Justification for the SMC Algorithm for Solving Fredholm Equations

We consider the Fredholm equation of the first kind

$$h(y) = \int f(x)\, g(y \mid x)\, dx,$$

where

- h is the observed data distribution (or its empirical version),
- f is the unknown density we wish to recover,
- $g(y \mid x)$ is the known forward kernel (e.g. a blurring operator).

1. Maximum-likelihood formulation

Define

$$h_f(y) = \int f(x)\, g(y \mid x)\, dx.$$

A natural way to estimate f is to maximize the similarity between h and h_f. The log-likelihood functional of f is

$$\ell(f) = \mathbb{E}_{Y\sim h}\big[\log h_f(Y)\big] = \int h(y)\, \log\!\left(\int f(x)\, g(y \mid x)\, dx\right) dy.$$

Direct maximization is intractable because $\ell(f)$ contains a logarithm of an integral.

2. Latent-variable construction and EM functional

Introduce a latent variable X with joint model

$$p_f(x, y) = f(x)\, g(y \mid x), \qquad h_f(y) = \int p_f(x, y)\, dx.$$

The complete-data log-likelihood is

$$\log p_f(X, Y) = \log f(X) + \log g(Y \mid X),$$

but only the term $\log f(X)$ depends on f.

Given a current iterate f_n, EM constructs the surrogate functional

$$Q(f \mid f_n) = \mathbb{E}_{Y\sim h}\Big[\mathbb{E}_{X\mid Y, f_n}[\log f(X)]\Big],$$

which removes the intractable $\log\int\cdot$ structure.

The posterior of the latent variable under f_n is

$$p_{f_n}(x \mid y) = \frac{f_n(x)\, g(y \mid x)}{\int f_n(z)\, g(y \mid z)\, dz}.$$

Hence

$$Q(f \mid f_n) = \int h(y) \left(\int p_{f_n}(x \mid y)\, \log f(x)\, dx\right) dy.$$

Swap integrals and define

$$r_n(x) = \int h(y)\, p_{f_n}(x \mid y)\, dy.$$

Then

$$Q(f \mid f_n) = \int r_n(x)\, \log f(x)\, dx.$$

3. Maximizing the EM functional

We maximize

$$Q(f \mid f_n) = \int r_n(x) \log f(x)\, dx$$

over all densities $f \geq 0$, $\int f = 1$.

Introduce the Lagrangian

$$\mathcal{L}(f, \lambda) = \int r_n(x) \log f(x)\, dx + \lambda\Big(1 - \int f(x)\, dx\Big).$$

Functional differentiation gives

$$\frac{\delta \mathcal{L}}{\delta f(x)} = \frac{r_n(x)}{f(x)} - \lambda = 0,$$

so

$$f_{n+1}(x) \propto r_n(x).$$

Since h is a probability density and $p_{f_n}(\cdot \mid y)$ integrates to 1,

$$\int r_n(x)\, dx = 1,$$

hence

$$f_{n+1}(x) = f_n(x) \int \frac{g(y \mid x)\, h(y)}{\int f_n(z)\, g(y \mid z)\, dz}\, dy.$$

This is the classical EM update for the Fredholm equation.

4. EMS: adding smoothing as regularization

The Fredholm equation is ill-posed: iterating the EM update leads to unstable, spiky, non-smooth estimates. To regularize the iteration, EMS (Expectation–Maximization Smoothing) applies a smoothing operator K after each EM step:

$$f_{n+1}(x) = \int K(x', x)\, f_n(x') \Big[\int \frac{g(y \mid x')\, h(y)}{\int f_n(z)\, g(y \mid z)\, dz} dy\Big]\, dx'.$$

This may be interpreted as:

- a projection of f_{n+1} onto a space of smooth densities,
- a form of roughness-penalized maximum likelihood,
- or an implicit Bayesian prior favoring smoothness.

5. Rewriting EMS as a Feynman–Kac flow

Define a joint density

$$\eta_n(x, y) = f_n(x)\, h(y).$$

The EMS update can be rewritten as

$$\eta_{n+1}(x, y) = \int \eta_n(x', y')\, G_n(x', y')\, M_{n+1}((x', y'), (x, y))\, dx'\, dy',$$

where

$$G_n(x, y) = \frac{g(y \mid x)}{\int f_n(z)\, g(y \mid z)\, dz}, \qquad M_{n+1}((x', y'), (x, y)) = K(x', x)\, h(y).$$

Thus $\{\eta_n\}$ obeys the classical Feynman–Kac recursion

$$\eta_{n+1} \propto \eta_n \, G_n \, M_{n+1}.$$

The marginal

$$f_n(x) = \int \eta_n(x, y) \, dy$$

recovers exactly the EMS iterates.

6. Sequential Monte Carlo approximation

Sequential Monte Carlo (SMC) methods are designed to approximate sequences of Feynman–Kac measures. Given the representation above, we may approximate η_n by a weighted particle system $\{(X_n^i, Y_n^i, W_n^i)\}_{i=1}^N$:

1. **Mutation:**
$$X_{n+1}^i \sim K(X_n^i, \cdot), \qquad Y_{n+1}^i \sim h.$$

2. **Weighting:**
$$W_{n+1}^i \propto G_n(X_n^i, Y_n^i) = \frac{g(Y_n^i \mid X_n^i)}{\frac{1}{N}\sum_{j=1}^N g(Y_n^i \mid X_n^j)}.$$

3. **Resampling:** Normalize and resample the particles.

4. **Approximation of** f_{n+1}:
$$f_{n+1}(x) \approx \sum_{i=1}^N W_{n+1}^i \, K(X_{n+1}^i, x),$$
possibly followed by kernel density estimation.

Thus SMC provides:

- a particle approximation of the EMS sequence $\{\eta_n\}$,
- an adaptive discretization of the domain of f,
- Monte Carlo evaluation of the integrals in the EMS operator,
- and provably convergent approximations of f_n as $N \to \infty$.

Therefore, SMC is a natural computational method for implementing the EMS recursion for solving Fredholm equations of the first kind.