A Quest for Knowledge*

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Abstract

Is more novelty in research desirable? We develop a model in which knowledge shapes society's policies and guides the search for discoveries. Researchers select which question to study and to what extent. The novelty of the question determines both the value and the difficulty of discovering its answer. We show that the benefits of discoveries are non-monotone in novelty. Nevertheless, due to a dynamic externality, it can be optimal to incentivize research on distant discoveries to improve the evolution of knowledge. One reason is that the probability of a discovery and the novelty of the question are endogenously linked. They can be complements or substitutes depending on existing knowledge. We analyze cost reductions and research awards as instruments to incentivize research. While a benefit-maximizing funder's optimal funding mix depends on her budget in general, distant discoveries can only be incentivized through awards.

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[Evolution] comes through asking the right questions, because the answer pre-exists. But it's the questions that we have to define and discover. ... You don't invent the answer. You reveal the answer.

Jonas Salk, inventor of the Polio vaccine.

1 Introduction

In his letter to Franklin D. Roosevelt (*Science, The Endless Frontier*), Vannevar Bush (1945) pleads with the president to preserve the freedom of inquiry by federally funding basic research—the "pacemaker of technological progress." That letter paved the way for the creation of the National Science Foundation (NSF) in 1950. The NSF today, like the vast majority of governments and scientific institutions,[1] cherishes scientific freedom and allows academic researchers to select research projects independently.

With scientific freedom comes the responsibility for "asking the right questions" that Jonas Salk advocates in the epigraph. However, what is the right question? Biologist and Nobel laureate Peter Medawar (1967) famously notes that "research is surely the art of the soluble. ... Good scientists study the most important problems they think they can solve." Finding the most important, yet soluble question is non-trivial. One reason is that both importance and solubility depend on the current state of knowledge (see, e.g., Iaria, Schwarz, and Waldinger, 2018).

In this paper, we propose and study a microfounded model of knowledge creation through research. Our model captures (i) the role of existing knowledge for the benefits and cost of research, (ii) the spillovers a discovery creates on the conjectures about similar questions, and (iii) the researcher's freedom to choose which question to study and to what extent. We show that the choice of a more novel research question does not always come at a reduction in the probability of discovery: depending on existing knowledge, novelty can substitute or complement the probability of discovery. We characterize the researcher's choices and address classical questions of science funding: Should we incentivize researchers to study questions far beyond the current knowledge frontier? Do such moonshots improve the evolution of knowledge? When and how should a budget-constrained funder incentivize innovative research?

We model the value of knowledge as the extent to which it improves decision making.[2] We represent society by a single decision maker who faces a set of problems. In her response to these problems she builds on the public good knowledge. Knowledge is the set of questions to which the answer has already been discovered. Because answers to similar questions are correlated, knowledge also provides the decision maker with conjectures regarding questions to which the answer is yet undiscovered. The precision of a conjecture depends on the question's location relative to existing knowledge.[3] We conceptualize the correlation by assuming that

[1] See, for example, OHCR (1966), Art 15., or, more recently, Bonn Declaration (2020).

[2] Following, e.g., Marschak (1974)'s "Knowledge is useful if it helps to make the best decisions."

[3] An example of such spillovers is the COVID-19 vaccine development by Moderna which "took all of one weekend" only. The speed was a direct consequence of researchers discovering how to replicate the spikes of another Corona virus in research on MERS. Details are discussed for instance in This American Life (2020).

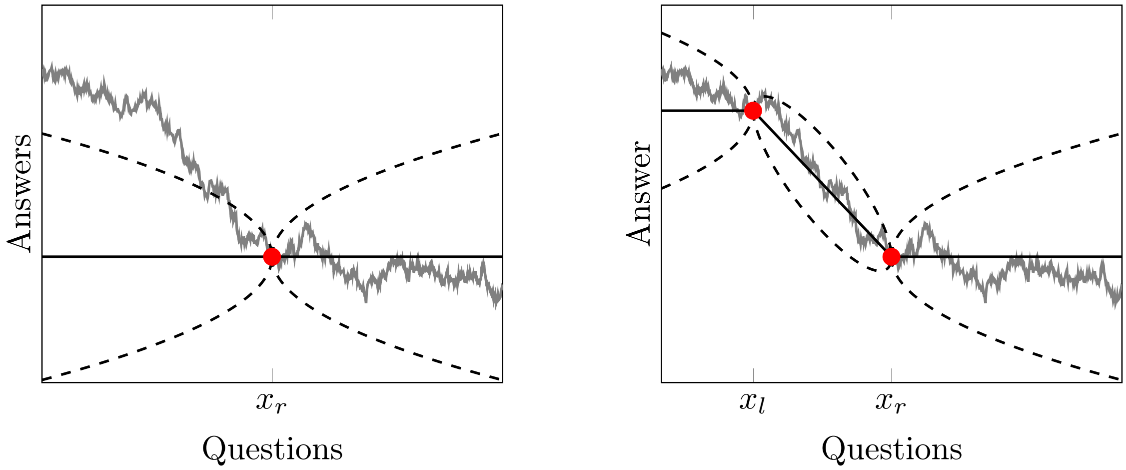


Figure 1: *Existing knowledge and conjectures.*

answers to questions follow the realization of a Brownian path. Figure 1 visualizes that idea. Questions are on the horizontal axis and the gray line depicts the answers to all questions. Dots (●) represent existing knowledge. Due to the Brownian assumption, all conjectures follow a normal distribution. The mean and the variance depend on existing knowledge. The solid black lines in Figure 1 represent the mean, the dashed lines provide the band of the 95-percent prediction interval.[4]

In a first step, we characterize the benefits of a discovery. To gain intuition, consider the left panel of Figure 1. Only the answer to question x_r is known. Assume that research discovers the answer to question x_l. We move to the right panel. How much does decision making improve? First, the decision maker has precise knowledge about the answer to x_l. Second, her conjectures about all questions to the left of x_r improve. Third, conjectures improve the most on the newly created area $[x_l, x_r]$ where the decision maker now has two pieces of knowledge that help her form the conjectures.

The benefit of discovering the answer to x_l depends on its distance to x_r. A discovery close to existing knowledge implies a short area $[x_l, x_r]$. There are only a few questions in the area, albeit with precise conjectures. The effect of an increase in the distance between x_l and x_r is similar to the effect of an output expansion by a monopolist. By increasing the distance of x_l from x_r more questions lie inside the area—a marginal gain. At the same time, the conjectures become less precise—an inframarginal loss. The benefit is maximized at an intermediate distance.

If both x_l and x_r are known initially, discoveries advance knowledge beyond the frontier if the discoveries are on questions $x \notin [x_l, x_r]$. Advancing beyond the frontier is identical to what we discuss in the paragraph above. Alternatively, discoveries deepen knowledge if they are on questions $x \notin [x_l, x_r]$. Depending on the distance between x_l and x_r both advancing knowledge and deepening knowledge can be optimal. If x_l and x_r are close, knowledge is dense: the conjecture about

[4]The 95-percent prediction intervals depend on existing knowledge and describe the following relation: For each question, the answer lies with a probability of 95 percent between the respective dashed lines given existing knowledge.

any question in $[x_l, x_r]$ is already precise. Advancing beyond the frontier provides larger benefits than deepening knowledge. If x_l and x_r are far apart, knowledge is sparse: conjectures about questions in $[x_l, x_r]$ are imprecise. Obtaining an answer to a question $x \in [x_l, x_r]$ divides this single area with imprecise conjectures into *two* areas with precise conjectures. Deepening knowledge provides larger benefits than advancing knowledge beyond the frontier.

Overall, the largest benefit comes from deepening knowledge between distant, yet not too far apart pieces of knowledge. Advancing knowledge beyond the frontier trumps deepening knowledge in an existing area only if all available areas are short.

In a second step, we conceptualize the research process as the search for an answer. We assume that it requires effort to search for the answer to a question and that the cost of effort are increasing and convex. We provide a partial solution to the search problem, which yields an expression for the cost of research on a particular question in terms of (i) *novelty*—the distance of the question to existing knowledge, and (ii) *output*—the probability that search results in discovery. That expression provides a link between the novelty of the question and the risk of failed search through the precision of the initial conjecture. The better that conjecture, the higher output for any given level of effort.

In a third step, we combine the benefits and cost of research to characterize (i) the researcher's choice of question, and (ii) the extent to which the researcher studies that question. We show that novelty and output are related non-trivially and that their relation depends on existing knowledge.

In particular, novelty and output can substitute or complement each other. If the question lies in a short area, novelty and output are complements. The more novel the question the higher the probability that the researcher discovers the answer. Benefits of discovery increase in novelty, yet the cost remain low. If the question lies in a larger area, novelty and output are substitutes for small levels of novelty, but complements for intermediate levels. At each of the boundaries the marginal cost of output increase fast in novelty causing the researcher to reduce output. However, any increase in distance to one end of the area comes at a decrease in distance to the other end. While the first effect reduces the precision of conjectures, the second effect attenuates that reduction in precision. As novelty increases, the attenuation becomes stronger which mitigates the marginal cost effect. Eventually, the marginal benefit increase dominates: novelty and output become complements. As the area size increases further, the region in which novelty complements output shrinks. Finally, if the researcher advances knowledge beyond the frontier, there is no 'other end': novelty and output are substitutes throughout.

In general, output is higher when the researcher deepens knowledge than when she advances beyond the frontier. The highest output is expected in areas of intermediate length. In such areas, the researcher deepens knowledge and pursues the question at the midpoint of the area.

In a fourth step, we use the insights from above to show under which circumstances "moonshot" discoveries—extremely novel discoveries far from existing knowledge—are desirable. Moonshot discoveries are myopically suboptimal. They create knowledge that is too disconnected from existing knowledge and therefore provides little immediate benefits. However, moonshots provide guidance for future researchers that aim to deepen knowledge in between the moonshot and previously existing knowledge. As a result of the moonshot, future researchers have higher output and knowledge created over time is more valuable than absent the moonshot. If society is patient and the cost of research are intermediate, the positive dynamic externality of moonshots on future researchers dominates the implied myopic loss—incentivizing a moonshot is beneficial.

In a fifth step, we introduce a budget-constrained funder with two funding instruments. The funder can reduce the researcher's cost ex ante and reward her for a novel discovery ex post. We characterize the set of implementable novelty-output pairs that the funder can implement. We show that output and novelty can be substitutes or complements from the funder's perspective. In particular, an increase in novelty comes with an increase in the induced output if cost reductions are expensive. Thus, for a funder who maximizes the myopic benefit of a discovery it may be optimal to incentivize excessive novelty only to increase output. However, if cost reductions are cheap, increases in novelty comes at the cost of a reduction in output. The same funder may incentivize excessive closeness to existing knowledge to keep the risk of failure low.

To summarize, we make three contributions. *First,* we propose a framework that endogenously links typical measures of research—novelty and output—to typical premises of research—selection from a large pool of questions, conjectures determined by existing knowledge, and the need of costly effort to obtain a discovery. We obtain this link by conceptualizing the discovery process of a Brownian path as a search for realized values guided by conjectures that build on known realizations.*Second,* we shed light on the non-trivial relation between novelty of research and expected research output. We show that whether the two complement or substitute each other crucially depends on existing knowledge. To derive this point, we characterize the researcher's optimal policy as a function of existing knowledge. *Third,* we provide new insights on two classical questions in the science of science funding: (i) Should society incentivize research far beyond the current frontier even if the immediate benefits of such a discovery are low?—Yes, if the cost of research are intermediate and society is patient. (ii) Which mode of funding provides larger expected benefits: ex ante cost reductions or ex post rewards?—Neither, in general. A funder that aims to maximize the benefits of research may strictly prefer to combine both measures. A funder that aims at incentivizing research far beyond the frontier has to offer ex post rewards.

1.1 Related Literature

Ample empirical literature in the science of science has documented the importance of novelty and output for progress in science. Fortunato et al. (2018) provide an extensive summary of it. The importance of (accessible) pre-existing knowledge for research purposes is documented, for example, in Iaria, Schwarz, and Waldinger (2018).[5] We aim to complement the (quasi-)experimental approach in these papers

[5]We want to stress that our notion of knowledge is orthogonal to that in the literature on epistemic game theory. Brandenburger (1992) provides an overview. Different to that literature knowledge is always fully transparent and there is no strategic interaction. However, it is possible

by providing a simple yet flexible formal model based on few parameters to make it identifiable and testable.

Several existing theoretical models in the science of science consider particular aspects of the scientific process we have in mind.[6] Aghion, Dewatripont, and Stein (2008) consider a setting in which progress has a predefined step-by-step sequential structure. To advance to the next question, a particular prior question has to be answered. We offer greater flexibility in that we posit that any question can—in principle—be addressed at any time. However, the benefits from a discovery and the effort needed for the discovery depend on previous work.[7] Bramoullé and Saint-Paul (2010) model the decision of a researcher to deepen knowledge in a given area or to advance the knowledge frontier. The main driver in their model is the assumption that as an area gets increasingly crowded, the reputation a researcher gains from new developments in that area declines.[8] We offer a decision-based microfoundation that provides a measure of uncertainty, in line with Frankel and Kamenica (2019). It reaches a similar conclusion: as the opportunities in the area become increasingly narrow, the informational content of an additional finding decreases, and hence its value does too. However, unlike in Bramoullé and Saint-Paul (2010), the researcher in our model has more discretion, as she chooses—in addition to the area—the degree of novelty and the level of research intensity which directly determines the probability of success. Both choices are continuous, and shrinking the research area may even be beneficial if it leads to better conjectures by closing the gap between existing pieces of knowledge. While our basic model is static, a simple dynamic extension could reproduce the core elements of either Bramoullé and Saint-Paul (2010) or Aghion, Dewatripont, and Stein (2008).[9]

The closest paper to ours in the literature on theretical models of innovation is Prendergast (2019), which is complementary to ours. He, too, studies a model of innovation in which the correlation between questions is determined by a Brownian motion. He focuses on an agency problem in a single exogenously given research area. While we abstract from agency concerns, the results in our mirofounded model come

to embed it in strategic settings to address alternative questions.

[6]There is a literature orthogonal to ours that views science as establishing links in a network between known answers (e.g., Rzhetsky et al., 2015). Our model is complementary, we consider research as the search for answers where the links in the network are known.

[7]To (ab)use Newton's metaphor: Any researcher can build a ladder to see farther, but the effort required depends on the existing giants' shoulders. Related ideas appear in Scotchmer (1991), Aghion et al. (2001), Bessen and Maskin (2009), and Bryan and Lemus (2017).

[8]Similar to Bramoullé and Saint-Paul (2010) innovation translates to a public good. That differentiates us from most models of R&D competition. Yet, similarly to, for example, Letina (2016), Letina, Schmutzler, and Seibel (2020), and Hopenhayn and Squintani (2021), we assume that progress corresponds to successful search in an ocean of possibilities. Unlike in those approaches, benefit and cost depend on the question's relation to existing knowledge in our setting.

[9]Other recent theoretical work studies frictions to the scientific process we abstract from. Bobtcheff, Bolte, and Mariotti (2017), Akerlof and Michaillat (2018), and Andrews and Kasy (2019) study inefficiencies due to the publication process, career concerns, or homophily. Hill and Stein (2019, 2020) provide empirical counterparts. Frankel and Kasy (2021) provide a normative jsutification. Similar to us, Liang and Mu (2020) look at (a sequence of) myopic researchers aiming to discover the truth. Different from us, they focus on the choice of the learning technology and show that depending on the complementarities between technologies, researchers may persistently select an inefficient technology.

from the researcher's choice between several distinct research areas and expanding knowledge beyond the frontier. In addition, we provide an endogenous relation between the cost of research, the chosen research question, and the probability of finding an answer that is based on the researcher decision on how much effort to invest into the search for an answer. While neither of the two models nests the other, there is a special case of our model that corresponds to a special case of his. We discuss the relationship in greater detail in Section 8.

Technically, we build on the literature that studies Gaussian search models, following Callander (2011a). Within that literature Callander and Clark (2017) is the closest. We discuss the relation in Section 8. Most other models assume that the payoffs are determined by a specific target of the stochastic process's realization (for example, Callander, 2011a) or the weighted sum of all realizations (for example, Bardhi, 2019). Garfagnini and Strulovici (2016) have a similar notion of frontiers and gaps as we do.[10] However, they are interested in finding the maximum value of the process. We differ in that we posit that the value of a discovery is determined by the reduction in the variance of conjectures. Moreover, our notion of cost is different. In the aforementioned literature cost are either exogenous, or depend only on the choice of question. We introduce an effort component. The more the researcher invests, the higher her chance to find the answer, yet the more costly her research. That effort component has important consequences. It is never optimal to sample the entire real line for answers and search fails with positive probability. Moreover, the effort choice non-trivially interacts with the question choice.

1.2 Roadmap

To emphasize the role of each model ingredient we chose a step-by-step approach. With each additional model feature, we analyze the respective consequences. In Section 2 we provide the basic model of knowledge and decision making. In Section 3 we derive the benefits of a discovery to society. In Section 4 we introduce the cost of searching for an answer to the model. We then derive the cost-of-research function and the notion of research output. In Section 5 we introduce a researcher and characterize her optimal decision. In Section 6 we analyze an extension of the model in which a long-run decision maker faces a series of short-lived researchers. Section 7 considers a setting in which the researcher's incentives are influenced by the decisions of a funding institution. In Section 8 we relate our findings to the two closest papers in the literature. Section 9 summarizes and provides an outlook on other applications.

[10] Jovanovic and Rob (1990), too, study the choice between expanding and deepening research. Unlike in our model expanding in Jovanovic and Rob (1990) implies an i.i.d. draw at a fixed cost, deepening is costless. In our model in contrast, all questions are connected and cost depend on both existing knowledge and the degree of novelty. Moreover, see Callander and Hummel (2014), Callander, Lambert, and Matouschek (2018), Callander and Matouschek (2019), and Bardhi and Bobkova (2021) for applications different from ours in a related framework. Some of the results in Section 6 are reminiscent of Callander (2011b) in a different context.

2 A Model of Knowledge

We set up a model of knowledge that captures the following desired properties:

(i.) Knowing the answer to a question informs conjectures about other questions.

(ii.) The distance between questions determines the impact that answering one question has on the conjecture about the other question.

(iii.) The set of available questions is unbounded.

(iv.) Knowledge informs decision making.

We first set up a formal model of knowledge and then introduce society as a decision maker that applies knowledge in its decision-making process.

Questions and answers. We represent the universe of questions by the real line, $\mathbb{R}$. A specific *question* is an element $x \in \mathbb{R}$. Each question x has precisely one *answer*, $y(x) \in \mathbb{R}$. A question-answer pair $(x, y(x))$ is thus a point in the two-dimensional Euclidean space.[11]

The answer $y(x)$ to question x is determined by the realization of a random variable, $Y(x) : \mathbb{R} \to \mathbb{R}$. We provide more structure on $Y(x)$ below.

Truth and knowledge. *Truth* is the collection of all question-answer pairs. It is the graph of the realization of the random variable $Y(x)$, over its domain $\mathbb{R}$. *Knowledge* is the finite collection of known question-answer pairs. We denote it by $\mathcal{F}_k = \{(x_i, y(x_i))\}_{i=1}^{k}$. For notational convenience, we assume that $\mathcal{F}_k$ is ordered such that $x_i < x_{i+1}$. We refer to x_1 and x_k as the *frontier* of current knowledge.

The key assumption of our model of knowledge concerns the truth-generating process $Y(x)$. We assume that $Y(x)$ follows a standard Brownian motion defined over the entire real line.[12] This assumption captures the notion that the answer to question x is likely to be similar to the answer to a close-by question x'. As the distance between x and x' increases, there remains a correlation. Yet the uncertainty increases in the distance.[13]

Knowledge $\mathcal{F}_k$ implicitly determines a partition of the real line consisting of $k+1$ elements

$$\mathcal{X}_k := \{(-\infty, x_1), [x_1, x_2), \cdots, [x_{k-1}, x_k), [x_k, \infty)\}.$$

In what follows, we make frequent use of the interval length X_i of an element of the partition that a particular question $x \in [x_i, x_{i+1})$ is part of.

[11] Our assumption implies that the relation between two questions can be obtained in a single dimension. While projecting all available questions across all disciplines onto a line might imply a sizeable loss, that loss is smaller if we think of our universe of questions as being within one specific and mature discipline.

[12] As in Callander (2011a), the realized truth, Y, is a random draw from the space of all possible paths, $\mathcal{Y}$, generated by a standard Brownian motion going through some initial knowledge point, $(x_0, y(x_0))$. While the process has fully realized at the beginning of time, knowledge is the filtration known to the observer, $\mathcal{F}_k$. We choose a standard Brownian path with zero drift and variance of one for convenience only. Our model extends naturally to other Gaussian processes. We want to emphasize that the x dimension should not be confused with a sequential structure of finding answers. Any question-answer pair $(x, y(x)) \notin \mathcal{F}_k$ is discoverable.

[13] We use the Euclidean distance on the x dimension, $|x - x'|$, throughout when we refer to distance.

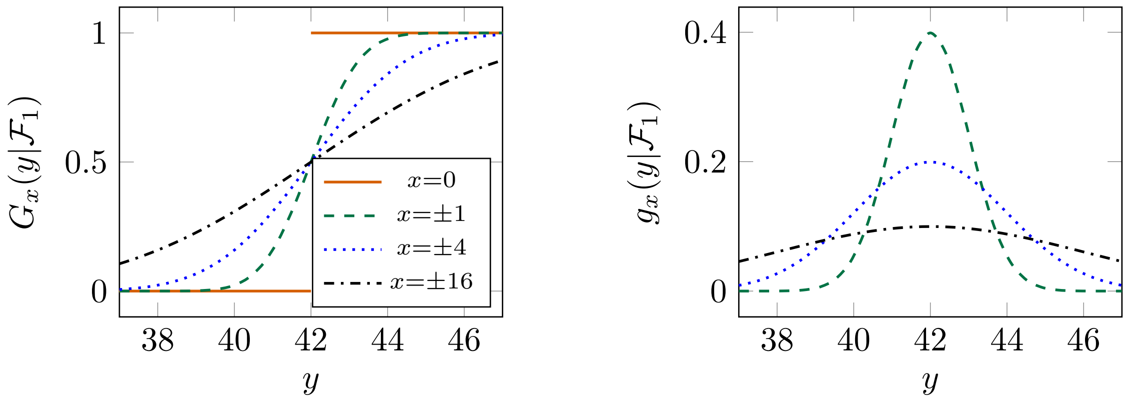


Figure 2: *Distributions of answers for different distances to knowledge when* $\mathcal{F}_1 = (0, 42)$. Given that the only question to which the answer is known is $x = 0$, we can determine knowlege about questions of distance 1,4, and 16 to $x = 0$. All answers have the same mean (42), but the variance and thus the precision of the conjecture differs. For $x = 0$, the answer is known and $G_0(y|\mathcal{F}_1)$ is a step function. Questions with longer distance have larger variances. The left panel depicts the respective distribution functions; the right panel depicts the densities.

We introduce the following terminology. We refer to each element of the partition $\mathcal{X}_k$ as an *area*. We call $(-\infty, x_1)$ area 0, $[x_1, x_2)$ area 1, and so on until area k which is $[x_k, \infty)$. The length of area $i \in \{1, .., k-1\}$ is $X_i := x_{i+1} - x_i$, and $X_0 = X_k = \infty$.

Conjectures. A *conjecture* is the cumulative distribution function $G_x(y|\mathcal{F}_k)$ of the answer $y(x)$ to question x given knowledge $\mathcal{F}_k$. Conjectures about questions to which the answer is known are trivial; if $(x_i, y(x_i)) \in \mathcal{F}_k$ then $G_{x_i}(y|\mathcal{F}_k) = \mathbf{1}_{y \geq y(x_i)}$, a right-continuous step function jumping to 1 at $y = y(x_i)$. The conjecture for a yet-to-be-discovered $y(x)$, $G_x(y|\mathcal{F}_k)$, is also well-defined. Because $Y(x)$ is determined by a Brownian motion, $G_x(y|\mathcal{F}_k)$ is a cumulative distribution function of a normal distribution with mean $\mu_x(Y|\mathcal{F}_k)$ and variance $\sigma_x^2(Y|\mathcal{F}_k)$. Both μ_x and σ_x^2 follow immediately from the properties of the Brownian motion. We differentiate between questions inside the knowledge frontier ($x \in [x_1, x_k]$) and outside that frontier ($x \notin [x_1, x_k]$).

Property 1 (Expected Value)**.** Given knowledge $\mathcal{F}_k$, the conjecture $G_x(y|\mathcal{F}_k)$ has the following mean:

$$\mu_x(Y|\mathcal{F}_k) = \begin{cases} y(x_1) & \text{if } x < x_1 \\ y(x_i) + \frac{x - x_i}{X_i}(y(x_{i+1}) - y(x_i)) & \text{if } x \in [x_i, x_{i+1}), i \in \{1, ..., k-1\} \\ y(x_k) & \text{if } x \geq x_k. \end{cases}$$

Property 2 (Variance)**.** Given knowledge $\mathcal{F}_k$, the conjecture $G_x(y|\mathcal{F}_k)$ has the following variance:

$$\sigma_x^2(Y|\mathcal{F}_k) = \begin{cases} x_1 - x & \text{if } x < x_1 \\ \frac{(x_{i+1} - x)(x - x_i)}{X_i} & \text{if } x \in [x_i, x_{i+1}), i \in \{1, ..., k-1\} \\ x - x_k & \text{if } x \geq x_k. \end{cases}$$

Figure 2 illustrates the distributions for different distances to existing knowledge assuming that knowledge is $\mathcal{F}_1 = (0, 42)$. In Appendix C we provide a graphical example that highlights the ingredients of our model of knowledge and how adding additional question-answer pairs to knowledge influences conjectures.

