Managing a Conflict: Supplementary Material

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Note: For hyperlinks to work you need *MaC.pdf* and *MaCSupp.pdf* in the same folder.

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D Proof of Lemma 2

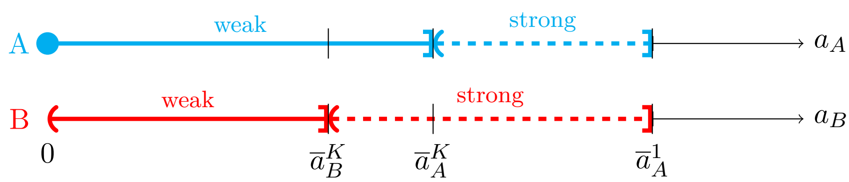

Figure S.1: Equilibrium strategies.

Proof. Bayes' plausibility implies that for any feasible belief system $\operatorname{sgn}(b_A(1)-b_A(K)) = \operatorname{sgn}(b_B(1)-b_B(K))$.

Let $[\underline{a}_i^{\theta_i}, \overline{a}_i^{\theta_i}]$ be the strategy support of disputant i, type θ_i.

Then, for any action choice a_i the continuation utility is

$$U_i(a_i, m_i = \theta_i; \theta_i, \mathcal{B}) = F_{-i}(a_i|\theta_i) - \theta_i a_i. \tag{S.1}$$

Moreover,

$$F_i(a|m) = b_i(m)F_i^1(a) + (1 - b_B m)F_i^K(a) = b_i(m)\left(F_i^1(a) - F_i^K(a)\right) + F_i^K(a), \tag{S.2}$$

and $F_i^{\theta_i}$ is strictly increasing on $(\underline{a}_i^{\theta_i}, \overline{a}_i^{\theta_i})$. By monotonicity of the on-path equilibrium any $F_i^1(a) < F_i^K(a)$ for all $a < a_A^1$, and $F_i(a|m)$ is linearly decreasing in $b_i(m)$ on that part of the domain. This implies first-order stochastic dominance, i.e.,

$$b_i(m) > b_i(m') \Rightarrow F_i(a|m) < F_i(a|m') \qquad \forall a < a_A^1.$$

Whenever $b_i(1) \neq b_i(K)$ at least one type of each disputant faces a stochastically dominated action distribution by the opponent after a deviation during ADR. Thus, that type must receive a strictly higher utility than on the equilibrium path.

To see the additional information advantage notice that players have to be indifferent between any action in their strategy support $[\underline{a}_i^{\theta_i}, \overline{a}_i^{\theta_i}]$ on the equilibrium path. That implies that for all $a_i \in [\underline{a}_i^{\theta_i}, \overline{a}_i^{\theta_i}]$

$$\frac{\partial U_i(a_i, m_i = \theta_i; \theta_i, \mathcal{B})}{\partial a_i} = b_i(\theta_i) \underbrace{\left(\frac{\partial F_i^1(a_i)}{\partial a_i} - \frac{\partial F_i^K(a_i)}{\partial a_i}\right)}_{d(a_i)} + \frac{\partial F_i^K(a_i)}{\partial a_i} - \theta_i = 0. \tag{S.3}$$

By the properties of a monotonic equilibrium, the term $d(a_i)$ is positive if $a_i > \overline{a}_{-i}^K$ and negative otherwise. Thus, for any on-path action a_i the first-order condition (S.3) fails off the equilibrium path. Moreover, since equilibrium action profiles have constant densities on any of the partitions $(0, \overline{a}_B^K], (\overline{a}_B^K, \overline{a}_A^K], (\overline{a}_A^K, \overline{a}_A^1]$, $d(a_i)$ is constant inside each of these intervals. The optimal action post deviation is thus any of the interval boundary points $0, \overline{a}_B^K, \overline{a}_A^K, \overline{a}_A^1$. Plugging into (S.1) shows that there always is a unique optimal point if $b_i(1) \neq b_i(K)$. $\square$

E Escalation Games with Ex-post Equilibrium and Proof of Proposition 3.

In this part we provide a general solution algorithm to arbitration if the escalation game has an ex-post equilibrium. We begin by describing how a simple lottery a la Hörner, Morelli, and Squintani (2015) differs from the litigation game we consider in Section 2. Then we generalize the problem to monotone lotteries and present our solution algorithm that implies the result from Proposition 3. A matlab implementation of the algorithm for monotone lotteries is available on our website.

E.1 Simple Lotteries

Setting. Consider the setup from Section 2. However, the escalation game is different. Participation requires a fixed cost $0 \leq c < 1/2$ and a player's strategy choice is constant in the information structure.[1] To emphasize this effect, we suppress action choices in the payoff function. It is

$$u(\theta_i, \theta_{-i}) = \begin{cases} 1/2 - c, & \text{if } \theta_i = \theta_{-i} \\ 1/2 - c + \xi, & \text{if } \theta_i < \theta_{-i} \\ 1/2 - c - \xi, & \text{if } \theta_i > \theta_{-i}, \end{cases}$$

where $\xi > c$ is the marginal payoff of a better case. Let $b_i(m_i) := \beta_i(1|m_i)$ be the probability mass on type 1, and $\mathbb{1}_1 = 1$ if θ_i=1 and 0 otherwise. Expected payoffs are

$$U_i(m_i; \theta_i, \mathcal{B}) = b_i(m_i)u(\theta_i, 1) + (1 - b_i(m_i))u(\theta_i, K) = 1/2 - c - b_i(m_i)\xi + \mathbb{1}_1\xi.$$

Optimal arbitration maximizes $\mathbb{E}[U|\mathcal{B}]+\mathbb{E}[\Psi|\mathcal{B}]$ subject to constraints C_F. K-types receive no virtual rent by definition, thus $\Psi_i(1, \mathcal{B}) = w(1)D_i(1; K, \mathcal{B})$.

Moreover, the ability premium, $D_i(1; K, \mathcal{B})=U_i(1; 1, \mathcal{B}) - U_i(1; K, \mathcal{B})=\xi$, is constant. Hence $\mathbb{E}[\Psi_i|\mathcal{B}] = \rho_i(1)w(1)\xi$ (linearly) increases in the post-escalation likelihood, $\rho_i(1)$. By design, the joint expected payoffs, $\mathbb{E}[U|\mathcal{B}]=1-2c$, are independent of $\mathcal{B}$. Thus, the optimal belief system is $(b_i(1) = 1, b_i(K) = 0)$ implying $\rho_i(1) = 1$. We denote this belief system by $\widetilde{\mathcal{B}}$.

The belief system $\widetilde{\mathcal{B}}$ *sorts* type profiles into the set of *easy-to-settle matches*, $\{(K, K), (1, K), (K, 1)\}$, and that of *difficult-to-settle matches*, $\{(1, 1)\}$. Under $\widetilde{\mathcal{B}}$ no type-1 player wants to mimic a type-K player. However, it is possible that the reduced-form mechanism implementing $\widetilde{\mathcal{B}}$ is not feasible. This happens if the worst escalation rule leading to $\widetilde{\mathcal{B}}$ does not satisfy the designer's resource constraint. By occasional escalation of $(1, K)$ and $(K, 1)$ the designer increases the escalation utility of type-1 players and, in turn, decreases their promised shares until the solution is resource feasible; (K, K)-matches remain settled for sure. The entire program is linear, and signals never improve.

E.2 Generalization: Monotone Lotteries

In this section we provide a general solution algorithm for *monotone lottery games* with an arbitrary number of types. The solution nests that of the previous paragraph, Fey and Ramsay (2011), Hörner, Morelli, and Squintani (2015), and the monotone cases in Bester

[1] In fact, we use the term *lottery* to emphasize that we could model the situation without the need of any formulation of a *game*. However, the application suggests existence of a *continuation game* with an ex-post equilibrium. The two are observationally equivalent.

and Wärneryd (2006). Let $U_i(m_i; \theta_i, \mathcal{B}|\theta_{-i})$ be θ_i's expected equilibrium payoff from the escalation game when her opponent is type θ_{-i}, the belief system is $\mathcal{B}$, and she holds m_i's belief.

Definition 10 (Lottery Game)**.** $\mathcal{E}$ is a lottery game if $U_i(\theta_i; \theta_i, \mathcal{B}|\theta_{-i})$ is constant in $\mathcal{B}$.

