A Quest for Knowledge*

Johannes Schneider† Christoph Wolf‡

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Abstract

Which discoveries improve decision making? Which question does a researcher pursue? How likely will a researcher discover an answer? We propose a model in which the answers to research questions are correlated. Revealing an answer endogenously improves conjectures on unanswered questions. A decision maker uses conjectures to address problems. We derive the benefits of a discovery on decision making. A researcher maximizes the benefits of discovery net the cost of research. We characterize the researcher's optimal choice of research question and research effort. Benefits and cost of research crucially depend on the structure of pre-existing knowledge. We find that discoveries are rare but advance knowledge beyond the frontier if existing knowledge is dense. Otherwise, discoveries are more likely and improve conjectures inside the frontier.

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†Carlos III de Madrid; E-mail: `jschneid@econ.uc3m.es`

‡Bocconi University, Department of Economics and IGIER; E-mail: `christoph.wolf@unibocconi.it`

[Evolution] comes through asking the right questions, because the answer pre-exists. But it's the questions that we have to define and discover. ... You don't invent the answer. You reveal the answer. Jonas Salk.

1 Introduction

In his letter to Franklin D. Roosevelt in 1945 (*Science, The Endless Frontier*[1]), Vannevar Bush pleads with the president to preserve the freedom of inquiry by federally funding basic research—the "pacemaker of technological progress." That letter paved the way for the creation of the National Science Foundation (NSF) in 1950. The NSF today, like the vast majority of governments and scientific institutions,[2] cherishes scientific freedom and allows academic researchers to select research projects independently. For example, the research ministers of the European Union emphasize that scientific freedom and access to knowledge are essential to progress in the Bonn Declaration (2020).[3]

With the power of scientific freedom comes the responsibility for "asking the right questions" that Jonas Salk advocates in the epigraph. However, what is the right question? In his guide for young researchers Varian (2016) suggests to become "a Wizard of Ahs"—to pursue questions that have "lots of implications." Thus we refine: Which question should a researcher pursue to produce the most implications? A question to advance our knowledge to new territories outside the current frontier? A question to explore the details around existing knowledge? Or a question to bridge the gap between established findings?

In this paper, we show that each strategy can outrank the others. The ranking depends on existing knowledge—that is, the set of questions to which the answers are known. In particular, the ranking depends on the density of existing knowledge—that is, the distance between questions in that set.

Consider the following setting. A decision maker faces a problem. In her response to the problem, the decision maker builds on the information that existing knowledge provides. This information stems from two sources. First, existing knowledge provides the answers to particular questions. Second, existing knowledge provides conjectures on questions to which the answers are yet undiscovered. The reason for the second source follows from answers being correlated: the precision of the conjecture depends on the question's location relative to existing knowledge.[4] A

[1] `https://www.nsf.gov/od/lpa/nsf50/vbush1945.htm`.

[2] See, for example, the International Covenant on Economic, Social and Cultural Rights, Office of the United Nations High Commissioner 1966, Art. 15; `https://www.ohchr.org/EN/ProfessionalInterest/Pages/CESCR.aspx`.

[3] "Scientific research benefits the people and society through the advancement of knowledge. Freedom of scientific research is a necessary condition for researchers to produce, share and transfer knowledge as a public good for the well-being of society." Bonn Declaration, 2020, `https://www.bmbf.de/files/10_2_2_Bonn_Declaration_en_final.pdf`

[4] An example of such spillovers is the COVID-19 vaccine development by Moderna which "took all of one weekend" only. The speed was a direct consequence of researchers discovering how to replicate the spikes of another Corona virus in research on MERS. See `https://www.thisamericanlife.org/727/boulder-v-hill/act-two-11?fbclid=IwAR1scBqt7cS-rB4TPZdE4NmenD3ng3HvlSyC7fQ_xOGIfGlwbsK9cj7K7w8`.

new discovery creates additional information. The benefit of the discovery measures the improvement of the decision maker's response due to that information.

We find the following relation between the density of the existing knowledge and the benefit of discovery.[5] If the existing knowledge is sparse—that is, any two questions in existing knowledge are far apart—it is best to narrow the gap between two adjacent questions asymmetrically. A researcher should pursue a discovery on a question closer to one of the two answered questions. If the existing knowledge is dense—that is, any two adjacent questions in existing knowledge are close—it is best to advance knowledge beyond but not too far from the frontier. A researcher should pursue a discovery on a question that does not lie between any answered questions. If the existing knowledge is neither sparse nor dense—that is, there are two adjacent questions of intermediate distance—it is best to bridge the gap between these questions. A researcher should pursue a discovery on the question equidistant to the two answered questions.

The intuition follows from how knowledge improves decision making through more precise conjectures. Take an unanswered question x and a question x_1 that is part of existing knowledge. Knowledge of x_1 shapes the conjecture about x. In particular, the closer x is to x_1 the more precise is the conjecture. Suppose now that x lies between x_1 and another question x_2 that is also in existing knowledge. Then, knowledge of x_2 shapes the conjecture about x as well. Analogously, the closer x is to x_2 the more precise the conjecture. Thus, x_1 and x_2 jointly determine the precision of the conjecture about x.

Suppose that initially only the answer to x_1 was known and the researcher discovers the answer to x_2. This discovery improves the precision of the conjecture about any question x between x_1 and x_2. The further x_2 is from x_1 the more questions have improved conjectures. However, the further x_2 is from x_1 the less precise is the conjecture about any question close to x_1. That relation defines a tradeoff: A discovery far apart from existing knowledge improves decision making. Many conjectures become more precise, albeit only mildly so. A discovery close to existing knowledge improves decision making as well. Some conjectures become very precise, albeit not many conjectures improve at all. When pursuing a question to advance knowledge beyond the frontier, a researcher resolves this tradeoff by selecting a question of intermediate distance.

Now, suppose that the answer to both x_1 and x_2 was known. Assume the researcher chooses to pursue a question x_3 that lies between x_1 and x_2. Here, the researcher balances the tradeoff from above on two sides simultaneously. If she discovers the answer to x_3, she replaces the initial area of questions spanned by x_1 and x_2 by two new areas: one area spanned by x_1 and x_3, another spanned by x_3 and x_2. In sum, the lengths of the two new areas (x_1 to x_3 and x_3 to x_2) equal the length of the initial area (x_1 to x_2). From the discussion of advancing knowledge beyond the frontier, we know that ideally the researcher creates two areas of intermediate length. However, the length of the initial area constrains the researcher. The following optimal strategies emerge: If the initial area is short enough, the researcher should bridge the gap and pursue a question x_3 equidistant to both x_1 and x_2. Otherwise, the researcher should narrow the gap: Pursue a question closer to one of the area's

[5] For now, we focus only on the benefit of research. We introduce the cost below.

bounds and thereby focus on creating one area of intermediate length.

Overall, the largest benefit comes from bridging the gap between distant, yet not too far apart pieces of knowledge. Advancing knowledge beyond the frontier trumps pursuing a question inside an existing area only if all available areas are short.

In reality, research comes at a cost. Naturally, the cost influence the researcher's choices. It takes time and effort to search for an answer. The more effort a researcher invests the higher the likelihood of a discovery. We find that this likelihood varies with existing knowledge because existing knowledge determines conjectures. The more precise the conjecture about a question the more likely it is to discover the answer with any given level of effort. This relationship influences the researcher's risk: Given effort, increasing the distance to existing knowledge reduces the likelihood of discovery. Mitigating that additional risk requires costly effort. Thus, introducing cost adds another dimension to the problem: the researcher balances risk and reward.

The qualitative statements on the optimal choice of question remain, even including the effort choice. In addition, we can address research productivity: how likely is discovery? We find the following relation between productivity and existing knowledge. Consider a researcher who chooses to bridge a gap in existing knowledge. If the area is short, the research productivity is low. The benefit of a discovery is low and hence the researcher refrains from exerting much effort. At first, research productivity increases in the area length. Both the benefit of a discovery and the cost increase, yet the former dominates the latter. However, if the area is large, research productivity decreases in the area length. The increase in the cost dominates.

If we combine the analysis of benefit and cost we obtain a clear picture: If available, the most productive and valuable research bridges the gap between distant, yet not too far apart pieces of knowledge. If such gaps do not exist, the researcher should aim at narrowing a large gap asymmetrically. Finally, if existing knowledge is dense, the researcher should take the risk to advance the knowledge frontier.