2.1 Society and decision making

We represent society by a single decision maker. That decision maker observes knowledge $\mathcal{F}_k$ and takes a continuum of actions—one on each question $x \in \mathbb{R}$. On each question she can either take a *proactive* action $a(x) \in \mathbb{R}$, or, alternatively select an outside option $a(x) = \varnothing$, for example the act of "doing nothing." The decision maker's choice is thus represented by a function $a : \mathbb{R} \to \mathbb{R} \cup \varnothing$.

The expected payoff of selecting the outside option $a(x) = \varnothing$ is finite, safe (that is, independent of $y(x)$), and question-invariant. We normalize it to 0. The choice of $\varnothing$ resembles the idea that on subject on which very little is known, it is not wise (in expectations) to take proactive actions.[14] The payoff of addressing a question x proactively is represented by a monotone transformation of the quadratic loss around the the true answer to question x, $y(x)$.

The decision maker's payoff on a particular question x from action $a(x)$ is

$$u(a(x); x) = \begin{cases} 1 - \frac{(a(x)-y(x))^2}{q} & \text{if } a(x) \in \mathbb{R} \\ 0 & \text{if } a(x) = \varnothing, \end{cases}$$

for a given q. The scaling parameter q measures the error tolerance of the decision maker: if the proactive choice $a(x)$ is less than $\sqrt{q}$ away from the optimal choice—the true answer $y(x)$—the decision maker prefers the proactive choice over the outside option.

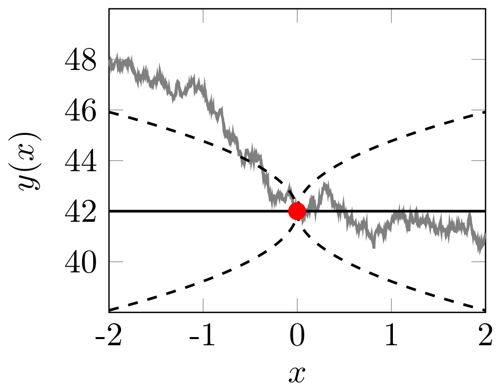
To keep the analysis focused, we abstract from any prioritization the decision maker might have among different x; that is, we assume the decision maker values all questions equally. If $a(x)$ is such that $u(a(x); x)$ is (locally) integrable, then total payoffs to the decision maker are given by

$$\int_{-\infty}^{\infty} u(a(x); x) dx.$$

Technically, it is the outside option $\varnothing$ and the finiteness of knowledge that ensure a bounded payoff at the optimum.[15] Thus, the outside option guarantees that

[14]What we have in mind as outside options are long-standing policies to which the expected payoff is finite. The question the decision maker asks on each question is whether she should revise her policies given the existing knowledge. Take the discussion about how to respond to climate change: Since the Kyoto Protocol, decision makers have reevaluated policies on several issues (transportation, energy, protection of nature etc). For each issue they use current knowledge and decide whether to continue with business as usual or to change policy.

[15]The decision maker's problem is straightforwardly solved pointwise resulting in a sufficiently well-behaved per-question payoff $u(\cdot)$ to ensure integrability. An alternative assumption to facing all questions is that the decision maker faces a single question at random. However, in the case of a uniform distribution—which would resemble equal weighting of questions—we need to restrict attention to draws from a large subset, $[\underline{x}, \overline{x}]$, of the set of all questions $\mathbb{R}$. If the subset from which the questions are drawn is large enough, the two assumptions are equivalent when the decision



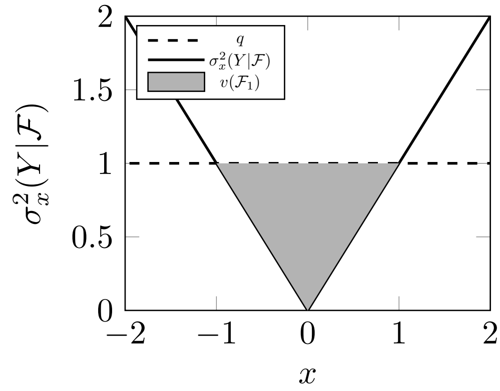


Figure 3: *The value of knowing* $\mathcal{F}_1$. The left panel depicts the same situation as the left panel in Figure 1. Only the answer to question 0 (which is 42) is known. The right panel depicts the The value of knowledge, $v(\mathcal{F}_1)$;: the shaded area. The variance $\sigma_x^2(Y|\mathcal{F}_k) = |x|$ is the Euclidean distance to 0. The expected payoff from taking an action equal to the mean of the conjecture, $a = \mu_x = 42$, is the vertical distance between the action and the dashed line. For $|x| \leq q$, $a = \mu_x = 42$ is preferred to $a = \varnothing$.

knowledge contributes in a quantifiable way to the decision maker's total payoff.[16]

3 The Benefits of Discovery

Discovery occurs whenever an answer is found and the new question-answer pair is added to the existing knowledge ($\mathcal{F}_k$). In this section, we derive a formulation that measures the benefits of such a discovery for the decision maker.

3.1 The Value of Knowledge

Knowledge informs decision making. For each question x, the decision maker uses the conjecture $G_x(y|\mathcal{F}_k)$ to decide on $a(x)$. Suppose first, the decision maker addresses a question x proactively, $a(x) \neq \varnothing$. Her expected payoff on that question is:

$$Eu(a \neq \varnothing; x|\mathcal{F}_k) = \int 1 - \frac{(a - y(x))^2}{q} \mathrm{d}G_x(y|\mathcal{F}_k).$$

maker acts optimally. Other weighting functions on questions are straightforward to incorporate; yet, they come at a significant cost of clarity in the analysis.

[16]While representing the universe of questions in a *circle* or a *bounded interval* would ensure that payoffs are bounded absent an outside option, it would violate the desired property of knowledge (iii.).

Due to the quadratic loss, the optimal action in that case corresponds to the mean of the distribution, $\mu_x(Y|\mathcal{F}_k)$ with payoff

$$\begin{aligned}Eu(\mu_x(Y|\mathcal{F}_k); x|\mathcal{F}_k) &= \int 1 - \frac{(\mu_x(Y|\mathcal{F}_k) - y(x))^2}{q} \mathrm{d}G_x(y|\mathcal{F}_k) \\ &= 1 - \frac{\sigma_x^2(Y|\mathcal{F}_k)}{q}.\end{aligned}$$

Addressing the question proactively is therefore optimal only if $\sigma_x^2(Y|\mathcal{F}_k) \leq q$; that is, only if the decision maker's conjecture is sufficiently precise. Otherwise, the decision maker prefers the outside option, $a(x) = \varnothing$, with payoff 0.

The decision maker's optimal policy is thus

$$a^*(x) = \begin{cases} \mu_x(Y|\mathcal{F}_k), & \text{if } \sigma_x^2(Y|\mathcal{F}_k) \leq q \\ \varnothing, & \text{if } \sigma_x^2(Y|\mathcal{F}_k) > q \end{cases}$$

which implies a total expected payoff of

$$v(\mathcal{F}_k) := \int_{-\infty}^{\infty} Eu(a^*(x); x|\mathcal{F}_k)\mathrm{d}x = \int_{-\infty}^{\infty} \max\left\{\frac{q - \sigma_x^2(Y|\mathcal{F}_k)}{q}, 0\right\} \mathrm{d}x.$$

We refer to $v(\mathcal{F}_k)$ as the *value of knowing* $\mathcal{F}_k$; that is, $v(\mathcal{F}_k)$ is the decision maker's gain from following the optimal policy under knowledge $\mathcal{F}_k$ over refraining from any proactive choices, $\forall x \in \mathbb{R}\ a(x) = \varnothing$.

The right panel of Figure 3 provides a graphical representation of $v(\mathcal{F}_1)$. The upper-right panel of Figure 4 represents $v(\mathcal{F}_2)$, the lower-right panel $v(\mathcal{F}_4)$.

3.2 The Benefits of Discovery

The benefits of a discovery measure how a discovery improves the value of knowledge. Formally, adding $(x, y(x))$ to $\mathcal{F}_k$ provides a benefit

$$V(x; \mathcal{F}_k) := v(\mathcal{F}_k \cup (x, y(x))) - v(\mathcal{F}_k).$$

The value of the discovery depends on the question that is answered, x, and on the existing knowledge. We distinguish two scenarios: *expanding* knowledge beyond the frontier and *deepening* knowledge in an area. A discovery $y(x)$ expands knowledge if $x \notin [x_1, x_k]$. A discovery $y(x)$ deepens knowledge in area i if $x \in [x_i, x_{i+1}]$.

We first state the benefit-of-discovery function. Three corollaries to that statement characterize its properties. The two main ingredients to determine this benefit are the *distance to knowledge*, which we formally define below, and the *research area*. It turns out that the area length X is a sufficient statistic for the research area.

Definition 1 (Distance)**.** The distance of a question x to knowledge $\mathcal{F}_k$ is the minimal Euclidean distance to a question to which the answer is known:

$$d(x) := \min_{\xi \in \{x_1, x_2, \ldots x_k\}} |x - \xi|$$

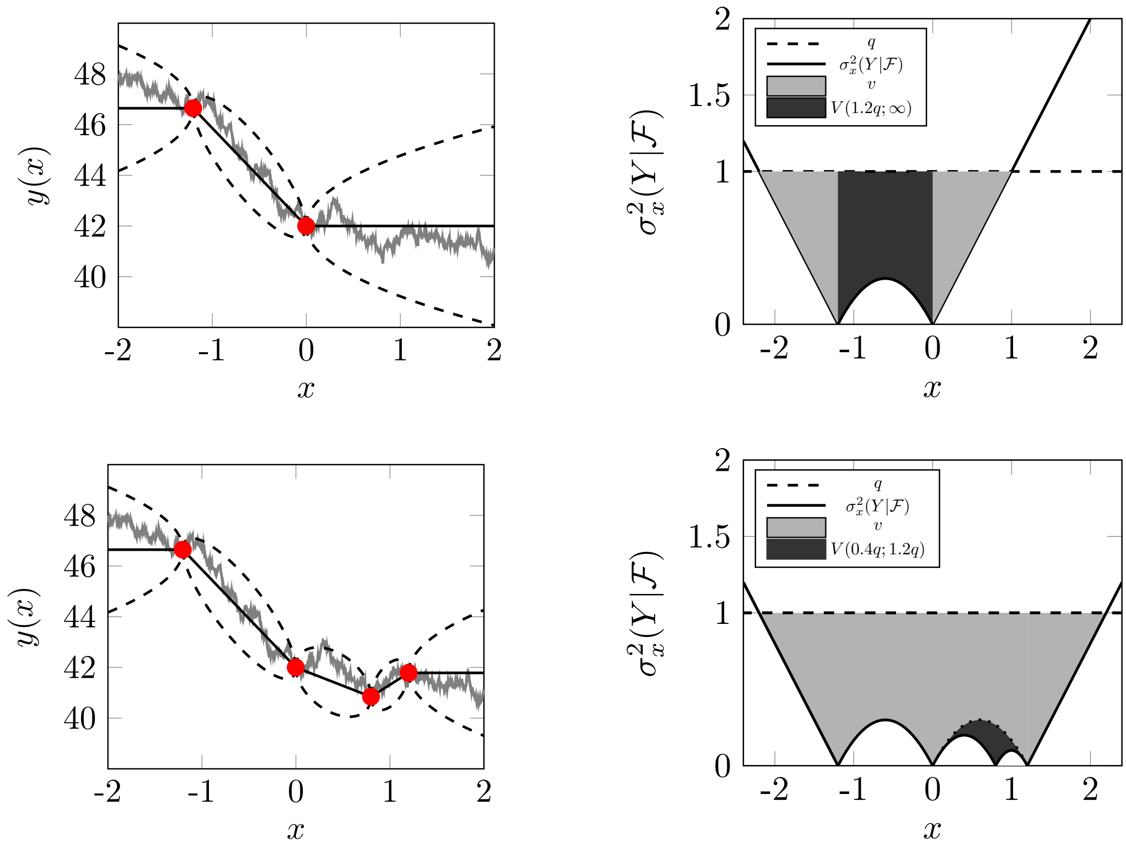


Figure 4: *The benefits of discovery.*

UPPER PANELS: BENEFIT OF A KNOWLEDGE-EXPANDING DISCOVERY. The left panel depicts The left panel depicts truth and knowledge under $\mathcal{F}_2 = \{(-1.2, 46.6), (0, 42)\}$. The right panel describes the benefits of discovering $(-1.2, 46.6)$ in addition to $(0, 42)$. Outside the frontier, $x \notin [-1.2, 0]$, the variance is $\sigma_x^2(Y|\mathcal{F}_k) = d(x)$. Inside, it is $\sigma_x^2(Y|\mathcal{F}_k) = d(x)(X - d(x))/X$, where $X = 1.2$ is the length of the interval $[-1.2, 0]$. That variance is smaller than $d(x)$ because of inference from *both* knowledge points. As in Figure 3, the value of $\mathcal{F}_2$ is the area (shaded in gray) below q but above $\sigma_x^2(Y|\mathcal{F}_k)$. The net benefit of discovering the answer to question $x = -1.2$ is the dark-gray area.

LOWER PANELS: BENEFIT OF KNOWLEDGE-DEEPENING DISCOVERY. The left panel depicts truth and knowledge under $\mathcal{F}_4$. The right panel shows the value of knowledge and the benefit of discovery when the research deepens knowledge by discovering $(x, y(x)) = (0.8, 40.8)$. The dark-gray area is the net benefit of that discovery over pre-existing knowledge $\mathcal{F}_3 = \{(-1.2, 46.6), (0, 42), (1.2, 41.8)\}$. As in the previous figures, the total value of $\mathcal{F}_4$ is the area above the variance and below q.

Definition 2 (Variance)**.** The variance of a question with distance d in an area of length X is

$$\sigma^2(d;X) := d(X-d)/X.$$

Note that $\sigma^2(d;X) = \sigma_x(Y|\mathcal{F}_k)$ whenever $d(x) = d$ and x is in an area of length X. This is allows us to simplify notation and to focus on the variables d and X exclusively rather than keeping track of the precise question x and its research area. We abuse of notation and state the benefit of discovery as $V(d;X)$.

Proposition 1. *Consider a discovery $(x, y(x))$ in an area of length $X \leq \infty$ with distance $d = d(x)$. The benefit of the discovery is*

$$\begin{aligned} V(d;X) = \frac{1}{6q}\Big(2X\sigma^2(d;X) &+ \mathbf{1}_{d>4q}\sqrt{d}(d-4q)^{3/2} \\ &+ \mathbf{1}_{X-d>4q}\sqrt{X-d}\ (X-d-4q)^{3/2}. \\ &- \mathbf{1}_{X>4q}\sqrt{X}(X-4q)^{3/2}\Big) \end{aligned}$$

Expanding knowledge beyond the frontier has a benefit of $V(d;\infty) := \lim_{X\to\infty} V(d;X)$.

Proposition 1 states that expanding knowledge is equivalent to the limiting case of deepening knowledge.

The terms in $V(d;X)$ without an indicator function measure the direct reduction in the variance due to a discovery and hence the effect on decision making conditional on a proactive action $a \neq \varnothing$. The terms with an indicator function, $\mathbf{1}$, become active whenever the corresponding area contains questions with too imprecise conjectures (see e.g., Figure 5, right panel). Such conjectures induce the decision maker to select the outside option $\varnothing$ rather than making a proactive decision to limit losses to 0. The terms with an indicator function that enter positively correspond to choices in the newly created areas. The term with an indicator function that enters negatively corresponds to choices in an old area that gets replaced. Therefore, the presence of the outside option cushions the decision maker against risky proactive actions when there is high uncertainty about the question's optimal answer.

Figure 4 illustrates the benefits of discovery for expanding knowledge (upper panels) and deepening knowledge (lower panels). The right panel of Figure 6 on page 17 illustrates the functions for different area lengths X. To gain intuition, it is useful to discuss expanding knowledge and deepening knowledge separately.

Expanding knowledge. Our first corollary states the closed form of $V(d;\infty)$.

Corollary 1. $V(d;\infty) = \frac{1}{6q}\left(6qd - d^2 + \mathbf{1}_{d>4q}\sqrt{d}(d-4q)^{3/2}\right)$.

We focus on discovering the answer to $x < x_1$, which is the case when moving from Figure 3 to the upper row of Figure 4. Case $x > x_k$ is analogous.

The benefit of expanding knowledge comes from the new area $[x, x_1)$ it creates. The discovery of $y(x)$ pushes the knowledge frontier to the left and creates a new

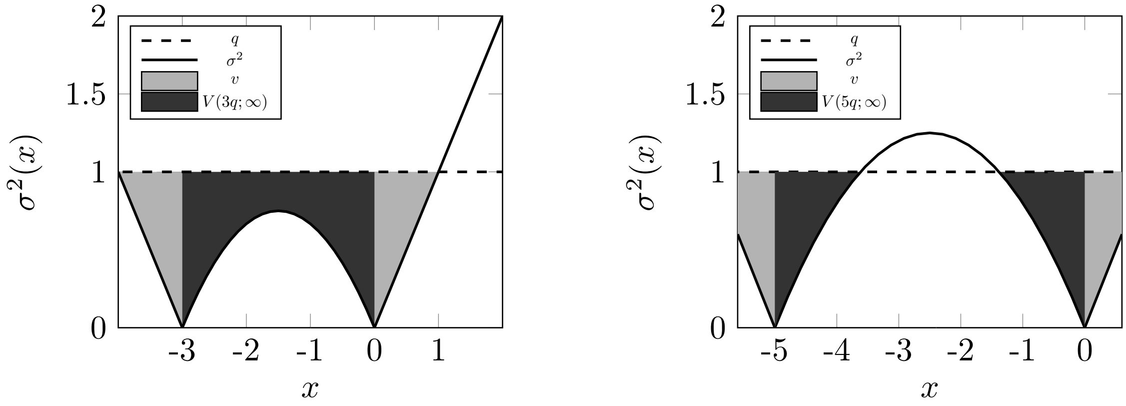


Figure 5: *Benefit-maximizing (left) and too large (right) distance of x given $\mathcal{F}_1$.* Given that the value of doing nothing is $-q = -1$, the benefit-maximizing distance when expanding knowledge is $d = 3$. The left panel depicts the benefit-maximizing choice, given $\mathcal{F}_1$, when expanding to the negative domain using $d = 3q$ and thus $x = -3$; the right panel shows the effect of a choice that is too far away ($x = -5, d = 5q$). The gain in knowledge $V(d;\infty)$ is the dark-shaded area. It is larger in the left panel than in the right panel.

research area $[x, x_1)$. The benefit of the discovery is the value of that new area (the dark-shaded area in the upper row of Figure 4).[17]

The value of adding an area depends on (i) the amount of questions in that area, and (ii) the degree of improvement in decision making over the outside option, $a = \varnothing$. The benefit-maximizing question solves a classical marginal-inframarginal tradeoff as in a monopoly pricing decision: Increasing the area length of the newly created area has two opposing effects on the value of discovery. The marginal gain is the increase the amount of questions to which the conjecture improves—an increase in the benefits of discovery. However, it comes at the cost because it decreases the precision of conjectures about all inframarginal questions in the area—a decrease in the benefits of discovery.

Figure 4 and 5 illustrate the benefits of discovery from creating a too short (upper right panel Figure 4), ideal(left panel Figure 5), and too large (right panel Figure 5) area. The largest benefits are derived at an intermediate level where all conjectures have a variance strictly smaller than q as the next corollary shows. The inframarginal losses outweigh the marginal gains already at a distance such that the decision maker refrains from using the outside option for all questions inside the new area. Define the benefit-maximizing distance in area X as

$$d^0(X) := \max_d V(d; X)$$

[17] To be precise, the conjectures about questions to the left of the old frontier are replaced by conjectures inside the new research area and conjectures to the left of the new frontier become also more precise. However, as can be seen in the upper right panel of Figure 4, the variance reduction to the left of the frontier is always of equal value. Hence the benefits are as if there was only the new area added.

Corollary 2. *The benefit of expanding knowledge is single peaked in d. The benefit-maximizing distance $d^0(\infty) = 3q$. The maximum benefit of expanding knowledge is $V^\infty := V(3q; \infty) = \frac{3}{2}q$.*[18]

Deepening knowledge is the process of discovering an answer $y(x)$ to a question x in an area i with two bounds, x_i and x_{i+1}. The answers $y(x_i)$ and $y(x_{i+1})$ are known. An illustration is the lower panel of Figure 4. The difference to expanding knowledge is that instead of creating a new area, deepening knowledge replaces the old area, $[x_i, x_{i+1})$, by two new areas, $[x_i, x)$ and $[x, x_{i+1})$.

The benefit of a discovery depends on the combination of improved decision making in either of the areas. We know from Corollary 2 that the largest benefits in a single area come from an area of length $3q$. Thus, if the old area i had length $X_i = 6q$ discovering the midpoint provides the largest benefits. However, if $X_i \neq 6q$ at least one of the two areas has a length different from $3q$.

If $X_i \neq 6q$ two forces are at play. First, there is a benefit to replacing the old area with two symmetric new areas each half the length of the old. The intuition follows the one from expanding knowledge: the inframarginal loss increases when an area becomes too large. Thus, choosing two areas with the same length reduces the inframarginal losses compared to the case of one large and one small area. Inspection of the lower right panel of Figure 4 provides a graphical intuition.

Second, benefits decline if an area length is larger than $3q$; conjectures inside the area become increasingly imprecise. Maintaining symmetry implies that newly created areas are larger than $3q$ if $X_i > 6q$ and thus of not maximizing the benefit.

If the initial area length X_i was small, the first force dominates. It is better to balance spillovers even if each area falls short of having length $3q$. If however, X_i was large, the tradeoff is solved in favor of having one high-value area. It is better to focus on creating one high-valued area at the cost of having imprecise conjectures in the other, larger, area. A cutoff $\widetilde{X}^0 \in [6q, 8q]$ exists such that it is benefit maximizing to create two symmetric areas if and only if $X_i < \widetilde{X}^0$

Which initial area length X_i provides the largest benefit of being transformed into two new areas? As explained above, two areas of length $3q$ provide the largest value. However, we have to take into account that the two new areas also replace an old area. The larger the area that gets replaced, the less its initial value. On the other hand, the larger the old area (beyond $6q$) the lower the value of the two new areas. The initial area length that provides the largest benefit when it is replaced is $\check{X}^0 \approx 6.2q$, i.e., is above $6q$.

Expanding vs deepening knowledge. On the one hand, creating new areas has the benefit that no knowledge needs to be replaced as all old areas remain. On the other hand, deepening knowledge has the benefit of creating two new areas with relatively precise conjectures. If an area is small, deepening knowledge in that area provides only a small benefit. Conjectures are already precise. If an area is large, conjectures are imprecise and deepening knowledge is more beneficial. Overall, there is a cutoff

[18] The results of this and the next corollary follow directly from an analysis of $V(\cdot;\cdot)$ derived in Proposition 1. However, their derivations are not entirely straightforward—hence, we provide them in the appendix.

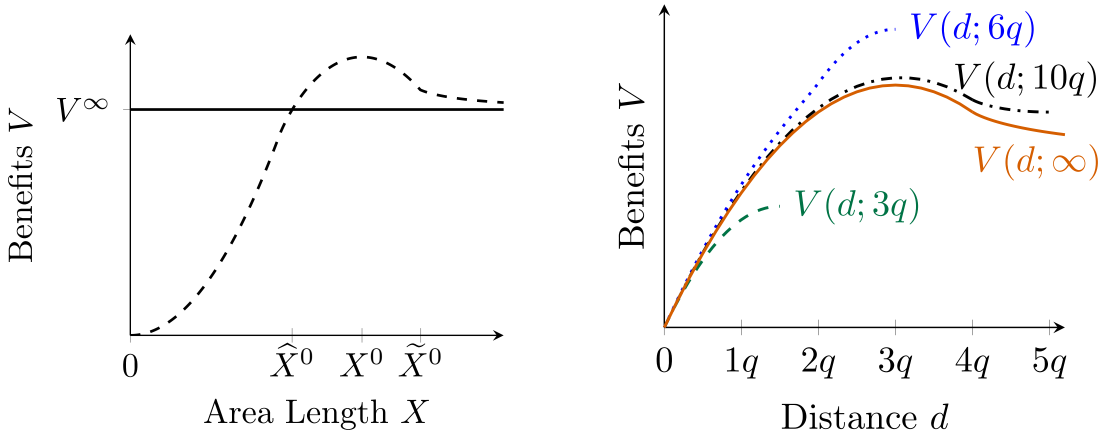


Figure 6: *The benefit of discovery.*
LEFT PANEL: BENEFIT OF DISCOVERY AS A FUNCTION OF THE AREA LENGTH X. The graph plots the benefit of discovery, $V(d^0(X);X)$ for areas of length $X < \infty$ (dashed line). The solid line is the maximum benefit of discovery when expanding knowledge V^∞. Deepening trumps expanding if $X > \widehat{X}^0 \approx 4.3q$. $V(d^0(X);X)$ is maximal at $\check{X}^0 \approx 6.2q$; $d^0(X) < X/2$ if $X > \widetilde{X}^0$.
RIGHT PANEL: BENEFIT OF DISCOVERY GIVEN X AS A FUNCTION OF DISTANCE d. Expanding knowledge (solid line) and deepening knowledge (nonsolid lines) for area lengths $X \in \{3q, 6q, 10q, \infty\}$.
Note: Plots for deepening knowledge end at the maximum distance in each area, $d = X/2$.