Definition 11 (Monotone Lottery Game)**.** A lottery game is monotone if $u(\theta_i, \theta_{-i}) - u(\theta_i{+}1, \theta_{-i})$ is weakly decreasing in θ_i and θ_{-i} and the prior p induces a weakly decreasing inverse hazard rate $w(\theta_i)$.

The lottery feature entails that expected continuation payoffs are linear in any $\beta_i(\cdot|m_i)$. Therefore, the problem is reminiscent of a standard mechanism-design problem with interdependent values. The solution is thus given by a linear program.

We define the ex-ante probability that the type profile (θ, k) will face each other in litigation as $\rho(\theta, k) := \rho_A(\theta)\beta_A(k|\theta)$ with $\sum_{(\theta,k)} \rho(\theta, k) = 1$. Naturally, $\rho(1, 1) = 1$ implies $\beta(1|1) = 1$ and 0 for all other beliefs. Substituting for $\beta_i(\theta_i|\theta_{-i})$ and rearranging yields the function $\widetilde{VV}(\theta_A, \theta_B)$, which is independent of any $\rho(\cdot, \cdot)$. The objective of $(\mathrm{P}^{\mathcal{B}}_{max})$ becomes

$$\Xi(\rho(\cdot, \cdot)) := 2u(K, K) + \sum_{\Theta^2 \setminus (K,K)} \rho(\theta_A, \theta_B)\widetilde{VV}(\theta_A, \theta_B). \tag{S.4}$$

We define the condition

$$(2V(1, (p, \rho^V)) - 1)\rho(1, 1) \leq (p(1))^2 \left(\Xi(\rho(\cdot, \cdot)) - 1\right). \tag{S.5}$$

Definition 12 (Top-Down Algorithm)**.** Denote the set of type pairs (k, θ) such that $\rho(k, \theta) > 0$ by Θ^2_+. Start with $\Theta^2_+ = \emptyset$.

1. Set $\rho(1, 1) = 1$ and check whether condition (S.5) is satisfied. If it is satisfied, then terminate. Otherwise continue at 2.
2. Identify the set $\Theta^2_N = \{(\theta_A, \theta_B)|(\theta_A, \theta_B) = \arg\max_{\Theta^2 \setminus \Theta^2_+} \widetilde{VV}(\theta_A, \theta_B)\}$.
   (a) Set $\rho(1, 1)$ to the solution of
   $$\sum_{\substack{(\theta_A,\theta_B)\in \\ \Theta^2_+ \cup \Theta^2_N}} \frac{p(\theta_A)p(\theta_B)}{(p(1))^2}\rho(1, 1) = 1. \tag{S.6}$$
   (b) Replace $\rho(\theta_A, \theta_B) = \frac{p(\theta_A)p(\theta_B)}{(p(1))^2}\rho(1, 1)\ \forall(\theta_A, \theta_B) \in \Theta^2_+ \cup \Theta^2_N$.
   (c) Check whether condition (S.5) is satisfied. If it is satisfied, decrease all ρ for the set Θ^2_N at the expense $\rho(1, 1)$ keeping the relation of 2(b) for $\Theta^2_+ \setminus \Theta^2_N$ until the condition holds with equality. Then, terminate. If (S.5) is violated, repeat step 2.

Proposition 9. *Suppose the escalation game is a monotone lottery. The optimal arbitration is the solution to the top-down algorithm.*

Proposition 3 is a corollary to Proposition 9.

Proof. Define

$$A_i(\theta_i, \theta_{-i}) := u(\theta_i, \theta_{-i}) - u(\theta_i{+}1, \theta_{-i}) \text{ and } W(\theta, k) = u(\theta_A, \theta_B) + u(\theta_B, \theta_A),$$

$$\textit{and } \widetilde{VV}(\theta, k) := \omega(\theta)A_A(\theta, k) + \omega(\theta)A_B(k, \theta) + W(\theta, k) - W(K, K).$$

The objective becomes

$$\sum_i \sum_{\theta_i \in \Theta} \rho(\theta_i) \left(\sum_{\theta_{-i} \in \Theta} \beta_i(\theta_{-i}|\theta_i) u(\theta_i, \theta_{-i}) + \sum_{\theta_{-i} \in \Theta} \beta_i(\theta_{-i}|\theta_i) \omega(\theta_i) A_i(\theta_i, \theta_{-i}) \right). \tag{S.7}$$

Substituting in for $\rho(\theta_A, \theta_B)$ we get

$$\max_{\rho(\cdot,\cdot)} \Big(2u(K,K) + \sum_{(\theta,k) \in \Theta \setminus (K,K)} \rho(\theta,k) \widetilde{VV}(\theta,k) \Big), \tag{S.8}$$

were we treat $A_i(K, \theta_{-i})$ as an arbitrary positive real number to avoid case distinctions. Simple algebra shows that $\widetilde{VV}$ is non-decreasing for monotone lotteries. Substituting $\rho(\theta_A, \theta_B) = (\gamma(\theta_A, \theta_B) p(\theta_A) p(\theta_B)) / Pr(\mathcal{E})$ and rearranging yields that conditions (S.5) and (S.6) are necessary and sufficient for the feasibility constraint (5).

Hence, the top-down algorithm point-wise maximizes (S.8) subject to (5) by construction. What remains is to show that any ignored constraint is satisfied. We show this using monotonicity, i.e., $p(\theta_i)\rho(\theta_i{-}1, \theta_{-i}) \geq p(\theta_i{-}1)\rho(\theta_i, \theta_{-i})$, for all θ_i, θ_{-i}.

Monotonicity trivially holds if $\gamma(1,1) \neq 1$, as it implies an optimum at $\rho(1,1) = 1$. Now, assume $\gamma(1,1) = 1$. By Bayes' rule

$$\gamma(\theta_A, \theta_B) = \frac{\rho(\theta_A, \theta_B)}{p(\theta_A) p(\theta_B)} Pr(\mathcal{E}).$$

When $\gamma(\theta_A, \theta_B) > 0$, then $\gamma(\theta_A{-}1, \theta_B) = 1$ and

$$\begin{aligned} Pr(\mathcal{E}) \rho(\theta_A{-}1, \theta_A) &= p(\theta_A{-}1) p(\theta_B), \\ Pr(\mathcal{E}) \rho(\theta_A, \theta_A) &\leq p(\theta_A) p(\theta_B). \end{aligned}$$

Thus, monotonicity holds since

$$p(\theta_A) \rho(\theta_A{-}1, \theta_A) \geq p(\theta_A{-}1) \rho(\theta_A, \theta_A).$$

Monotonicity implies that all but the local downward incentive constraints are redundant as

$$\sum_{\theta_{-i}=1}^{K} \left(p(\theta_i') \rho(\theta_i, \theta_{-i}) - p(\theta_i) \rho(\theta_i', \theta_{-i}) \right) \left[u(\theta_i, \theta_{-i}) - u(\theta_i', \theta_{-i}) \right] \geq 0, \tag{S.9}$$

for all θ_i and $\theta_i' > \theta_i$. Moreover, $z_i(\theta_i) - z_i(\theta_i{-}1) = y_i(\theta_i; \theta_i{-}1) - y_i(\theta_i; \theta_i) \geq 0$ which implies that $z_i(\theta_i) \geq 0$. Finally, we claim that only the lowest types' participation constraint binds at the optimum. It is straightforward to verify that this implies that downward local incentive constraints hold with equality. We verify the claim by induction. We first show that $\Pi_i(2;2) \geq V(2, (p, \rho^V))$. By local incentive compatibility and $\Pi_i(1;1) \geq V(1, (p, \rho^V))$,[2] we know that $\Pi_i(2;2) = V(2, (p, \rho^V)) - y_i(1;1) + y_i(1;2)$. We want to show that $V(1, (p, \rho^V)) - V(2, (p, \rho^V)) \geq y_i(1;1) - y_i(1;2)$. The game is a simple lottery, so the condition reduces to

$$\sum_{\theta_{-i}} [p(\theta_{-i}) - \gamma_i(1) \beta_i(\theta_{-i}|1)][u(1, \theta_{-i}) - u(2, \theta_{-i})] \geq 0.$$

[2]In the optimal mechanism it holds that $\Pi_i(1;1) = V(1, (p, \rho^V))$.

Moreover,

$$[p(\theta_{-i}) - \gamma_i(1)\beta_i(\theta_{-i}|1)] = \frac{1}{p(1)}[Pr(1,\theta_{-i}) - Pr(1,\theta_{-i},\mathcal{E})] \geq 0.$$

Assume that $\Pi_i(\theta_i;\theta_i) \geq V(\theta_i,(p,\rho^V))$ for all $\theta_i < K$. Using a similar argument verifies that $V(K-1,(p,\rho^V)) - V(K,(p,\rho^V)) \geq y_i(K-1,K-1) - y_i(K-1,K)$.