We provide a microfounded framework to study decisions in research and their consequences. We make the following basic assumptions: (i) the set of potential research questions is large and given by the real line. (ii) The answer to one question is informative about the answers to other questions. We assume that answers are predetermined by the realization of a Brownian motion. (iii) A decision maker has access to the public good *knowledge*. Applying this knowledge, the decision maker chooses whether and how to pro-actively address a problem or to "do nothing," e.g., continue with business as usual. (iv) The value of knowledge corresponds to the quality of decision making. Finally, (v) we conceptualize research as choosing a question and an interval of possible answers that the researcher samples. If the answer lies in the sampled interval, discovery is successful. The larger the sample interval, the larger the cost to the researcher.

We derive both an endogenous benefit-of-discovery function and an endogenous cost-of-research function based on those assumptions.[6] Our microfoundation implies

[6]While we derive these functions with an academic researcher in mind, we believe they are also interesting in other settings. For instance, they may be relevant when firms buy innovative start-ups simply to obtain knowledge and to improve conjectures relevant for their own questions. Our model may be helpful to differentiate among the different effects in the ongoing debate about killer applications (see Letina, Schmutzler, and Seibel, 2020; Cunningham, Ederer, and Ma, 2021).

a set of non-standard properties of both functions.[7] Despite these peculiarities, the researcher's optimization problem remains tractable. Due to these peculiarities, however, we obtain rich and intuitive results regarding the researcher's choices. Therefore, we believe that our model is widely applicable.

To illustrate our model's applicability beyond the researcher's problem, we embed it into a setting of science funding. A funder who respects the researcher's scientific freedom can spend a fixed budget both on reducing the researcher's cost and on rewards for successful research. We construct the funder's feasible set of implementable choices. We show that—from the funder's perspective—novelty and output can be both complements or substitutes depending on the effective price ratio between cost reductions and rewards. In the analysis, we combine standard tools from consumer theory with our previous results.

1.1 Related Literature

Ample empirical literature in the science of science has documented the importance of novelty and output for progress in science. Fortunato et al. (2018) provide an extensive summary of it. The importance of (accessible) pre-existing knowledge for research purposes is documented, for example, in Iaria, Schwarz, and Waldinger (2018).[8] That literature studies the connection between benefits of discoveries, cost of search, and existing knowledge. Yet its findings are mainly based on (quasi)experiments; a formal model connecting these aspects is missing. We aim at closing this gap. The formal model we provide is simple yet flexible enough to address several issues. The model is based on few (in the baseline model, two) parameters, which makes it identifiable and testable.

Several existing theoretical models in the science of science consider particular aspects of the scientific process we have in mind.[9] Aghion, Dewatripont, and Stein (2008), for example, consider a setting in which progress has a predefined step-by-step sequential structure. To advance to the next question, a particular prior question has to be answered. Our model offers greater flexibility in that it posits that any question can—in principle—be addressed at any time. However, the benefits from a discovery and the effort needed for the discovery depend on previous work.[10] Bramoullé and Saint-Paul (2010) model the decision of a researcher to deepen knowledge in a given area or to advance the knowledge frontier. The main driver in their model is the assumption that as an area gets increasingly crowded,

[7] The benefit of a discovery is neither single-peaked in distance to existing knowledge nor globally concave in area length; the cost are concave in distance at any point.

[8] We want to stress that our notion of knowledge is orthogonal to that in the literature on epistemic game theory. Brandenburger (1992) provides an overview. Different to that literature knowledge is always fully transparent and there is no strategic interaction. However, it is possible to embed it in strategic settings to address alternative questions.

[9] There is a literature orthogonal to ours that views science as establishing links in a network between known answers (e.g., Rzhetsky et al., 2015). Our model is complementary, we consider research as the search for answers where the links in the network are known.

[10] To (ab)use Newton's metaphor: Any researcher can build a ladder to see farther, but the effort required depends on the existing giants' shoulders. Related ideas appear in Scotchmer (1991), Aghion et al. (2001), and Bessen and Maskin (2009).

the reputation a researcher gains from new developments in that area declines.[11] We offer a decision-based microfoundation that provides a measure of uncertainty, in line with Frankel and Kamenica (2019). It reaches a similar conclusion: as the opportunities in the area become increasingly narrow, the informational content of an additional finding decreases, and hence its value does too. However, unlike in Bramoullé and Saint-Paul (2010), the researcher in our model has more discretion, as she chooses—in addition to the area—the degree of novelty and the level of research intensity which directly determines the probability of success. Both choices are continuous, and shrinking the research area may even be beneficial if it leads to better conjectures by closing the gap between existing pieces of knowledge. While our model is static, a simple dynamic extension could reproduce the core models of either Bramoullé and Saint-Paul (2010) or Aghion, Dewatripont, and Stein (2008).[12]

The closest paper to ours in the literature on innovation is Prendergast (2019), which is complementary to ours. He, too, studies a model of innovation in which the correlation between questions is determined by a Brownian motion. He focuses on an agency problem in a single exogenously given research area. While we abstract from agency concerns, the results in our mirofounded model come from the researcher's choice between several distinct research areas and expanding knowledge beyond the frontier. In addition, we provide an endogenous relation between the effort invested and the probability of a discovery. While neither of the two models nests the other, there is a special case of our model that corresponds to a special case of his. We discuss the relationship in greater detail in Section 7.

Technically, we build on the literature that uses Brownian models to analyze R&D choices, following Callander (2011).[13] Most of that literature assumes that the payoffs are determined by a specific target of the stochastic process's realization (for example, Callander, 2011) or the weighted sum of all realizations (for example, Bardhi, 2019). We differ from it in that we posit that the value of a discovery is determined by the reduction in the variance of conjectures. Importantly, the objective of our model is not to compare the expected realizations to a safe outcome. Instead, we care about the precision of conjectures about all realizations. Callander and Clark (2017) is the closest work to ours within this literature. We discuss its relation to our model in detail in Section 7.

[11] Similar to Bramoullé and Saint-Paul (2010) innovation fully translates to a public good in our setting. That differentiates us from most models of R&D competition. Yet, similarly to, for example, Letina (2016), Letina, Schmutzler, and Seibel (2020), and Hopenhayn and Squintani (2021), we assume that progress corresponds to successful search in an ocean of possibilities. Unlike in those approaches, benefit and cost depend on the question's relation to existing knowledge in our setting.

[12] Other recent work in that area includes Akerlof and Michaillat (2018), Andrews and Kasy (2019), and Frankel and Kasy (2020). Andrews and Kasy (2019) and Frankel and Kasy (2020) study the (potentially) distorting affects of the publication process—a friction we abstract from. Akerlof and Michaillat (2018) study how false paradigms survive due to homophily. Our focus is different: we differentiate not quality of research but selection of topics. It is straightforward to extend our analysis to research-specific preferences by introducing additional parameters; for example, homophily in the research field.

[13] See also Callander and Hummel (2014), Garfagnini and Strulovici (2016), Callander and Clark (2017), Callander, Lambert, and Matouschek (2018), Bardhi (2019), and Callander and Matouschek (2019) and references therein.

2 Model

We build a parsimonious model of knowledge that captures the following real-world aspects:

(i.) Knowledge informs decision making.

(ii.) Knowing the answer to certain questions spills over onto conjectures about other questions.

(iii.) The set of questions available is infinite.

(iv.) The impact of answering one question on another one depends on how close the two questions are.

Questions and answers. We represent the universe of questions by the real line, $\mathbb{R}$. A specific question is an element $x \in \mathbb{R}$. Each question x has precisely one answer, $y(x) \in \mathbb{R}$. A question-answer pair $(x, y(x))$ is thus a point in the two-dimensional Euclidean space.[14]

The answer $y(x)$ to question x is determined by the realization of a random variable, $Y(x) : \mathbb{R} \to \mathbb{R}$. We provide more structure for $Y(x)$ below.

Truth and knowledge. *Truth* is the collection of *all* question-answer pairs. It is the realization of all random variables $Y(x)$ over the entire domain, $\mathbb{R}$. *Knowledge* is the collection of *known* question-answer pairs. We denote it by $\mathcal{F}_k = \{(x_i, y(x_i)\}_{i=1}^{k}$. For notational convenience, we assume that $\mathcal{F}_k$ is *ordered* such that $x_i < x_{i+1}$.

