965.
\end{aligned}$$

5.1.3 Nonhomogeneous Processes

The rate λ in a Poisson process $X(t)$ is the proportionality constant in the probability of an event occurring during an arbitrarily small interval. To explain this more precisely,

$$\begin{aligned}
\Pr\{X(t+h) - X(t) = 1\} &= \frac{(\lambda h)e^{-\lambda h}}{1!} \\
&= (\lambda h)\left(1 - \lambda h + \frac{1}{2}\lambda^2 h^2 - \cdots\right) \\
&= \lambda h + o(h),
\end{aligned}$$

where $o(h)$ denotes a general and unspecified remainder term of smaller order than h.

It is pertinent in many applications to consider rates $\lambda = \lambda(t)$ that vary with time. Such a process is termed a *nonhomogeneous* or *nonstationary* Poisson process to distinguish it from the stationary, or homogeneous, process that we primarily consider. If $X(t)$ is a nonhomogeneous Poisson process with rate $\lambda(t)$, then an increment $X(t) - X(s)$, giving the number of events in an interval $(s,t]$, has a Poisson distribution with parameter $\int_s^t \lambda(u)\,\mathrm{d}u$, and increments over disjoint intervals are independent random variables.

Example Demands on a first aid facility in a certain location occur according to a nonhomogeneous Poisson process having the rate function

$$\lambda(t) = \begin{cases} 2t & \text{for } 0 \le t < 1, \\ 2 & \text{for } 1 \le t < 2, \\ 4 - t & \text{for } 2 \le t \le 4, \end{cases}$$

where t is measured in hours from the opening time of the facility. What is the probability that two demands occur in the first 2 h of operation and two in the second 2 h? Since demands during disjoint intervals are independent random variables, we can answer the two questions separately. The mean for the first 2 h is $\mu = \int_0^1 2t\,\mathrm{d}t + \int_1^2 2\mathrm{d}t = 3$, and thus

$$\Pr\{X(2) = 2\} = \frac{\mathrm{e}^{-3}(3)^2}{2!} = 0.2240.$$

For the second 2 h, $\mu = \int_2^4 (4-t)\,\mathrm{d}t = 2$, and

$$\Pr\{X(4) - X(2) = 2\} = \frac{\mathrm{e}^{-2}(2)^2}{2!} = 0.2707.$$

Let $X(t)$ be a nonhomogeneous Poisson process of rate $\lambda(t) > 0$ and define $\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u$. Make a deterministic change in the time scale and define a new process $Y(s) = X(t)$, where $s = \Lambda(t)$. Observe that $\Delta s = \lambda(t)\Delta t + o(\Delta t)$. Then

$$\begin{aligned}\Pr\{Y(s+\Delta s) - Y(s) = 1\} &= \Pr\{X(t+\Delta t) - X(t) = 1\} \\ &= \lambda(t)\Delta t + o(\Delta t) \\ &= \Delta s + o(\Delta s),\end{aligned}$$

so that $Y(s)$ is a homogeneous Poisson process of unit rate. By this means, questions about nonhomogeneous Poisson processes can be transformed into corresponding questions about homogeneous processes. For this reason, we concentrate our exposition on the latter.

5.1.4 Cox Processes

Suppose that $X(t)$ is a nonhomogeneous Poisson process, but where the rate function $\{\lambda(t),\, t \ge 0\}$ is itself a stochastic process. Such processes were introduced in 1955 as models for fibrous threads by Sir David Cox, who called them *doubly stochastic Poisson processes*. Now they are most often referred to as *Cox processes* in honor of their discoverer. Since their introduction, Cox processes have been used to model a myriad of phenomena, e.g., bursts of rainfall, where the likelihood of rain may vary with the season; inputs to a queueing system, where the rate of input varies over time, depending on changing and unmeasured factors; and defects along a fiber, where the rate and type of defect may change due to variations in material or manufacture. As these applications suggest, the process increments over disjoint intervals are, in general, statistically dependent in a Cox process, as contrasted with their postulated independence in a Poisson process.

Let $\{X(t);\, t \geq 0\}$ be a Poisson process of constant rate $\lambda = 1$. The very simplest Cox process, sometimes called a *mixed Poisson process*, involves choosing a single random variable Θ, and then observing the process $X'(t) = X(\Theta t)$. Given Θ, then X' is, conditionally, a Poisson process of constant rate $\lambda = \Theta$, but Θ is random, and typically, unobservable. If Θ is a continuous random variable with probability density function $f(\theta)$, then, upon removing the condition via the law of total probability, we obtain the marginal distribution

$$\Pr\{X'(t) = k\} = \int_0^\infty \frac{(\theta t)^k e^{-\theta t}}{k!} f(\theta)\mathrm{d}\theta. \tag{5.3}$$

Problem 5.1.12 calls for carrying out the integration in (5.3) in the particular instance in which Θ has an exponential density.

Chapter 6, Section 6.7 develops a model for defects along a fiber in which a Markov chain in continuous time is the random intensity function for a Poisson process. A variety of functionals are evaluated for the resulting Cox process.

Exercises

5.1.1 Defects occur along the length of a filament at a rate of $\lambda = 2$ per foot.
- **(a)** Calculate the probability that there are no defects in the first foot of the filament.
- **(b)** Calculate the conditional probability that there are no defects in the second foot of the filament, given that the first foot contained a single defect.

5.1.2 Let $p_k = \Pr\{X = k\}$ be the probability mass function corresponding to a Poisson distribution with parameter λ. Verify that $p_0 = \exp\{-\lambda\}$, and that p_k may be computed recursively by $p_k = (\lambda/k)p_{k-1}$.

5.1.3 Let X and Y be independent Poisson distributed random variables with parameters α and β, respectively. Determine the conditional distribution of X, given that $N = X + Y = n$.

5.1.4 Customers arrive at a service facility according to a Poisson process of rate λ customer/hour. Let $X(t)$ be the number of customers that have arrived up to time t.
- **(a)** What is $\Pr\{X(t) = k\}$ for $k = 0, 1, \ldots$?
- **(b)** Consider fixed times $0 < s < t$. Determine the conditional probability $\Pr\{X(t) = n + k | X(s) = n\}$ and the expected value $E[X(t)X(s)]$.