$\widehat{X}^0 \approx 4.3q$ such that deepening knowledge in X_i is more beneficial than expanding knowledge beyond the frontier if and only if $X_i > \widehat{X}^0$.

Our next corollary and the associated Figure 6 summarize the discussion.

Corollary 3. *There are three cutoff area lengths, $4q < \widehat{X}^0 < 6q < \check{X}^0 < \widetilde{X}^0 < 8q$, such that the following holds:*

- *The benefit of expanding knowledge by $3q$ dominates the benefit of deepening knowledge in area i if and only if $X_i < \widehat{X}^0$.*
- *The maximum benefit of deepening knowledge in an area i is increasing in the area length if $X_i < \check{X}^0$; it is decreasing if $X_i > \check{X}^0$.*
- *The distance $d^0(X_i)$ of the benefit-maximizing discovery, is increasing in X_i for $X_i < \widetilde{X}^0$ and decreasing for $X_i > \widetilde{X}^0$. If $X_i < \widetilde{X}^0$, $d^0(X_i) = X_i/2$. Otherwise $d^0(X_i) \in (3q, X_i/2)$. As $X \to \infty$, $d^0(X) \to d^0(\infty)$ and $V(d;X) \to V(d;\infty)$ uniformly.*

4 The Cost of Research

In this section, we introduce a cost of research. The cost imply an endogenous measure of the productivity of research. We conceptualize research as the *search for an answer*. That is, we model research as sampling a set of candidates for answers to question x with the goal of discovering the actual answer $y(x)$.

Formally, we assume that the sampling decision consists of selecting an interval

$[a, b] \in \mathbb{R}$. If the true answer lies inside the chosen interval, $y(x) \in [a, b]$, research succeeds and a discovery is obtained. If $y(x) \notin [a, b]$ research fails and no discovery is obtained. Thus, the choice of the research interval will lead to an ex-ante success probability of research. We assume that the cost of sampling are proportional to $(a - b)^2$.

Restricting the sampling decision to a single interval $[a, b]$ is without loss for our purposes as conjectures $G_x(y|\mathcal{F}_k)$ follow a normal distribution. The quadratic formulation is for convenience only. What matters for our results qualitatively is that the cost is *increasing and convex* in the length of the interval sampled.

We now characterize the cost of research in terms of three variables of interest: the research area, the novelty of the question and the expected output. The area length, X, and the novelty $d(x)$ of a research question are the familiar concepts from Section 3. Output describes the expected probability that search is successful and leads to discovery. We denote that probability by ρ.

The cost function $c(d, \rho, X)$ that we derive describes the cost of discovering an answer with probability ρ to a question with distance d in an area of size X. We begin by defining a prediction interval.

Definition 3 (Prediction Interval)**.** The prediction interval $\alpha(x, \rho)$ is the shortest interval $[a, b] \subseteq \mathbb{R}$ such that the answer to question x lies within $[a, b]$ with probability ρ.

Next, we describe the prediction interval $\alpha(x, \rho)$ based on the conjecture $G_x(y|\mathcal{F}_k)$.

Proposition 2. *Suppose $\alpha(x, \rho)$ is the prediction interval for probability ρ and question x when the answer $y(x)$ is normally distributed with mean μ and standard deviation σ. Then, any prediction interval has the following two features:*

1. *The interval is centered around μ.*
2. *The length of the prediction interval is $2^{3/2} erf^{-1}(\rho)\sigma$, where erf^{-1} is the inverse of the Gaussian error function.*

The dashed line in the figures containing the Brownian path depict the $\rho = 95$-percent-prediction interval (e.g., in Figure 7). The right panel of Figure 7 indicates that the prediction interval depends on the location of the question. Two questions with the same distance to existing knowledge (that is, distance to question $x = 0.6$) have different 95-percent prediction intervals depending on whether research deepens knowledge or whether research expands it. That difference translates to different cost. The cost-of-research function (see Figure 8) follows from a simple corollary to Proposition 2. To abbreviate notation let $\tilde{c}(\rho) := (erf^{-1}(\rho))^2$.

Corollary 4. *Fix knowledge $\mathcal{F}_k$, a probability ρ and a question x. The minimal cost of obtaining an answer to question x with probability ρ is proportional to*

$$c(\rho, d; X) = \tilde{c}(\rho)\sigma^2(d; X).$$

Corollary 4 intuitively links the cost of research effort to the probability of a discovery. Because the inverse error function is increasing and convex, the cost of finding an answer with probability ρ is increasing and convex in ρ. Discovering

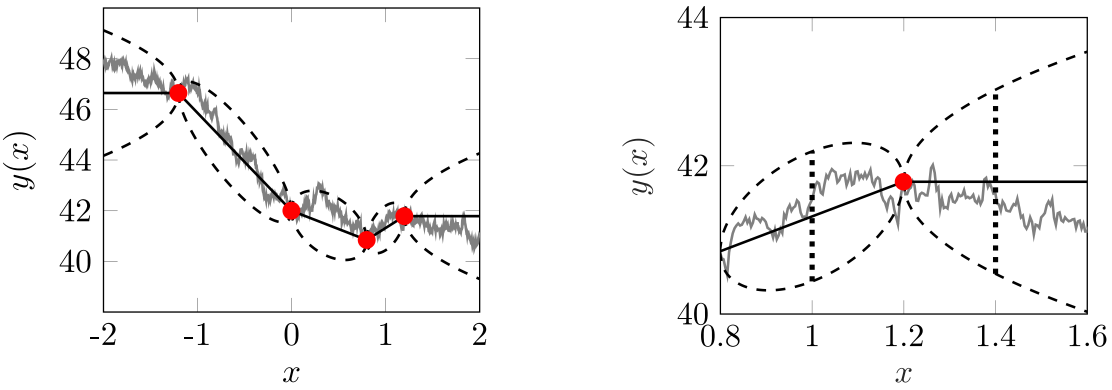


Figure 7: *Cost of research and interference.* The left panel is the same as the lower left in Figure 4. The right panel is a close-up of the left panel on $[0.8, 1.6]$. The dotted lines represent the 95-percent prediction intervals for the answers to questions $x = 1$ and $x = 1.4$ assuming $\mathcal{F}_4 = \{(-1.6, 46.6), (0, 42), (0.8, 40.8), (1.2, 41.8)\}$. Both questions have the same distance to $\mathcal{F}_4$ ($d(x) = 0.2$). However, the 95-percent prediction interval at question $x = 1$ is shorter because the variance is smaller as researching $x = 1$ deepens knowledge. Research on question $x = 1.4$ expands knowledge which implies a larger variance.

an answer with certainty implies infinite cost, as there is always a chance that the answer is outside the sampled interval.

A convenient feature of the cost function $c(\rho, d; X)$ is that it is multiplicatively separable in $\tilde{c}(\rho)$ and $\sigma^2(d; X)$. That implies that fixing ρ, the changes in the cost with respect to novelty, d, and area length, X, are proportional to changes in the variance $\sigma^2(d; X)$—they are increasing in d and X; concavity in d decreases in X with the limiting case of linear cost in d as $X \to \infty$ (see left panel of Figure 8). Similarly, holding distance and area length constant, changes in ρ translate into cost changes according to $\tilde{c}(\rho)$ (see right panel of Figure 8).

Importantly, the cost function also links output and novelty of research: for a given level of effort, the probability of a success depends on the precision of the conjecture about this question. Research on a more novel question inside the same research area with the same level of effort entails a higher risk.

5 The Researcher's Choices

In this section, we introduce a researcher to our model of knowledge. We combine the ingredients from the previous sections to characterize her optimal choice.

5.1 The Researcher's Objective

Consider a researcher that can search for a discovery. Her expected payoff is composed of the benefits of discovery (that is, the function from Section 3) and her own cost of research (that is, the cost function from Section 4). She is unconstrained in her choice of the research question and her effort, but if she fails to obtain an

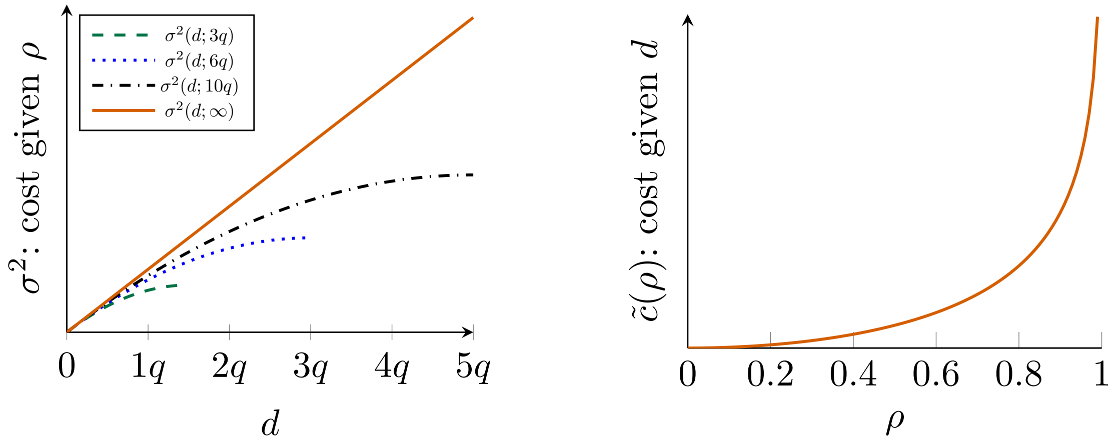


Figure 8: *Cost of research as a function of distance to knowledge (left) and probability of discovery (right).* As the area length X decreases, cost diminish given (d, ρ). The cost are linear in d when expanding knowledge but strictly concave when deepening knowledge. The cost function is convex in ρ. The left panel plots $\sigma^2(d; X)$ for different X; the right panel $\tilde{c}(\rho)$. In the left panel, plots for an area length $X < 10q$ end at the maximum possible distance $(d = X/2)$.

answer, the benefit of her research is zero.[19]

Using our results from Section 3 and 4, the researcher's choices of question x and research interval $[a, b]$ imply a distance (d) in an area of length X, and an expected probability of discovery (ρ). Together with an exogenous cost parameter $\eta > 0$, these three variables are sufficient to describe the researcher's payoff:

$$u_R(d, \rho; X) := \rho V(d; X) - \eta c(\rho, d; X).$$

From the previous discussion, it follows that the researcher resolves a tension between the cost and benefit of research: short distances allow the researcher to find answers with a high probability at a relatively low cost. However, such answers provide little benefit to society. By increasing the distance, the researcher increases the benefit but at the same time either the cost increase or the researcher has to accept a lower probability of discovery. This tradeoff from the researcher's perspective is at the heart of many discussions on research funding.[20]

We abstract from any motivations other than the researcher's desire to improve the value of knowledge. That is, we implicitly assume that the market for research

[19] A rationale for discarding nonfindings comes from moral hazard concerns: science is complex, and it is impossible to distinguish the absence of a finding from the absence of proper search. Our model can easily account for the possibility of publishing nonfindings; unsurprisingly, these increase the value of knowledge as well. The difficulty of publishing the absence of evidence, however, has been long recognized in the literature. See, for example, Sterling (1959). In principle, it is relatively straightforward to compute updated answer distributions based on null results in our setting. Including this in our researcher model, however, is beyond the scope of this paper.

[20] See, e.g., the emphasis on high risk/high gain research by the European Research Council `https://ec.europa.eu/research/participants/data/ref/h2020/call_ptef/ef/h2020-call-ef-erc-stg-cog-2015_en.pdf` or the National Institute of Health `https://commonfund.nih.gov/highrisk`.

is frictionless and rewards researchers only for their relative contributions to the value of knowledge. We make this assumption to provide a clean analysis of the researcher's trade-offs.

In the following, we characterize the researcher's optimal choice and elaborate on the resolution of the novelty-output tradeoff.

5.2 The Researcher's Decision

The researcher solves

$$\max_{X \in \{X_0, \ldots, X_k\}} \underbrace{\max_{\substack{d \in [0, X/2], \\ \rho \in [0,1]}} \rho V(d; X) - \eta c(\rho, d; X)}_{=: U_R(X)}$$

If there is no cost ($\eta = 0$), we can apply Proposition 1 do derive the optimum. For any X, the researcher selects $\rho = 1$. For $X < \widetilde{X}^0$ she selects $d = X/2$ and $d \in (3q, X/2)$ for $\widetilde{X}^0 < X < \infty$. If $X = \infty$, the researcher chooses $d = 3q$. She prefers to expand knowledge if and only if $X_i \leq \widehat{X}^0$ for any area $X_i < \infty$ defined by $\mathcal{F}_k$.

However, for $\eta > 0$ the researcher's decision on effort is non-trivial and interlinked with her decision on the research question. Choosing a question with small distance allows for a high probability of discovery at a low cost. Her initial conjecture about the answer is already precise. Nevertheless, her payoff is low as such a discovery provides little benefit.

By increasing the distance, the researcher increases the benefit of a discovery, but also increases the cost ceteris paribus. The effect on the optimal probability of discovery is ambiguous: Depending on which effect dominates, the distance and the probability of discovery are substitutes or complements.

Definition 4. Let $\rho^*(d; X)$ be the success probability ρ that maximizes the researcher's payoff given d and X. Output ρ and novelty d are complements (substitutes) if $\rho^*(d; X)$ is strictly increasing (decreasing) in d.

It turns out that how output and novelty behave jointly depends crucially on both the length of the research area and the level of novelty.

If novelty is too high, the benefit of a discovery is declining in novelty (see Proposition 1). In this case, novelty and output are substitutes. Reducing novelty increases the marginal benefit and reduces the marginal cost of increasing output.

Optimal choice within a research area. First, we consider how the researcher's choice of distance and probability of discovery interact within an area of length X. The following proposition summarizes the behavior of d and ρ.

Proposition 3. *Suppose $\eta > 0$.*

1. *When the researcher expands knowledge, distance, d, and probability of discovery, ρ, are substitutes. The optimal choice of $d^\infty \in (2q, 3q)$.*
2. *When the researcher deepens knowledge in an area of length X, d and ρ are*

- *independent if* $X \leq 4q$,
- *complements if* $X \in (4q, \frac{5}{2-\sqrt{\frac{3}{2}}}q)$,
- *substitutes for* $d \in (0, \hat{d})$ *and complements for* $d \in (\hat{d}, \frac{X}{2})$ *if* $X \in (\frac{5}{2-\sqrt{\frac{3}{2}}}q, 8q)$
- *substitutes if* $X > 8q$.

The researcher's optimal choice of d *is at its maximum value* $d = \frac{X}{2}$ *if* $X < \widetilde{X}$ *and interior with* $d < \frac{X}{2}$ *if* $X \geq \widetilde{X}$ *with* $\widetilde{X} \in (6q, 8q)$.

Whenever the researcher expands knowledge, an increase in distance increases the marginal cost of the success probability more than it increases the marginal benefit. Thus, any increase in novelty comes at the cost of a reduction in output. The optimal distance is intermediate: a too short distance provides only small benefits. However, the value-maximizing distance is $3q$ and the researcher will never select a question further than $3q$ outside the frontier. As research comes with the risk of failure and at a cost, the optimal distance is shorter than the benefit-maximizing distance.

Whenever the researcher deepens knowledge, the effect that distance has on the optimal choice of ρ changes in contrast to expanding knowledge. The variance of the conjecture is concave in the distance if $X < \infty$ because the researcher uses information from two answered questions to form a conjecture. Thus, the negative effect from moving further from one question in existing knowledge is mitigated by moving closer to another question in existing knowledge. Increasing distance to existing knowledge is therefore less costly in terms of the probability of discovery than when expanding knowledge. Distance and probability of discovery can become complements.

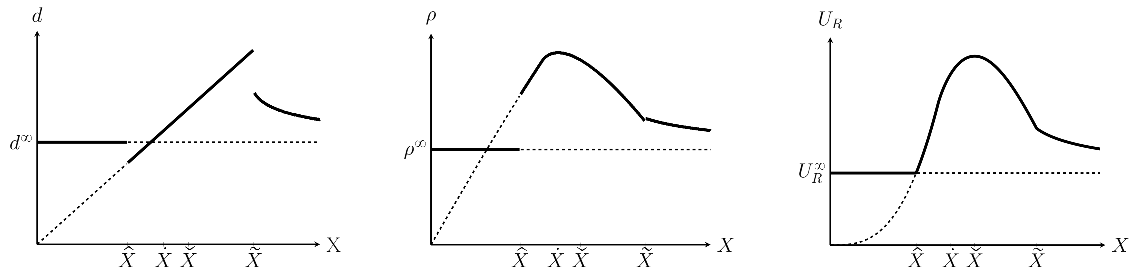
Observe that for any choice of distance, the optimal choice of probability solves

$$\eta \tilde{c}(\rho) = \frac{V(d; X)}{\sigma^2(d; X)}.$$

Thus, the probability of discovery depends on the ratio of the benefit of a discovery (the marginal benefit of increasing ρ) and on the variance of the question considered (to which the marginal cost of increasing ρ is proportional). Therefore, the probability adjusts to distance depending on how this ratio changes. If the relative increase in the benefit is larger than the relative increase in the variance, the researcher will increase the probability and vice versa.

For short research areas—areas in which all questions inside the area are proactively addressed by the decision maker—the benefit of a discovery is proportional to the variance of the question (see Proposition 1): the probability of a discovery is independent of the choice of the distance. However, assume now that the area prior to the researcher's discovery is larger. The decision maker's losses were limited by the decision maker's choice of the outside option on some questions. An increase in distance of the new discovery now leads to a larger relative increase in the benefit than in the variance: d and ρ become complements.

Once the research area becomes even larger, the benefit of an increase in the distance shrinks for small distances compared to before. The benefit from reducing



(a) Distance d (b) Probability ρ (c) Payoff U_R

Figure 9: *Outcomes of the researcher's choices in areas of different length.* The graphs indicate the researcher's choices conditional on area length, X. We compare them with optimal choices of expanding knowledge (the horizontal line in each graph). On the horizontal axis we indicate the cutoffs $\widehat{X}, \check{X}, \dot{X}$, and $\widetilde{X}$ from Proposition 4.
The *solid graph* plots the optimal choice conditional on X being the best available area. For small areas ($X < \widehat{X}$), the researcher prefers to expand knowledge. For areas of length $X > \widehat{X}$, deepening knowledge is preferred to expanding knowledge. If the area has length $X < \widetilde{X}$, the researcher selects the largest distance possible in area X—that is, $d = X/2$. If $X > \widetilde{X}$, it is optimal to select a distance $d < X/2$ closer to one of the end points of the area. For small areas ($X < \dot{X}$), $\rho(X)$ increases with X. For large areas ($X > \dot{X}$), $\rho(X)$ decreases in X (apart from a discontinuous jump at $\widetilde{X}$). The researcher's payoff increases in area length for $X < \check{X}$ and decreases for $X > \check{X}$. The order of the cutoffs is independent of the value of the cost parameter, η.

the length of the larger of the two resulting areas is smaller for small initial distances. Distance and probability become substitutes again. However, for larger distances in the same research area this effect disappears—distance and probability of discovery return to being complements.

Finally, as the area grows further, the set of distances for which d and ρ are complements shrinks. Eventually, distance and probability become substitutes throughout as in the case of expanding knowledge.

The researchers optimal choices inside each area are as follows. The researcher chooses the largest possible distance if the area is not too large. If, in addition distance and probability are complements, the maximum distance is chosen together with the largest probability of discovery in this interval. In such research areas, there is no tradeoff between novelty and output of research: any increase in novelty, d, is accompanied by an increase in output, ρ. However, for large research areas and for the case of expanding knowledge this tradeoff arises—novelty and output are substitutes.

Optimal choice between intervals. We now characterize the researcher's choice of the research area X. We take the optimal choice inside each area as given. Let $d(X)$ and $\rho(X)$ be the researcher's choices conditional on an area of length X, and let $U_R(X)$ be the associated payoff. Analogously, d^{∞}, ρ^{∞}, and U_R^{∞} are the respective objects for expanding research. The following proposition summarizes the findings.

Figure 9 illustrates the following proposition.

Proposition 4. *Suppose $\eta > 0$. There is a set of cutoff values $\widehat{X} \leq \dot{X} \leq \check{X} \leq \widetilde{X} < 8q$ such that the following holds:*

- *The researcher expands knowledge if and only if all available research areas are shorter than $\widehat{X}$.*
- *The researcher's payoffs, $U_R(X)$ are single peaked with a maximum at $\check{X}$.*
- *The optimal choices of distance, $d(X)$, and probability of discovery, $\rho(X)$, are non-monotone in X. The probability $\rho(X)$ has a maximum at $\dot{X}$, the distance $d(X)$ at $\widetilde{X}$.*

Proposition 4 shows that the pattern in the choices of distance are qualitatively the same as in Corollary 3. However, the cost add another dimension to the problem: The optimal probability depends on the size of the area. Consider a short area X. The scope for improvement of the decision maker's policies is small as conjectures are already precise. Thus, investing into discovery has a small expected payoff.The researcher does not invest much into the search for an answer albeit the small cost—the probability of discovery is small. Now, consider a large area. The benefit of deepening research is greater than in the small area. However—because conjectures are imprecise—the cost are larger too. The researcher does not invest much into a discovery—the probability of discovery is small. In an area of intermediate length, the benefit of a discovery is relatively high, yet conjectures are relatively precise and limit the cost of research. The return on investment is largest—the probability of discovery is highest.

Moreover, the researcher only trades off $d(X)$ against $\rho(X)$ if X is of intermediate length. If X is small, an increase in X increases the benefit of research. Cost are small, and the researcher has an incentive to increase $d(X)$ and $\rho(X)$. As X becomes larger, the marginal increase in the benefit of research declines, yet the marginal cost of research increase both for $d(X)$ and $\rho(X)$. Eventually the researcher faces a tradeoff: should she lower $\rho(X)$ to maintain $d(X) = X/2$? It turns out that she should. While the researcher wants to remain at a boundary in her choice of $d(X)$ she mitigates the increased cost by lowering $\rho(X)$. As X increases further, the researcher eventually also lowers $d(X)$. After a discrete jump upwards at $\widetilde{X}$ of $\rho(X)$ following the jump downwards of $d(X)$ from the midpoint $X/2$ to an interior point, d and ρ co-move again; both decline in X.

The researcher's most preferred area length, $\check{X}$, is in a region in which a trade-off between $\rho(X)$ and $d(X)$ exists. While the researcher would prefer a larger research area to increase the benefit of research, she would prefer a smaller research area to reduce the cost of finding an answer. Thus, $d(X)$ is increasing and $\rho(X)$ decreasing at the point at which $U_R(X)$ is maximal.

Note that we have only characterized the researcher's decision *conditional* on an area length X and compared these values so far. An explicit analytical characterization of the researcher's choice depends on the available research areas which is determined by the existing knowledge $\mathcal{F}_k$. Given our characterization, computing the optimal area is straightforward.[21]

[21] A computer program to numerically calculate all choices given $\mathcal{F}_k$ is available from the authors.

6 Application: Incentivizing Moonshots

In this section, we apply the findings from Section 5 to a setting in which a long-lived decision maker (e.g. society) faces a sequence of short-lived researchers.

Thusfar, our focus has been on the static problem of a researcher who has access to some exogenous existing knowledge and decides how to advance it. In reality, existing knowledge is, however, endogenously created by decisions of previous generations of researchers. By creating additional knowledge, a discovery today entails spillovers onto future researchers. A long-lived decision maker may therefore want to incentivize researchers to aim for what we call a *moonshot*—expanding knowledge further beyond the frontier than statically optimal.

Recall the following insights from our previous analysis:

(i) too distant discoveries provide lower benefits than those of intermediate distance (Proposition 1 and Corollary 2 and 3), and
(ii) benefits and cost depend on existing knowledge (Proposition 1 to 4)
(iii) the probability that the researcher discovers an answer is small if the chosen question is close or distant. It is largest if the chosen question is of intermediate distance (Proposition 3 and 4).

While (i) implies that incentivizing a moonshot is statically suboptimal, (ii) and (iii) together imply that current researchers can impose a dynamic externality on future researchers. In the following, we show that—while such spillovers never compensate the short-run losses if the researcher's cost are too high or too low—incentivizing moonshots is beneficial for intermediate cost levels.

Setup. Time is discrete and indexed by $t = 1, 2, ...$ with initial knowledge at the beginning of $t = 1$ being $\mathcal{F}_1 = \{(0, y(0))\}$. In each period there is a single, short-lived researcher. Moreover, there is one long-lived decision maker who discounts the future by $\delta \in (0, 1]$. Within a period t the timing is as follows. The researcher is born and observes $\mathcal{F}_t$. Then, she decides upon a question x and a research interval $[\underline{y}, \overline{y}]$. If $y(x) \in [\underline{y}, \overline{y}]$, knowledge updates to $\mathcal{F}_{t+1} = \mathcal{F}_t \cup (x, y(x))$ otherwise $\mathcal{F}_{t+1} = \mathcal{F}_t$. At the end of the period, the decision maker decides on her actions $a(x) \in \mathbb{R} \cup \varnothing$ for each x conditioning on $\mathcal{F}_{t+1}$ only. Then, the next period, $t + 1$, begins.

We define a moonshot as an advancement beyond the frontier with larger distance than the myopic optimum of Corollary 2.

Definition 5 (Moonshot)**.** Given $\mathcal{F}_k$, a question $x \in \mathbb{R}$ is a moonshot if

1. $x \notin [x_1, x_k]$ and
2. $d(x) > 3q$.

To keep the focus on the main trade-off we make two simplifying assumptions.