The key assumption of our model concerns the truth-generating process $Y(x)$. We assume that $Y(x)$ follows a standard Brownian motion defined over the entire real line.[15] This assumption captures the notion that the answer to question x (for example, what is the efficacy of a vaccine in a trial group?) is similar to the answer to a close-by question x' (for example, what is the efficacy of that vaccine in a general population similar to the trial group?). As the distance to the original question increases (for example, what is the vaccine's efficacy against other viruses or in subpopulations not represented in the trial?), there remains a correlation. Yet the noise increases the larger the distance to the closest piece of knowledge.[16]

Knowledge $\mathcal{F}_k$ determines a partition of the real line consisting of $k+1$ elements

$$\mathcal{X}_k := \{(-\infty, x_1), [x_1, x_2), \cdots, [x_{k-1}, x_k), [x_k, \infty)\}.$$

[14]Our assumption implies that the relation between two questions can be obtained in a single dimension. While projecting all available questions *across all disciplines* onto a line might imply a sizeable loss, that loss is smaller if we think of our universe of questions as being *within one specific discipline.*

[15]As in Callander (2011), the realized truth, Y, is a random draw from the space of all possible paths, $\mathcal{Y}$, generated by a standard Brownian motion going through some initial knowledge point, $(x_0, y(x_0))$. While the process has fully realized at the beginning of time, knowledge is the filtration known to the observer, $\mathcal{F}_k$. We choose a standard Brownian path with zero drift and variance of one for convenience only. Our model extends naturally to other (Gaussian) processes. We want to emphasize that the x dimension should not be confused with a sequential structure of finding answers. To the contrary, the entire process $Y(X)$ has realized at the beginning and any question-answer pair $(x, y(x)) \notin \mathcal{F}_k$ is both discoverable and yet to be discovered.

[16]We use the Euclidean distance on the x dimension, $|x - x'|$, throughout when we refer to distance.

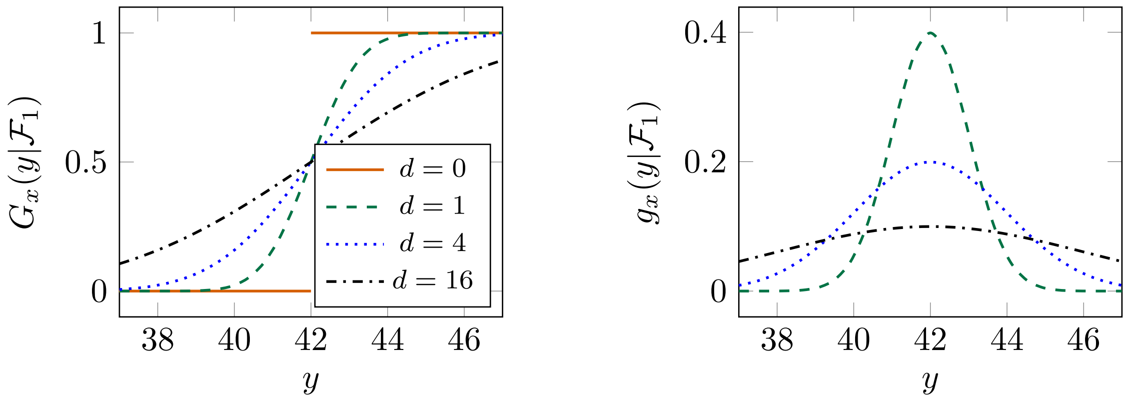


Figure 1: *Distributions of answers for different distances, d, to knowledge when $\mathcal{F}_1 = (0, 42)$* Given that the known question is $x = 0$, $d = 1$ depicts the distribution of answers to questions $x = -1$ and $x = 1$, $d = 4$ the distribution of answers to questions $x = -4$ and $x = 4$, and so on. All answers have the same mean (42), but the variance and thus the precision of the conjecture differs with the various distances, d, to the only known point, $\mathcal{F}_1$. For $d = 0$, the answer is known and $G_0(y|\mathcal{F}_1)$ is a step function. Questions with longer distance have larger variances. The left panel depicts the respective distribution functions; the right panel depicts the densities.

We refer to each element of the partition as an *area* and define it by the index of its lower bound. Thus, question $x \in [x_i, x_{i+1})$ is in area i of length $X_i := x_{i+1} - x_i$. For $x < x_1$ we say x is in area 0 of length $X_0 = \infty$, and for $x > x_k$ it is in area k of length $X_k = \infty$.

Conjectures. Based on the commonly known truth-generating process, $Y(x)$, and the existing knowledge, $\mathcal{F}_k$, it is straightforward to form a *conjecture*—that is, to compute the *distribution of the answer*—for each question x. We denote the conjecture about question x by the conditional distribution function $G_x(y|\mathcal{F}_k)$, which is defined over the answer domain, $\mathbb{R}$. Conjectures about questions to which the answer is known are trivial. The conjecture $G_{x_i}(y|\mathcal{F}_k) = \mathbf{1}_{y \geq y(x_i)}$ is a right-continuous step function jumping to 1 at $y = y(x_i)$ if $(x_i, y(x_i)) \in \mathcal{F}_k$. The conjecture for a yet-to-be-discovered $y(x)$, $G_x(y|\mathcal{F}_k)$, is a well-defined cumulative distribution function. Because $Y(x)$ is determined by a Brownian motion, any $G_x(y|\cdot)$ follows a normal distribution with mean $\mu_{\mathcal{F}_k}(x)$ and variance $\sigma^2_{\mathcal{F}_k}(x)$. Figure 1 depicts the distributions for different distances to the existing piece of knowledge, assuming it is $\mathcal{F}_1 = (0, 42)$.

Both $\mu_{\mathcal{F}_k}$ and $\sigma^2_{\mathcal{F}_k}$ follow immediately from the properties of the Brownian motion. We have to differentiate between settings in which we are on a Brownian bridge ($x \in [x_1, x_k]$) and those in which we are outside the current frontier ($x \notin [x_1, x_k]$).

Property 1 (Expected Value)**.** Given knowledge $\mathcal{F}_k$, the answer to question x has the following expectations:

$$\mu_{\mathcal{F}_k}(x) = \begin{cases} y(x_1) & \text{if } x < x_1 \\ y(x_i) + \frac{x - x_i}{X_i}(y(x_{i+1}) - y(x_i)) & \text{if } x \in [x_i, x_{i+1}), i \in \{1, ..., k-1\} \\ y(x_k) & \text{if } x \geq x_k \end{cases}$$

Property 2 (Variance)**.** Given knowledge $\mathcal{F}_k$, the answer to question x has the following variance:

$$\sigma^2_{\mathcal{F}_k}(x) = \begin{cases} x_1 - x & \text{if } x < x_1 \\ \frac{(x_{i+1}-x)(x-x_i)}{X_i} & \text{if } x \in [x_i, x_{i+1}), i \in \{1, ..., k-1\} \\ x - x_k & \text{if } x \geq x_k \end{cases}$$

Decision making. When a decision maker faces a question $x \in \mathbb{R}$, she has to take an action $a \in \mathbb{R} \cup \varnothing$. The symbol $\varnothing$ represents the act of "doing nothing", or continuing with the status quo action. Any number $a \in \mathbb{R}$ represents a proactive decision regarding question x.

We assume that the expected payoff of selecting $a = \varnothing$ is finite, safe (that is, independent of the true answer y), and question-invariant. We denote it by $-q$. The payoff of addressing x with some action $a \neq \varnothing$ is represented by a quadratic loss around the answer, $y(x)$, to question x.

$$u(a; x) = \begin{cases} -(a - y(x))^2 & \text{if } a \neq \varnothing \\ -q & \text{if } a = \varnothing \end{cases}$$

When facing the question x, the decision maker has access to $\mathcal{F}_k$ only.

The purpose of the safe option $a = \varnothing$ is the following: If a question is far from existing knowledge, the expected payoff for any pro-active address $a \in \mathbb{R}$ becomes arbitrarily negative because of a lack of knowledge. The safe option guarantees that the expected payoff is well defined nonetheless.[17]

Graphical example. Before moving to the analysis, we present a short graphical example that highlights our main model features and fosters intuition for the remainder of the paper. Suppose the following snapshot of the realization of the Brownian path constitutes the truth on $[-2, 2]$.