Hence ignoring all participation constraints but those of type 1 is without loss. As $z_i(\theta_i)$ is weakly increasing, downward local incentive constraints are binding. $\square$

F Sufficiency of MDR

Proposition 10. *Suppose (MDR) holds. Local incentive constraints imply (global) downward incentive constraints.*

The two following lemmas proof Proposition 10.

Lemma 5. *Local incentive constraints and MDR imply that $\gamma_i(m)D_i(m,k;\mathcal{B})$ is non-increasing in m for any $m < k$.*

Proof. Take i and any m and $m+1$. Local incentive compatibility implies that

$$y_i(m,m) - y_i(m,m+1) \geq z_i(m+1) - z(m) \geq y_i(m+1,m) - y_i(m+1,m+1), \text{ thus,}$$

$$\gamma_i(m)D_i(m;m,\mathcal{B}) \geq \gamma_i(m{+}1)D_i(m+1;m,\mathcal{B}) \Leftrightarrow \frac{\gamma_i(m)}{\gamma_i(m{+}1)} \geq \frac{D_i(m{+}1;m,\mathcal{B})}{D_i(m;m,\mathcal{B})} \tag{S.10}$$

The term $\gamma_i(m)D(m,k,\mathcal{B})$ increases in k if

$$\gamma_i(m)D_i(m;k,\mathcal{B}) \geq \gamma_i(m{+}1)D_i(m{+}1;k,\mathcal{B}) \Leftrightarrow \frac{\gamma_i(m)}{\gamma_i(m{+}1)} \geq \frac{D_i(m{+}1,k;\mathcal{B})}{D_i(m,k;\mathcal{B})} \tag{S.11}$$

which holds by MDR and (S.10) if $m < k$. $\square$

Lemma 6. *If $\gamma_i(m)D_i(m;k,\mathcal{B})$ is non-increasing in m on some interval $[\underline{m},\overline{m}]$ and $k \in [\underline{m},\overline{m}]$, then local incentive compatibility for type k implies incentive compatibility for any report in that interval.*

Proof. Take k and m. Incentive compatibility holds iff

$$\begin{aligned} & z_i(k) + \gamma_i(k)U_i(k;k,\mathcal{B}) \geq z_i(m) + \gamma_i(m)U_i(m;k,\mathcal{B}) \\ \Leftrightarrow \quad & -\gamma_i(m)U_i(m;k,\mathcal{B}) \geq z_i(m) - z_i(k) - \gamma_i(k)U_i(k;k,\mathcal{B}) \end{aligned} \tag{S.12}$$

Assume first that $m < k$. Adding $\sum\limits_{v=m}^{k-1} \gamma_i(m)U_i(m;v,\mathcal{B}) - \gamma_i(m)U_i(m;v,\mathcal{B})$ to the LHS of (S.12) turns it into

$$\gamma_i(m)\left(\sum_{v=m}^{k-1} D_i(m;v,\mathcal{B}) - U(m;m,\mathcal{B})\right).$$

Adding $\sum\limits_{v=m+1}^{k-1} z_i(v) - z_i(v)$ to the RHS of (S.12) and using local upward incentive compatibility, i.e., $z_i(v) - z_i(v+1) \le y_i(v+1, v+1) - y_i(v+1, v)$, leads

$$\sum_{v=m+1}^{k-1} (z_i(v) - z_i(v+1)) - \gamma_i(k) U_i(k; k, \mathcal{B}) \le \sum_{v=m}^{k-1} \gamma_i(v+1) D_i(v+1; v, \mathcal{B}) - \gamma_i(m_i) U_i(m; m, \mathcal{B}).$$

The RHS of the above equation is an upper bound on the RHS of (S.12). Thus, (S.12) holds if

$$\sum_{v=m}^{k-1} \gamma_i(m) D_i(m; v, \mathcal{B}) \ge \sum_{v=m}^{k-1} \gamma_i(v+1) D_i(v+1; v, \mathcal{B}) \tag{S.13}$$

which holds since $\gamma_i(m) D_i(m; v, \mathcal{B})$ is non-increasing in m.

For $m > k$, take equation (S.12), add and subtract $\sum_{v=k+1}^{m} \gamma_i(m) U_i(m; v, \mathcal{B})$ from the LHS. Iteratively applying local downward incentive compatibility to $z_i(m)$ and simplify to

$$\sum_{v=k}^{m-1} \gamma_i(m) D_i(m; v, \mathcal{B}) \ge \sum_{v=k}^{m-1} \gamma_i(v+1) D_i(v+1; v, \mathcal{B}).$$

which again holds since $\gamma_i(m) D_i(m; v, \mathcal{B})$ is non-increasing in m. □

The special case of Lemma 6 with $\underline{m} = 1, \overline{m} = k$ for any k concludes the proof of Proposition 10. In addition Lemma 6 proves Proposition 7.

G Statement of the General Problem and of the Lagrangian Objective

For any i, θ, the constraints to the minimization problem are

$$\begin{aligned}
\forall \theta \ne \theta' \quad -(z_i(\theta) - z_i(\theta')) - y_i(\theta;\theta) + y_i(\theta';\theta) &\le 0, && (IC)\\
-z_i(\theta) - y_i(\theta;\theta) + V_i(\theta, (p, \rho^V)) &\le 0, && (PC_i)\\
-1 + \sum_i \sum_\theta p(\theta) z_i(\theta) + Pr(\mathcal{E}) &\le 0, && (RC)\\
-z_i(\theta) &\le 0, && (EPI)\\
\gamma(\theta_A, \theta_B) - 1 &\le 0, && (F)\\
\text{and } \forall Q \subset \Theta &&&\\
\sum_i \sum_{\theta \in Q_i} z_i(\theta) p(\theta) + \sum_{(\theta_A, \theta_B) \in \bar{Q}} (1 - \gamma(\theta_A, \theta_B)) p(\theta_A) p(\theta_B) - 1 + Pr(\mathcal{E}) &\le 0. && (EP)
\end{aligned}$$

We now derive the Lagrangian representation of the optimization problem. First, we state the complementary slackness conditions and the respective Lagrangian multipliers

$$\begin{aligned}
[z_i(\theta) - z_i(\theta') + y_i(\theta;\theta) - y_i(\theta';\theta)] \nu^i_{\theta,\theta'} &= 0, & \nu^i_{\theta,\theta'} &\ge 0;\\
[z_i(\theta) + y_i(\theta;\theta) - V_i(\theta, (p, \rho^V))] \lambda^i_\theta &= 0, & \lambda^i_\theta &\ge 0;\\
\Big[1 - \sum_i \sum_\theta p(\theta) z_i(\theta) - Pr(\mathcal{E})\Big] \delta &= 0, & \delta &\ge 0;\\
z_i(\theta) \zeta^i_\theta &= 0, & \zeta^i_\theta &\ge 0;\\
[1 - \gamma(\theta_A, \theta_B)] \mu_{\theta_A,\theta_B} &= 0, & \mu_{\theta_A,\theta_B} &\ge 0;
\end{aligned}$$

$$\Big[-\sum_i \sum_{\theta\in Q_i} z_i(\theta)p(\theta) - \sum_{\substack{(\theta_A,\theta_B)\\ \in\bar{Q}}} (1-\gamma(\theta_A,\theta_B))p(\theta_A)p(\theta_B) + 1 - Pr(\mathcal{E})\Big]\eta_Q = 0, \qquad \eta_Q \geq 0.$$

For any Lagrangian multiplier, say t, we introduce the following notation $\tilde{t} \equiv \frac{t}{\delta}$. Let Q^2 be the set of all combinations of Q. Let $\tilde{e}^i_\theta := p(\theta) \sum_{Q\in Q^2|\theta\in Q_i} \tilde{\eta}_Q$ and define

$$\tilde{\Lambda}^i(\theta) := \sum_{k=1}^{\theta} \tilde{\lambda}^i_k, \qquad \tilde{E}^i(\theta) := \sum_{k=1}^{\theta} \tilde{e}^i_k, \qquad \tilde{Z}^i(\theta) := \sum_{k=1}^{\theta} \tilde{\zeta}^i_\theta \tag{S.14}$$

