Spin

$$m_e = e = \hbar = \frac{1}{4\pi\epsilon_0} = k_B = 1.$$

State space $\mathcal{H}$

Inner product space on $\mathbb{C}$.

$|\psi\rangle \in \mathcal{H}$. ket vector

basis $|+\rangle_z$, $|-\rangle_z$

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle, \quad c_1, c_2 \in \mathbb{C}.$$

adjoint, $\langle\psi|$. bra vector.

$\langle\psi|\varphi\rangle := (\psi, \varphi)$ inner product.

$\langle +_z|+_z\rangle = \langle -_z|-_z\rangle = 1$, $\langle +_z|-_z\rangle = 0$.

$\mathcal{H} \simeq \mathbb{C}^2$

Operator. Linear operator on $\mathcal{H}$. $\cong \mathbb{C}^{2\times 2}$

Ex. SG_z filter.

$|+_z\rangle \equiv |\uparrow\rangle \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|-_z\rangle \equiv |\downarrow\rangle \sim \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|0\rangle = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Adjoint

$$\langle \varphi | \hat{A} \psi \rangle := (\varphi, \hat{A}\psi) = (\hat{A}^* \varphi, \psi \rangle =: \langle \hat{A}^* \varphi | \psi \rangle$$

self adjoint $\hat{A} = \hat{A}^*$

Postulation : all physical observables are represented by self adjoint operators.

Eigen-decomposition.

$$\hat{A}|\varphi_i\rangle = a_i|\varphi_i\rangle, \quad a_i \in \mathbb{R}.$$

$$\langle\varphi_i|\varphi_j\rangle = \delta_{ij}$$

postulation: (Dirac quote)

1) Measurement interact w. quantum system.

2) Outcome of a single measurement is a single eigenvalue.

3) After measurement, wavefunction collapse into corresponding eigenvector.

$$|\psi\rangle = \sum_{i=1}^{n} c_i |\varphi_i\rangle \qquad c_i \in \mathbb{C}. \quad \sum_i |c_i|^2 = 1.$$

<u>Probability interpretation</u>:

After measurement related to $\hat{A}$, probability of obtaining $|\varphi_i\rangle$ is $|c_i|^2$.

<u>Ex</u>. spin-$\frac{1}{2}$.

$$|+_x\rangle = \frac{1}{\sqrt{2}}\left[|+_z\rangle + e^{i\alpha}|-_z\rangle\right]$$

$$|-_x\rangle = \frac{1}{\sqrt{2}}\left[|+_z\rangle - e^{i\alpha}|-_z\rangle\right]$$

$$|\pm_y\rangle = \frac{1}{\sqrt{2}}\left[|+_z\rangle \pm e^{i\beta}|-_z\rangle\right]$$

$$\langle \pm_x | \pm_y \rangle = \frac{1}{2} \Rightarrow \alpha - \beta = \pm\frac{\pi}{2} + 2\pi n \quad (\text{exer})$$

$$n \in \mathbb{Z}.$$

Choose $\alpha = 0$, $\beta = \frac{\pi}{2}$.

$$|\pm_x\rangle \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}, \quad |\pm_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

$$\hat{\vec{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)^T \quad \text{spin operator.}$$

$$\hat{S}_z = \frac{1}{2}\left(|+_z\rangle\langle +_z| - |-_z\rangle\langle -_z|\right) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\hat{S}_x = \frac{1}{2}\left(|+_x\rangle\langle +_x| - |-_x\rangle\langle -_x| \right) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{S}_y = \frac{1}{2}\left(|+_y\rangle\langle +_y| - |+_y\rangle\langle -_y| \right) = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

(exer)

Pauli matrices.

$$\hat{\vec{S}} = \frac{1}{2}\vec{\sigma} \quad . \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$$

$$\hat{S}_{\vec{n}} = \vec{n}\cdot\hat{\vec{S}} = n_1\hat{S}_x + n_2\hat{S}_y + n_3\hat{S}_z \quad . \quad |\vec{n}| = 1.$$

$$\sigma_{\vec{n}} = \vec{n}\cdot\vec{\sigma}$$

Ex. $\hat{\vec{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3}{4}I.$

Ex. $\sigma_{\vec{n}}^2 = I$. $|\vec{n}|=1$.

Pf: $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$. $\{\sigma_x, \sigma_y\} = \{\sigma_x, \sigma_z\} = \{\sigma_x, \sigma_y\} = 0$.

$$\sigma_{\vec{n}} \cdot \sigma_{\vec{n}} = n_1^2 \sigma_1^2 + n_2^2 \sigma_2^2 + n_3^2 \sigma_3^2 = I. \quad \square.$$

$\hat{A}$, $\hat{B}$ on $\mathcal{H}$.

compatible $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$.

in compatible $\neq 0$.