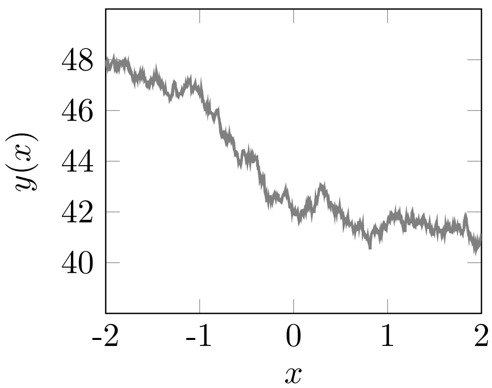
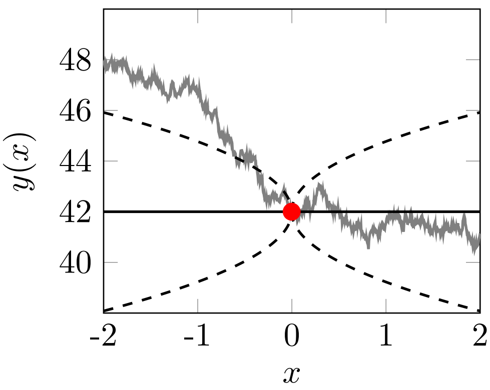


Figure 2: *The color of the truth is gray*

[17]Making q stochastic and dependent on x is straightforward and does not alter our arguments (see also footnote 18). The only substantial assumption we make is that an action with a finite expected payoff exists.

The next two graphs depict the situations in which the answer to a single question is known, $\mathcal{F}_1 = \{(0, 42)\}$, and in which two answers, $\mathcal{F}_2 = \{(-1.2, 46.6), (0, 42)\}$, are known.



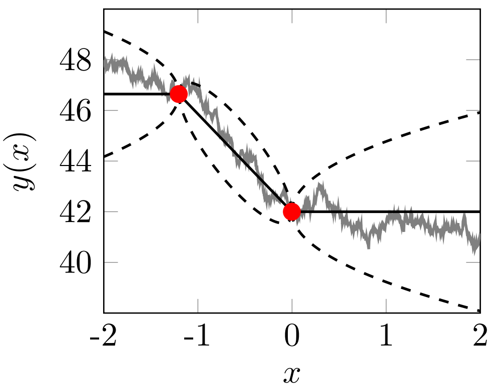


Figure 3: *Conjectures and their precision under $\mathcal{F}_1$ (left) and $\mathcal{F}_2$ (right)*
The red dots represent known question-answer pairs. The solid lines represent the expected answer to each question x given the existing knowledge. The dashed line represents the 95-percent prediction interval—that is, the interval in which the answer to question x lies, with a probability of 95 percent, given $\mathcal{F}_k$.

In the situation represented in the left panel of Figure 3, under $\mathcal{F}_1$, only the answer to question 0, which is 42, is known. We represent that knowledge by a dot (●). Given the martingale property of a Brownian motion, the current conjecture is that the answer to all other questions is normally distributed with mean 42. We represent the mean of the conjecture by the solid lines. However, the farther a question is from 0, the less precise is the conjecture (see Figure 1). We depict the level of precision by the dashed 95-percent prediction interval. For each question x, the truth lies, with a probability of 95 percent, between the two dashed lines given the knowledge $\mathcal{F}_1$.

In the right panel of Figure 3, in addition to $\mathcal{F}_1$, the answer to question $x = -1.2$, which is 46.6, is known. The additional knowledge changes the conjectures for questions in the negative domain compared to the left panel. The conjecture about questions between -1.2 and 0 is represented by a Brownian bridge. The expectation of answers is decreasing from -1.2 to 0 and is 46.6 to the left of -1.2. Moreover, uncertainty decreases for all questions in the negative domain, and the prediction bands become narrower. The positive domain is unchanged because of the martingale property of Brownian motion.

Now, consider moving to knowledge $\mathcal{F}_3 = \{(-1.2, 46.6), (0, 42), (1.2, 41.8)\}$ (left panel of Figure 4) and then to $\mathcal{F}_4 = \{(-1.6, 46.6), (0, 42), (0.8, 40.8), (1.2, 41.8)\}$ (right panel of Figure 4).

Moving from $\mathcal{F}_2$ to $\mathcal{F}_3$, the change is similar to that from $\mathcal{F}_1$ to $\mathcal{F}_2$, but this time in the positive domain. All conjectures in the positive domain become more precise, but the negative domain is unaffected. Further, a Brownian bridge between the known points $(0, 42)$ and $(1.2, 41.8)$ arises.

Moving from $\mathcal{F}_3$ to $\mathcal{F}_4$, knowledge of an answer to a question that lies between two already-answered questions is added. That implies that conjectures about answers

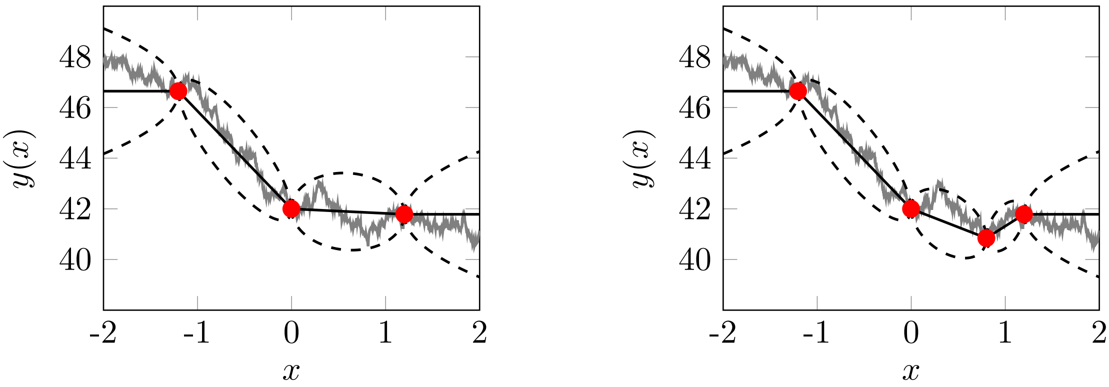


Figure 4: Conjectures and their precision under $\mathcal{F}_3$ (left) and $\mathcal{F}_4$ (right)

to questions between 0 and 1.2 become more precise. Further, since $40.8 < 41.8$, answers to all questions between 0 and 1.2 are expected to be lower compared to the conjecture based on knowledge $\mathcal{F}_3$. Moreover, the expected answers are decreasing in x from 0 to 0.8 and increasing from 0.8 and 1.2.

3 The Benefits of Discovery

Discovery occurs if a new question-answer pair is found and added to the existing knowledge ($\mathcal{F}_k$). In this section, we derive a formulation that measures the benefits of such additions for the decision maker.

3.1 Preliminaries: Decision Making

Knowledge is important for decision making. When facing a question x, the decision maker uses her conjecture about the answer to x to evaluate her options. Suppose the decision maker takes a particular action $a \neq \varnothing$. Her expected utility, given $\mathcal{F}_k$, is determined as follows:

$$U(a \neq \varnothing; x|\mathcal{F}_k) = -\int (a-y)^2 dG_x(y|\mathcal{F}_k)$$

Whenever the decision maker addresses the question, she selects

$$a^* := \arg\max_{a\in\mathbb{R}} U(a \neq \varnothing; x|\mathcal{F}_k) = \mu_{\mathcal{F}_k}(y)$$

and her expected payoff is

$$U(\mu_{\mathcal{F}_k}(x); x|\mathcal{F}_k) = -\int (\mu_{\mathcal{F}_k}(x) - y)^2 dG_x(y|\mathcal{F}_k) = -\sigma^2_{\mathcal{F}_k}(x).$$

Addressing the question is only optimal if $\sigma^2_{\mathcal{F}_k}(x) \leq q$. Otherwise, the decision maker exercises the safe option $a = \varnothing$ with payoff $-q$.[18] We denote the optimal action to question by $a^*(x) := \arg\max_{a\in\{\mathbb{R}\cup\varnothing\}} U(a; x|\mathcal{F}_k)$.