Next, we characterize the solution in terms of the Lagrangian objective. As stated directly above Corollary 3 we can maximize with respect to $\{Pr(\sigma), \boldsymbol{\rho}(\sigma)\}_{\sigma\in\Sigma}$. Further, consider the symmetrized problem, i.e., the problem in which for each signal σ and associated probabilities $\rho(\theta_A,\theta_B|\sigma)$ there exists signal realization σ' such that $Pr(\sigma) = Pr(\sigma')$ and $\rho(\theta_A,\theta_B|\sigma) = \rho(\theta_B,\theta_A|\sigma')$ for all (θ_A,θ_B).[3]

Lemma 7. *The lottery $\{Pr(\sigma), \boldsymbol{\rho}(\sigma)\}_\sigma$ is an optimal solution to the designers problem if and only if there are Lagrangian multipliers that satisfy complementary slackness given the lottery and the lottery includes every $\boldsymbol{\rho}(\sigma)$ that maximizes*

$$\begin{aligned}\widehat{\mathcal{L}}(\mathcal{B}(\sigma)) :=& \mathcal{T}(\mathcal{B}(\sigma)) + \sum_i \Bigg[\sum_{\theta=1}^{K} \rho_i(\theta|\sigma)(\frac{\mathsf{m}^{\mathsf{i}}_\theta}{p(\theta)})U_i(\theta;\theta,\mathcal{B}(\sigma)) \\ &+ \sum_{\theta=1}^{K-1}\sum_{\theta'=\theta+1}^{K} \frac{\tilde{\nu}^i_{\theta,\theta'} + \mathsf{M}^{\mathsf{i}}(\theta) - \tilde{\nu}^i(\theta,\theta')}{p(\theta)} \rho_i(\theta|\sigma)\{U_i(\theta;\theta,\mathcal{B}(\sigma)) - U_i(\theta;\theta',\mathcal{B}(\sigma))\} \\ &- \sum_{\theta=1}^{K-1}\sum_{\theta'=\theta+1}^{K} \frac{\tilde{\nu}^i_{\theta,\theta'}}{p(\theta')}\rho_i(\theta'|\sigma)\left\{U_i(\theta';\theta,\mathcal{B}(\sigma)) - U_i(\theta';\theta',\mathcal{B}(\sigma))\right\}\Bigg],\end{aligned} \tag{S.15}$$

where $\mathsf{m}^{\mathsf{i}}_\theta := p(\theta) + \tilde{e}^i_\theta - \tilde{\zeta}^i_\theta$, $\mathsf{M}^{\mathsf{i}}(\theta) := \tilde{\Lambda}^i(\theta) - \sum_{k=1}^{k=\theta} p(k) - \tilde{E}^i(\theta) + \tilde{Z}^i(\theta)$, $\tilde{\nu}_i(\theta,\theta') := \sum_{k=1}^{\theta}\sum_{\hat{\theta}>\theta}^{K}(\tilde{\nu}^i_{\hat{\theta},\theta} - \tilde{\nu}^i_{\theta,\hat{\theta}}) - (\tilde{\nu}^i_{\theta',\theta} - \tilde{\nu}^i_{\theta,\theta'})$.

$$\begin{aligned}\mathcal{T}(\mathcal{B}(\sigma)) :=& \sum_{Q\in Q^2}\sum_{(\theta_A,\theta_B)\in\tilde{Q}} [\rho(\theta_A|\sigma)\beta_1(\theta_B|\theta_A,\sigma)]\tilde{\eta}_Q \\ &- \sum_{\theta_A\times\theta_B} \frac{\rho_1(\theta_A|\sigma)\beta_1(\theta_B|\theta_A,\sigma)}{p(\theta_A)p(\theta_B)}\tilde{\mu}_{\theta_A,\theta_B}.\end{aligned} \tag{S.16}$$

Hence, $\boldsymbol{\rho} = \sum_\sigma Pr(\sigma)\boldsymbol{\rho}(\sigma)$ is a maximizer of the concave hull of the above function. Moreover, the following is true at the optimum:

- *Constraint* (5) *is always binding, i.e., $\delta > 0$.*
- $\mathsf{M}^i(\theta) = \tilde{\nu}^i(\theta,\theta') + \tilde{\nu}_{\theta',\theta} - \tilde{\nu}^i_{\theta,\theta'}$ *for any θ'.*
- *If the Border constraints are redundant, then $\tilde{e}^i_\theta = \tilde{E}_i(\theta) = 0 = \tilde{Z}^i(\theta) = \tilde{\zeta}^i_\theta$.*
- *If $\tilde{\Lambda}^i(\theta) + \tilde{Z}^i(\theta) - \sum_{v=1}^{v=\theta} p(v) - \tilde{E}^i(\theta) > 0$, then the* downward incentive constraints

[3] We can symmetrize the problem in this way without loss of generality as players are symmetric ex-ante.

are binding. *If in addition the upward incentive constraints are redundant, then* $\tilde{\nu}^i_{\theta,\theta'} = 0$ *for all* $\theta' \geq \theta$.

- *If* $\tilde{\Lambda}^i(\theta) + \tilde{Z}^i(\theta) - \sum_{v=1}^{v=\theta} p(v) - \tilde{E}^i(\theta) < 0$*, the* upward incentive constraints *are* binding. *If in addition the downward incentive constraints are redundant, then* $\tilde{\nu}^i_{\theta,\theta'} = 0$ *for all* $\theta' < \theta$.

- *If local incentive constraints are sufficient, then* $\tilde{\nu}^i_{\theta,\theta'} = 0$ *for any* θ *such that* $\theta' > \theta{+}1$ *or* $\theta' < \theta{-}1$. *Moreover,* $\tilde{\nu}^i(\theta, \theta') = \mathsf{M}^i(\theta)$ *for any* θ, θ' *such that* $\theta' \neq \{\theta{-}1, \theta{+}1\}$.

Proof. The first part of the proof is along the heuristics below Proposition 6. We manipulate the Lagrangian, $\mathcal{L}$, and derive a tractable dual problem. The second part verifies that the optimum is on the concave hull of the objective. The Lagrangian takes the form

$$\begin{aligned}
\mathcal{L} &= Pr(\mathcal{E}) + \delta[-1 + \sum_i \sum_{\theta=1}^{K} p(\theta) z_i(\theta) + Pr(\mathcal{E})] \\
&+ \sum_i \sum_{\theta=1}^{K} [-z_i(\theta) - y_i(\theta;\theta) + V_i(\theta, (p, \rho^V))]\lambda^i_\theta \\
&+ \sum_i \sum_{\theta=1}^{K} \sum_{\theta' \in \Theta \setminus \theta}^{K} [-z_i(\theta) + z_i(\theta') - y_i(\theta;\theta) + y_i(\theta';\theta)]\nu^i_{\theta,\theta'} \\
&+ \sum_{Q \in Q^2} \Big[\sum_i \sum_{\theta \in Q_i} z_i(\theta) p(\theta) + \sum_{(\theta_A,\theta_B) \in \bar{Q}} (1 - \gamma(\theta_A, \theta_B)) p(\theta_A) p(\theta_B) - 1 + Pr(\mathcal{E})\Big]\eta_Q \\
&+ \sum_{\theta_A \times \theta_B} [\gamma(\theta_A, \theta_B) - 1]\mu_{\theta_A,\theta_B} - \sum_i \sum_\theta z_i(\theta)\zeta^i_\theta
\end{aligned} \tag{S.17}$$

Using Theorem 1 we optimize over $\{z_i(\cdot), \gamma^\sigma(1,1), \boldsymbol{\rho}(\sigma)\}$, with $\gamma^\sigma(1,1){:=}Pr(\mathcal{E}, \sigma|\theta_A{=}1, \theta_B{=}1)$.
Step 1: Eliminating $z_i(\cdot)$ using First-order Conditions. Define $\nu^i_{K+1,K} := 0 =: \nu^i_{1,0} = \nu^i_{0,1}$ for ease of notation. The FOC w.r.t. $z_i(\theta)$ are

$$p(\theta)\delta - \lambda^i_\theta - [\underset{\rightarrow}{\nu}^i(\theta) - \underset{\leftarrow}{\nu}^i(\theta)] + p(\theta) \sum_{Q \in Q^2 | k \in Q_i} \eta_Q - \zeta^i_\theta = 0. \tag{S.18}$$