Assumption 1. All generations of researchers starting from $t = 2$ have the same cost type η. Moreover, researchers cannot condition their strategy on calender time t, but only on knowledge $\mathcal{F}_t$. We focus on symmetric pure strategies, that is all researchers select the same strategy given the same knowledge.

Assumption 2. In $t = 1$ the decision maker fully controls the researcher's decision $(\hat{x}, \hat{\rho})$ at no cost.

The assumptions ensure clarity of the argument, yet neither is, per se, realistic. We provide a discussion and interpretation at the end of the section where we show how to embed them in a more realistic setting.

Here, we briefly describe their role in the analysis. Assumption 1 implies that the continuation game is as simple as possible. If in any period t, a researcher fails to discover an answer, science has ended. All future generations will replicate the decision made in t and therefore will also fail to discover an answer. Assumption 2 removes both the cost of incentivizing a moonshot and the potential risk involved. Consequently, the decision maker always selects $\hat{\rho} = 1$. Assumption 2 allows us to focus on the *implications* of the moonshot. Our results do not rely on $\hat{\rho} = 1$ or on the absence of cost to the decision maker. We address the cost side of research funding in Section 7.

No-cost Benchmark. As a benchmark, assume $\eta = 0$ for every researcher. In that case, researchers always search along the entire real line for an answer, $(-\infty, \infty)$, and discover an answer with certainty. In addition, every researcher finds it optimal to expand the frontier and select a question with distance $d = 3q$. Recall from Corollary 3 for areas of length $X < \widehat{X}^0$ further expansion is optimal. Furthermore, $\widehat{X}^0 > 3q$. Thus there are no spillovers between periods. Output is maximized with $\rho = 1$ and areas of length $3q$ are benefit maximizing among all area lengths (see Corollary 2). Thus the myopic sequence is dynamically optimal too. The following statement is therefore a corollary to Proposition 1.

Corollary 5. *If research is costless, incentivizing a moonshot is not beneficial. Moreover, the static optimum is also dynamically optimal for any δ.*

Including Cost. With cost included, $\eta > 0$, preferences between the decision maker and the researchers are no longer aligned for two reasons. First, the cost of research enter the researcher's payoff function directly, but that of the decision maker only through the researcher's decision.[22] Second, findings in period $t = 1$ influence the cost of future generations of researchers in periods $t > 1$. The long-run decision maker internalizes that dynamic externality.

Analysis. We now demonstrate why incentivizing moonshots can be beneficial to the decision maker. Figure 10 sketches the evolution of knowledge for two different initial discoveries.

Because of Assumption 2 the decision maker chooses $\hat{x}$. Her ex-ante payoff is

$$\sum_{t=1}^{\infty} \delta^{t-1} \left(\prod_{k=1}^{t} \rho_k \right) v(\mathcal{F}_t),$$

where $\rho_1 = 1$ and $\rho_{t>2}$ is the researchers choice of ρ in period t.

The myopic benchmark for $\delta = 0$ is given by Corollary 2 and implies $\hat{x} = 3q$. However, if $\delta > 0$ selecting $\hat{x} > 3q$ has two effects on the expected payoff.

[22] If the decision maker were to internalize the researchers' cost as well, it is straightforward to generate an analogous result. Indeed, moonshots would become even more beneficial.

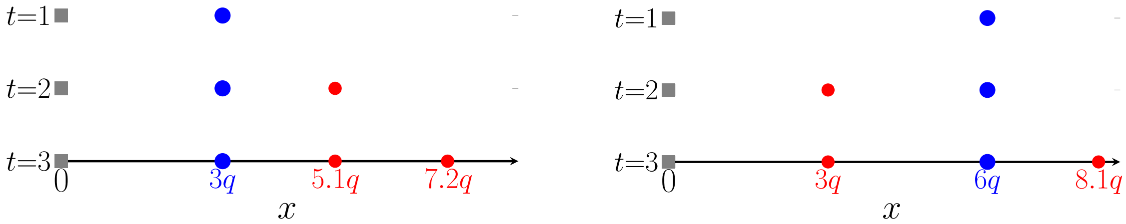


Figure 10: *Evolution of knowledge from $t = 1$ to $t = 3$ for different initial $\hat{x}$.* We assume $\mathcal{F}_1 = \{(0, y(0))\}$ (■). The dots describe which questions have known answer at each point in time t assuming that discovery is successful in all periods up to and including t. Apart from the initial discovery (●) all question choices (●) are optimal choices by identical researchers with cost parameter $\eta = 1$. The left panel depicts an initial choice of $\hat{x} = 3q$ the right panel one of $\hat{x} = 6q$.

One effect comes from the cost function of future researchers that aim to deepen knowledge between $\hat{x}$ and 0. Suppose $|\hat{x}|$ is large enough to make the (t=2)-researcher pursue a question in $[0, \hat{x}]$ with a distance d_2 to $\mathcal{F}_2$. That researcher's cost are determined by the variance $\sigma(d_2; |\hat{x}|)$ which also depends on the choice of $|\hat{x}|$. A properly chosen moonshot $\hat{x} > 3q$ reduces the cost of research for future generations of researchers while they work on connecting the initial body of knowledge with the moonshot. This induces a higher productivity during that period.

A second effect comes from persistently shaping knowledge. After some τ periods of successful discoveries, knowledge is sufficiently dense such that all researchers from $t = \tau + 1$ onwards move to expanding knowledge with d^∞. However, the knowledge created during these τ periods is generically different with and without the initial moonshot $\hat{x}$. Figure 10 illustrates two scenarios with persistently different knowledge. A properly chosen moonshot $\hat{x} > 3q$ induces a more valuable landscape of knowledge for the decision maker as distance-choices are improved due to the cost reduction.

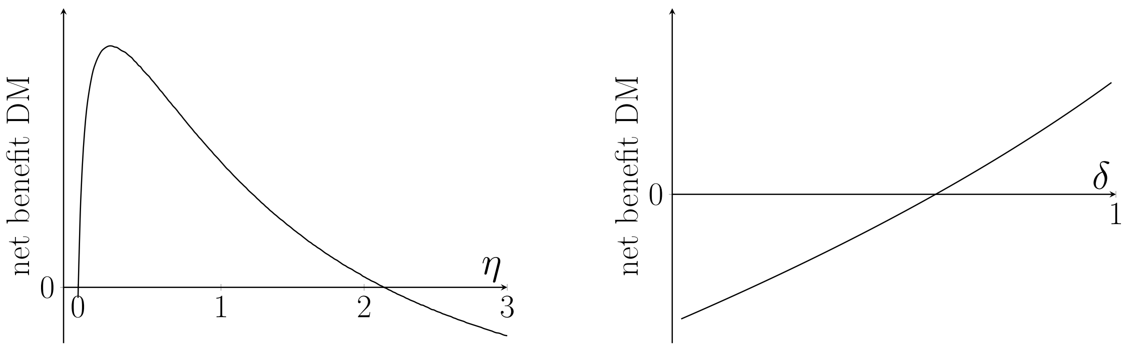


Figure 11: *Moonshot of $d = 6q$ vs the myopic optimum $d = 3q$ for different parameters.* The *left panel* plots the difference between the first-period net-present value of a $6q$-moonshot and that of the myopic optimum $d = 3q$ against different η. The discount factor is $\delta = 0.9$. The moonshot is strictly preferred for the interval $[\underline{\eta}, \overline{\eta}] \approx (0.01, 2.13)$.
The *right panel* plots the difference between the first-period net-present value of a $6q$-moonshot and that of the myopic optimum $d = 3q$ against different δ. The cost parameter is $\eta = 1$. The moonshot is strictly preferred for $\delta > \underline{\delta} \approx 0.6$.

Overall, an optimal moonshot can increase future output and future novelty. The next proposition shows, that these benefits may outweigh the short-run cost of a too distant discovery for a sufficiently patient decision maker provided that the cost of research are in an intermediate range.

Proposition 5. *There is a non-empty range of cost parameters* $(\underline{\eta}, \overline{\eta})$ *such that the decision maker strictly prefers a moonshot in* $t = 1$ *for any* $\eta \in (\underline{\eta}, \overline{\eta})$ *provided* δ *is larger than a critical discount factor* $\underline{\delta}(\eta) < 1$.

Proposition 5 states that moonshots are optimal if the decision maker is patient enough and cost are not extreme. Figure 11 provides an illustration.

Indeed, if the cost are very low, the reasoning of the no-cost benchmark above applies. A moonshot has little benefit because cost do not distort the researchers' decisions much while the cost of a suboptimal choice in the first period remains. If, instead, the cost are very high, it is optimal for future generations to limit search to small intervals. The probability of a discovery is low and it is unlikely that future generations will eventually succeed in closing the gap after a moonshot. In both cases, the decision maker's optimal choice is the myopically optimal $\hat{x} = 3q$.

For intermediate cost, moonshots are beneficial. The positive externality on future researchers is large, and research is generally productive. Future generations profit from the cost advantage and produce more valuable knowledge at a higher likelihood.

The length of the *optimal moonshot* depends on the cost parameter. If the cost are low and the decision maker is patient, the optimal moonshot can be such that several future generations work on closing the gap. The less patient the decision maker and the more costly the research, the shorter the time targeted to closing the gap. While the effect of the discount factor is as expected, the reason for the cost effect is the following. If cost are high, the chance that a researcher fails to obtain an answer increases. As a result, the *effective* discount factor decreases. The decision maker prefers moonshots that entail less future risk. Figure 12 shows the ex-ante value of different moonshots to the decision maker for two cost scenarios.

Discussion. The stylized model above illustrates a friction caused by the different time preferences of researchers and the long-run decision maker. However, the analysis relies on Assumption 1 and 2 both of which are not very realistic. We provide a short discussion how the analysis and our findings change once we relax these assumptions.

Assumption 1 eliminates any heterogeneity between researchers. In particular, it implies that there is no uncertainty on the behavior of future researchers with the result that if one researcher fails to obtain a discovery, all future researchers will follow her footsteps and fail too. Research reaches a dead end.

There are several ways to relax this assumption. The simplest is to allow researchers to use asymmetric strategies. Whenever the optimal choice of the researcher is not the midpoint of a research area, there is a second analogous alternative to the researcher's optimal point. Allowing researchers to choose between such alternatives can prevent research reaching a dead end after one failed attempt.

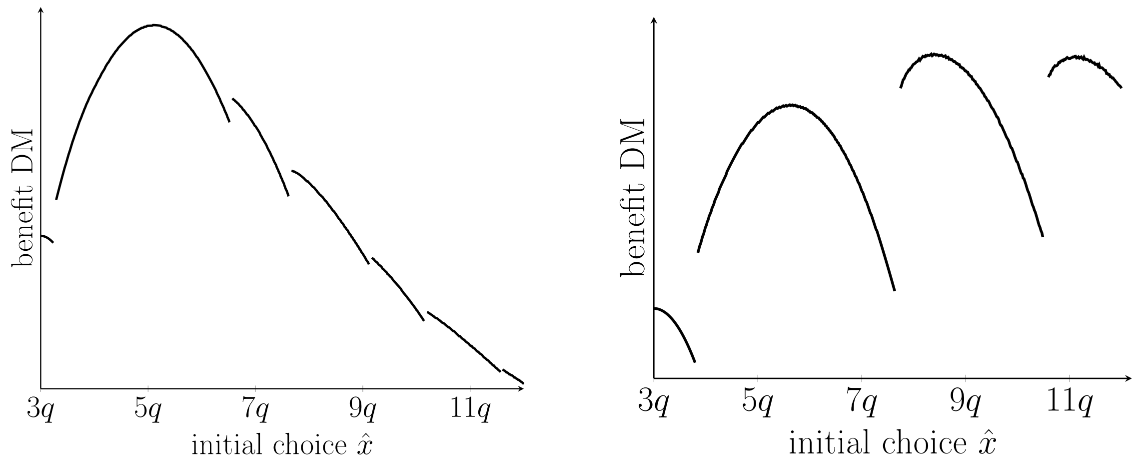


Figure 12: *Optimal moonshots.* First-period net present value against the initial moonshot. In the *left panel* $\eta = 1$ and the optimal moonshot $\hat{x}^*$ is between $5q$ and $6q$. After the initial moonshot, the researcher in $t = 2$ bridges the gap by selecting $x = \hat{x}^*/2$. In the *right panel* $\eta = 0.1$ and the optimal moonshot is between $8q$ and $9q$. After the initial moonshot it takes two researchers until the gap is bridged. In both figures $\delta = 0.9$.

However, if search fails on both alternatives the same situation as under Assumption 1 arises: Research has reached a dead end.

One option to guarantee the eventual progress of science is to assume that the researcher's cost parameter η is a random variable representing differences in talent, opportunity cost, research environment, etc. If, for example, the lower bound of the support of η is zero, dead ends are broken almost surely in finite time: Every new discovery changes the existing knowledge $\mathcal{F}_k$ and thus conjectures. Hence, making η a random variable allows low-cost types to remove a knowledge block of other types. If research reaches a dead end and many cost types fail to discover answers, science slows down. However, a single discovery can overturn that blockage. It changes the questions the types optimally pursue and may lead to discoveries by all types again.

In principle, such a model can create 'science cycles'. Over time regular types reach a dead end. If, however, a rare genius arrives that obtains a new discovery, the regular types change focus and are now able to discover answers again.[23]

A third alternative to relax the assumption is to allow researchers to observe previous failures too. However, changing the model in that respect implies changing also the results from Proposition 3 and 4. We leave it for future research.

Assumption 2 implies that the decision maker can, at no cost, obtain a particular finding. One interpretation of that assumption is that the decision maker can award a prestigious prize or a powerful position if she finds an answer to $y(\hat{x})$. If the benefit to the researcher is large, she will pursue question $\hat{x}$ and find an answer with high probability. In principle, the decision maker may be budget constrained and awarding prizes may become more costly the further away $\hat{x}$ is. Here, we abstract from this by implicitly assuming that incentives for researchers can be provided at

[23]In such an enhanced model a full algebraic characterization of all potential paths is difficult. However, Proposition 4 offers a complete characterization of any potential stage. A matlab code to simulate potential paths (including the simplified ones that lead to the graphs in this section) is available from the authors.

no cost.[24]

However, there is a second part in the assumption not captured in this interpretation. Assumption 2 assumes that *any* initial choice comes with the *same* probability ($\hat{\rho} = 1$). We know from Proposition 3 and 4 that this is not the case and that these differences are important. However, as we show in Section 7 the difference in ρ for different levels of novelty when the incentive provision is modeled explicitly are sensitive to the funding parameters. We thus chose to deliberately ignore them for the sake of clarity. However, it is straightforward to numerically verify that Proposition 5 holds even when endogenizing $\hat{\rho}$, e.g., by explicitly modeling a large prize.

7 Application: Science Funding

In this section, we derive the feasible set of novelty-output pairs a budget-constrained funder can implement. Specifically, we ask which combination of distance to knowledge, d, and probability of discovery, ρ, a funding institution can implement. We assume that the institution has two instruments: ex ante cost reductions for the researcher (by, for example, providing grants to reduce the researcher's cost) and ex post rewards (by, for example, handing out prizes for seminal contributions).[25]

We assume that the funder respects scientific freedom and spends her budget on any combination of cost reductions and rewards. In principle, we are agnostic about the funder's objective. As we have seen in Section 6 even if we assume the funder wants to maximize the value of the knowledge created, her objective depends on the time horizon and the expected cost-type of future researchers. Instead, we characterize the feasible set that the funder can implement and discuss stylized examples of funding objectives.

Setup. Knowledge consists of a single question-answer pair, $\mathcal{F}_0 = (x_0, y(x_0))$. The researcher's cost parameter is η^0. The funder has a fixed budget K to invest into a funding scheme with two technologies: ex ante cost reductions, h, and ex post rewards, ζ. The funder's budget constraint is

$$K = \zeta + \kappa h$$

with parameter κ capturing the ratio of the marginal cost of the instruments. A cost reduction of h implies that the researcher faces the new cost parameter $\eta \equiv \eta^0 - h$. We assume $\kappa > K/\eta^0$. Receiving an ex post reward gives the researcher additional utility of ζ. We assume rewards come for "seminal contributions," that is, the more novel and difficult the problem, the larger the chances to receive the reward. We proxy that relation by a function $f(\sigma_{\mathcal{F}_k}) : \mathbb{R} \to [0, 1]$ that determines the probability

[24] Many prestigious prizes such as the John Bates Clark medal in economics or the Fields medal in mathematics come with negligible direct monetary benefits, but provide large reputation gains. See Kosfeld and Neckermann (2011) for an experimental study on prizes without monetary rewards.

[25] There is a large literature debating different forms of science funding. For recent contributions, see Price (2019) and Azoulay and Li (2020) and references therein.

of a reward. To keep the analysis simple, we assume a piecewise linear relationship

$$f(\sigma) = \begin{cases} \frac{\sigma^2}{s} & \text{if } \sigma^2 < s \\ 1 & \text{otherwise,} \end{cases}$$

for some $s \geq \max\{3q, 0.1\}$. The parameter s determines the level of difficulty that guarantees the reward.[26] The researcher's problem becomes the following:

$$\max_{d,\rho} \rho\Big(V(d;\infty) + \frac{\sigma^2(d;\infty)}{s}\zeta\Big) - \eta\tilde{c}(\rho)\sigma^2(d;\infty).$$

The feasible set. We characterize the feasible set of choices (d, ρ) that a funder can induce with a given budget. Computing the set provides a useful tool to analyze the optimal funding scheme given a particular preference relation over (d, ρ)-bundles. As in a standard consumer problem, it can be readily applied with this set being the analogue of a budget set. We provide an example below. Define $\tilde{c}_\rho(\rho) := \partial\tilde{c}/\partial\rho(\rho)$.

Definition 6. The research-possibility frontier $d(\rho; K)$ describes the largest distance a funder with budget K can implement for a given level of ρ. If $d(\rho; K)$ is increasing (decreasing) in ρ, then d and ρ are complements (substitutes) from the funder's perspective.

Proposition 6. *The set of implementable (d, ρ)-combinations for a given cost ratio κ and a budget K is described by the research-possibility frontier $d(\rho; K)$ defined over $[\underline{\rho}, \overline{\rho}]$, which are the endogenous upper and lower bounds of ρ. These bounds are determined by the extreme funding schemes $(\zeta = 0, \eta = \eta^0 - K/\kappa)$ and $(\zeta = K, \eta = \eta^0)$. The research-possibility frontier is*

$$d(\rho; K) = \min\{6q(K + s - \kappa\eta^0)\frac{\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)}{2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) - \kappa\rho}, s\}. \tag{1}$$

Novelty, d, and output, ρ, can be substitutes or complements from the funder's perspective.

Besides the characterization of the research possibility frontier, the main takeaway from Proposition 6 is that the slope of the research-possibility frontier can be positive or negative. If the slope is positive, changing the allocation of funds to increase output, ρ, implies a simultaneous increase in novelty, d—output and novelty are complements from the funder's perspective. If the slope is negative, an increase in output implies a decrease in novelty—output and novelty are substitutes from the funder's perspective.

We illustrate the feasible set for two levels of the cost parameter in Figure 13. In the left panel, cost reductions are relatively cheap. As a consequence, novelty

[26] We aim to avoid unintelligible case distinctions. Parameter s being large enough is sufficient to guarantee this. Extensions to small s are straight-forward but tedious. The crucial assumption here is that f is a bounded function, which is true whenever f is indeed a probability. If f instead was unbounded, the researcher would naturally (for any $\zeta > 0$) choose to select $d = \infty$. We select the linear form purely for convenience.

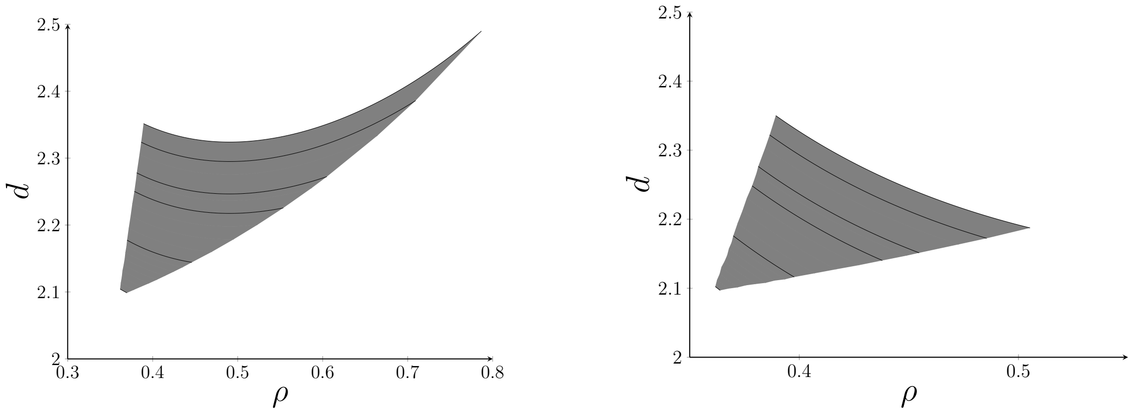


Figure 13: *Feasible set for different reward standards.* The shaded area shows the implementable (ρ, d) combinations of a funder for a given budget K. All points on a solid line require the same amount of funding, K. In both panels, the funder has a budget of $K = 6$, the reward technology $s = 100$, and the baseline cost factor is $\eta^0 = 1$. The status quo parameter has value $q = 1$. In the left panel, the relative cost $\kappa = 7$; in the right panel, $\kappa = 14$.

and output are complements for the funder whenever $\rho > \tilde{\rho} \approx 49\%$. A funder with monotone preferences in (d, ρ) would select the north-east corner which maximizes both d and ρ. The highest implementable level $\overline{\rho} \approx 82\%$. In the right panel, the relative price for cost reductions has doubled. That implies that the funder can only implement more risky strategies, the highest implementable level is $\overline{\rho} \approx 51\%$. Moreover, ρ and d are substitutes. A funder with monotone preferences therefore has to solve a trade-off between incentivizing greater output ρ and greater novelty d.

Novelty and output can become complements if ρ is large because an increase in ρ has two effects: (i) it increases the marginal benefit of distance, $V_d(d, X) + \frac{\zeta}{s}\sigma_d^2(d, X)$, by increasing the probability that the researcher finds an answer, and (ii) it increases the marginal cost of distance, $(\eta^0 - h)\tilde{c}(\rho)\sigma_d^2(d, X)$. The uncertainty of the conjecture about questions, $\sigma^2(d, X)$, is increasing in distance. Moreover, for any distance, the interval that has to be covered to find an answer with probability ρ is increasing in this probability. For certain parameter constellations, it happens that an increase in ζ by one unit increases the weight placed on the marginal-benefit effect relatively more than the resulting increase in η (by κh) increases the marginal-cost effect. As a result, d and ρ are complements. Straightforwardly, the larger κ, the stronger the increase in η when shifting one unit of the budget from cost reductions to rewards.

Incentivizing Researchers. We begin with a derivative of Corollary 2.

Corollary 6. *If $\zeta = 0$, the researcher's optimal choice satisfies $d \leq 3q$.*

The benefit of a discovery is maximized for $d = 3q$. Cost reductions reduce the friction that keep the researcher from targeting $d = 3q$, but will never incentivize her to go beyond. An immediate consequence of Corollary 6 is that if the funder wants to incentivize a moonshot as in Section 6, cost reductions alone cannot achieve that goal. However, even if the funder aims at maximizing the myopic value, the funder may need to combine the two policy instruments as the next proposition shows.

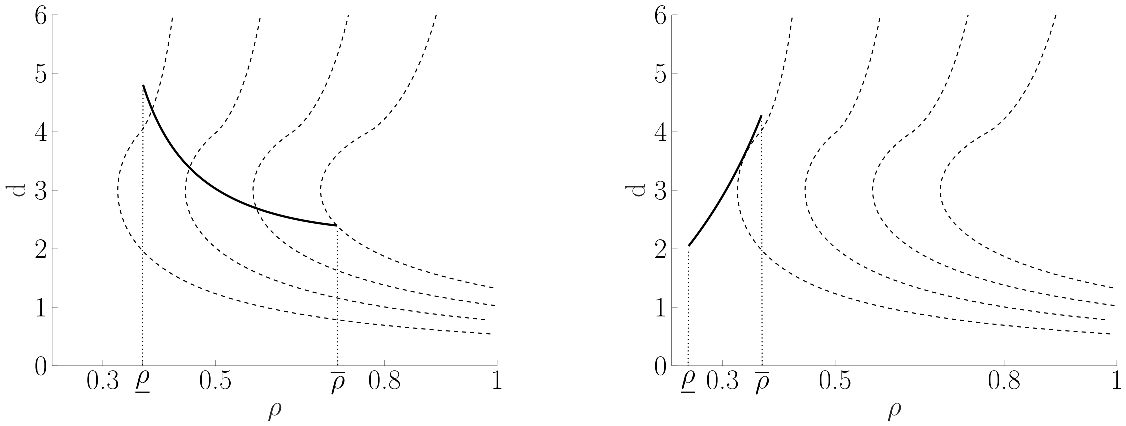


Figure 14: *Funding schemes that maximize immediate benefits.* The dashed elliptic curves depict all points that deliver the same expected value $\rho V(d;\infty)$. The solid line is the funder's budget line. In the left panel, the relative price of cost reductions is $\kappa = 4$ on the right that price is $\kappa = 20$. The optimal solution on the left is to exclusively fund through cost reductions, $h = K/\kappa$. The optimal solution on the right is to provide a funding mix between cost reductions and rewards, $(\zeta, h) > 0$. In both examples, $K = 7, s = 6, q = 1, \eta^0 = 2$.