[18]For simplicity, we abstract from any dependency of the payoff on x. Regarding the status quo, we have in mind long-standing policies to which the expected payoff is finite. The question

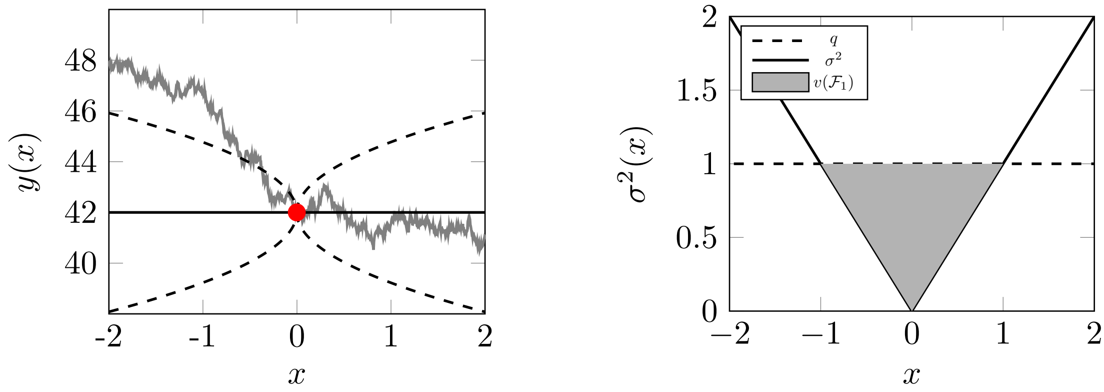


Figure 5: *The value of knowing* $\mathcal{F}_1$
The left panel is identical to the left panel in Figure 3. The right panel shows the variance of the conjectures about questions. The expected payoff from taking an action equal to the mean of the question, $a = \mu(x)$, is the area above the variance but below q. We assume the value of $a = \varnothing$ is $-q = -1$. The variance $\sigma^2(x) = d(x)$. For $d \leq 1$, $a = \mu(x)$ is preferred to $a = \varnothing$. The net payoff from $a = \mu(x)$ relative to $a = \varnothing$ is $q - d(x)$. The total value of $\mathcal{F}_1$ is the shaded area.

3.2 The Benefits of Discovery

The value of knowing $\mathcal{F}_k$. If $y(x)$ is known, the decision maker addresses x optimally. She selects $a = y(x)$ and obtains $u(y(x); x) = 0$. All other questions are addressed through the imperfect $a^*(x) \in \{\mu_{\mathcal{F}}(x), \varnothing\}$. The resulting expected utility to the decision maker is $U(a^*; x|\mathcal{F}_k) = \max\{-q, -\sigma^2_{\mathcal{F}_k}(x)\}$. The total value of knowing $\mathcal{F}_k$, $v(\mathcal{F}_k)$, captures how much the decision maker gains compared to not knowing anything and thereby responding with the safe option $a = \varnothing$ regardless of the problem. We use the (normalized) utility

$$\frac{q - U(a^*(x); x|\mathcal{F}_k)}{q}$$

to describe how much the decision maker benefits from the knowledge $\mathcal{F}_k$ in her response to x. To keep the analysis focused, we abstract from any exogenous prioritization. The decision maker cares about the improvement in her actions in response to any question in the same way, that is, there is no exogenous ranking of questions. Thus, the *value of knowing* $\mathcal{F}_k$ is[19]

$$v\left(\mathcal{F}_k\right) := \int \max\left\{\frac{q - \sigma^2_{\mathcal{F}_k}(x)}{q}, 0\right\} dx.$$

the decision maker asks is whether she should revise her policies given the the existing knowledge. An example is the discussion about how to respond to climate change. Since the Kyoto Protocol, decision makers have reevaluated their policies based on the evolution of knowledge by constantly trying to decide whether to continue with business as usual or to change policy in different areas (for example, transportation, energy, protection of nature).

[19]It is straightforward to (numerically) incorporate a weighting function for questions; however, it comes at the cost of clarity.

The right panel of Figure 5, using our previous example, provides a graphical representation of the value of knowing $\mathcal{F}_1$. The upper-right panel of Figure 6 represents the value of knowing $\mathcal{F}_2$, the lower-right panel that of knowing $\mathcal{F}_4$.

The benefits of discovery. The benefits of a discovery describe how much the value of knowledge improves due to the discovery. Formally, adding an additional point $(x, y(x))$ provides the benefit:

$$V := v(\mathcal{F}_k \cup (x, y(x))) - v(\mathcal{F}_k)$$

The size of V depends on the choice of x. We distinguish two scenarios: *expanding* knowledge and *deepening* knowledge. We say a discovery expands knowledge if it is on a question $x \notin [x_1, x_k]$ outside the current frontier. We say a discovery deepens knowledge if it is on a question $x \in [x_1, x_k]$ inside the current frontier.

We first derive the benefit-of-discovery function. We then state two corollaries that characterize properties of the benefit of discovery. The two main ingredients to determine this benefit are the *distance to knowledge*, which we formally define below, and the *research area.* It turns out that the length X is a sufficient statistic for the research area. Recall that $X = \infty$ for areas outside the current frontier.

Definition 1. The distance of a question x to knowledge $\mathcal{F}_k$ is the smallest Euclidean distance to a question to which the answer is known:

$$d(x) := \min_{\xi \in \{x_1, x_2, \dots x_k\}} |x - \xi|$$

We now state the benefits of discovery.

Proposition 1. *Take a discovery $(x, y(x))$ with distance $d = d(x)$ and in a research area of length X. The benefit of discovery, $V(d, X)$ is*

$$V(d; \infty) = -\frac{d^2}{6q} + d + \mathbf{1}_{d>4q} \frac{(d-4q)^{3/2}\sqrt{d}}{6q},$$

if knowledge is expanded, and

$$\begin{aligned} V(d; X) = \frac{1}{6q}\Big(2dX - 2d^2 &+ \mathbf{1}_{d>4q}\sqrt{d}(d-4q)^{3/2} \\ &- \mathbf{1}_{X>4q}\sqrt{X}(X-4q)^{3/2} \\ &+ \mathbf{1}_{X-d>4q}\sqrt{X-d}\ (X-d-4q)^{3/2}\Big) \end{aligned}$$

if knowledge is deepened.

The terms in $V(d; \cdot)$ without an indicator function measure the direct change in variance and hence the effect on decision making conditional on a proactive action $a \neq \varnothing$. The terms with an indicator function, $\mathbf{1}$, become active whenever the corresponding area contains questions with conjectures sufficiently imprecise (see e.g., Figure 7, right panel). The imprecise conjectures induce the decision maker to refrain from a proactive action. Instead the decision maker chooses the safe option $\varnothing$ which limits losses to $-q$. The terms with an indicator function that enter positively correspond to newly created areas. The term with an indicator function that enters negatively corresponds to an old replaced area.