Summing over all K conditions in (S.18) and recalling definition (S.14) yields

$$1 = \tilde{\Lambda}^i(K) - \tilde{E}^i(K) + \tilde{Z}^i(K), \tag{S.19}$$

(S.18) holds for all θ if and only if

$$\sum_{k=1}^{\theta} \sum_{\hat{\theta} > \theta}^{K} (\tilde{\nu}^i_{\hat{\theta},\theta} - \tilde{\nu}^i_{\theta,\hat{\theta}}) = -\sum_{v=1}^{v=\theta} p(v) + \tilde{\Lambda}^i(\theta) - \tilde{E}^i(\theta) + \tilde{Z}^i(\theta). \tag{S.20}$$

Thus, $\sum_{k=1}^{\theta} \sum_{\hat{\theta}>\theta}^{K} \tilde{\nu}^i_{\hat{\theta},\theta} > 0$ if $\mathsf{M}^{\mathsf{i}}(\theta) > 0$ and vice versa for $\sum_{k=1}^{\theta} \sum_{\hat{\theta}>\theta}^{K} \tilde{\nu}^i_{\theta,\hat{\theta}}$. We solve (S.18) for λ^i_θ and substitute into (S.17). We also substitute $\tilde{\nu}^i_{\theta',\theta} = \mathsf{M}^{\mathsf{i}}(\theta) + \tilde{\nu}(\theta, \theta') - \tilde{\nu}^i\theta', \theta$ for all $\theta' > \theta$ into (S.17) and sort terms. Moreover, all terms involving $z_i(\cdot)$ cancel out from (S.17) via (S.18).

Step 2: Reformulating the Lagrangian Objective. Given the above necessary conditions, we manipulate the Lagrangian objective to derive a more tractable maximization problem. Define $\eta := \sum_{Q\in Q^2} \eta_Q$ and $\tilde{\eta} := \sum_{Q\in Q^2} \tilde{\eta}_Q$. Next, using Bayes' rule together with the homogeneity established in the proof of Theorem 1 (step 1), applying algebra and using the first-order-conditions it is straightforward to show that (S.17) admits the following representation

$$\mathcal{L} = Pr(\mathcal{E})(1+\delta+\eta) - \delta C - \delta \sum_{\sigma} \Pr(\mathcal{E},\sigma)\widehat{\mathcal{L}}(\mathcal{B}(\sigma)), \tag{S.21}$$

where C is a constant that is independent of the choice variables,

$$C{:=}1+\tilde{\eta} - \sum_{Q\in Q^2} \sum_{(\theta_A,\theta_B)\in\tilde{Q}} p(\theta_A)p(\theta_B)\tilde{\eta}_Q - \sum_i \sum_\theta \tilde{\lambda}_\theta V_i(\theta,(p,\rho^V)) + \sum_{\theta_A\times\theta_B} \tilde{\mu}_{\theta_A,\theta_B} < 0.$$

Define $\gamma^\sigma(\theta_A,\theta_B) := Pr(\mathcal{E},\sigma|\theta_A,\theta_B)$. From the proof of Theorem 1 (step 1) with $\alpha^\sigma = \gamma^\sigma(1,1)$ it follows that $\gamma^\sigma(\theta_A,\theta_B) = f(\mathcal{B}(\sigma),\theta_A,\theta_B)\gamma^\sigma(1,1)$, where $f(\mathcal{B}(\sigma),\theta_A,\theta_B)$ is a positive real number. Thus, $Pr(\mathcal{E},\sigma) = \gamma^\sigma(1,1)R(\mathcal{B}(\sigma))$ with $R(\mathcal{B}(\sigma)) := \sum_{\theta_A\times\theta_B} p(\theta_A)p(\theta_B)f(\mathcal{B}(\sigma),\theta_A,\theta_B)$. Plugging into (S.21) yields

$$\mathcal{L} = \sum_\sigma \gamma^\sigma(1,1)R(\mathcal{B}(\sigma))(1+\delta+\tilde{\mu}) - \delta C - \delta\sum_\sigma \gamma^\sigma(1,1)R(\mathcal{B}(\sigma))\widehat{\mathcal{L}}(\mathcal{B}(\sigma)). \tag{S.22}$$

The FOC of (S.22) w.r.t. $\gamma^\sigma(1,1)$ is

$$R(\mathcal{B}(\sigma))\Big((1+\delta+\eta) - \delta\widehat{\mathcal{L}}(\mathcal{B}(\sigma))\Big) = 0, \tag{S.23}$$

for each signal. By Assumption 1 $R(\mathcal{B}(\sigma)) > 0$ and thus, $\widehat{\mathcal{L}}(\mathcal{B}(\sigma)) - 1 > 0$ if $\gamma^\sigma(1,1) > 0$. Therefore, $\delta = (1+\eta)(\widehat{\mathcal{L}}(\mathcal{B}(\sigma)) - 1)^{-1}$. As δ is independent of σ, $\widehat{\mathcal{L}}(\mathcal{B}(\sigma))$ takes the same value for each signal realization. Substituting into (S.22) and simplifying yields

$$\mathcal{L} = \frac{-C(1+\eta)}{\widehat{\mathcal{L}}(\mathcal{B}(\sigma)) - 1} \tag{S.24}$$

which is minimized if and only if $\widehat{\mathcal{L}}(\mathcal{B}(\sigma))$ is maximized.[4]

Thus, for the optimal multipliers one constructs the concave hull of $\widehat{\mathcal{L}}$ by taking spreads over those $\mathcal{B}(\sigma)$ that are a global maximum of $\widehat{\mathcal{L}}$. If there are multipliers and a unique maximizer, $\mathcal{B}(\sigma)$, that satisfies the complementary slackness conditions, signals do not improve. If there are multiple global optima and there is a spread that satisfies the complementary slackness conditions, signals improve. □

H Binding Constraints for Games with Linearity in Types

In this part we state a more general version of Lemma 3 if games satisfy linearity in types, but Assumption 3 and MDR may not hold. The proof is analogous to that of Lemma 3. We then characterize the set of binding constraints C_R under this relaxed condition.

[4]The Lagrangian multipliers are necessarily such that C is negative at the optimum. Otherwise (S.17) and (S.24) imply that $Pr(\mathcal{E})$ is negative, a contradiction to Assumption 1 and /or the fact that a feasible solution to the minimization problem always exists: take a degenerate signal distribution and set $\gamma(\theta_A,\theta_B) = 1$ for all type profiles.

Lemma 8. *The following holds for the optimal mechanism*

i. all incentive constraints not concerning adjacent types are redundant,

ii. if both local incentive constraints are redundant for type θ_i, then her participation constraint is satisfied with equality or $z_i(\theta_i) = 0$,

iii. the participation constraint for at least one type of every player is binding.

Although the set of binding constraints may depend on the exact location of the optimum, Lemma 8 provides enough structure to determine settlement values as a function of escalation values. This reduces the dimensionality of the choice set.

Corollary 4. *Consider the escalation values $\{y_i(\theta_i;\theta_i), y_i(\theta_i{+}1;\theta_i), y_i(\theta_i{-}1;\theta_i)\}$ for all $\theta_i \in \Theta$ of the optimal mechanism. There is a partition of $P(\Theta) = \{\Theta^{z_i=0}, \Theta^{PC}, \Theta^{IC^+}, \Theta^{IC^-}\}$ such that*

$$z_i(\theta_i) = z_i(\tilde{\theta}_i) + y_i(\tilde{\theta}_i;\theta_i) - y_i(\theta_i;\theta_i), \quad \forall \theta_i \in \{\Theta^{IC^+}, \Theta^{IC^-}\} \tag{Z}$$

with

$$\tilde{\theta}_i = \begin{cases} \theta_i + 1 & \textit{if } \theta_i \in \Theta^{IC^+} \\ \theta_i - 1 & \textit{if } \theta_i \in \Theta^{IC^-}. \end{cases}$$

Moreover $z_i(\theta_i) = 0$ for $\theta_i \in \Theta^{z_i=0}$ and $z_i(\theta_i) = V_i(\theta_i, (p, \rho^V)) - y_i(\theta_i;\theta_i)$ for $\theta_i \in \Theta^{PC}$.

I Consistent Belief System

We first confine ourselves to interior belief systems, i.e., belief systems that do not have zero entries.

We derive a representation that links the belief system, $\mathcal{B}(\sigma)$ to the escalation rule, γ^σ. We interpret every type profile as a node in a network. Node (θ_A, θ_B) has the value $\gamma^\sigma(\theta_A, \theta_2)$. The (values of the) nodes are linked using a *transition* function $q_i(\theta_A, \theta_B)$.