Proposition 7. *Suppose the funder aims at maximizing the myopic expected benefit from research, $\rho V(d;\infty)$. In general, the optimal funding scheme can be to focus on one of the two instruments ($\zeta = 0, h > 0$ or $\zeta > 0, h = 0$) or a combination of the two ($\zeta > 0, h > 0$). Moreover,*

1. *there is a $\overline{K}$ such that if $K < \overline{K}$, output and novelty are complements from the funder's perspective throughout and*
   - (a) *novelty is maximized by investing all funding into ex post rewards,*
   - (b) *output is maximized by investing all funding into ex post rewards, and*
   - (c) *maximizing $\rho V(d;\infty)$ can involve a combination of the two instruments and induce a more-than-optimal degree of novelty, $d > 3q$.*
2. *if output and novelty are substitutes from the funder's perspective throughout*
   - (a) *novelty is maximized by investing all funding into ex post rewards,*
   - (b) *output is maximzied by investing all funding into ex ante cost reductions, and*
   - (c) *maximizing $\rho V(d;\infty)$ can involve a combination of the two instruments and never induces a more-than-optimal degree of novelty, $d \leq 3q$.*

Figure 14 illustrates Proposition 7. Figure 14 highlights that the fundamental difference between the case when cost reductions and rewards are substitutes and the case when they are complements. In the left panel, the two are substitutes. Moreover, the budget line is flatter than the iso-benefit curves. Despite the trade-off, the funder can increase output without much loss in novelty. The funder prefers a corner solution in which she invests her entire budget into cost reductions.

In the right panel, the relationship reverses: Novelty and output are complements. However, the funder's preferences are non-monotone in novelty. Her optimal solution involves overshooting in novelty—the novelty induced is larger than the value-maximizing $d = 3q$. The reason for such overshooting is that more novelty comes with a higher output in this case. The researcher's desire to win the award induces her to work harder on finding a solution—the output increases.

8 Relation to Existing Models

In this section, we relate our model and our findings to the two closest papers in the literature, Callander and Clark (2017) and Prendergast (2019).

Callander and Clark (2017) consider judicial decision making. Judges are interested in learning the realization of the entire path of a Brownian motion. As in our model, a decision maker (a lower court in their framework) adjusts decisions to the unknown state of the world. The decision maker's knowledge is the realizated value of the Brownian path for certain cases that a higher court reveals. Aside from the application, the main differences between our model and that in Callander and Clark (2017) are the following. First, we assume that the researcher selects, in addition to the research question, her intensity of research. We model the discovery of the realization as a costly and risky search. The effort that a researcher invests into the search affects both the cost of research and the probability of finding an answer. Second, the decision maker's and therefore the researcher's objective is to learn the exact realization of the Brownian path, in contrast to learning whether it lies above or below a threshold as in Callander and Clark (2017). Unlike in their model, the value of a discovery in our framework does not lie in how the expectations relate to the threshold but in how they relate to other known points: as the conjectures become more precise and finding an answer comparatively easy, the benefits of research shrink too. Third, in Callander and Clark (2017), it is eventually optimal to stop discovery. In our model, research never stops. Although the benefit of asking a specific question decreases when similar questions have a known answer, there is always a question worth researching.

Prendergast (2019) considers a model of creative innovation with a different focus. He studies contracts that directly condition on an agent's choice. The benefit-of-discovery function Prendergast (2019) assumes is a special case of ours in which the area length of the best available area is larger than $\widehat{X}$ (so that expanding knowledge is not beneficial) but smaller than $\widetilde{X}$ (so that selecting the midpoint is always optimal). The novelty-dependent probability-of-success function he assumes in an extension matches the properties that we derive for our endogenous probability of success. Thus, we can construct a special case of our model in which predictions coincide with a special case of his model.

Our focus is on the microfoundation of the functions while his is on agency concerns in a reduced-form model. Yet, in terms of modeling approaches our model of knowledge differs at least along two crucial dimensions. First, we assume that it is possible to expand research beyond the frontier and show that it can also be optimal to do so. Prendergast (2019) instead assumes that research always takes place between two existing findings. Second, our decision maker has an outside option that limits her expected losses if conjectures are too imprecise. Existence of an outside option implies that there are bounds on the benefit of novelty. Once newly created areas become too large, conjectures in that area become too imprecise which mitigates the value of the area.[27] Therefore, and different from Prendergast (2019), we obtain a nonmonotonicity in the value of novelty.

Finally, Prendergast (2019) assumes an exogenous cost that maps the likelihood of obtaining a solution to a cost of research. In an extension, he assume that this mapping depends exogenously on the distance to knowledge. We assume that cost are exogenous at a lower level: the effort that the researcher invests into the search for an answer is costly with a quadratic functional form. Starting from that assumption, we derive an *endogenous* mapping from the probability of successful discovery, ρ, the distance of the research question to existing knowledge, d, and the size of the area the question lies in, X, to the cost of research.

9 Final Remarks

We propose and study a tractable model of knowledge and research. The starting points of our model are that (i) finding the answer to one question spills over onto the conjectures about other questions; (ii) questions in close proximity to existing knowledge are easier to answer than questions that are far away from existing knowledge; (iii) the benefit of a discovery depends on its effect on decision making; and (iv) researchers are motivated by this benefit but bear a cost of searching for an answer.

Building on these four elements, we derive the benefit and the cost of research and characterize the researcher's optimal choice. The characterization reveals that novelty and research output are non-trivially linked with each other and the existing knowledge a researcher builds on.

We apply our model of research to address to two classical questions in the economics of science funding. First, we show that it may be optimal for society to incentivize moonshots—highly novel discoveries. While moonshots are myopically suboptimal, they provide guidance for future discoveries. Thus, the evolution of knowledge suffers from a dynamic externality when researchers do not take the effect of their discoveries on future generations of researchers into account. Properly chosen moonshots induce an increase in both the research productivity and the value of knowledge generated in future periods.

Our results are in line with recent empricial work—for example, Rzhetsky et al. (2015)—that analyzes the impact of findings on future developments. That work suggests that scientists choose a dynamically suboptimal strategy when selecting their research questions. Rzhetsky et al. (2015) identifies researcher myopia as one of the drivers of that observation. While our model is consistent with this sentiment, we also raise a note of caution. Whether incentivizing moonshots is beneficial, depends crucially on the current state of knowledge and moonshots should be chosen carefully—too much novelty can hurt the evolution of knowledge.

Second, we study the interaction of two potential instruments a funding institution

[27] That result also formalizes the argument in Price (2020) that focusing on providing incentives that maximize novelty can backfire as knowledge may consist of disconnected islands. In our model, research that is too far from the current body of knowledge has little positive spillovers on society's conjectures about unanswered questions.

can use to incentivize researchers: an ex-ante reduction of the cost of research or an ex-post reward for novel discoveries. We provide a full characterization of the set of feasible novelty-output pairs that a budget-constrained funder can implement and illustrate that the optimal funding mix can be found using standard tools of consumer theory.

Another dimension present in the analysis of Rzhetsky et al. (2015) is to consider research as establishing links between questions. In contrast to that, the implicit assumption in our model is that links are ex-ante known and given by a line network. A natural direction for future research is to combine our conceptualization of the search for answers with the task of discovering the network of questions.

Our paper starts by emphasizing the role of scientific freedom. Preserving that freedom remains a challenging task for science-funding institutions when designing a funding architecture that provides researchers the support to engage in research activities that might not be undertaken otherwise. The NSF emphasizes that it aims at funding high-risk/high-reward research to advance the knowledge frontier. Our findings in Section 6 and 7 illustrate two particular trade-offs that funding institutions face in that context. Thus, while the question of optimal market design is beyond the scope of this paper, we hope that our modeling framework will serve as a stepping stone toward developing a structural model of science funding which allows an evaluation of funding incentives and provides meaningful counterfactuals to inform decision makers about the optimal provision of research incentives.

Appendix

A Notation and Properties of $\tilde{c}$

Notation: We use argument subscripts to denote the partial derivatives with respect to the argument. We omit function argument whenever it is convenient. we use the notation $\frac{\mathrm{d}f(x,y)}{\mathrm{dx}}$ to indicate the total derivative $(f_x + f_y y_x)$.

Properties of $\tilde{c}$. Some of the proofs rely on the properties of the inverse error function or more specifically the representation $\tilde{c}(\rho) = (erf^{-1}(\rho))^2$. The function $\tilde{c}(\rho)$ is convex and increasing on $[0,1)$ with $\tilde{c}(0) = 0$ and $\lim_{\rho\to 1}\tilde{c}(\rho) = \infty$.[28] The derivative

$$\tilde{c}_\rho(\rho) = \sqrt{\pi}e^{\tilde{c}(\rho)}erf^{-1}(\rho)$$

is increasing and convex with the same limits.

We make use of the fact that for $\rho \in (0,1)$ $\tilde{c}(\rho)$ has a convex and increasing elasticity bounded below by 2 and unbounded above. Its derivative $\tilde{c}_\rho(\rho)$ has an increasing elasticity bounded below by 1 and unbounded above. We want to

[28]Due to this limit and the researcher's ability to choose $\rho = 1$, we augment the support of the cost function to include $\rho = 1$ with $\tilde{c}(1) = \infty$. However, the optimal ρ is always strictly interior unless the cost parameter η is chosen to be zero in which case we assume that $\eta\tilde{c}(\rho = 1) = 0$.

emphasize that these properties are not special to our quadratic cost assumption. To the contrary, $erf^{-1}(x)^k$ for any $k \geq 2$ admits similar properties with only the lower bounds changing. Formally, the following properties are invoked in the proofs:

$$\begin{aligned}
\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} &\in (2,\infty) \text{ and increasing,}\\
\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)} &\in (1,\infty) \text{ and increasing,}\\
\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho) &\in (0,\infty) \text{ and increasing,}\\
\tilde{c}_\rho^{-1}(x) &= erf\left(\sqrt{\frac{W(2x^2/\pi)}{\pi}}\right).
\end{aligned}$$

with $W(\cdot)$ the principal branch of the Lambert W function.

B Proofs

At various points we make use of inequality relations the proof of which we relegate to supplementary Appendix D. In each of these cases proving the inequalities is done via straightforward algebra that produces little additional insight.

B.1 Proof of Proposition 1

Proof. The value of knowing $\mathcal{F}_k$ is

$$\int \max\left\{\frac{q-\sigma_x^2(y|\mathcal{F}_k)}{q}, 0\right\} \mathrm{d}x.$$

No matter which point of knowledge $(x, y(x))$ is added to $\mathcal{F}_k$, the value of knowledge outside the frontier is identical for both $\mathcal{F}_k$ and $\mathcal{F}_k \cup (x, y(x))$. Area lengths $X_1 = X_k = \infty$ do not depend on $\mathcal{F}_k$ and neither does the variance for a question $x < x_1$ or $x > x_k$ with a given distance d to $\mathcal{F}_k$. The conjectures about all questions outside $[x_1, x_k]$ deliver a total value of

$$2\int_0^q \frac{q-x}{q}\mathrm{d}x = q,$$

which is independent of $\mathcal{F}_k$. Moreover, answering question $\hat{x} \in [x_i, x_{i+1}]$ with $(x_i, y(x_i)), (x_{i+1}, y(x_{i+1})) \in \mathcal{F}_k$ only affects questions in the area $[x_i, x_{i+1}]$, i.e., $G(x|\mathcal{F}_k) = G(x|\mathcal{F}_k \cup (\hat{x}, y(\hat{x})) \;\forall\; x \notin (x_i, x_{i+1})$.

To simplify notation, let us consider the points in terms of distance to the lower bound of the area with X, $d \equiv x - x_i$.

The value of the area $[x_i, x_{i+}]$ is (with abuse of notation)

$$v(X) = \int_0^X \max\left\{\frac{q - \frac{d(X-d)}{X}}{q}, 0\right\} \mathrm{d}d.$$

Note that whenever $X \leq 4q$, $\frac{d(X-d)}{X} \leq q$. Hence, we can directly compute the value of any area with length $X \leq 4q$ as

$$v(X) = X - \frac{X^2}{6q}.$$

Whenever $X > 4q$, value is only generated on a subset of points in the area. As the variance is a symmetric quadratic function with $X/2$ as midpoint, there is a symmetric area centered around $X/2$ which has a variance exceeding q. The points with variance equal to q are given by $\overline{d}_{1,2} = \frac{X}{2} \pm \frac{1}{2}\sqrt{X}\sqrt{X-4q}$. Hence, the value of an area with $X > 4q$ is (due to symmetry)

$$\begin{aligned} v(X) &= 2\int_0^{\overline{d}_1} \frac{q - \frac{d(X-d)}{X}}{q} \mathrm{d}d \\ &= X - \frac{X^2}{6q} + \frac{X-4q}{6q}\sqrt{X}\sqrt{X-4q}. \end{aligned}$$

If knowledge expands beyond the frontier, a new area is created and no area is replaced. The value created is thus

$$V(d;\infty) = v(d) = d - \frac{d^2}{6q} + \begin{cases} 0, & \text{if } d \leq 4q \\ \frac{d-4q}{6q}\sqrt{d}\sqrt{d-4q}, & \text{if } d > 4q. \end{cases}$$

If a knowledge point is added inside an area with length X with distance d to the closest existing knowledge, it generates two new areas with length d and $X-d$ that replace the old area with length X. The total value of the two intervals new is

$$\begin{aligned} v(d) + v(X-d) = d - \frac{d^2}{6q} &+ \begin{cases} 0, & \text{if } d \leq 4q \\ \frac{d-4q}{6q}\sqrt{d}\sqrt{d-4q}, & \text{if } d > 4q \end{cases} \\ +(-\frac{(X-d)^2}{6q}) + X - d &+ \begin{cases} 0, & \text{if } X-d \leq 4q \\ \frac{X-d-4q}{6q}\sqrt{X-d}\sqrt{X-d-4q}, & \text{if } X-d > 4q \end{cases}. \end{aligned}$$

The benefit of discovery is then $V(d;X) = v(d) + v(X-d) - v(X)$. Noticing that $\sigma^2(d;X) = d(X-d)/X$ and replacing accordingly results in the expression from the proposition. Taken the limit of $X \to \infty$ corresponds to the value of expanding reasearch beyond the frontier. $\square$

B.2 Proof of Corollary 2

Proof. The first-order condition for $d \leq 4q$ is

$$\frac{\partial V(d;\infty | d \leq 4q)}{\partial d} = -\frac{d}{3q} + 1 = 0.$$

Moreover, the benefit is decreasing in d for $d > 4q$ which can be seen from the derivative with respect to d which is

$$\frac{\partial V(d;\infty|d>4q)}{\partial d} = -\frac{d}{3q} + 1 + \sqrt{\frac{d-4q}{d}}\frac{d-q}{3q} < 0.$$

The inequality holds by Lemma 21 in Appendix D. □

B.3 Proof of Corollary 3

We prove Corollary 3 via a series of lemmata.

- In Lemma 1 and 2 we first show that the distance that maximizes deepening knowledge is $d^0(X) = X/2$ for small $X$and $d^0(X) < X/2$ for large X.
- Lemma 3 shows that $d^0(X) < X/2$ implies decreasing benefits in X.
- Lemma 4 shows that once $d^0(X) < X/2$ for some X it is true for all $X' > X$ and thus establishes $\widetilde{X}$.
- Lemma 5 shows our convergence and $d^0(X > 6q) > 3q$.
- Lemma 6 and 7 establishe singl peakedness and determine $\check{X}$ and $\widehat{X}$.
- Lemma 8 determines the order of the cutoffs.

Throughout, we refer to the distance d that maximizes $V(d;X)$ as $d^0(X)$.

Proof.

Lemma 1. *$d^0(X) = X/2$ if $X \leq 6q$.*

Proof. **1. Assume** $X \leq 4q$.

The benefits of discovery are

$$V(d;X|X\leq 4q) = \frac{1}{3q}(Xd - d^2)$$

which is increasing in d and hence maximized at $d = X/2$. Moreover, $V(X/2;X) = X^2/(12q)$ which is increasing in X.

2. Assume $X \in (4q, 6q]$

(i) $d \geq X - 4q$ implies (since $d \leq 3q$)

$$V(d;X|d\geq X-4q, X\in(4q,6q])) = \frac{1}{6q}\left(2dX - 2d^2 - \sqrt{X}(X-4q)^{3/2}\right)$$

which is the same as in the first case up to constant $-\sqrt{X}(X-4q)^{3/2}$. Thus the optimal d conditional on $d \geq X - 4q$ is $d = X/2$.

(ii) For $d \leq X - 4q$ the benefit becomes

$$V(d;X|d\leq X-4q, X\in(4q,6q])) = \\ \frac{1}{6q}\left(2dX - 2d^2 + \sqrt{X-d}(X-d-4q)^{3/2} - \sqrt{X}(X-4q)^{3/2}\right),$$

with derivative

$$V_d = \frac{1}{3q}\left(X - 2d - (X - d - q)\sqrt{\frac{X - d - 4q}{X - d}}\right).$$

which is positive for $d \leq X - 4q, X \in (4q, 6q]$ by Lemma 22 from Appendix D. Hence, $V_d(d; X|d \leq X - 4q, X \in [4q, 6q]) > 0$ for all d and X in the considered domain. Thus, $d = X - 4q$ maximizes $V(d; X|d \leq X - 4q, X \in (4q, 6q]))$ and hence $d = X/2$ maximizes $V(d; X|X \in (4q, 6q])$. □

Lemma 2. *If $X > 8q$ then $d^0(X) \neq X/2$.*

Proof. Take $\overline{d} = 4q < X/2$. That implies

$$V(\overline{d}; X|X > 8q) = \frac{1}{6}\left(8Xq - 32q^2 - \sqrt{X}(X - 4q)^{3/2} + \sqrt{(X - 4q)}(X - 8q)^{3/2}\right).$$

By comparison

$$V(X/2; X|X > 8q) = \frac{1}{6}\left(\frac{X^2}{2} - \sqrt{X}(X - 4q)^{3/2} + \frac{1}{2}\sqrt{X}(X - 8q)^{3/2}\right)$$

The difference of the two is thus

$$\begin{aligned} &V(\overline{d}; X|\cdot) - V(X/2; X|\cdot) \\ &\qquad = \frac{1}{6q}\left(\sqrt{X - 4q}(X - 8q)^{3/2} - \frac{\sqrt{X}}{2}(X - 8q)^{3/2} - \frac{(X - 8q)^2}{2}\right) \\ &\qquad = \frac{1}{6}\frac{(X - 8q)^{3/2}}{2}\left(2\sqrt{X - 4q} - \sqrt{X} - \sqrt{(X - 8q)}\right), \end{aligned}$$

which is positive if

$$4(X - 4q) > 2X - 8q \Leftrightarrow X > 4q$$

and holds by assumption. □

Lemma 3. $d^0(X) < X/2 \Rightarrow \frac{dV(d^0(X);X)}{dX} < 0.$

Proof. By the envelope theorem,

$$\frac{dV(d^0(X); X)}{dX} = V_X(d^0(X); X).$$

This derivative is negative for $X \geq 4q$ and for all $d \in [0, X - 4q]$ by Lemma 23 in Appendix D. If $X \geq 8q$ that claim is sufficient. By Lemma 1 we know that $X \geq 6q$ whenever $d^0(X) \neq X/2$. In 2.(i) in the proof of Lemma 23, page 64, we show that $V_d > 0$ for $d \in [X - 4q, X/2)$ if $X \leq 8q$. Hence if $d^0(X) \neq X/2$, then $d^0(X) \leq X - 4q$ and the inequality proved in Lemma 23 proves the lemma. □

Lemma 4. *$d^0(X) < X/2$ for some $X \in [6q, 8q] \Rightarrow d^0(X) < X/2$ for all $X' > X$, $X' \in [6q, 8q]$.*

Proof. We prove the claim by showing that $V(d^0(X);X)$ for $d(X|X<2)<X/2$ cuts $V(X/2;X)$ from below at *any* intersection point. Thus, there is at most one switch from $d^0(X)=X/2$ to $d^0(X)<X/2$ and no switch back. By continuity that implies the statement.

$V(d;X)$ is a continuously differentiable function in X and d. Thus any interior (local) optimum $d^0(X)$ is continuous as well and so are $V(d^0(X);X)$ and $V(X/2;X)$. We now show that if $V(d^0(X);X)=V(X/2;X)$ for some $d^0(X)<X/2$ and $X\in[6q,8q]$, then $\mathrm{d}V(d^0(X);X)/\mathrm{d}X>\mathrm{d}V(X/2;X)/\mathrm{d}X$. Note that $\mathrm{d}V(d^0(X),X)/\mathrm{d}X<0$ by Lemma 3. The first intersection therefore can occur only in a region when $V(X/2,X)$ is decreasing and must be such that $\mathrm{d}V(X/2,X)/\mathrm{d}X<\mathrm{d}V(d^0(X),X)/\mathrm{d}X$. We prove that this is the only potential intersection in Lemma 24 in Appendix D where we show that $\mathrm{d}^2V(X/2,X)/(\mathrm{d}X)^2<0$ and $\mathrm{d}^2V(d^0(X),X)/(\mathrm{d}X)^2>0$. □

Lemma 5. *$V(d^0(X);X)$ is continuous in X. As $X\to\infty$, it converges uniformly to $V(d;X)$ and $d^0(X)\to d^0(\infty)$. For any $X>6q$ we have $d^0(X)>3q$ and $V(d^0(X),X)>V(3q,\infty)$.*

Proof. Continuity follows because $V(d^0(X);X)=\max_d V(d;X)$ with $V(d;X)$ continuous in both $d\in[0,X/2]$ and X. Now take any sequence of increasing X_n with $\lim_{n\to\infty}X_n=\infty$. For any $\delta(d),\exists n$ such that $V_n(d;X)-V(d;\infty)<\delta(d)$. Hence $V(d;X)$ converges uniformly to $V(d;\infty)$. By uniform convergence the maximizer $d^0(X)$ converges too. To see convergence from above observe that $V(3q;X)>V(3q;\infty)$ for any $6q<X<\infty$.

Finally, from Corollary 2 and the proof of Proposition 1 we know that $V(d;\infty)$ describes the value of an area of length d. That value is increasing for $d<3q$ and decreasing for $d>3q$. Now suppose $X>6q$ and $d^0(X)<3q$. Then by increasing d both areas created become closer to $3q$ and are thus increasing in value. A contradiction to $d^0(X)$ being the maximizer. □

Lemma 6. *$V(d^0(X);X)$ is single peaked with an interior peak.* $\check{X}\approx 6.204q$

Proof. Follows from continuity of $V(X/2;X)$ (by Lemma 5) and Lemma 1 to 4.

The peak can be computed. It is the solution to

$$\frac{X}{X-q}=2\frac{\sqrt{X-4q}}{\sqrt{X}}. \tag{2}$$

Replacing $X\equiv mq$ the above reduces to

$$\frac{m}{m-1}>2\sqrt{\frac{(m-4)}{m}}$$

For $m>4$, the LHS decreases in m while the RHS increases in m for $m>4$. Moreover, both sides are equal at $m\approx 6.204$. □

Lemma 7. $\widehat{X}\approx 4.338q$.

Proof. $V(3q;X)>V(3q;\infty)$ for $X>6q$. For $X\in[0,6q]$ all we need to consider is $d^0(X)=X/2$ by Lemma 6. We compare

$$V(X/2;X) = \frac{X^2}{12q} - \frac{\sqrt{X}(X-4q)^{3/2}}{6q}$$

with $V(3q;\infty) = \frac{3q}{2}$. Replacing $X \equiv \ell q$ and simplifying, the two intersect if

$$q\left(\frac{\ell^2}{12} - \frac{\sqrt{\ell}}{6}(\ell-4)^2 - 3/2\right) = 0$$

which has a unique solution for $\ell < 6$ with $\ell \approx 4.338$. □

Lemma 8. $4q < \widehat{X}^0 < 6q < \check{X}^0 < \widetilde{X}^0 < 8q$.

Proof. The first two inequalities follow from Lemma 7, the third from Lemma 6. The fourth follows from Lemma 1. Lemma 4 implies that $\widetilde{X}^0$ exists and Lemma 2 provides the last inequality. □

□

B.4 Proof of Proposition 2

Proof. The normal distribution is symmetric around the mean with a density decreasing in both directions, it follows directly that the smallest interval that contains the realization with a particular likelihood is centered around the mean.

Take an interval of length $Z < \infty$ that is symmetric around the mean μ and assume a total mass of ρ is inside the interval. Then $(1-\rho)/2$ lies to the left of the interval by the symmetry of the normal. Moreover, the left bound z_l of the interval has (by symmetry of the interval) distance $\mu - Z/2$ from the mean. From the properties of normal distributions

$$\Phi(z_l) = 1/2\left(1 + erf\left(\frac{z_l - \mu}{\sigma\sqrt{2}}\right)\right) = 1/2\left(1 + erf\left(\frac{-Z/2}{\sigma\sqrt{2}}\right)\right).$$

Solving (using symmetry of erf)

$$1/2\left(1 - erf\left(\frac{Z}{\sigma 2^{3/2}}\right)\right) = \frac{1-\rho}{2}$$

or equivalently

$$erf\left(\frac{Z}{\sigma 2^{3/2}}\right) = \rho$$
$$\Leftrightarrow Z = 2^{3/2} erf^{-1}(\rho)\sigma.$$

□

B.5 Proof of Proposition 3

Proof. The optimal level of ρ is determined by the first order condition

$$\eta\tilde{c}_\rho(\rho) = \frac{V(d;X)}{\sigma^2(d;X)}. \tag{3}$$

The left-hand side is increasing in ρ for any ρ and independent of d. Thus, to determine whether ρ and d are substitutes or complements, we need to determine whether the right-hand side of this equation is increasing or decreasing.