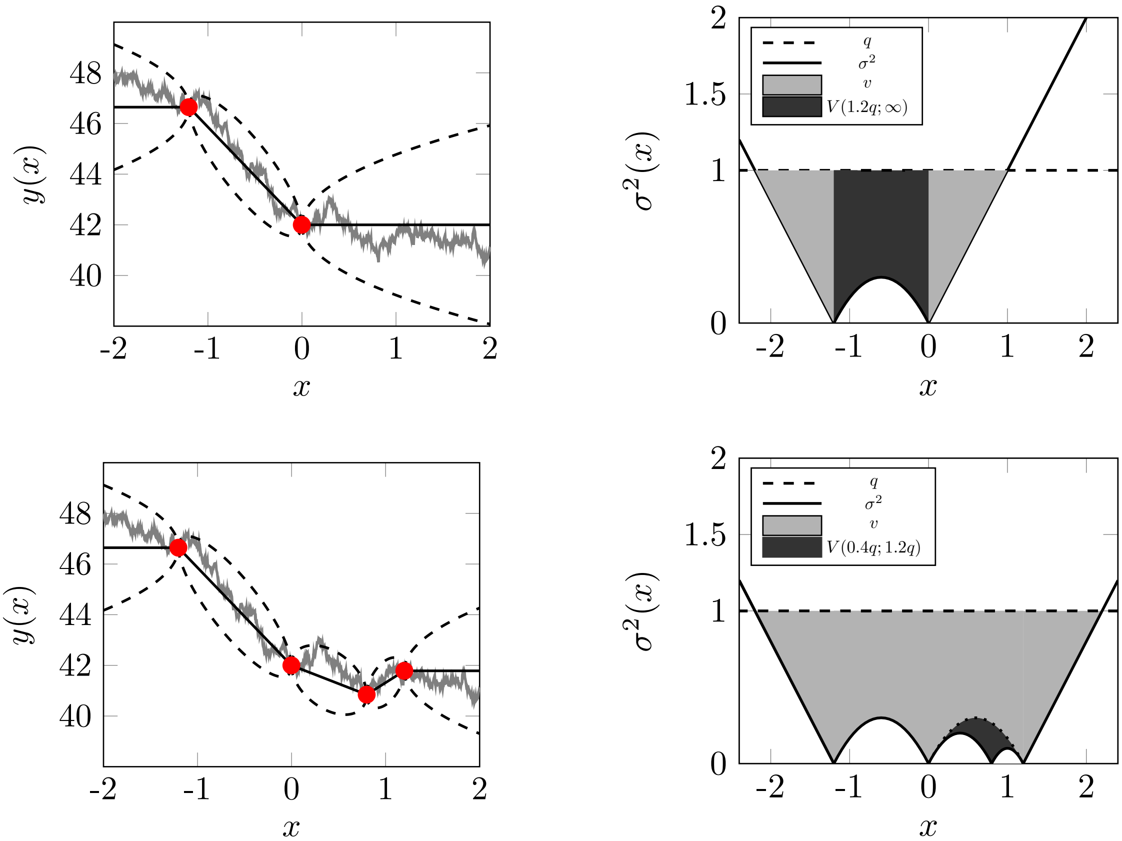


Figure 6: *The benefits of discovery*

Upper panels: Benefit of a knowledge-expanding discovery. The left panel is identical to the right panel in Figure 3. Outside the frontier, $x \notin [-1.2, 0]$, the variance is $\sigma^2(x) = d(x)$. Inside, it is $\sigma^2(x) = d(x)(X - d(x))/X$, where $X = 1.2$ is the length of the interval $[-1.2, 0]$. That variance is smaller than $d(x)$ because of inference from *both* knowledge points. As in Figure 5, the benefit of $\mathcal{F}_2$ is the area (shaded in gray) below q but above $\sigma^2(x)$. The net benefit of discovering the answer to question $x = 1.2$ is the dark-gray area.

Lower panels: Benefit of knowledge-deepening discovery. The left panel is identical to the right panel in Figure 4. The right panel shows the value of knowledge and the benefit of discovery when research deepens knowledge. The dark-gray area is the net benefit of discovery of point $(x, y(x)) = (0.8, 40.8)$ over existing knowledge $\mathcal{F}_3 = \{(-1.2, 46.6), (0, 42), (1.2, 41.8)\}$. As in the previous figures, the total value of $\mathcal{F}_4$ is the area above the variance and below q.

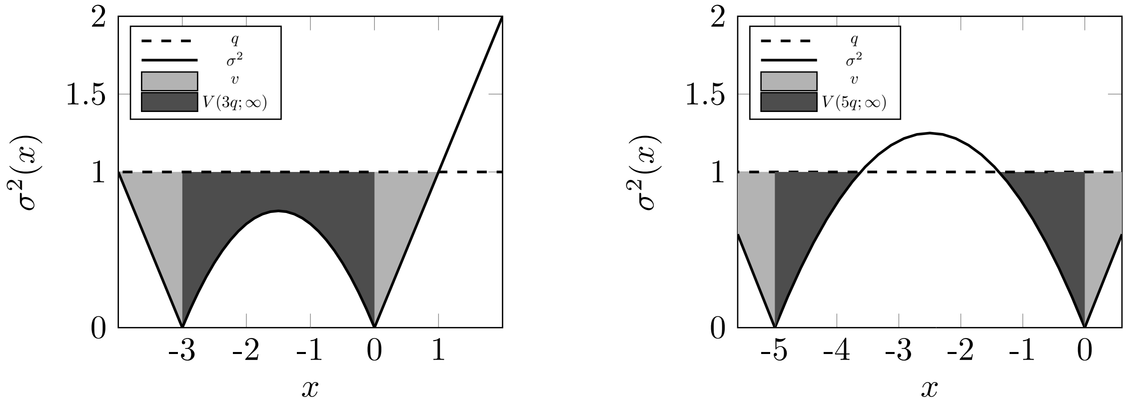


Figure 7: *Benefit-maximizing (left) and too large (right) distance of x given $\mathcal{F}_1$*
Given that the value of doing nothing is $-q = -1$, the benefit-maximizing distance when expanding knowledge is $d = 3$. The left panel depicts the benefit-maximizing choice, given $\mathcal{F}_1$, when expanding to the negative domain using $d = 3q$ and thus $x = -3$; the right panel shows the effect of a choice that is too far away ($x = -5, d = 5q$). The gain in knowledge $V(d;\infty)$ is the dark-shaded area. It is larger in the left panel than in the right panel.

Figure 6 illustrates the benefits of discovery for expanding knowledge (upper panels) and deepening knowledge (lower panels). The right panel of Figure 8 illustrates the functions for different area lengths X. To gain intuition, it is useful to discuss expanding knowledge and deepening knowledge separately.

Expanding knowledge is the process of discovering an answer outside the current domain of knowledge ($[x_1, x_k]$). We focus on discovering the answer to $x < x_1$, which is the case when moving from Figure 5 to the upper row of Figure 6. Case $x > x_k$ is analogous. The benefit of expanding knowledge comes from the new research area it creates. A discovery of $y(x)$ with $x < x_1$ pushes the boundary to the left and creates a new research area $[x, x_1)$. The benefit of that discovery is precisely the value of that area (the dark-shaded area in the upper row of Figure 6).[20]

The value of adding an area depends on (i) the amount of questions in that area, and (ii) the degree of improvement in decision making compared to the default $a = \varnothing$. The latter depends on the precision of the conjectures. The precision of the conjecture to each question in $[x, x_1)$ is determined by its distance to both bounds of the area. Increasing either of these distances decreases the precision of the conjecture. Consider two areas $[x, x_1)$ and $[x', x_1)$ with $x' < x$ and a question $z \in [x, x_1)$ with $d(z) = x_1 - z$. The conjectures about z depend on whether x or x' is discovered. Since x' is further apart from z than x, the conjecture becomes is precise, $\sigma^2_{\mathcal{F}_k \cup [x', x_1)}(x) > \sigma^2_{\mathcal{F}_k \cup [x, x_1)}(z)$. A less precise conjecture implies a less improved decision. Thus, increasing the area length implies a decrease in conjectures for questions of a given distance d.

[20] To be precise, the conjectures about questions to the left of the old frontier are replaced by conjectures inside the new research area and conjectures to the left of the new frontier become also more precise. However, as can be seen in the upper right panel of Figure 6, the variance reduction to the left of the frontier is always of equal value. Hence the benefits are as if there was only a new area added.

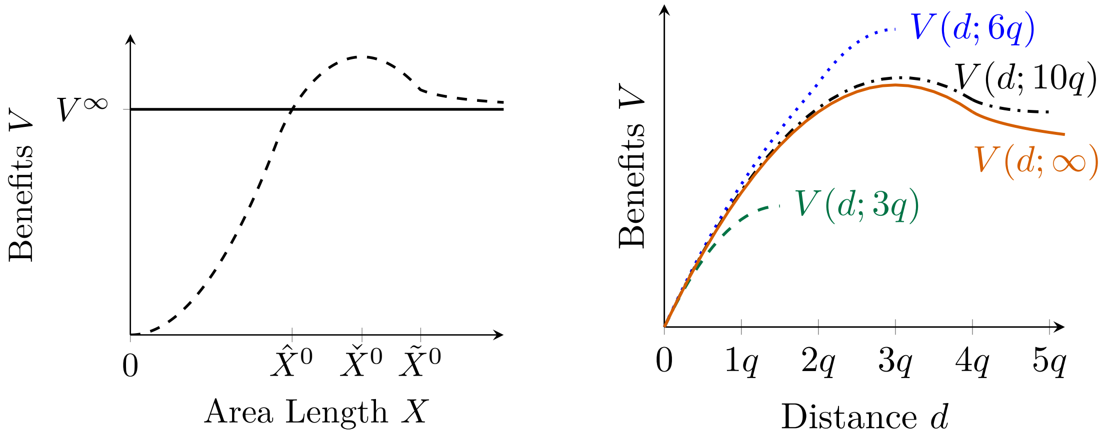


Figure 8: *The benefit of discovery*
Left panel: Benefit of discovery as a function of the area length. The graph plots the benefit of discovery in each interval when choosing the optimal distance $d^0(X)$ therein (dashed line) and the benefit of choosing the optimal distance $d^0(\infty)$ on an expanding interval. Deepening is preferred to expanding if $X > \hat{X}^0 \approx 4.3q$. The maximum benefit is at $\check{X}^0 \approx 6.2q$; $d^0(X) < X/2$ if $X > \tilde{X}^0$.
Right panel: Benefit of discovery given X as a function of distance d. Expanding knowledge (solid line) and deepening knowledge (nonsolid lines) for area lengths $X \in \{3q, 6q, 10q\}$. The benefits are nonmonotone in X.
Note: Plots for deepening knowledge end at the maximum distance in each area, $d = X/2$.