Observation 1. Consider nodes (θ_i, θ'_{-i}), (θ_i, θ_{-i}) and the *transition* function $q_i(\theta'_{-i}, \theta_{-i}|\theta_i) := \frac{p(\theta_{-i})}{p(\theta'_{-i})} \frac{\beta_i(\theta'_{-i}|\theta_i,\sigma)}{\beta_i(\theta_{-i}|\theta_i,\sigma)}$. Then,

$$\gamma^\sigma(\theta'_1, \theta_B) = q_B(\theta'_A, \theta_A|\theta_B)\gamma^\sigma(\theta_A, \theta_B),$$

$$\gamma^\sigma(\theta_A, \theta'_B) = q_A(\theta'_B, \theta_B|\theta_A)\gamma^\sigma(\theta_A, \theta_B).$$

The result follows using Bayes' rule as

$$q_i(\theta'_{-i}, \theta_{-i}|\theta_i) = \frac{p(\theta_{-i})}{p(\theta'_{-i})} \frac{\gamma^\sigma(\theta_i, \theta'_{-i})}{\gamma^\sigma(\theta_i, \theta_{-i})} \frac{p(\theta'_{-i})}{p(\theta_{-i})}.$$

Fix two nodes in the network, say (θ_A, θ_B) and (k_A, k_B). There are several paths that connect the two nodes. Starting from (θ_A, θ_B) we can go to (k_A, θ_B) and then to (k_A, k_B). Equivalently we can approach (k_A, k_B) through (θ_A, k_B). Bayes' rule implies that both paths have the same length, or, the values of the nodes are the same. Using Observation 1 we have that

$$\gamma^\sigma(k_A, k_B) = q_A(k_B, \theta_B|k_A)q_B(k_A, \theta_A|\theta_B)\gamma^\sigma(\theta_A, \theta_B),$$

$$\gamma^\sigma(k_A, k_B) = q_B(k_A, \theta_A|k_B)q_A(k_B, \theta_B|\theta_A)\gamma^\sigma(\theta_A, \theta_B).$$

Definition 13 (Bayes' consistency)**.** A belief system is Bayes' consistent if for every (θ_A, θ_B), (k_A, k_B)

$$q_A(k_B, \theta_B|k_A)q_B(k_A, \theta_A|\theta_B) = q_B(k_A, \theta_A|k_B)q_A(k_B, \theta_B|\theta_A). \tag{S.25}$$

Definition 14 ($(1,1)$-Consistent)**.** A belief system is $(1,1)$-consistent if for every (θ_A, θ_B),

$$q_A(\theta_B, 1|\theta_A)q_B(\theta_A, 1|1) = q_B(\theta_A, 1|\theta_B)q_A(\theta_B, 1|1). \tag{S.26}$$

Lemma 9. *An interior belief system is Bayes' consistent if and only if it is* $(1,1)$-*consistent.*

Proof. Bayes' consistency trivially implies (1,1)-consistency. For the reverse direction take any $\gamma^\sigma(k_A, k_B)$ and $\gamma^\sigma(\theta_A, \theta_B)$. We want to show that (1,1)-consistency implies

$$q_A(k_B, \theta_B|k_A)q_B(k_A, \theta_A|\theta_B) = q_B(k_A, \theta_A|k_B)q_A(k_B, \theta_B|\theta_A).$$

By (1,1)-consistency we know that

$$q_A(\theta_B, 1|k_A)q_B(k_A, 1|1) = q_B(k_A, 1|\theta_B)q_A(\theta_B, 1|1), \tag{S.27}$$
$$q_A(k_B, 1|\theta_A)q_B(\theta_A, 1|1) = q_B(\theta_A, 1|k_B)q_A(k_B, 1|1), \tag{S.28}$$
$$q_A(\theta_B, 1|\theta_A)q_B(\theta_A, 1|1) = q_B(\theta_A, 1|\theta_B)q_A(\theta_B, 1|1), \tag{S.29}$$
$$q_A(k_B, 1|k_A)q_B(k_A, 1|1) = q_B(k_A, 1|k_B)q_A(k_B, 1|1). \tag{S.30}$$

Plugging into $q_i(\theta_{-i}, 1|\theta_i)q_{-i}(\theta_i, 1|1) = q_{-i}(\theta_i, 1|\theta_{-i})q_i(\theta_{-i}, 1|1)$, and rearranging yields

$$\frac{\beta_i(\theta_{-i}|\theta_i, \sigma)}{\beta_{-i}(\theta_i|\theta_{-i}, \sigma)} = \frac{\beta_i(1|\theta_i, \sigma)}{\beta_{-i}(1|\theta_{-i}, \sigma)} \frac{\beta_i(\theta_{-i}|1, \sigma)}{\beta_i(1|1, \sigma)} \frac{\beta_{-i}(1|1, \sigma)}{\beta_{-i}(\theta_i|1, \sigma)}. \tag{S.31}$$

Arrange all equations (S.27) to (S.30) according to (S.31). Observe on these transformations that the RHS of (S.27) times that of (S.28) is the same as the RHS of (S.29) times that of (S.30). Using the respective LHS of the equations yields

$$\begin{aligned} & \frac{\beta_A(k_B|k_A, \sigma)}{\beta_A(\theta_B|k_A, \sigma)} \frac{\beta_B(k_A|\theta_B, \sigma)}{\beta_B(\theta_A|\theta_B, \sigma)} = \frac{\beta_B(k_A|k_B, \sigma)}{\beta_B(\theta_A|k_B, \sigma)} \frac{\beta_A(k_B|\theta_A, \sigma)}{\beta_A(\theta_B|\theta_A, \sigma)} \\ \Leftrightarrow \quad & q_A(k_B, \theta_B|k_A)q_B(k_A, \theta_A|\theta_B) = q_B(k_A, \theta_A|k_B)q_A(k_B, \theta_B|\theta_A). \end{aligned}$$

$\square$

Using Lemma 9 consistency is reduced to (1,1)-consistency. Thus, any consistent belief system is implemented by some escalation rule only if every node (θ_A, θ_B) has a value weakly below 1. The value of a node is given by the length of the path connecting the node with the initial node, i.e.,

$$\gamma^\sigma(\theta_A, \theta_B) = q_A(\theta_B, 1|\theta_A)q_B(\theta_A, 1|1)\gamma^\sigma(1, 1). \tag{S.32}$$

Note that the above exposition implies the following:

Observation 2. An interior belief system can be attained via some escalation rule if and only if it is Bayes' consistent.

Finally, observe that Bayes' consistency is equivalent to the definition of consistency given in the main text.

Finally, we show that it is without loss of generality to focus on interior belief systems.

Lemma 10. *A belief system can be implemented if and only if it can be approximated by a convergent sequence of implementable interior-belief systems.*

Proof. Take a sequence of consistent $\mathcal{B}_n(\sigma) \to \mathcal{B}(\sigma)$. $\mathcal{B}_n(\sigma)$ is consistent, thus Observation 2 implies some function $f : \mathcal{B}(\sigma) \to [0,1]^{K \times K}$, such that $f(\mathcal{B}_n(\sigma)) = \gamma_n^\sigma$ with γ_n^σ implementing $\mathcal{B}_n(\sigma)$. Since f is continuous, $\lim_{n\to\infty} f(\mathcal{B}_n(\sigma)) = f(\lim_{n\to\infty} \mathcal{B}_n(\sigma)) = \gamma^\sigma$. Equation (S.26) can be rewritten as

$$g_L(\mathcal{B}(\sigma)) = g_R(\mathcal{B}(\sigma)), \tag{S.33}$$

where both g_L and g_R are continuous functions from belief systems to $\mathbb{R}$. We can conclude that $g_L(\mathcal{B}(\sigma)) - g_R(\mathcal{B}(\sigma)) = \lim_{n\to\infty}[g_L(\mathcal{B}_n(\sigma)) - g_R(\mathcal{B}_n(\sigma))] = 0$ and $\mathcal{B}(\sigma)$ satisfies equation (S.26). This holds because $g_L(\mathcal{B}_n(\sigma)) - g_H(\mathcal{B}_n(\sigma)) = 0$.