First, consider the expanding area. In this case,

$$\frac{V(d;X)}{\sigma^2(d;X)} = \frac{1}{6q}\left(-d + 6q + \mathbf{1}_{d>4q}\frac{(d-4q)^{3/2}}{\sqrt{d}}\right) \tag{4}$$

which has derivative

$$\frac{1}{6q}\left(-1 + \mathbf{1}_{d>4q}\frac{(d+2q)\sqrt{d-4q}}{d^{3/2}}\right) < 0. \tag{5}$$

Thus, output and novelty are substitutes in the expanding area.

Next, consider deepening knowledge when $X < 4q$. In this case,

$$\frac{V(d;X)}{\sigma^2(d;X)} = \frac{2X}{6q}. \tag{6}$$

Thus, output and novelty are independent in short research areas.

Suppose $X \in (4q, 8q)$ and $d < 4q$ and $X - d < 4q$. In this case,

$$\frac{V(d;X)}{\sigma^2(d;X)} = \frac{1}{6q}\left(2X - \frac{\sqrt{X}(X-4q)^{3/2}}{\sigma^2(d;X)}\right) \tag{7}$$

with derivative

$$\sigma_d^2(d;X)\frac{\sqrt{X}(X-4q)^{3/2}}{\sigma^4(d;X)} > 0. \tag{8}$$

Thus, output and novelty are complements in intermediate research areas with high degrees of novelty.

Suppose $X \in (4q, 8q)$ and $d < 4q$ and $X - d > 4q$. In this case,

$$\frac{V(d;X)}{\sigma^2(d;X)} = \frac{1}{6q}\left(2X - \frac{\sqrt{X}(X-4q)^{3/2}}{\sigma^2(d;X)} + \frac{\sqrt{X-d}(X-d-4q)^{3/2}}{\sigma^2(d;X)}\right) \tag{9}$$

with derivative

$$\sigma_d^2(d;X)\left(\frac{\sqrt{X}(X-4q)^{3/2}}{\sigma^4(d;X)} - \frac{\sqrt{X-d}(X-d-4q)^{3/2}}{\sigma^2(d;X)}\right) - \frac{2(x-d-q)x\sqrt{x-d-4q}}{d(x-d)^{3/2}}. \tag{10}$$

Note that evaluated at $d = 0$ and $d = X - 4q$, this derivative is

$$\frac{\partial}{\partial d}\left(\frac{V(d;X)}{\sigma^2(d;X)}\right)\Bigg|_{d=0} = -\frac{X^2 - 8qX - 10q^2}{X^{3/2}\sqrt{X-4q}} \tag{11}$$

$$\frac{\partial}{\partial d}\left(\frac{V(d;X)}{\sigma^2(d;X)}\right)\Bigg|_{d=X-4q} = -X^{3/2}\frac{X-8q}{16q^2\sqrt{X-4q}} > 0. \tag{12}$$

Thus, for $d \to X - 4q$, output and novelty are always complements. However, for $d = 0$, the derivative is positive (negative) if $X < (>)\frac{5}{2-\sqrt{3/2}}$. Moreover, the derivative has at most one root for $d \in (0, X - 4q)$ which is given by

$$\hat{d} = \frac{1}{X-6q}\left(2(X^2 - 6qX + 6q^2) - (X-2q)\sqrt{\frac{18q^2 - 8qX + X^2}{2}}\right). \tag{13}$$

This root exists only for $X > \frac{5}{2-\sqrt{3/2}}$.

Thus, output and novelty are substitutes for small d when $X > \frac{5}{2-\sqrt{3/2}}$ and complements for large d. They are complements when $X < \frac{5}{2-\sqrt{3/2}}$.

Finally, whenever $X > 8q$ and $d < 4q$, we are in the same case as above for large X only that the root does not exist for $d < 4q$ and output and novelty are substitutes. When $d > 4q$, the roots of the derivative is outside $[0, X/2]$

$$\hat{d} = \frac{X(1296q^4 - 1152q^3X + 384q^2X^2 + 3X^4}{2(X-6q)^2(36q^2 - 20qX + 3X^2)}$$
$$\pm\frac{(12q^2 - 8qX + X^2)\sqrt{(68q^2 - 28qX + 3X^2)(36q^2 - 20qX + 3X^2)}}{2(X-6q)^2(36q^2 - 20qX + 3X^2)}$$

and at $d = 4q$, the derivative is negative.[29] This concludes the proof. □

B.6 Proof of Proposition 4

We prove Proposition 4 via a series of lemmata.

- Lemma 9 proves existence of an optimum for any area length.
- Lemma 10 characterizes the choices for expanding research.
- Lemma 11 and 12 establishes a local maximum at $\check{X}$.
- Lemma 13 establishes $\widetilde{X}$.
- Lemma 14 shows that $\check{X}$ is also a global maximum.
- Lemma 15 establishes $\dot{X}$.
- Lemma 16 establishes $\widehat{X}$.
- Lemma 17 derives the order of the cutoffs.

At the end of this part we provide a (technical) corollary useful in later results.

Proof.

[29] A Mathematica file verifying the computations is available from the authors.

Lemma 9. *There is a non-trivial optimal choice with $\infty > d > 0, 1 > \rho > 0$ on any interval with positive length, $X \in (0, \infty)$.*

Proof. Recall that the researcher can always guarantee a non-negative payoff by choosing either $d = 0$ or $\rho = 0$. Hence, the researcher's value is bounded from below, $U_R(X) \equiv \max_{d,\rho} u_R(d, \rho; X) \geq 0$. Next, note that $u_R(\rho = 0, d > \varepsilon; X) = 0$ for some small $\varepsilon > 0$ and that $\frac{\partial u_R(\rho=0,d>\varepsilon;X)}{\partial \rho} = V(\varepsilon, X) > 0$ by Proposition 1. Hence, on any interval X there is a maximum with $d > 0, \rho > 0$.

Moreover, by Corollary 3 the value of knowledge is bounded $V(d, X) \leq M < \infty$ and $\lim_{\rho \to 1} \tilde{c}(\rho) = \infty$. Therefore the optimal $\rho < 1$. Finally, $V(d, \infty)$ is decreasing in d for d large enough while the cost $\eta \tilde{c}(\rho)\sigma^2(d, \infty)$ is increasing in d. Hence, the optimal distance is bounded $d \leq D < \infty$. □

Lemma 10. *Beyond the frontier, $X = \infty$, the optimal choice is characterized by the first-order conditions (FOCs). The FOCs are sufficient and the optimal $d^\infty \in (2q, 3q)$. The researcher's value is strictly positive $U_R(X = \infty) > 0$.*

Proof. Fix any $\rho \geq 0$. Since $\sigma^2(d; \infty)$ is increasing it is immediate that the researcher's utility is non-increasing in d if $V(d; \infty)$ decreases in d. Thus, it is sufficient to restrict attention to $d \leq 3q$.

By Lemma 9, the researcher's optimal choice is interior and, hence, characterized by the first-order conditions. That the value is positive follows immediately from the choice being strictly interior and $X > 0$ by Lemma 9. To see sufficiency of the first-order conditions with a unique solution note first that the first principal minor of Hessian is $\rho V_{dd} - \eta c \sigma^2_{dd} = -\rho \frac{1}{3q} < 0$ as $\sigma^2_{dd} = 0$ and the second is given by the determinant of the Hessian which is positive

$$
\begin{aligned}
& -\rho V_{dd} \eta \tilde{c}_{\rho\rho}(\rho)\sigma^2(d; \infty) - (V_d - \eta \tilde{c}_\rho(\rho)\sigma^2_d(d; \infty))^2 \\
=& \rho \frac{1}{3q} \eta \tilde{c}_{\rho\rho}(\rho) d - \left(-\frac{d}{3q} + 1 - \eta \tilde{c}_\rho(\rho) \right)^2 \\
=& \rho \frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)} \frac{V(d; \infty)}{3q} - \left(-\frac{d}{3q} + 1 - \frac{V(d; \infty)}{\sigma^2(d; \infty)} \right)^2
\end{aligned}
\tag{14}
$$

where the last equality follows from substituting using the first-order conditions

$$\rho V_d(d; \infty) - \eta \tilde{c}(\rho)\sigma^2_d(d; \infty) = 0 \tag{15}$$

$$V(d; \infty) - \eta \tilde{c}_\rho(\rho)\sigma^2(d; \infty) = 0; \tag{16}$$

in particular, $\eta \sigma^2(d; \infty) = \frac{V(d;\infty)}{\tilde{c}_\rho(\rho)}$, $\eta \tilde{c}_\rho(\rho) = \frac{V}{\sigma^2}$. Rearranging the last term of equation (14) and substituting for V under the assumption that $d \leq 3q$.

$$\rho \frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_{\rho}(\rho)} > \left(-\frac{d}{3q} + 1 - \frac{\frac{-d^2}{6q} + d}{d}\right)^2 \frac{3q}{-\frac{d^2}{6q} + d}$$
$$\Leftrightarrow 2\rho \frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_{\rho}(\rho)} > \frac{d}{(6q - d)}.$$

where the inequality follows by the properties of $\tilde{c}(\rho)$ from Appendix A implying $LHS \geq 2$ and the observation that $RHS \in [0, 1]$ for $d \leq 3q$. Distance $3q$ is an upper bound for every critical d because the d-first-order condition yields

$$d^{\infty} = 3q \left(1 - \eta \frac{\tilde{c}(\rho)}{\rho}\right) < 3q. \tag{17}$$

Replacing η via equation (16) and solving for d we obtain

$$d^{\infty} = 3q \left(1 - \frac{\tilde{c}(\rho)}{2\tilde{c}_{\rho}(\rho)\rho - \tilde{c}(\rho)}\right) \in (2q, 3q)$$

where the lower bound follows from the properties of the error function. □

Lemma 11. *Fix $d = X/2$ and a assume that an interior optimum exists. Then $U_R(X|d = X/2)$ is maximal only if the total differential $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} \geq 0$.*

Proof. Under the assumption $d = X/2$, $U_R(X)$ is defined and continuously differentiable for all $X \in [0, \infty)$ despite the indicator functions. Because $X = 0$ implies $U_R(X = 0) = 0$ and Lemma 9 holds, there is an interior X at which $U_R(X)$.

Then, if $U_R(X)$ is maximal for some interior and differentiable X it needs to satisfy

$$\frac{\partial U_R}{\partial X} = 0$$

By assumption we have $d(\check{X}) = X/2$ and the first order condition with respect to ρ holds. At an interior and differentiable X we need that

$$\rho \frac{\mathrm{d}V(d = X/2; X)}{\mathrm{d}X} = \frac{\eta}{4}\tilde{c}(\rho).$$

The right hand side is non-negative, which implies the desired result.[30] □

Lemma 12. *The value of the deepening boundary solution $U_R(d \equiv \frac{X}{2}; X)$ peaks in X at $\check{X} \in (4q, \check{X}^0]$.*

Proof. Note that $U_R'(d \equiv X/2; X) > 0$ for $X \in [0, 4q]$ as in this case $U_R(d \equiv X/2, X) = \rho\frac{X^2}{12q} - \eta\tilde{c}(\rho)\frac{X}{4}$ and, hence, $U_R'(d \equiv X/2, X) = \rho\frac{X}{6q} - \eta\tilde{c}(\rho)\frac{1}{4}$. Using

[30] The inequality is weak as for $\eta = 0$, $\rho(X) = 1$ and $U_R(X) = V(X)$.

optimality of ρ via the FOC

$$\frac{X}{3q} = \eta\tilde{c}_\rho(\rho) \Rightarrow \frac{X}{6q} = \frac{\eta\tilde{c}_\rho(\rho)}{2}$$

which yields

$$\begin{aligned} U_R'(X) =& \rho\frac{\eta\tilde{c}_\rho(\rho)}{2} - \eta\tilde{c}(\rho)\frac{1}{4} \\ =& \frac{\tilde{c}_\rho(\rho)}{4}\rho\eta\left(2 - \frac{\tilde{c}(\rho)}{\rho\tilde{c}_\rho(\rho)}\right) > 0 \end{aligned}$$

where the inequality follows again from the properties of $\tilde{c}(\rho)$.

Moreover, $U_R(X)$ is strictly concave on $[4q, 8q]$ as $V(d = X/2, X)$ is concave on this interval (see Appendix B.3) and $\sigma^2_{XX}(d = X/2, X) = 0$ implying[31]

$$U_R''(X) = \rho\frac{\mathrm{d}\mathrm{d}V(d = X/2; X)}{\mathrm{d}X\mathrm{d}X} < 0.$$

For $X > \check{X}^0$, $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} < 0$ by the definition of $\check{X}^0$. By Lemma 11, it follows that the maximizing $X \in (4q, \check{X}^0]$ □

Lemma 13. *The researcher's optimal choice of distance is on the midpoint of the area, $d = \frac{X}{2}$, for $X \leq \widetilde{X}$ and interior, $d < \frac{X}{2}$, for $X > \widetilde{X}$ with $\widetilde{X} > \check{X}$. It converges from to d^∞, $\lim_{X\to\infty} d(X) = d^\infty$. Any optimal distance choice satisfies $d \leq 4q$.*

Proof. Note first that the choice $d = \frac{X}{2}$ always constitutes a local maximum as the marginal cost of distance is zero at this point, $\frac{\partial\sigma^2(d,X)}{\partial d} = 1 - \frac{2d}{X}$, and, for any choice of d, there is a unique ρ that solves the first-order condition with respect to ρ because the first-order condition with respect to ρ for any d, $\frac{V(d,X)}{\sigma^2(d,X)} = \eta\tilde{c}_\rho(\rho)$, has a continuous, strictly increasing, unbounded right-hand side that starts at $\tilde{c}_\rho(0) = 0$ and a constant left-hand side. Hence, the boundary solution with $d = \frac{X}{2}$ is always a candidate solution.

We first show that for $X \leq 4q$, the optimal choice will always be the boundary solution with $d = X/2$

The first-order conditions for an interior solution are given by

$$\begin{aligned} \rho V_d(d, X) - \eta\tilde{c}(\rho)\sigma^2_d(d, X) = 0 \\ V(d, X) - \eta\tilde{c}_\rho(\rho)\sigma^2(d, X) = 0. \end{aligned}$$

Replacing η from the second (ρ's) first-order condition in the first (d's) first-order

[31] Note that we totally differentiate the value twice and all $\rho'(X)$ and $\rho''(X)$ terms drop out by optimality of ρ by applying the first-order condition directly and total differentiation of the first-order condition.

condition, we obtain

$$\frac{\frac{V_d(d,X)}{\sigma_d^2(d,X)}}{\frac{V(d,X)}{\sigma^2(d,X)}} = \frac{\frac{\tilde{c}(\rho)}{\rho}}{\tilde{c}_\rho(\rho)}.$$

It follows from the properties of $\tilde{c}(\rho)$ that the $RHS \in [0, 1/2]$ and decreasing. Thus, whenever the $LHS > 1/2$ for all ρ, the boundary choice $d = \frac{X}{2}$ will be optimal. For $X \leq 4q$

$$\frac{\frac{V_d}{\sigma_d^2}}{\frac{V}{\sigma^2}} = \frac{\frac{2(X-2d)}{\frac{X-2d}{X}}}{\frac{2(dX-d^2)}{\frac{d(X-d)}{X}}} = 1.$$

Hence, for small areas, the boundary choice is indeed optimal.

Next, we show that for $X > 8q$, the boundary solution is suboptimal to some interior solution. Note first that the variance of the question on the boundary is always larger than for any interior question as $\sigma^2 = \frac{d(X-d)}{X}$ is increasing in d. Hence, if the benefit of research V is larger for an interior question than for the boundary question, the researcher can obtain a higher payoff by choosing an interior question with the same ρ as for the boundary question: the cost will be lower, the success probability the same and the benefit upon success higher. The benefit of finding an answer on the boundary of an area with $X > 8q$ is always smaller than for some interior distance by Lemma 2 from the proof of Corollary 3. Hence, an interior choice is optimal for $X > 8q$.

For $X \in (4q, 8q)$ and $X - d < 4q$,

$$\frac{\frac{V_d(d,X)}{\sigma_d^2(d,X)}}{\frac{V(d,X)}{\sigma^2(d,X)}} = \frac{2d(X-d)}{-2d^2 + 2dX - \sqrt{X}(X-4q)^{3/2}}$$

which is decreasing in d with limit

$$\lim_{d \to X/2} \frac{2d(X-d)}{-2d^2 + 2dX - \sqrt{X}(X-4q)^{3/2}} = \frac{X^2/2}{X^2/2 - \sqrt{X}(X-4q)^{3/2}}$$

which, in turn, is increasing in X and 1 for $X = 4q$. Hence, any interior solution must be such that $X - d > 4q$ as otherwise, the first-order condition with respect to d is always positive.

Thus, we know that (i) in areas with $X < 4q$, the researcher's distance choice on the deepening area will be a boundary solution, (ii) in areas with $X > 8q$ the researcher's distance choice will be interior, (iii) in areas with $X \in [4q, 8q]$ the researcher's distance choice may be interior or on the boundary, and (iv) any interior choice has to satisfy $X - d > 4q$ and $d < 4q$.[32]

It remains to show that the values, U_R of $d_1 = X/2$ and $d_2(X) < X/2$ with $d_2(X)$ solving the first-order condition of $d_2(X)$ and $\rho_i(d_i, X)$ chosen optimally cross only

[32]From Lemma 2, 4 and 5 any interior choice that maximizes V (ignoring cost) satisfies $X - d > 4q$ and $d < 4q$.

once. We use three observations to show this.

1. First, at the area length X for which $U_R(d_1; X) = U_R(d_2; X)$, the payoff at the boundary must be decreasing faster than the payoff in the interior as the first switch is from the boundary solution to the interior solution by continuous differentiability of all terms and the observation from above that $d(X) = X/2$ for $X < 4q$.
2. Second, on the interval $[4q, 8q]$ the payoff of the boundary solution has a strictly lower second derivative with respect to X for all X than the interior solution. Hence, the two values can cross at most once on this interval.
3. Third, the value of the boundary solution is bounded from above by the value of the interior solution for all $X \geq 8q$.

The first observation is immediate.

For the second observation follows from totally differentiating U_R for the two types of local maxima. Using envelope conditions we obtain that the payoff is concave in the boundary solution and convex in the interior solution which implies the second observation. Define $\varphi(X) := \max_\rho u(d = X/2, \rho, X)$ for the boundary; we show in Lemma 25 in Appendix D that $\varphi(X)$ is concave. In Lemma 26 in Appendix D we in turn show that $U_R(X) = \max_{\rho,d} u(d, \rho, X)$ is convex in X provided that the maximizer $d(X) < X/2$. The result follows.

For the third observation, note that when $X \to \infty$, $V(d, X)$ converges to $V(d, \infty)$ and $\sigma^2(d, X)$ to $\sigma^2(d, \infty)$ and the researcher's optimization on the deepening interval converges to the optimization on the expanding interval which has a unique and interior maximum at (d^∞, ρ^∞). In particular, if such an interior optimum exists, the envelope condition implies that

$$U_R(X) = \rho V_X(d, X) - \eta \tilde{c}(\rho) \sigma_X^2(d, X) < 0$$

as $V_X(d, X) < 0$ according to Corollary 3 for $X > 4q$ and $X - d > 4q$ and $\sigma_X^2(d, X) > 0$.

Hence, the value of any optimal interior choice is decreasing in X. Because the payoff is continuous in X it follows that $\widetilde{X} > \check{X}$ where $\widetilde{X}$ denotes the first area length such that the interior value with $d < X/2$ is equal to the boundary value with $d = X/2$. □

Lemma 14. *The researcher's payoff $U_R(X)$ is single-peaked in X with the maximum attained at $\check{X}$.*

Proof. The result follows from 3 observations: First, $\widetilde{X} > \check{X} > 4q$ by Lemma 12 and 13. Second, $V_X(d; X) < 0$ if $X > 4q$ and $d < X/2$ by Lemma 3. Third, by the envelope theorem, if $d(X) < X/2$ it holds that $\partial U_R(X)/\partial X = \rho(X) V_X(d(X); X) - \eta \tilde{c}(\rho(X)) \sigma_X(d(X); X) < \rho(X) V_X(d(X); X)$. Thus, the payoff of the interior solution cuts the payoff of the boundary solution from below at an area where both payoffs are decreasing. The single peak is at $\check{X}$. □

Lemma 15. *Suppose $d = X/2$ is optimal for a range $[\underline{X}, \overline{X}]$. Then the optimal $\rho(X)$ is single peaked in that range. It is highest at $\dot{X} = \frac{8\cos\left(\frac{\pi}{18}\right)}{\sqrt{3}}$*

Proof. By Lemma 11 we know that $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} \geq 0$ and by Lemma 12 $\overline{X} > \widehat{X}^0$. Moreover, recall $\sigma^2(d = X/2; X) = X/4$. The first-order condition with respect to ρ becomes

$$\frac{V(X/2;X)}{X} = \frac{\eta}{4}\tilde{c}_\rho(\rho),$$

With

$$\frac{V(X/2;X)}{X} = \frac{X}{12q} - \mathbf{1}_{X>4q}\frac{(X-4q)^{3/2}}{\sqrt{X}6q}.$$

The latter is continuous and concave. Since $\tilde{c}(\rho)$ is an increasing, twice continuously differentiable and convex function, ρ increases in X if and only if $V(X/2;X)/X$ increases in X. By concavity of $V(X/2;X)/X$ that implies single peakedness.

Thus, $\dot{X}$ is independent of η and given by $\dot{X} = \frac{8\cos\left(\frac{\pi}{18}\right)}{\sqrt{3}} \approx 4.548q$. □

Lemma 16. *$\widehat{X}$ exists, $\lim_{X \searrow \widehat{X}} \rho(X) > \rho^\infty$, and $\widehat{X}$ decreases in η.*

Proof. As $X \to 0$, $d(X) \to 0$ and thus $U_R(X) \to 0$. By Lemma 10, $U_R(\infty) > 0$. Thus, by continuity of $U_R(X)$ (from Lemma 3 and 13) $\exists \widehat{X}$ such that expanding reaseach dominates deepening research for all $X < \widehat{X}$. Cost are increasing in X and by Corollary 3 $V(d; X \in (\widehat{X}^0, \infty)) > V(d;\infty)$ which implies $U_R(\infty > X > \widehat{X}^0) > U_R(\infty)$. By Lemma 14 and again continuity of $U_R(X)$ that payof is maximal at $\check{X}$. Thus, we obtain that $\widehat{X} < \check{X}$ exists.

We now show that $\lim_{X \searrow \widehat{X}} \rho(X) > \rho^\infty$ holds if $\widehat{X} < 6q$, then we show $\widehat{X}$ decreases in η which together with the observation that $\widehat{X}^0 < 6q$ is sufficient to prove the lemma.

At $\widehat{X}$ we have

$$\begin{aligned} U_R(\widehat{X}) &= U_R(\infty) \\ \rho(\widehat{X})V(\widehat{X}/2;\widehat{X}) - \eta\tilde{c}(\rho(\widehat{X}))\frac{\widehat{X}}{4} &= \rho^\infty V(d^\infty;\infty) - \eta\tilde{c}(\rho^\infty)d^\infty. \end{aligned} \tag{18}$$

where the fact that $d(\widehat{X}) = \widehat{X}/2$ follows from Lemma 12 to 14. Moreover, the following has to hold by optimality

$$V(d^\infty;\infty) = \eta\tilde{c}_\rho(\rho^\infty)d^\infty \tag{FOC ρ^∞}$$

$$V(\widehat{X}/2;\widehat{X}/2) = \eta\tilde{c}_\rho(\rho(\widehat{X}))\frac{\widehat{X}}{4} \tag{FOC $\rho^{\widehat{X}}$}$$

Step 1: $\rho^\infty < \rho(\widehat{X})$ if $\widehat{X} < 6q$. Using (FOC ρ^∞) and (FOC $\rho^{\widehat{X}}$) we obtain that by the properties of the error function $\rho(\widehat{X}) > \rho^\infty$ if and only if

$$4\frac{V(\widehat{X}/2;\widehat{X}/2)}{X} > \frac{V(d^\infty;\infty)}{d^\infty}.$$

Case 1: $\widehat{X} > 4q$. Substituting for the $V(\cdot)$'s the above becomes[33]

[33]Since $\widehat{X} \leq \check{X} \leq 8q$ that case is irrelevant.

$$
\begin{aligned}
& \frac{\widehat{X}}{3q} - \frac{2}{3q}\frac{(\widehat{X}-4q)^{3/2}}{\sqrt{\widehat{X}}} > 1 - \frac{d^\infty}{6q} \\
\Leftrightarrow \qquad & d^\infty + 2\widehat{X} - 4\underbrace{\frac{(\widehat{X}-4q)^{3/2}}{\sqrt{\widehat{X}}}}_{<(\widehat{X}-4q)} > 6q
\end{aligned}
$$

A sufficient condition for the above to hold is thus that

$$d^\infty - 2\widehat{X} + 10q > 0$$

Using that $d^\infty > 2q$ by Lemma 10 we obtain that sufficient condition for $\rho(\widehat{X}) > \rho^\infty$ is that $\widehat{X} < 6q$.

Case 2: $\widehat{X} \in (2q, 4q]$. Performing the same steps only assuming that $\widehat{X} \in [2q, 4q]$ we

$$
\begin{aligned}
\frac{\widehat{X}}{3q} &> 1 - \frac{d^\infty}{6q} \\
\Leftrightarrow 2\widehat{X} &> 6q - d^\infty > 4q
\end{aligned}
$$

which implies the desired result.