Increasing the area length of the newly created area has two opposing effects on the value of discovery. It increases the amount of questions to which the conjecture improves—an increase in the benefits of discovery. It decreases the precision of all questions in the area—a decrease in the benefits of discovery. The upper right panel of Figure 6 illustrates the benefits of discovery of creating a (too) short area, the left panel of Figure 7 illustrates the largest attainable benefits of discovery, and the right panel of Figure 7 illustrates the benefits of discovery of creating a (too) large area. The largest benefits are derived at an interior level ($d(x) = 3q$) where all conjectures have a variance strictly smaller than q.

Corollary 1. *The benefit of expanding knowledge is single peaked in d. The benefit-maximizing distance in the expanding area is $d^0(\infty) = 3q$. The benefit of optimally expanding knowledge is $V(3q; \infty) = \frac{3}{2}q$.*[21]

Deepening knowledge is the process of discovering an answer $y(x)$ to a question x in an area i with two bounds, x_i and x_{i+1}. The answers $y(x_i)$ and $y(x_{i+1})$) are known. An illustration is the lower panel of Figure 6. The difference to expanding knowledge is that instead of creating a new area, deepening knowledge replaces an old area ($[x_i, x_{i+1})$) by two new areas $[x_i, x)$ and $[x, x_{i+1})$.

[21] The results of this and the next corollary follow directly from the analysis of $V(\cdot;\cdot)$ derived in Proposition 1. However, their derivations are not entirely straightforward—hence, we provide them in detail in the appendix.

The benefit of a discovery depends on the combination of improved decision making in either of the areas. We know from Corollary 1 that the largest benefits come from an area of length $3q$. Thus, if the old area i had length $X_i = 6q$ discovering the midpoint provides the largest benefits. However, if $X_i \neq 6q$ at least on of the two areas has to have a length different from $3q$ and thus provide smaller benefits.

If $X_i \neq 6q$ two forces are at play. First, there is a benefit of replacing the old area with two symmetric new areas each half the length of the old. Increasing the length of an area comes at the cost of decreasing precision in conjectures. However, the larger the area the greater that loss in precision. The reason is that spillovers from the boundaries decline. As a result, it is beneficial to (marginally) decrease the length of the larger area at the expense of an increase the length of the smaller area. Inspection of the lower right panel of Figure 6 provides a graphical intuition.

Second, benefits decline if an area length is far from $3q$. Maintaining symmetry would imply that the length of both newly created areas is far from $3q$ if the length of the old X_i is far from $6q$.

If the old X_i was small, that tradeoff is solved optimally in favor of symmetry. It is better to balance spillovers even if each area falls short of having length $3q$. If however, the old X_i was large, the tradeoff is solved in favor of having one high-value area. It is better to suffer from low spillovers onto questions in the middle of the larger area than to create two areas with imprecise conjectures each. As a result the smaller area is close to yet above length $3q$, the other area is much larger. As the length of area grows large, $X_i \to \infty$, the impact of the larger area becomes less and less important and the length of the smaller area converges to $3q$. A cutoff $\tilde{X}^0 \in [6q, 8q]$ exists such that it is most beneficial to create two symmetric areas if and only if $X_i < \tilde{X}^0$

Finally, one might ask which initial area length X_i provides the largest benefit of being transformed into 2 new areas. As explained above, two areas of length $3q$ provide the largest value. However, we have to take into account that the two new areas also replace an old area. The larger the old area the lower the value it initially provided. Thus, there is a benefit to replace large areas. Yet, the larger the old area (beyond $6q$) the lower the benefit the two new areas provide. As a result the area length whose replacement provides the largest benefits, $\check{X}^0 \approx 6.2q$, is above $6q$.

Expanding vs deepening knowledge. A natural question is to ask when are the benefits of expanding knowledge larger than those of deepening knowledge. On the one hand, creating new areas has the benefit that no knowledge is replaced as all old areas remain. On the other hand, deepening knowledge has the benefit of creating two new areas with relatively precise conjectures. If an area is small, the benefits of replacing that area by two areas with more precise conjectures is small. Conjectures and hence decisions do not improve much. If an area is large before knowledge is deepened, it does not contribute much to the value of knowledge. In particular, if the area contains questions to which conjectures are so imprecise that decision remain at $a = \varnothing$ deepening may be beneficial. We find that indeed deepening is only beneficial if the initial area contains such questions with imprecise conjectures and are hence larger than $4q$. We determine a cutoff $\hat{X}^0 \approx 4.3q$ such that deepening is more beneficial than expanding if $X_i > \hat{X}^0$.

Below, we summarize all findings in a corollary to Proposition 1. Figure 8

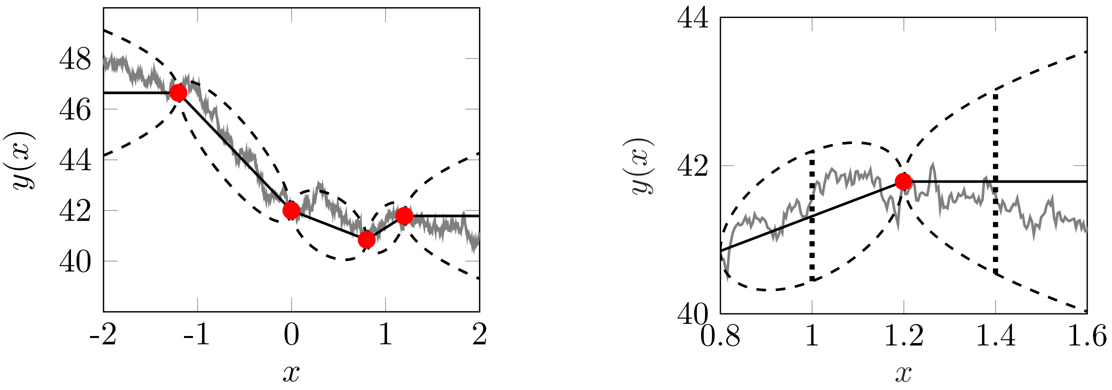


Figure 9: *Cost of research and interference*
The left panel corresponds to the right panel in Figure 4. The right panel is a close-up of the left panel. On the right, the two dotted lines represent the 95-percent prediction intervals for the answers to questions $x = 1$ and $x = 1.4$ respectively if the knowledge is $\mathcal{F}_4 = \{(-1.6, 46.6), (0, 42), (0.8, 40.8), (1.2, 41.8)\}$. Both questions have the same distance to $\mathcal{F}_4$ ($d(x) = 0.2$). However, the 95-percent prediction interval at question $x = 1$ is shorter because the variance is smaller, in turn because researching $x = 1$ implies deepening knowledge. Research on question $x = 1.4$ expands knowledge which implies a larger variance.

provides the graphical illustration.

Corollary 2. *There are three cutoff area lengths, $4q < \hat{X}^0 < 6q < \check{X}^0 < \tilde{X}^0 < 8q$, such that the following holds:*[22]

- *If $\mathcal{F}_k$ is such that $X_i < \hat{X}^0$ for all intervals i, then expanding knowledge maximizes the benefit of discovery. Otherwise, deepening knowledge maximizes the benefit of discovery.*
- *The benefit of discovery is largest for area length $\hat{X}^0$. It is increasing in area length for $X < \hat{X}^0$ and decreasing in area length for $X > \hat{X}^0$.*
- *The benefit-maximizing distance to knowledge (given area X), $d^0(X)$, is increasing for $X < \tilde{X}^0$ and decreasing for $X > \tilde{X}^0$. It is benefit-maximizing to select the midpoint $d^0(X) = X/2$ if $X < \check{X}^0$ and to select $d^0(X) < X/2$ if $X > \check{X}^0$.*
- *As $X \to \infty$, $d^0(X)$ converges to $d^0(\infty)$ and $V(d; X)$ converges uniformly to $V(d; \infty)$.*

4 The Cost of Research

So far, we have only considered the benefits of a discovery. However, not all research leads to discovery and there are costs of research. Searching for an answer requires time and effort. We assume—in line with our assumption that research is a discovery process—that having selected a question, the researcher samples an interval of her

[22]The values of the cutoffs $\hat{X}^0, \check{X}^0$ are analytically yet implicitly computed in Appendix B.5. We obtain $\hat{X}^0 \approx 4.338q, \check{X}^0 \approx 6.204q$. We are not able to analytically compute $\tilde{X}$ in a meaningful way, but we can bound its value by $\check{X}^0$ from below and by $8q$ from above.

choice on the real line for the answer. If the answer is within that interval, the researcher has found the answer; otherwise, the researcher has not found the answer.