Thm. $[\hat{A}, \hat{B}] = 0$.

$$\hat{A}|\varphi_i\rangle = a_i|\varphi_i\rangle, \quad \hat{B}|\varphi_i\rangle = b_i|\varphi_i\rangle.$$

Measure values of $\hat{A}, \hat{B}$ for same state

$|\varphi_i\rangle \longrightarrow$ deterministic.

In general not possible.

Minimal uncertainty $\rightarrow$ uncertainty principle.

Average.

$$\langle\psi|\hat{A}|\psi\rangle = \sum_i |c_i|^2 \langle\varphi_i|\hat{A}|\varphi_i\rangle = \sum_i |c_i|^2 a_i$$

expectation value.

$$\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle I := \hat{A} - \langle\psi|\hat{A}|\psi\rangle I.$$

$$\langle\Delta\hat{A}\rangle = 0 .$$

Uncertainty.

$$\langle\Delta\hat{A}^2\rangle := \langle\psi|\Delta\hat{A}^2|\psi\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2.$$

<u>Thm</u>. $|\psi\rangle \in \mathcal{H}$.

$$\langle\Delta\hat{A}^2\rangle\langle\Delta\hat{B}^2\rangle \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]\rangle\right|^2.$$

Ex. $\hat{A}|\psi\rangle = a|\psi\rangle$, $\hat{B}|\psi\rangle = b|\psi\rangle$.

$[\hat{A},\hat{B}] = 0$. $\langle\Delta\hat{A}^2\rangle = \langle\Delta\hat{B}^2\rangle = 0$.

Pf: $\Delta\hat{A}\Delta\hat{B} = \frac{1}{2}\{\Delta\hat{A}, \Delta\hat{B}\} + \frac{1}{2}[\Delta\hat{A}, \Delta\hat{B}]$

real   imag.

$$\frac{1}{4}|\langle[\Delta\hat{A}, \Delta\hat{B}]\rangle|^2 \le |\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2$$

$$\le \langle\Delta\hat{A}^2\rangle\langle\Delta\hat{B}^2\rangle.$$

$[\Delta\hat{A}, \Delta\hat{B}] = [\hat{A}, \hat{B}]$ $\square$.

Equation of motion

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$$

① $1 = \langle \psi(t)|\psi(t)\rangle = \langle \psi(t_0)|\,U^*(t, t_0)U(t, t_0)\,|\psi(t_0)\rangle$

$\longrightarrow \quad U^*(t, t_0)\,U(t, t_0) = I.$

② $t_0 < t_1 < t_2$.

$$|\psi(t_2)\rangle = U(t_2, t_1)\,U(t_1, t_0)\,|\psi(t_0)\rangle$$

$$= U(t_2, t_0)$$

$U(t_2, t_0) = U(t_2, t_1)\,U(t_1, t_0)$ semigroup.

$t_0 < t < t + \Delta t$.

$$U(t,t) = \lim_{\Delta t \to 0^+} U(t+\Delta t, t) = I.$$

$$U(t+\Delta t) = I - i\hat{H}(t)\Delta t + O(\Delta t^2)$$

$\hat{H}(t) = \hat{H}^*(t)$ Hamiltonian.

$$|\psi(t+\Delta t)\rangle = |\psi(t)\rangle - i\hat{H}(t)\Delta t\,|\psi(t)\rangle + O(\Delta t^2)$$

$$\Rightarrow i\partial_t |\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle.$$

Stationary state.

$\hat{H}(t) \equiv \hat{H}$. $\mathcal{H} \cong \mathbb{C}^n$

$\hat{H}|\varphi_i\rangle = E_i|\varphi_i\rangle$, $E_0 \le E_1 \le \cdots \le E_n$

$|\psi(t_0)\rangle = |\varphi_i\rangle$. $|\psi(t)\rangle = e^{-iE_it}|\psi(t_0)\rangle$.

E_0: ground

E_1: 1st excited

E_2: 2nd excited

...

In general.

$$|\psi(t)\rangle = \sum_i c_i e^{-iE_i(t-t_0)}|\psi(t_0)\rangle.$$

Ex. $\hat{A}(t) = \hat{A}$.