Case 3: $\widehat{X} < 2q$ We show that case 3 never occurs, that is $\widehat{X} > 2q$. To do so we compare $U_R(d = 2q; \infty)$ with $U_R(d = 1q; X = 2q)$ and show that the former is always larger. Hence $X = 2q < \widehat{X}$ for any η.

For $X = d = 2q$ we have that

$$\frac{\widehat{X}}{3q} = 1 - \frac{d}{6q},$$

and thus $\rho(X = 2q) = \rho(d; \infty) = \rho$ (cf. case 2). Moreover we have that

$$V(1q; 2q) = q/3 \qquad\qquad V(2q; \infty) = 4/3q,$$

and (FOC ρ^X) implies

$$4V(1q; 2q)/2q = 2/3 = \eta\tilde{c}_\rho(\rho)$$

Since $\tilde{c}_\rho(\rho) > \tilde{c}(\rho)/\rho$ for any $\rho > 0$ that implies $\eta\tilde{c}(\rho)/\rho < 2/3$.

Now take

$$\begin{aligned} U_R(d = 2q; \infty) &- U_R(X = 2q) \\ \rho \frac{4q}{3} &- \eta \tilde{c}(\rho) - \rho \frac{q}{3} + \eta \tilde{c}(\rho) \frac{q}{2} \\ q \left(\rho - \frac{3}{2} \eta \tilde{c}(\rho) \right), & \end{aligned}$$

which is positive whenever $\eta \tilde{c}(\rho)/\rho < 2/3$ which we know has to hold. Thus $U_R(d = 2q; \infty) > U_R(X = 2q)$ and therefore $\widehat{X} < 2q$.

Step 2: If $\rho^\infty < \rho(\widehat{X})$ then $\widehat{X}$ decreases in η.

Using (FOC ρ^∞) and (FOC $\rho^{\widehat{X}}$) to replace the $V(\cdot)$'s in equation (18) and dividing by η we obtain

$$d^\infty \left(\rho^\infty \tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty) \right) = \widetilde{X}/4 \left(\rho(\widehat{X}) \tilde{c}_\rho(\rho(\widehat{X})) - \tilde{c}(\rho(\widehat{X})) \right)$$

from which we get

$$\widehat{X}/4 = d^\infty \frac{\left(\rho^\infty \tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty) \right)}{\left(\rho(\widehat{X}) \tilde{c}_\rho(\rho(\widehat{X})) - \tilde{c}(\rho(\widehat{X})) \right)}.$$

Now we use the envelope theorem to calculate

$$\frac{\partial U_R(\widehat{X}) - U_R(\infty))}{\partial \eta} = \tilde{c}(\rho(\widehat{X})) \frac{\widehat{X}}{4} - \tilde{c}(\rho^\infty) d^\infty.$$

Replacing for $\widehat{X}$ implies that the RHS is positive if and only if

$$(\tilde{c}(\rho^\infty)) - \tilde{c}(\rho(\widehat{X})) \frac{\rho^\infty \tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty)}{\rho(\widehat{X}) \tilde{c}_\rho(\rho(\widehat{X})) - \tilde{c}(\rho(\widehat{X}))} > 0.$$

Using that $\rho \tilde{c}_\rho(\rho) > \tilde{c}(\rho)$ by the properties of the error function and factoring out he denominator of the first term, the above holds if and only if

$$\begin{aligned} \tilde{c}(\rho^\infty) \rho^\infty \tilde{c}_\rho(\rho(\widehat{X})) - \tilde{c}(\rho(\widehat{X}) \rho^\infty \tilde{c}_\rho(\rho^\infty) &> 0 \\ \frac{\rho(\widehat{X}) \tilde{c}_\rho(\rho(\widehat{X}))}{\tilde{c}(\rho(\widehat{X}))} &> \frac{\rho^\infty \tilde{c}_\rho(\rho^\infty)}{\tilde{c}(\rho^\infty)} \end{aligned}$$

which holds if and only if $\rho(\widehat{X}) > \rho^\infty$ by the properties of the error function. Thus, $\widehat{X}$ decreases if $\rho(\widehat{X}) > \rho^\infty$.

Conclusion: Since $\widehat{X}^0 \in [2q, 6q]$, $\Rightarrow \rho^\infty < \rho(\widehat{X}) \Rightarrow \widehat{X}$ is decreasing in η. □

Lemma 17. $\widehat{X} \leq \dot{X} < \check{X} \leq \widetilde{X}$

Proof. **Step 1: $\check{X} > \dot{X}$.** By the envelope theorem we need for $X = \check{X}$

$$\frac{\partial U_R(\check{X})}{\partial X} = \rho \frac{\mathrm{d}V(d = \check{X}/2; \check{X})}{\mathrm{d}X} - \frac{\eta}{4} \tilde{c}(\rho) = 0. \tag{19}$$

The FOC for ρ implies

$$\frac{V}{\check{X}} = \frac{\eta}{4}\tilde{c}_\rho(\rho)$$

Now assume for a contradiction that $\rho(\check{X})$ is increasing, then by Claim 11 $V(\cdot)/\check{X}$ must be increasing which holds if and only if

$$\frac{\mathrm{d}V(d=\check{X}/2;\check{X})}{\mathrm{d}X}\check{X} > V(d=\check{X}/2;\check{X}).$$

But then we obtain the following contradiction to $U_R(\check{X})$ being maximal

$$\frac{\mathrm{d}V(d=\check{X}/2;\check{X})}{\mathrm{d}X} > \frac{V(d=\check{X}/2;\check{X})}{\check{X}} = \frac{\eta}{4}\tilde{c}_\rho(\rho) > \frac{\eta}{4}\frac{\tilde{c}(\rho)}{\rho}.$$

The first inequality follow because $V(d=\check{X}/2;\check{X})/\check{X}$ must be increasing, the equality follows by equation (19). The last inequality is a consequence of the properties of erf. By Lemma 15, $\rho(X)$ is single peaked which proves the claim.

Step 2: Ordering. By Lemma 16 we know that $\widehat{X} < \widehat{X}^0$. Thus because $\widehat{X}^0 < \dot{X} \Rightarrow \widehat{X} < \dot{X}$. Moreover, $\widehat{X} > \check{X}$ by Lemma 13 which concludes the proof. □

□

Corollary 7. *d^∞ is linear in q and ρ^∞ is constant in q.*

Proof. The corollary follows because $\sigma^2(mq;\infty) = mq$ and thus (by Proposition 1) the functions $f(m,q) := V(mq;\infty)/\sigma^2(mq;\infty)$ and $g(m,q) := V_d(mq;\infty)$ are homogenous of degree 0 in q.

It is then immediate from (15) and (16) that d^∞ is homogenous of degree 1 in q and ρ^∞ is homogenous of degree 0. Noticing that $d^\infty(q=0)=0$ implies the result. □

B.7 Proof of Proposition 5

Proof. To prove the claim, we show that selecting a moonshot of length $6q$ is preferred to selecting the myopically optimal interval $3q$ for some $(\underline{\eta},\overline{\eta})$ and $\delta(\eta) < 1$.

We first list the respective data. We restrict attention to η-levels such that $d(6q) = 3q$. These levels exist by continuity of the cost term and the fact that $\widehat{X}^0 > 6q$ by Lemma 8.

MOONSHOT:

- *Values created in the first period:* $V(6q;\infty) = \frac{2}{\sqrt{3}}q$
- *Value created in the second period (if successful):* $V(3q;6q) = \left(3 - \frac{2}{\sqrt{3}}\right)q$
- *Probability of discovery in the second period:* Solution to researcher's first-order condition

$$\frac{4V(3q;6q)}{6q\eta} = \tilde{c}_\rho(\rho(6q))$$

which implies

$$\rho(6q) = erf\left(\frac{\sqrt{W\left(2\frac{\left(\frac{4V(3q;6q)}{6q\eta}\right)^2}{\pi}\right)}}{\sqrt{2}}\right) = erf\left(\sqrt{\frac{W\left(\frac{8\left(3-2\sqrt{3}\right)}{9\eta^2\pi}\right)}{2}}\right)$$

where $W(\cdot)$ is the Lambert W function.

- *Continuation payoff:* Conditional on discovery in $t = 2$, the continuation payoff from $t = 3$ onwards is $\rho^\infty V(d^\infty; \infty)$. The node (discovery in $t = 2$) is reached with probability $\rho(6q)$. Hence from an ex-ante point of view the continuation payoff from $t = 3$ onwards has $t = 1$ net present value

$$\frac{\delta^2 \rho^\infty}{1 - \delta\rho(6q)} V(d^\infty; \infty).$$

Myopic Optimum:

- *Values created in the first period:* $V(3q; \infty) = \frac{3}{2}q$
- *Value created in the second period (if successful):* $V(d^\infty; \infty) = d^\infty - (d^\infty)^2/6q$
- *Probability of discovery in the second period:* Given d^∞ it is the solution to

$$\frac{V(d^\infty; \infty)}{d^\infty} = c_\rho(\rho^\infty)$$

which implies

$$\rho^\infty = erf\left(\sqrt{\frac{W\left(\frac{2\left(\frac{6q-d^\infty}{6q\eta}\right)^2}{\pi}\right)}{2}}\right)$$

where $W(\cdot)$ is the Lambert-W (or product log) function.

- *Distance chosen by the researcher in period 2:* Solution to

$$d^\infty = 3q - \eta\frac{c(\rho^\infty)}{\rho^\infty}$$

- *Continuation payoff:* Conditional on discovery in $t = 2$, the continuation payoff from $t = 3$ onwards is $\rho^\infty V(d^\infty; \infty)$. The node (discovery in $t = 2$) is reached with probability ρ^∞. Hence from an ex-ante point of view the continuation payoff from $t = 3$ onwards has $t = 1$ net present value

$$\frac{\delta^2 \rho^\infty}{1 - \delta\rho^\infty} V(d^\infty; \infty).$$

The values follow directly from Proposition 1, the first-order conditions are discussed in the proof of Proposition 4.

Notice that $\rho(6q) > \rho^\infty$ by construction (and Proposition 4). A sufficient condition for our proposition to hold is to show that moonshots can be optimal even under a stronger condition. To that extend, we treat the continuation payoffs as identical—biasing the problem *against* moonshots.[34] Thus, whenever we find that a $6q$-moonshot is preferred to a myopic $3q$-strategy, some moonshot (not necessarily $6q$) is optimal under the stronger condition.

Given our bound the losses of a moonshot in the $t = 1$ are

$$\left(3/2 - \frac{2}{\sqrt{3}}\right) q. \tag{20}$$

For the gains in $t = 2$ notice that $\rho(6q) > \rho^\infty$ by construction (and Proposition 4). Thus in expectations it is without loss to consider the following events and probabilities:

1. both types of search lead to discovery (probability: ρ^∞)
2. only the deepening research leads to discovery (probability: $\rho(6q) - \rho^\infty$)
3. none leads to discovery (probability: $1 - \rho(6q)$)

In case 3 there is no value generated. In case 2 the $t = 2$ gains are

$$\left(3 - \frac{2}{\sqrt{3}}\right) q.$$

In case 1 the $t = 2$ gains are

$$\left(3 - \frac{2}{\sqrt{3}}\right) q - \left(d^\infty - \frac{(d^\infty)^2}{6q}\right).$$

Total gains (including discounting) are

$$\delta\left(\rho(6q)\left(3 - \frac{2}{\sqrt{3}}\right) q - \rho^\infty d^\infty\left(1 - \frac{d^\infty}{6q}\right)\right) \tag{21}$$

By continuity in η and δ it suffices to show that for $\delta = 1$ and some $\eta > 0$ we have that (20)<(21). (Numerically) solving for $d^\infty, \rho^\infty, \rho(6q)$ using, e.g., $\eta = 1$ verifies that this is the case.[35] □

B.8 Proof of Proposition 6

Proof. We make use of the Marginal Rate of Substitution (MRS) between ζ and η for the probability ρ and the distance d. The MRS describes the slope of the iso-ρ and iso-d line, respectively, in the (ζ, η)-space. The MRS for ρ is thus

$$MRS^{\rho}_{\zeta\eta} := -\frac{\frac{\partial \rho}{\partial \eta}}{\frac{\partial \rho}{\partial \zeta}},$$

and $MRS^{d}_{\zeta\eta}$ analogously.

[34]The moonshot creates two interval of size $3q$ which is better than the myopic optimum $3q$ and d^∞. Thus, the continuation payoff of the moonshot is (weakly) larger, as the decision maker benefits from better knowledge in all periods $t > 2$ as well. See Figure 10 for an illustration.

[35]Note that by Corollary 7, both (20) and (21) are linear in q. It suffices to validate the results for $q = 1$.

Lemma 27 in Appendix D on page 68 shows

$$MRS^{\rho}_{\zeta\eta} = s\left(2\tilde{c}_{\rho}(\rho) - \tilde{c}(\rho)/\rho\right), \tag{22}$$

and

$$MRS^{d}_{\zeta\eta} = \tilde{c}_{\rho}\frac{\tilde{c}/\rho - \tilde{c}_{\rho} + \frac{\tilde{c}}{\tilde{c}_{\rho}}\tilde{c}_{\rho\rho}}{\tilde{c}/\rho - \tilde{c}_{\rho} + \rho\tilde{c}_{\rho\rho}}. \tag{23}$$

Deriving the Research Possibility Frontier: Using the two first order conditions of the researcher and solving for ζ and η we obtain

$$\begin{aligned} \eta &= \frac{d}{6q}\frac{\rho}{\rho\tilde{c}_{\rho} - \tilde{c}} \\ \zeta &= \left(\frac{d}{3q} - 1 + \frac{d}{6q}\frac{\tilde{c}}{\rho\tilde{c}_{\rho} - \tilde{c}}\right)s. \end{aligned} \tag{24}$$

Using the calculated $MRS^{\rho}_{\zeta\eta}$ and $MRS^{d}_{\zeta\eta}$ we observe that any (ρ, d) can at most be implemented through one (ζ, η) combination because each iso-ρ curve crosses each iso-d curve at most once: both slopes are positive and the slope of the iso-ρ curves is steeper throughout than the slope of the iso-d curves because $s > 0.1$.

Given budget K $\zeta \in [0, K]$, and $\eta = \eta^0 - h \in [\check{\eta}, \eta^0]$ where $\check{\eta} = \eta^0 - K/\kappa$. Moreover the budget line is $K = \kappa h + \zeta$.

The polar solutions induce $(\zeta = 0, \check{\eta})$ is a direct application of Proposition 4. More generally, we can plug conditions (24) into the the budget line and rearrange to obtain

$$d(\rho) = 6q(K + s - \kappa\eta^0)\frac{\rho\tilde{c}_{\rho}(\rho) - \tilde{c}(\rho)}{2s\rho\tilde{c}_{\rho}(\rho) - s\tilde{c}(\rho) - \kappa\rho}. \tag{25}$$

our research possibility frontier.

Deriving the Boundary ρ's on the Research Possibility Frontier The first term in brackets, $K + s - \kappa\eta^0$ can be both positive or negative depending on the chosen parameters. Observe, however, that the minimum attainable cost factor $\check{\eta} = \eta^0 - K/\kappa$. Thus we can rewrite

$$K + s - k\eta^0 = s(1 - \check{\eta}\kappa).$$

Replacing $\check{\eta} = 1/(2\tilde{c}_{\rho}(\check{\rho}) - \tilde{c}(\check{\rho})/\check{\rho}$ which is the solution of equation (16) from the proof of Lemma 10 performing the same steps as when solving for equation (17) (which is possible because $\zeta = 0$ in that case) that implies

$$K + s - \kappa\eta^0 = s\left(1 - \frac{\kappa}{\underbrace{s(2\tilde{c}_{\rho}(\check{\rho}) - \tilde{c}(\check{\rho})/\check{\rho})}_{=MRS^{\rho}_{\zeta\eta}(\check{\rho})}}\right) = s\left(\frac{MRS^{\rho}_{\zeta\eta}(\check{\rho}) - \kappa}{MRS^{\rho}_{\zeta\eta}(\check{\rho})}\right).$$

Thus,

$$K + s > \kappa\eta^0 \Leftrightarrow MRS^{\rho}_{\zeta\eta}(\check{\rho}) > \kappa.$$

Now consider the last term, $(\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho))/(2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) - \kappa\rho)$. It is positive if and only if

$$2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) > \kappa\rho$$
$$\kappa < \underbrace{s\left(\frac{2\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)}{\rho}\right)}_{=MRS^{\rho}_{\zeta\eta}}.$$

If $MRS^{\rho}_{\zeta\eta}(\check{\rho}) > \kappa$ then any iso-ρ curve that crosses the budget line must satisfy $\rho > \check{\rho}$ and thus $MRS^{\rho}_{\zeta\eta}(\rho) > \kappa$ by Lemma 18. Moreover, the largest implementable $\overline{\rho}$ on the research possibility frontier comes from the polar case $\zeta = K, \eta = \eta^0$. Similarily, if $MRS^{\rho}_{\zeta\eta}(\check{\rho}) < \kappa$ then all iso-ρ curves that cross the budget line must satisfy $\rho < \check{\rho}$ and the lowest implementable ρ on the research possibility frontier comes from the polar case $\zeta = K, \eta = \eta^0$

Deriving the slope of the Research Possibility Frontier: Let $n(\rho)$ be the numerator of the last term and $dn(\rho)$ the denominator. Then, the last term is increasing in ρ if and only if

$$n'(\rho)dn(\rho) > n(\rho)dn'(\rho)$$

or equivalently using that $n'(\rho) = \rho\tilde{c}_{\rho\rho}(\rho) > 0$, $dn'(\rho) = s\left(2\rho\tilde{c}_{\rho\rho} + \tilde{c}_\rho\right) - \kappa$ if and only if

$$\frac{\kappa}{s} < \underbrace{\frac{\tilde{c}_\rho(\rho)\tilde{c}(\rho) + \rho\tilde{c}(\rho)\tilde{c}_{\rho\rho}(\rho) - \rho\left(\tilde{c}_\rho(\rho)\right)^2}{\tilde{c}_{\rho\rho}(\rho)\rho^2 - \rho\tilde{c}_\rho(\rho) + \tilde{c}(\rho)}}_{=MRS^{d}_{\zeta\eta}(\rho)}.$$

Thus $d(\rho)$ is increasing if and only if $(MRS^{\rho}_{\zeta\eta}(\check{\rho}) - \kappa)(MRS^{d}_{\eta\zeta}(\rho) - \kappa/s) > 0$. □

B.9 Proof of Proposition 7

Proof. We begin with three lemmas that prove large parts of the proposition.

Lemma 18. *Generically, output is maximized by focusing on only one of the two funding options. Specifically, there is a cutoff budget $\overline{K}$ such output is maximized by focusing on ex post rewards if $K < \overline{K}$ and by focusing on ex ante cost reductions if $K > \overline{K}$.*

Proof. Given ρ, $MRS^{\rho}_{\zeta\eta}(\rho)$ is constant and thus a straight line. Moreoever it is increasing and convex in ρ with $\lim_{\rho\to 1} MRS^{\rho}_{\zeta\eta} = \infty$ by the properties of $\tilde{c}(\rho)$.

Using standard arguments from consumer theory, we obtain that $\zeta{=}K$ is optimal if $MRS^{\rho}(\zeta\eta)|_{\zeta=K} < \kappa$ and $\zeta{=}0$ is optimal if $MRS^{\rho}(\zeta\eta)|_{\zeta=K} > \kappa$. As K increases the highest implementable ρ increase and so does the slope. □

Lemma 19. *Generically, novelty is maximized by focusing on only one of the two funding options. If output is maximized by focusing on ex post rewards, so is novelty. The reverse does not hold.*

Proof. For $s > 0.1$ equations (22) and (23) imply that $MRS^{\rho}_{\zeta\eta} > MRS^{d}_{\zeta\eta}$. Thus $MRS^{\rho}_{\zeta\eta} < \kappa \Rightarrow MRS^{d}_{\zeta\eta} < \kappa$. The result follows. □

Lemma 20. *Asumme a funder that maximizes $\rho V(d;\infty)$. If $d(\rho;K)$ is monotonically decreasing then the optimal induced $d^* \leq 3q$.*

Proof. Suppose $d^* > 3q$. Then reducing $3q$ increases benefits $V(d;\infty)$ by Corollary 2 and ρ because $d(\rho;K)$ is decreasing. □

If $K < \overline{K}$ exclusively into ex-post rewards is maximizes both output and novelty by Lemma 18 and 19. Because $sMRS^{d}_{\zeta\eta} < MRS^{\rho}_{\zeta\eta} < \kappa$ we have that $d(\rho;K)$ is increasing (see the last line of Appendix B.8) and hence d and ρ are complements.

If d and ρ are substitutes we must have that $K > \overline{K}$ by the above, so output is maximized by setting $h = K/\kappa$. Since d and ρ are substitutes throughout d is maximized at the other polar point (by Proposition 6), $\zeta = K$.

The remaining statements follow from these examples. In particular,

1. parameter values ($K = 7, s = 6, q = 1, \eta^0 = 2, \kappa = 4$) implies substitutes, and $K > \overline{K}$. Thus the funder's optimal solution is to focus on cost reductions (see Figure 14);
2. parameter values ($K = 7, s = 6, q = 1, \eta^0 = 2, \kappa = 20$) imply complements, and a solution $\rho \in (\underline{\rho}, \overline{\rho})$ which (by equation (1)) implies $(\zeta, h) > 0$;
3. parameter values ($K = 3, s = 6, q = 1, \eta^0 = 2, \kappa = 20$) imply complements, and $K < \overline{K}$ since the solution is $\rho = \overline{\rho}$ that implies $h = K/\kappa$. Moreover, in this case $d(\overline{\rho};K) > 3q$.
4. parameter values ($K = 3, s = 6, q = 1, \eta^0 = 2, \kappa = 4$) imply substitutes, and a solution $\rho \in (\underline{\rho}, \overline{\rho})$ which (by equation (1)) implies $(\zeta, h) > 0$. □

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Supplementary Material

C Graphical example

Here, we present a short graphical example to highlight our model ingredients and fosters intuition. Suppose the following snapshot of the realization of the Brownian path constitutes the truth on $[-2, 2]$.

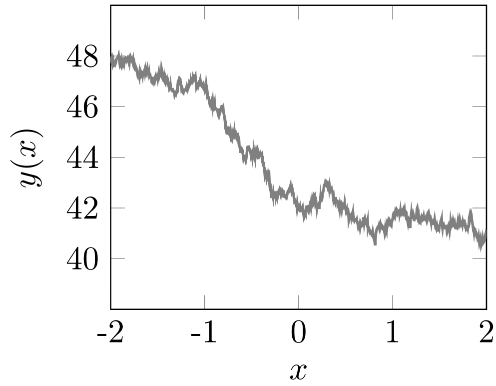

Figure 15: *The color of the truth is gray.*

The next graphs depict knowledge if the answer to a single question is known, $\mathcal{F}_1 = \{(0, 42)\}$, and in if two answers are known, $\mathcal{F}_2 = \{(-1.2, 46.6), (0, 42)\}$.

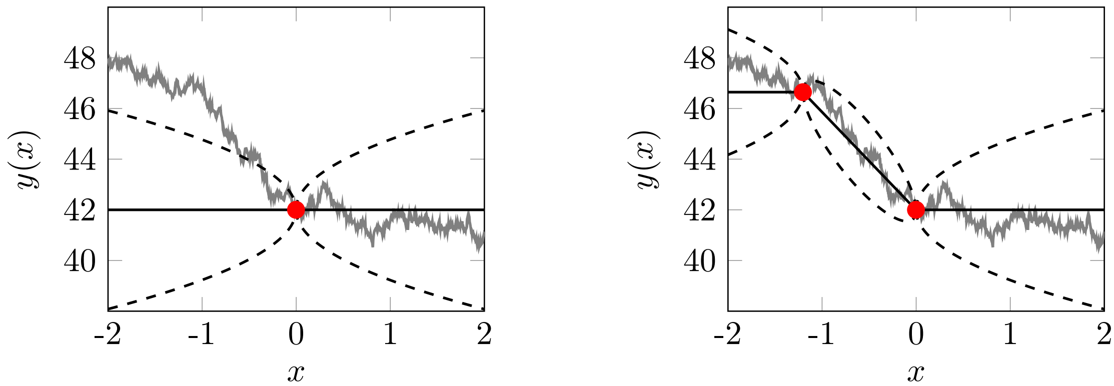

Figure 16: *Conjectures and their precision under $\mathcal{F}_1$ (left) and $\mathcal{F}_2$ (right).* The red dots represent known question-answer pairs. The solid lines represent the expected answer to each question x given the existing knowledge. The dashed line represents the 95-percent prediction interval—that is, the interval in which the answer to question x lies, with a probability of 95 percent, given $\mathcal{F}_k$.

In the situation represented in the left panel of Figure 16, under $\mathcal{F}_1$, only the answer to question 0, which is 42, is known. We represent that knowledge by a dot (●). Given the martingale property of a Brownian motion, the current conjecture is that the answer to all other questions is normally distributed with mean 42. We represent the mean of the conjecture by the solid lines. However, the farther a question is from 0, the less precise is the conjecture (see Figure 2). We depict the level of precision by the dashed 95-percent prediction interval. For each question x, the truth lies, with a probability of 95 percent, between the two dashed lines given the knowledge $\mathcal{F}_k$.