Conversely, take any $\mathcal{B}(\sigma)$ being implemented by some γ^σ. We show that we can find a sequence of interior belief systems that are consistent and converge to $\mathcal{B}(\sigma)$: Let $\hat{\gamma}^\sigma$ be the escalation rule so that $\mathcal{B}(\sigma)$ is implemented by $\hat{\gamma}^\sigma$. Choose a sequence of escalation rules in the interior that converges to $\hat{\gamma}^\sigma$. By Bayes' rule every element of the sequence, γ_n^σ, corresponds to some belief system $\mathcal{B}_n(\sigma)$. Moreover, Observation 2 implies that there exists a continuous function, say $f^{-1} : [0,1]^{K\times K} \to [0,1]^{K\times K}$, such that $f^{-1}(\gamma_n^\sigma) = \mathcal{B}_n(\sigma)$ and $\mathcal{B}_n(\sigma)$ satisfies equation (S.26). Note that f^{-1} is continuous which implies that $\lim_{n\to\infty} B_n(\sigma) = \lim_{n\to\infty} f^{-1}(\gamma_n^\sigma) = f^{-1}(\hat{\gamma}^\sigma) = \mathcal{B}(\sigma)$. $\square$

Lemma 11. *Let $\mathcal{O}$ be a continuous function defined on the domain of $\gamma \in [0,1] \cap C$ where C consists of those γ's that satisfy a given set of weak inequality constraints, each of which is continuous in γ. Then,* $\arg\max_{\gamma\in[0,1]|\gamma\in C} \mathcal{O}(\gamma) = \arg\sup_{\gamma\in(0,1)|\gamma\in C} \mathcal{O}(\gamma)$.

Proof. Without loss of generality suppose the argument that maximizes $\mathcal{O}$, γ^*, gives rise to a non-interior belief system, $\mathcal{B}^*$. Then, Lemma 10 implies that we can approximate $\mathcal{B}^*$ by a convergent sequence of consistent interior belief systems. It follows that $\lim_{n\to\infty} \mathcal{O}(\mathcal{B}_n) = \mathcal{O}(\mathcal{B}^*)$ because $\mathcal{O}$ is continuous in γ (and through Observation 1 continuous in $\mathcal{B}$). Moreover, because the constraints are inequality constraints and continuous in γ (and $\mathcal{B}$), there is n' such that every element $\mathcal{B}_n$ with $n > n'$ satisfies the constraints. Therefore, $\max_{\gamma\in[0,1]|\gamma\in C} \mathcal{O}(\gamma) = \sup_{\gamma\in(0,1)|\gamma\in C} \mathcal{O}(\gamma)$ and $B^* = \lim_{n\to\infty} \mathcal{B}_n$. Using Lemma 10 we note that for every $\mathcal{B}_n$ there is γ_n so that $\lim_{n\to\infty} \gamma_n = \gamma^*$. $\square$

J Private Information about Valuation

Here, we assume that players private information is about their valuation from winning the conflict. We argue that the model is identical to the main model, where players' private information is about their strength in the escalation game.

An Auxiliary Model. Assume valuations are reversely ordered, that is, $\theta_1 > \theta_2 > ... > \theta_K$. Define $\check{U}_i(m_i; \theta_i, \mathcal{B}) := U_i(m_i; \theta_i, \mathcal{B})/\theta_i$, where U is the (continuation) payoff from the escalation game. Similarly transform the outside options $\check{V}_i(1, (p, \tilde{\rho})) := V_i(1, (p, \tilde{\rho}))/\theta_1$ and the values from participating in the mechanism $\check{\Pi}_i(m_i; \theta_i) := \Pi(m_i; \theta_i)/\theta_i = z_i(m_i) + \gamma_i(m_i)\check{U}_i(m_i; \theta_i, \mathcal{B})$. Note that when replacing U by $\check{U}$, V by $\check{V}$ and Π by $\check{\Pi}$ the model becomes identical to that in the main text. We call it the auxiliary model. Below we argue that the auxiliary model's solution determines that of the correct model.

Participation Constraints. The auxiliary model's solution satisfies players' participation constraints by design. These constraints also hold in the correct model:

$$\check{\Pi}_i(\theta_i;\theta_i) \geq \check{V}_i(\theta_i,(p,\tilde{\rho})) \Leftrightarrow \theta_i \cdot \check{\Pi}_i(\theta_i;\theta_i) \geq V_i(\theta_i,(p,\rho^V)) \Leftrightarrow \Pi_i(\theta_i;\theta_i) \geq V_i(\theta_i,(p,\rho^V)).$$

Next, we determine the full-settlement condition in the auxiliary model. Under full settlement type 1's continuation value from participating reads

$$\check{\Pi}_i(1;1) \geq \check{V}_i(1,(p,\rho^V)) \Leftrightarrow z_i(1) \geq \check{V}_i(1,(p,\rho^V)) \Leftrightarrow \theta_1 \cdot z_i(1) \geq V_i(1,(p,\rho^V)).$$

Thus, resource feasibility requires $\sum_i \check{V}_i(1,(p,\rho^V)) \leq 1$. Hence full-settlement in the correct model is feasible if and only if it is feasible in the auxiliary model.

Incentive Constraints. Next, we focus on incentive constraints. Suppose that an allocation is incentive compatible in the auxiliary model. Then that allocation is also incentive compatible in the correct model:

$$\check{\Pi}_i(\theta_i;\theta_i) - \check{\Pi}_i(m_i;\theta_i) \geq 0 \ \Leftrightarrow \theta_i\Big(\Pi_i(\theta_i;\theta_i) - \Pi_i(m_i;\theta_i)\Big) \geq 0 \Leftrightarrow \Pi_i(\theta_i;\theta_i) - \Pi_i(m_i;\theta_i) \geq 0.$$

Optimal Solution. Consider the auxiliary problem, that is, $\min_{\gamma,X,\Sigma} Pr(\mathcal{E})$ *s.t. participation constraints and incentive constraints hold.* Similar, the correct model reads $\min_{\gamma,X,\Sigma} Pr(\mathcal{E})$ *s.t. participation constraints and incentive constraints hold.* The objective is the same, and, as demonstrated above, the constraints are equivalent. Therefore, both optimal solutions coincide.

Transforming a Private Cost Model to a Private Valuation Model. Fix an escalation game. Suppose the game has the structure $u_i(a_i,a_{-i},\theta) = \phi(a_i,a_{-i}) - c_i(a_i)\theta_i$. Moreover, assume that $\theta_k = 1/\theta_{K+1-k}$. Optimal arbitration when private information is about the costs is the mirror image of that when private information is about the valuations, that is, when the game has the structure $\phi(a_i,a_{-i})/\theta_i - c_i(a_i)$.

K Monotone Mechanisms

Here we provide details behind the results obtained for type-separable escalation games at the end of section Section 4.5. Recall that these escalation games feature the following payoff structure.

$$u(a_i;a_{-i},\theta_i) = \phi(a_i,a_{-i}) - \zeta(\theta_i)c(a_i,a_{-i}). \tag{S.34}$$

Let a^*_{-i} be the equilibrium action of player $-i$ and define $c(a_i) := \mathbb{E}[c(a_i,a^*_{-i})|m_i,\mathcal{B}]$.

K.1 Proof of Proposition 8

Proof. Assume without loss that $\zeta(\theta_i) = \theta_i$. A player's best-response to her opponent's action, $a(m_i;\theta_i,\mathcal{B})$, satisfies first-order conditions. The envelope theorem implies

$$U(m_i,\theta_i,\mathcal{B}) = U(m_i;1,\mathcal{B}) - \int_1^{\theta_i} c(a_i(m_i;s'\mathcal{B}))ds = U(m_i;1,\mathcal{B}) - h(m_i;\mathcal{B})\int_1^{\theta_i} g(s)ds,$$

where we used that $c(a_i(m_i;\theta_i,\mathcal{B})) = h(m_i;\mathcal{B})g(\theta_i)$. Thus, $D_i(m_i;\theta_i,\mathcal{B}) = h_i(m_i;\mathcal{B})\int_{\theta_i}^{\theta_i+1}\tilde{g}(s)ds$. Moreover, for any m_i and m_i' we have that

$$\frac{D_i(m_i;\theta_i,\mathcal{B})}{D_i(m_i';\theta_i,\mathcal{B})} = \frac{h_i(m_i;\mathcal{B})\int_{\theta_i}^{\theta_i+1} g(s)ds}{h_i(m_i';\mathcal{B})\int_{\theta_i}^{\theta_i+1} g(s)ds} = \frac{h_i(m_i;\mathcal{B})}{h_i(m_i';\mathcal{B})},$$

which is independent of θ_i. □

K.2 Non-Constant Difference Ratios

Next, we turn to non-constant difference ratios. First, we derive the designer's objective if ϕ only distributes the pie without destroying any surplus.