$$i\frac{d}{dt}\langle\hat{A}\rangle(t) = \langle[\hat{A},\hat{H}]\rangle(t)$$

Tensor product

2 spin-$\frac{1}{2}$ particles $\mathcal{H} = \text{span}\{|\uparrow\rangle, |\downarrow\rangle\} \cong \mathbb{C}^2$

$$\mathcal{H} \otimes \mathcal{H} = \text{span}\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \cong \mathbb{C}^4.$$

$$\mathcal{H}_A = \text{span}\{|\varphi_i^A\rangle\}_{i=1}^{N_A}.$$

$$\mathcal{H}_B = \text{span}\{|\varphi_j^B\rangle\}_{j=1}^{N_B}.$$

$$\mathcal{H}_A \otimes \mathcal{H}_B = \text{span}\{|\varphi_i^A \varphi_j^B\rangle, \ i=1,\cdots,N_A, \ j=1,\cdots,N_B\}$$

$$\langle \varphi_i^A \varphi_j^B | \varphi_{i'}^A \varphi_{j'}^B \rangle = \langle \varphi_i^A | \varphi_{i'}^A \rangle \langle \varphi_j^B | \varphi_{j'}^B \rangle$$

$$= \delta_{ii'} \delta_{jj'}$$

$$(\hat{A} \otimes \hat{B}) | \varphi_i^A \varphi_j^B \rangle = | (\hat{A} \varphi_i^A)(\hat{B} \varphi_j^B) \rangle$$

N spin-$\frac{1}{2}$ particles.

$$\mathcal{F} = \bigotimes_{i=1}^{N} \mathbb{C}^2 \cong \mathbb{C}^{2^N}$$

binary code $|\uparrow\rangle \leftarrow 0$. $|\downarrow\rangle \leftarrow 1$.

$|a_1 \cdots a_N\rangle$, $a_i \in \{0, 1\}$. basis.

Operator. $\mathbb{C}^{2^n \times 2^n}$.

$$\sigma_\alpha^{(i)} = I \otimes \cdots \otimes \underset{\substack{\uparrow \\ \text{ith position}}}{\sigma_\alpha} \otimes I \otimes \cdots \otimes I. \qquad \begin{aligned} i &= 1, \cdots, n \\ \alpha &= x, y, z. \end{aligned}$$

Ex. 1D Ising model w. transverse field.

$$\hat{H} = -\sum_{i=1}^{N-1} \sigma_z^{(i)} \sigma_z^{(i+1)} - g \sum_{i=1}^{N} \sigma_x^{(i)}$$

Heisenberg model.

$$\hat{H} = -\sum_{i=1}^{N-1} \left(J_x \sigma_x^{(i)} \sigma_x^{(i+1)} + J_y \sigma_y^{(i)} \sigma_y^{(i+1)} + J_z \sigma_z^{(i)} \sigma_z^{(i+1)} + h \sigma_z^{(i)} \right)$$

$\mathbb{C}^2 \otimes \mathbb{C}^2$.

$\hat{S}_z^{(1)} = \hat{S}_z \otimes I$. $\quad \hat{S}_z^{(2)} = I \otimes \hat{S}_z$

$$\hat{S}_z^{tot} = \hat{S}_z^{(1)} + \hat{S}_z^{(2)}$$

$$\hat{S}_z^{tot} |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle \quad , \quad \hat{S}_z^{tot} |\downarrow\downarrow\rangle = - |\downarrow\downarrow\rangle$$

$$\hat{S}_z^{tot} |\downarrow\uparrow\rangle = \hat{S}_z^{tot} |\uparrow\downarrow\rangle = 0 .$$

$$\left(\hat{\vec{S}}^{tot}\right)^2 = \hat{S}_x^{tot\,2} + \hat{S}_y^{tot\,2} + \hat{S}_z^{tot\,2}$$

$$\hat{S}_z^{tot\,2} = \left(\hat{S}_z \otimes I\right)^2 + 2\,\hat{S}_z \otimes \hat{S}_z + \left(I \otimes S_z\right)^2$$

$$= \frac{1}{2} I \otimes I + 2\,\hat{S}_z \otimes \hat{S}_z\,.$$

$$\left(\hat{\vec{S}}^{tot}\right)^2 = \frac{3}{2} I \otimes I + 2\left(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z\right)$$

Ex. $\left[\left(\hat{\vec{S}}^{tot}\right)^2,\ \hat{S}_z^{tot}\right] = 0\,.$

$$\left(\hat{\vec{S}}^{tot}\right)^2 |\uparrow\uparrow\rangle = 2\,|\uparrow\uparrow\rangle\,.$$

$$\left(\hat{\vec{S}}^{tot}\right)^2 |\downarrow\downarrow\rangle = 2\,|\downarrow\downarrow\rangle\,.$$

$$\hat{\vec{S}}^{tot^2} \left(\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \right) = 2 \left(\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \right)$$

$$\hat{\vec{S}}^{tot^2} \left(\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right) = 0$$

		$\hat{\vec{S}}^{tot^2}$	$\hat{S}_z$
$\frac{1}{\sqrt{2}}(\|\uparrow\downarrow\rangle - \|\downarrow\uparrow\rangle)$	singlet	0	0
$\|\uparrow\uparrow\rangle$			1
$\frac{1}{\sqrt{2}}(\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle$	triplet	2	0
$\|\downarrow\downarrow\rangle$			-1