In the right panel of Figure 16, in addition to $\mathcal{F}_1$, the answer to question $x = -1.2$, which is 46.6, is known. The additional knowledge changes the conjectures for questions in the negative domain compared to the left panel. The conjecture about questions between -1.2 and 0 is represented by a Brownian bridge. The expectation of answers is decreasing from -1.2 to 0 and is 46.6 to the left of -1.2. Moreover, uncertainty decreases for all questions in the negative domain, and the prediction bands become narrower. The positive domain is unchanged because of the martingale property of the Brownian motion.

Now, consider moving to knowledge $\mathcal{F}_3 = \{(-1.2, 46.6), (0, 42), (1.2, 41.8)\}$ (left panel of Figure 17) and then to $\mathcal{F}_4 = \{(-1.6, 46.6), (0, 42), (0.8, 40.8), (1.2, 41.8)\}$ (right panel of Figure 17).

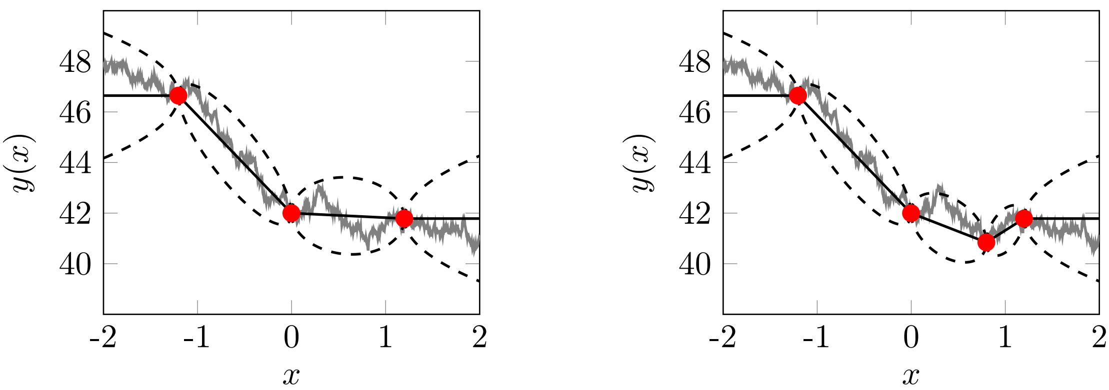

Figure 17: *Conjectures and their precision under $\mathcal{F}_3$ (left) and $\mathcal{F}_4$ (right).*

Moving from $\mathcal{F}_2$ to $\mathcal{F}_3$, the change is similar to that from $\mathcal{F}_1$ to $\mathcal{F}_2$, but this time in the positive domain. All conjectures in the positive domain become more precise, but the negative domain is unaffected. Further, a Brownian bridge between the known points $(0, 42)$ and $(1.2, 41.8)$ arises.

Moving from $\mathcal{F}_3$ to $\mathcal{F}_4$, knowledge of an answer to a question that lies between two already-answered questions is added. Conjectures about answers to questions between 0 and 1.2 become more precise. Further, since $40.8 < 41.8$, answers to all questions between 0 and 1.2 are expected to be lower compared to the conjecture based on knowledge $\mathcal{F}_3$. Moreover, the expected answers are decreasing in x from 0 to 0.8 and increasing from 0.8 and 1.2.

D Omitted Parts of the Proofs

Here we provide the steps that we have omitted in the proofs because they involve cumbersome algebraic manipulation with little economic or mathematical insight.

Lemma 21. $\frac{\partial V(d;\infty|d>4q)}{\partial d} < 0$.

Proof.

$$\frac{\partial V(d;\infty|d>4q)}{\partial d} = -\frac{d}{3q} + 1 + \sqrt{\frac{d-4q}{d}}\frac{d-q}{3q}$$

Letting $\tau := d/q$ the statement is negative if

$$\frac{3-\tau}{3} + \sqrt{\frac{\tau-4}{\tau}}\frac{\tau-1}{3} < 0$$

The left-hand side is increasing in τ and converges to 0 as $\tau \to \infty$. $\square$

Lemma 22. $\frac{1}{3q}\left(X - 2d - (X-d-q)\sqrt{\frac{X-d-4q}{X-d}}\right) > 0$ *if* $d \in [0, X-4q]$.

Proof. We show that the derivative V_d is a convex function which is positive at its minimum on $[0, X-4q]$ and hence throughout on that domain.

The relevant derivatives to consider are

$$V_d = \frac{1}{3q}\left(X - 2d - (X-d-q)\sqrt{\frac{X-d-4q}{X-d}}\right).$$
$$V_{dd} = \frac{1}{3q}\left(-2 + \frac{1}{\sqrt{X-d-4q}(X-d)^{3/2}}\left((X-d-4q)(X-d) + (X-d-q)2q\right)\right).$$
$$V_{ddd} = \frac{4q^2}{(X-d)^{5/2}(X-d-4q)^{3/2}} > 0.$$

where $V_{ddd} > 0$ follows immediately from $(X-d) > 0$ and $(X-d-4q) > 0$. It follows that, V_d is strictly convex over the relevant range. The maximal distance in this range, $d = X-4q$, $V_d|_{d=X-4q} = \frac{8q-X}{3q} > 0$.

Hence, the minimum of the first derivative is either at $d = 0$ or at some interior d such that $V_{dd} = 0$. Suppose the minimum is at $d = 0$, then $V_d|_{d=0} = \frac{1}{3q}\left(X - (X-q)\sqrt{\frac{X-4q}{X}}\right) > 0$ because $\frac{X-4q}{X} < 1$.

Hence, the only remaining case is when V_d attains an interior minimum. In this case, $V_{dd} = 0$ must hold at the minimum and hence

$$\sqrt{X-d-4q}(X-d)^{3/2} = \frac{(X-d-4q)(X-d) + (X-d-q)2q}{2}.$$

The first derivative can be rewritten as

$$V_d = \frac{1}{3q}\left(X - 2d - \frac{1}{\sqrt{X-d-4q}(X-d)^{3/2}}(X-d-q)(X-d-4q)(X-d)\right)$$

and plugging in for the minimum condition we obtain

$$
\begin{aligned}
&V_d|_{V_{dd}=0}\\
&= \frac{1}{3q}\left(X - 2d - \frac{2(X-d-q)(X-d-4q)(X-d)}{(X-d-4q)(X-d)+(X-d-q)2q}\right)\\
&= \frac{1}{3q}\frac{(X-2d)((X-d-4q)(X-d)+(X-d-q)2q)-2(X-d-q)(X-d-4q)(X-d)}{(X-d-4q)(X-d)+(X-d-q)2q}.
\end{aligned}
$$

As the denominator and $\frac{1}{3q}$ are both positive, the sign of V_d at its minimum is determined by the sign of its numerator only. Note that the numerator is increasing in d because its derivative is $2(X-6q)(X-d-q) > 0$. Thus, the numerator of the derivative of V_d evaluated at the interior minimum d such that $V_{dd} = 0$ is greater than

$$
-X(X^2 - 8qX + 10q^2) = -X((X-4q)^2 - 6q^2) > 0.
$$

□

Lemma 23. *$V_X(d^0(X); X) < 0$ if $X \geq 4q$ and $d \in [0, X-4q]$.*

Proof. Observe that for any $X \geq 4q$ and $d \leq X - 4q$

$$
V_{Xd} = \frac{1}{24q}\left(8 - 3\sqrt{\frac{X-d}{X-d-4q}} - (5(X-d)+4q)\frac{\sqrt{X-d-4q}}{(X-d)^{3/2}}\right).
$$

Denote $a := X - d$, this is an increasing function in a as

$$
\frac{dV_{Xd}}{da} = \frac{4q^2}{a^{5/2}(a-4q)^{3/2}} > 0.
$$

Hence, the highest value of V_{Xd} is attained for $a \to \infty$ and

$$
\lim_{a\to\infty} \frac{1}{24q}\left(8 - 3\underbrace{\sqrt{\frac{a}{a-4q}}}_{\to 1} - 5\underbrace{\frac{a\sqrt{a-4q}}{a^{3/2}}}_{\to 1} + 4q\underbrace{\frac{\sqrt{a-4q}}{a^{3/2}}}_{\to 0}\right) = 0.
$$

It follows that the V_{Xd} converges to zero from below implying that $V_{Xd} < 0$. Thus,

$V_X(d^0(X), X) < V_X(d = 0, X)$ and we obtain

$$\begin{aligned}
&V_X(d, X|d \leq 4q, X-d \geq 4q)\\
&\qquad = \frac{1}{3q}\left(d + (X-d-q)\sqrt{\frac{X-d-4q}{X-d}} - (X-q)\sqrt{\frac{X-4q}{X}}\right)\\
&\qquad < V(d=0, X|d \leq 4q, X-d \geq 4q)\\
&\qquad = \frac{1}{3q}\left((X-q)\sqrt{\frac{X-4q}{X}} - (X-q)\sqrt{\frac{X-4q}{X}}\right)\\
&\qquad = 0
\end{aligned}$$

as desired. □

Lemma 24. $\mathrm{d}^2V(X/2, X)/(\mathrm{d}X)^2 < 0$ *and* $\mathrm{d}^2V(d^0(X), X)/(\mathrm{d}X)^2 > 0$.

Proof. Considering the boundary solution we obtain

$$\begin{aligned}
\frac{d^2V(X/2, X)}{dX^2} &= -\frac{X^2 - 2qX - 2q^2}{3qX^{3/2}\sqrt{X-4q}} + \frac{1}{6q}\\
\frac{d^3V(X/2, X)}{dX^3} &= \frac{4q^2}{X^{5/2}(X-4q)^{3/2}} > 0
\end{aligned}$$

implying that $\frac{d^2V(X/2,X)}{dX^2} \leq \frac{d^2V(4q,8q)}{dX^2}$ with

$$\frac{d^2V(4q,8q)}{dX^2} = -\frac{64q^2 - 16q^2 - 2q^2}{3q8^{3/2}q^{3/2}2q^{1/2}} + \frac{1}{6q} = -\frac{46q^2}{96\sqrt{2}q^3} + \frac{1}{6q} = \frac{8 - 23/\sqrt{2}}{48q} < 0.$$

Next, consider the value of any interior solution and apply the envelope and implicit function theorem to obtain

$$\begin{aligned}
\frac{dV(d^0(X), X)}{dX} &= V_X + d'(X)\underbrace{V_d}_{=0 \text{ by optimality of } d} = V_X\\
\frac{d^2V(d^0(X), X)}{dX^2} &= V_{XX} + d'(X)V_{dX} + d'(X)\underbrace{(V_{Xd} + V_{dd}d'(X))}_{=0 \text{ by IFT on FOC}} + d''(X)\underbrace{V_d}_{=0 \text{ by optimality}}\\
&= V_{XX}(d^0(X), X) + d'(X)V_{dX}\\
&= V_{XX}(d^0(X), X) \underbrace{-\frac{V_{dX}^2}{V_{dd}}}_{>0 \text{ as } V_{dd}<0}.
\end{aligned}$$

Observing that

$$V_{XXd}(d, X|d \leq 4q, X-d \geq 4q) = \frac{24q^3}{(X-d)^{5/2}(X-d-4q)^3/2} > 0$$

we can compute as lower bound for

$$
\begin{aligned}
V_{XX}(d^0(X),X) &= \frac{1}{24q}\left(3\left(\sqrt{\frac{X-d}{X-d-4q}}-\sqrt{\frac{X}{X-4q}}\right)+6\left(\sqrt{\frac{X-d-4q}{X-d}}-\sqrt{\frac{X-4q}{X}}\right)\right.\\
&\quad \left. +\left(\frac{X-4q}{X}\right)^{3/2}-\left(\frac{X-d-4q}{X-d}\right)^{3/2}\right)\\
&\geq V_{XX}(d=0,X)\\
&=0
\end{aligned}
$$

implying that $\mathrm{d}^2V(d^0(X),X)/(\mathrm{d}X^2)\geq 0$ and concluding the proof.

□

Lemma 25. *Assume* $X\in[4q,8q]$*, then* $\mathrm{d}^2U_R(d=X/2;X)/(\mathrm{d}X)^2<0$.

Proof. Take the case of the boundary solution: we are analyzing a one-dimensional optimization problem with respect to ρ. Denote the objective $f(\rho;X)$ and the optimal value by $\varphi(X)=\max_\rho f(\rho;X)$. Then, the optimal ρ solves $f_\rho=0$. We obtain

$$
\begin{aligned}
\varphi'(X) &= \underbrace{f_\rho}_{=0\text{ by optimality}}\rho'(X)+f_X\\
\varphi''(X) &= \underbrace{f_\rho}_{=0\text{ by optimality}}\rho''(X)+\underbrace{(f_{\rho\rho}\rho'(X)+f_{X\rho})}_{=0\text{ by total differentiation of FOC}}\rho'(X)+f_{XX}+\rho'(X)f_{X\rho}\\
&= f_{XX}-\frac{f_{X\rho}^2}{f_{\rho\rho}}\\
&= \rho(X)V_{XX}(X/2;X)+\frac{(V_X-\frac{V}{X})^2}{V\frac{c''}{c'}}
\end{aligned}
$$

which yields

$$
\rho(X)\frac{c''}{c'}>-\frac{(V_X-\frac{V}{X})^2}{V_{XX}V}.
$$

Note that at the boundary solution the right-hand side simplifies to

$$
\frac{X^{3/2}-2(X+2q)\sqrt{X-4q}}{X^{3/2}-2(X-4q)\sqrt{X-4q}}\,\frac{16q^2+4qX-2X^2+X^{3/2}\sqrt{X-4q}}{8q^2+8qX-4X^2+2X^{3/2}\sqrt{X-4q}}
$$

where both fractions are less than one. Finally, we know that the right-hand side is above two by the properties of the inverse error function. Hence, the optimal value at the boundary solution is strictly concave as $\sigma^2_{XX}(X/2;X)=0$ and $V_{XX}<0$ in the region considered by Corollary 3 □

Lemma 26. *Let* $d^i<X/2$ *be a local maximum of* $u_R(\rho,d,X)$*. If* $d^i(X)$ *exists on* $X\in[4q,8q]$*, then* $\mathrm{d}^2U_R(d=d^i(X);X)/(\mathrm{d}X)^2>0$.

Proof. The implicit function theorem yields for $d'(X)$ and $\rho'(X)$

$$\begin{pmatrix} d'(X) \\ \rho'(X) \end{pmatrix} = -\frac{1}{f_{dd}f_{\rho\rho} - f_{\rho d}^2} \begin{pmatrix} f_{dX}f_{\rho\rho} - f_{\rho X}f_{d\rho} \\ f_{\rho X}f_{dd} - f_{dX}f_{d\rho} \end{pmatrix}.$$

Note that $-\frac{1}{f_{dd}f_{\rho\rho} - f_{\rho d}^2} < 0$ as this is $-\frac{1}{det(\mathcal{H})}$ and the determinant of the second principal minor being positive is a necessary second order condition for a local maximum given that the first ($f_{\rho\rho}$) is negative.

Denote the objective $f(\rho, d; X)$ and the optimal value by $\varphi(X) = \max_{\rho,d} f(d, \rho; X)$. Then, the optimal (d, ρ) solves $f_\rho = 0$ and $f_d = 0$. Differentiating the value of the researcher twice with respect to X yields

$$\begin{aligned}
\varphi'(X) = &\underbrace{f_\rho}_{=0 \text{ by optimality}} \rho'(X) + \underbrace{f_d}_{=0 \text{ by optimality}} d'(X) + f_X \\
\varphi''(X) = &\underbrace{f_\rho}_{=0 \text{ by optimality}} \rho''(X) \\
&+ d'(X) \underbrace{(f_{dX} + f_{dd}d'(X) + f_{d\rho}\rho'(X))}_{=0 \text{ by total differentiation of foc wrt } d} \\
&+ \rho'(X) \underbrace{(f_{\rho X} + f_{\rho d}d'(X) + f_{\rho\rho}\rho'(X))}_{=0 \text{ by total differentiation of foc wrt } \rho} \\
&+ f_{dX}d'(X) + f_{\rho X}\rho'(X) + f_{XX} \\
&= f_{dX}d'(X) + f_{\rho X}\rho'(X) + f_{XX}.
\end{aligned}$$

Observe first that $f_{XX} > 0$ as $f_{XX} = \rho V_{XX}(d; X) - \eta\tilde{c}(\rho)\sigma_{XX}^2(d; X)$ and $V_{XX} > 0$ by proof of Corollary 3 and $\sigma_{XX}^2(d; X) = -\frac{2d^2}{X^3}$. Next, we show $f_{dX}d'(X) + f_{\rho X}\rho'(X) > 0$ using the implicit function theorem together with the property of the local maximum that $f_{\rho\rho}f_{dd} > f_{\rho d}^2$.

$$f_{dX}d'(X) + f_{\rho X}\rho'(X) = -f_{dX}\left(\frac{f_{dX}f_{\rho\rho} - f_{\rho X}f_{d\rho}}{f_{dd}f_{\rho\rho} - f_{\rho d}^2}\right) - f_{\rho X}\left(\frac{f_{\rho X}f_{dd} - f_{dX}f_{d\rho}}{f_{dd}f_{\rho\rho} - f_{\rho d}^2}\right).$$

As we only need the sign of this expression we can ignore the positive denominator to verify

$$\begin{aligned}
-f_{dX}(f_{dX}f_{\rho\rho} - f_{\rho X}f_{d\rho}) - f_{\rho X}(f_{\rho X}f_{dd} - f_{dX}f_{d\rho}) &> 0 \\
f_{dX}^2 f_{\rho\rho} + f_{\rho X}^2 f_{dd} - 2f_{dX}f_{\rho X}f_{d\rho} &< 0 \\
\frac{f_{dX}}{f_{\rho X}}\frac{f_{\rho\rho}}{f_{d\rho}} + \frac{f_{\rho X}}{f_{dX}}\frac{f_{dd}}{f_{\rho d}} &> 2.
\end{aligned}$$

where we used the signs of the terms that follow because

$$\begin{aligned}
f_{\rho\rho} &= -\eta\tilde{c}_{\rho\rho}(\rho)\sigma^2(d;X) < 0 \\
f_{\rho X} &= V_X - \eta\tilde{c}_\rho(\rho)\sigma_X^2(d;X) \\
&< V_X - \eta\frac{\tilde{c}(\rho)}{\rho}\sigma_X^2(d;X) < 0 \\
f_{d\rho} &= V_d - \eta\tilde{c}_\rho(\rho)\sigma_d^2(d;X) \\
&< V_d - \eta\frac{\tilde{c}(\rho)}{\rho}\sigma_d^2(d;X) = 0 \\
f_{dX} &= \rho V_{dX} - \eta\tilde{c}(\rho)\sigma_{dX}^2 < 0
\end{aligned}$$

which in turn follow from the first-order conditions and Corollary 3.

Because $f_{\rho\rho}f_{dd} - f_{\rho d}^2 > 0$, we can replace $\frac{f_{\rho\rho}}{f_{d\rho}}$ with $\frac{f_{d\rho}}{f_{dd}}$ as $\frac{f_{\rho\rho}}{f_{d\rho}} > \frac{f_{d\rho}}{f_{dd}}$ yielding

$$2 < \frac{f_{dX}}{f_{\rho X}}\frac{f_{d\rho}}{f_{dd}} + \frac{f_{\rho X}}{f_{dX}}\frac{f_{dd}}{f_{\rho d}}$$

which is true as the right-hand side can be written as $g(a) = a + \frac{1}{a}$ with $a = \frac{f_{dX}}{f_{\rho X}}\frac{f_{d\rho}}{f_{dd}} > 0$. Note that $g(a)$ is a strictly convex function for $a > 0$ and minimized at $a = 1$ with $g(a = 1) = 2$. □

Lemma 27. *$MRS_{\zeta\eta}^{\rho} = s(2\tilde{c}_\rho(\rho) - \tilde{c}(\rho)/\rho)$ and $MRS_{\zeta\eta}^{d} = \tilde{c}_\rho\frac{\tilde{c}/\rho - \tilde{c}_\rho + \frac{\tilde{c}}{\tilde{c}_\rho}\tilde{c}_{\rho\rho}}{\tilde{c}/\rho - \tilde{c}_\rho + \rho\tilde{c}_{\rho\rho}}$.*

Proof. For any (η, ζ) the system of first-order conditions for a non-boundary choice is given by

$$\begin{aligned}
V_d(d,\infty) + \zeta\sigma_d^2(d,\infty)/s &= \eta\tilde{c}(\rho)/\rho \\
\frac{V(d,\infty) + \zeta\sigma_d^2(d,\infty)/s}{d} &= \eta\tilde{c}_\rho(\rho)
\end{aligned}$$

For an interior optimal choice of (d, ρ), we obtain using $\sigma^2(d,X) = d, \sigma_d^2(d,X) = 1$ and $\sigma_{dd}^2(d,X) = 0$[36]

$$\begin{pmatrix} \frac{\mathrm{d}d}{\mathrm{d}\eta} \\ \frac{\mathrm{d}d}{\mathrm{d}\zeta} \\ \frac{d\rho}{\mathrm{d}\eta} \\ \frac{d\rho}{\mathrm{d}\zeta} \end{pmatrix} = -\frac{1}{det(\mathcal{H})}
\begin{pmatrix} d(\tilde{c}_\rho(V_d + \zeta/s - \eta\tilde{c}_\rho) + \eta\tilde{c}\tilde{c}_{\rho\rho}) \\ -d(V_d + \zeta/s - \eta\tilde{c}_\rho + \rho\eta\tilde{c}_{\rho\rho}) \\ -\rho\sigma^2\tilde{c}_\rho V_{dd} + \tilde{c}(V_d + \zeta/s - \eta\tilde{c}_\rho) \\ -\rho/s(V_d + \zeta/s - \eta\tilde{c}_\rho - dV_{dd}) \end{pmatrix}.$$

where $det(\mathcal{H})$ is the determinant of the Hessian matrix of the objective function

[36] We suppress arguments of the functions for readability.

which is given by

$$-\eta\sigma^2\tilde{c}_{\rho\rho}\rho V_{dd} - (V_d + \zeta/s - \eta\tilde{c}_\rho)^2 > 0.$$

Note that the determinant of the Hessian matrix for a local maximum is positive as the Hessian is negative semidefinite and the first principal minor $-\eta\tilde{c}_{\rho\rho}\sigma^2 < 0$ by convexity of the inverse error function.[37]

It follows that the sign of the derivatives are determined only by the negative of the sign of the respective terms in the matrix. Using the first-order conditions to rewrite these equations yields

$$\begin{aligned}\frac{\mathrm{d}d}{\mathrm{d}\eta} &= -\frac{d\eta}{det(\mathcal{H})}\left(\tilde{c}_\rho\left(\frac{\tilde{c}}{\rho} - \tilde{c}_\rho\right) + \tilde{c}\tilde{c}_{\rho\rho}\right) < 0\\ \frac{\mathrm{d}d}{\mathrm{d}\zeta} &= \frac{d\eta}{det(\mathcal{H})}\left(\frac{\tilde{c}}{\rho} - \tilde{c}_\rho + \rho\tilde{c}_{\rho\rho}\right) > 0\end{aligned}$$

where the inequalities hold due to the properties of the inverse error function.

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}\eta} = -\frac{\rho\eta}{det(\mathcal{H})}\\ (2\tilde{c}_\rho - \tilde{c}/\rho)\,(\tilde{c}_\rho - \tilde{c}/\rho) < 0\end{aligned}$$

where we have used that $\sigma^2 V_{dd} = -\frac{d}{3q}$ and from the first-order conditions we know that $\frac{d}{3q} = 2\eta(\tilde{c}_\rho - \tilde{c}/\rho)$. The properties of $\tilde{c}$ imply that $\tilde{c}_\rho > \tilde{c}/\rho$. Finally,

$$\frac{\mathrm{d}\rho}{\mathrm{d}\zeta} = \frac{\rho\eta/s}{det(\mathcal{H})}\,(\tilde{c}_\rho - \tilde{c}/\rho) > 0$$

where the analogous reasoning as for the previous inequality applies. To conclude, we have

$$\begin{array}{ll}\frac{\mathrm{d}d}{\mathrm{d}\eta} < 0 & \frac{\mathrm{d}d}{\mathrm{d}\zeta} > 0\\ \frac{\mathrm{d}\rho}{\mathrm{d}\eta} < 0 & \frac{\mathrm{d}\rho}{\mathrm{d}\zeta} > 0.\end{array}$$

We obtain for the marginal rate of substitution between ζ and η on the expanding interval

$$-\frac{\frac{\mathrm{d}\rho}{\mathrm{d}\eta}}{\frac{\mathrm{d}\rho}{\mathrm{d}\zeta}} = MRS^{\rho}_{\zeta\eta} = s(2\tilde{c}_\rho - \tilde{c}/\rho)$$

where we used the simplifications from above.

[37] In our case, one can actually show that this has to hold given that $d < \infty$. Plugging in from the first-order conditions yields

$$\eta^2(\tilde{c}_\rho - \tilde{c}/\rho)2(\rho\tilde{c}_{\rho\rho} - \tilde{c}_\rho + \tilde{c}/\rho) > 0$$

where the inequality follows from the properties of $\tilde{c}$.

Similarily, we obtain

$$-\frac{\frac{\mathrm{d}d}{\mathrm{d}\eta}}{\frac{\mathrm{d}d}{\mathrm{d}\zeta}} = MRS^{d}_{\zeta\eta} = \tilde{c}_{\rho}\frac{\tilde{c}/\rho - \tilde{c}_{\rho} + \frac{\tilde{c}}{\tilde{c}_{\rho}}\tilde{c}_{\rho\rho}}{\tilde{c}/\rho - \tilde{c}_{\rho} + \rho\tilde{c}_{\rho\rho}}.$$

□