The envelope theorem implies

$$U_i(m_i;\theta_i,\mathcal{B}) = \mathbb{E}[\phi(a_i(m_i;1,\mathcal{B}),a^*_{-i})|m_i,\mathcal{B}] - C(m_i,\theta_i;\mathcal{B}),\ \textit{where} \tag{S.35}$$

$C(m_i;\theta_i,\mathcal{B}) := \int_1^{\theta_i} c(m_i;s,\mathcal{B})ds + c(m_i;1,\mathcal{B})$. Optimal arbitration maximizes $\sum_i (\mathbb{E}[U_i|\mathcal{B}] + \mathbb{E}[\Psi_i|\mathcal{B}])$ subject to constraints C_F. Since $\phi(a_i,a_{-i})+\phi(a_{-i},a_i) = 1$, $\mathbb{E}[U_i|\mathcal{B}] = 1-\sum \rho_i(\theta_i)c(\theta_i;\theta_i,\mathcal{B})$. Moreover, (S.35) implies that $D_i(m_i;\theta_i,\mathcal{B}) = C(m_i;\theta_i,\mathcal{B}) - C(m_i;\theta_i+1,\mathcal{B})$.

Let $\tilde{S}(\theta_i,\theta_{-i};\mathcal{B}) := \sum_{i\in\{A,B\}}\omega(\theta_i)D_i(\theta_i;\theta_i,\mathcal{B}) - \zeta(\theta_i)c(\theta_i;\theta_i,\mathcal{B})$. The objective is $\Xi(\rho(\cdot,\cdot)) := \sum \rho(\theta_1,\theta_2)\tilde{S}(\theta_1,\theta_2;\mathcal{B})$. If the type space is sufficiently dense, we can set up an auxiliary problem. We replace $\tilde{S}$ with S being defined as

$$S(\theta_i,\theta_{-i};\mathcal{B}) := (\omega(\theta_i)-\zeta(\theta_i))D_i(\theta_i;\theta_i,\mathcal{B}) + (\omega(\theta_{-i})-\zeta(\theta_{-i}))D_i(\theta_{-i};\theta_{-i},\mathcal{B}).$$

Sufficiency of the Auxiliary Problem. We show that the solution to the auxiliary problem solves the original problem if $\Delta\theta := \zeta(\theta)-\zeta(\theta-1)$, for any two adjacent types, is sufficiently small. By the intermediate value theorem we have that

$$\tilde{S}(\theta_i,\theta_{-i};\mathcal{B}) = \sum_{i\in\{A,B\}}\omega(\theta_i)\Delta\theta c(\theta_i;\tilde{\theta},\mathcal{B}) - \zeta(\theta_i)c(\theta_i;\theta_i,\mathcal{B}) \tag{S.36}$$

for some $\tilde{\theta}_i \in [\theta_i,\theta_i+1]$. Define $\tilde{p} := p/\Delta\theta$ and $\tilde{\omega}(\theta_i) := (1-\int_1^{\theta_i}\tilde{p}(s)ds)/\tilde{p}(\theta_i)$. The objective becomes $\sum_{i\in\{A,B\}}\tilde{\omega}(\theta_i)c(\theta_i;\tilde{\theta}_i,\mathcal{B}) - \theta_i c(\theta_i;\theta_i,\mathcal{B})$. If $\Delta\to 0$, then $c(\theta_i;\tilde{\theta},\mathcal{B})\to c(\theta_i;\theta_i,\mathcal{B})$ and the scores of the auxiliary problem and the original problem coincide. Hence the solutions coincide.

Next, we show that we can parameterize the conflict game such that part (i) in Definition 5 is satisfied.

Part (i) of Definition 5. We treat the function V as a primitive. We fix $V_i(1,(p,\rho^V)) \in (1/2,1)$. Then, a sufficient condition is that $V_i(2,(p,\rho^V)) < 1/2$.

Lemma 12. *Suppose that p is non-decreasing. Then, part (i) of Definition 5 holds if and only if*

$$p(1)V_i(1,(p,\rho^V)) + p(2)V_i(2,(p,\rho^V)) \le \frac{p(1)+p(2)}{2} \tag{S.37}$$

Proof. Note that $p(\theta)(1-V_i(\theta,(p,\rho^V)))$ is weakly increasing in θ because both $1-V_i(\theta,(p,\rho^V))$ and $p(\theta)$ are weakly increasing in θ. □

Holding fix $p(1)$ and $p(2)$, that are weakly increasing, we can find some reduced-form outside option that satisfies the above condition.

Results. Suppose $p(\theta)$ is non-decreasing in θ. Moreover, assume the distance between any two adjacent types, Δ, is sufficiently small and the game is parameterized such that part (i) of (5) holds. If the conflict game features strategic complements, the solution to Theorem 2 is directly applicable. In particular, (8) is satisfied. Now, we state an algorithm that solves the problem.

We use the general Lagrangian approach from Section G to develop a solution algorithm for our class of games. We apply it to the the auxiliary problem. We first relax that problem by ignoring all global incentive constraints. Then, the Lagrangian of the reduced-form problem becomes

$$\sum_{\theta_i \times \theta_{-i}} \rho(\theta_i, \theta_{-i}) \Big(\{ \sum_{i \in \{A,B\}} (\omega(\theta_i) - \zeta(\theta_i)) D_i(\theta_i, \theta_i; \mathcal{B}) \} - \frac{\mu(\theta_i, \theta_{-i})}{p(\theta_i) p(\theta_{-i})} \Big), \tag{S.38}$$

where $\mu(\theta_i, \theta_{-i})$ is the Lagrangian multiplier on the feasibility constraints, i.e.,

$$(2V(1, (p, \underline{\rho}^V)) - 1)\rho(1,1) \leq (p(1))^2 \left(\Xi(\rho(\cdot, \cdot))\right).$$

If that constraint does not bind, then the optimal solution features $\rho(1,1) = 1$. This follows from the complementary nature of the conflict, together with the non-decreasing virtual valuations.

Assume that $\rho(1,1) = 1$ is not feasible, that is, $\mu(1,1) > 0$. Then, the least-constrained solution is not feasible and signals may improve.

We state an algorithm, a top-down version with information revelation, and then argue that this algorithm is optimal.

Algorithm. Define the score of a type profile the following way:

$$\hat{S}(\theta_i, \theta_{-i}) = (\omega(\theta_i) - \zeta(\theta_i)) D_i(\theta_i; \theta_i, \mathcal{B}^{\theta_i, \theta_{-i}}) + (\omega(\theta_{-i}) - \zeta(\theta_{-i})) D_{-i}(\theta_{-i}; \theta_{-i}, \mathcal{B}^{\theta_i, \theta_{-i}}),$$

where $\mathcal{B}^{\theta_i, \theta_{-i}}$ is the belief system that results if each match receives full information. We order type profiles according to their score. If the highest type profile is (θ_i, θ_{-i}), then the next highest type profile is either $(\theta_i + 1, \theta_{-i})$ or $(\theta_i, \theta_{-i} + 1)$ by complements.

Signals might improve because the least-constrained problem is not feasible. Hence given active type profiles, implement the optimal information revelation policy, i.e., that which maximizes the objective given the active type profiles. Check whether the objective satisfies condition (F) on page S.7 of the supplementary material. If not, continue to the next highest type profile and repeat the maximization.

Optimality of the Algorithm. The decreasing hazard rate and strategic complements imply that higher type profiles have a higher score. Moreover, strategic complements imply that, given the optimal information disclosure policy, type profiles with the highest ex-post scores are the most beneficial ones.

Optimal Information Revelation. To construct the concave hull of the Lagrangian objective, (S.38), we have to distinguish two cases. Define $c(a_i^*(a_{-i}))$ as that part of the cost that is independent of $-i$'s action. If that function is concave, then we disclose no information. In contrast, if that function is convex, then we disclose full information.

Secondly, observe that the form of the Lagrangian objective (S.38) implies that given $\rho(\cdot, \cdot)$ the information disclosure that maximizes the least-constrained objective is optimal.

Incentive Constraints. We need to verify that this algorithm satisfies the incentive constraints. That is, $\gamma_i(m_i) D_i(m_i; \theta_i, \mathcal{B})$ is weakly decreasing in m_i. Given part (i) of Definition 5 what remains is to show that $\gamma_i(\theta_i)$ is non-decreasing in θ_i, which is satisfied by construction.

L Solution Programs

Click on either below for the respective Matlab Programs on the authors homepages.

- Solution Program for ADR
- Solution Program for Monotone Lotteries