We make the standard assumption that the marginal cost of search are increasing in the interval length covered. Formally, we assume that if a researcher samples an interval $[a, b]$ on the real line to see whether the answer $y(x)$ to question x lies in that interval, she incurs a cost proportional to $(b - a)^2$.[23]

We now characterize the (endogenous) cost of research incurred by a researcher who aims at obtaining the answer to question x with a particular likelihood ρ. For any ρ the researcher selects the shortest interval $[a, b]$ that contains an answer with probability ρ. There are three steps leading to our characterization: First, that shortest interval is centered around the expected answer. Second, the length of the interval is proportional to the standard deviation and thus a function of the distance to existing knowledge $d(x; \mathcal{F}_k)$ and the length of the research area, X. Third, fixing distance d and X, the length of the interval is proportional to the inverse Gaussian error function, $erf^{-1}(\rho)$. Combining the three steps, we obtain a simple, reduced-form cost function that is separable in d and ρ. We begin with a definition.

Definition 2 (Prediction Interval)**.** The prediction interval $\alpha(x, \rho)$ is the smallest interval $[a, b] \subseteq \mathbb{R}$ such that the answer to question x lies within $[a, b]$ with a probability of at least ρ.

In Figure 3 and Figure 4, the dashed lines plot the 95-percent prediction interval for each question x as a function of existing knowledge. We now characterize the prediction interval analytically. The cost function is a corollary to the following simple proposition regarding normal distributions.

Proposition 2. *Suppose $\alpha(x, \rho)$ is the prediction interval for probability ρ and question x, where the answer $y(x)$ is normally distributed with mean μ and standard deviation σ. Then any prediction interval has the following two features:*

1. *The interval is centered around μ.*
2. *The length of the prediction interval is $2^{3/2} erf^{-1}(\rho)\sigma$, where erf^{-1} is the inverse of the Gaussian error function.*

Figure 9 illustrates how the length of the prediction interval depends on the location of the question. Two questions with the same distance to existing knowledge (that is, distance to question $x = 0.6$) have different 95-percent prediction intervals depending on whether the research deepens knowledge or expands it. That difference translates to a difference in the cost function. The cost-of-research function (see Figure 10) follows from a simple corollary to Proposition 2.

Corollary 3. *For given knowledge $\mathcal{F}_k$, fix a probability ρ and a question x. The minimal cost of obtaining an answer to question x with probability ρ is proportional to*

$$c(\rho, x; \mathcal{F}_k) = 8(erf^{-1}(\rho))^2 \sigma^2_{\mathcal{F}_k}(x)$$

[23]The quadratic formulation is for convenience only. It allows us to describe closed-form solutions. What matters for our results is that the cost is *increasing and convex* in the length of the interval sampled. See Proposition 2.

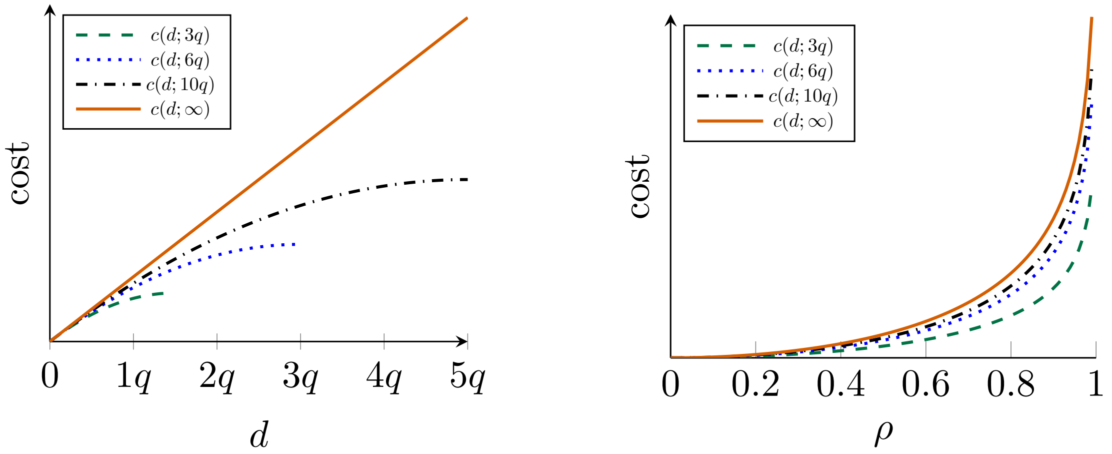


Figure 10: *Cost of research as a function of distance to knowledge (left) and probability of success (right)*
The cost become smaller for a given (d, ρ) as interval length X decreases. Moreover, the cost are linear in distance when expanding knowledge but concave when deepening knowledge. The cost function is always convex in ρ. The left panel holds $\rho = 95\%$ fixed; the right panel holds $d = 3q/2$ fixed. In the left panel, plots for an area length $X < 10q$ end at the maximum possible distance ($d = X/2$). Numerical values on the y-axis are omitted, as costs are only proportional to the depicted functions (see Corollary 3).

Corollary 3 yields an intuitive characterization of the cost of discovering an answer to question x given initial knowledge $\mathcal{F}_k$ with probability ρ. Because the inverse error function is increasing and convex, the cost is increasing and convex in the probability of finding an answer. Finding an answer with certainty implies infinite cost, as there is always a chance that the answer is outside the sampled interval.

Moreover, the more imprecise the conjecture about question x, the higher the cost of discovering an answer with a certain probability (ρ). Hence, if a more distant question is to be answered, the ceteris paribus cost of doing so is higher.

Finally, when comparing two questions with the same distance but different research areas, the costs are different. To illustrate this, consider a question that, if answered, would expand knowledge (for example, some $x_e = x_1 - d$) and a question that, if answered, would deepen knowledge (for example, some $x_d = x_1 + d = x_2 - d$). Distance d is the same for both questions. However, the conjecture is more precise in the deepening interval, as information from two known answers can be used to form that conjecture. As a consequence, for the same distance (d) and probability (ρ), the cost is lower in the deepening interval.

5 The Researcher's Choices

In this section, we introduce a researcher to our model of knowledge and characterize her optimal choice.

5.1 The Researcher's Objective

Consider a researcher that can search for a discovery. Her expected payoff is composed of the benefits she provides to the decision maker if she finds an answer (that is, the function from Section 3) and her own cost of research (that is, the cost function from Section 4). She is unconstrained in her choice of the research question and her effort, but whenever she fails to obtain an answer, the benefit of her research is zero.[24]

The closest real-world analogue to our model researcher is perhaps a tenured university professor. She is free to select a research question and the intensity with which to work on it. The quality of her research is mainly evaluated through peer review and depends on the (expected) impact of her research on decision making. The more the researcher invests in discovery, the more resources she needs in such forms as research assistants or time taken away from other projects. To capture the relative weighting of cost and benefits, we introduce a cost parameter, $\eta > 0$.

Using our results from Section 3 and 4, the researcher's choices of question x and research interval $[a, b]$ imply a distance (d), an expected probability of finding an answer (ρ), and an area length (X). These three variables are sufficient to describe the researcher's payoff:

$$u_R(d, \rho; X) := \rho V(d; X) - \eta c(\rho, d; X)$$

For the above formulation and for what follows, we abuse notation slightly to facilitate understanding. We (re)define the cost $c(\rho, d, X) := c(\rho, x; \mathcal{F}_k)/8$, where $d(x) = d$ and X is the length of the area containing x. In addition, to ease computation, we divide the function from Corollary 3 by 8 and capture the constant within η.

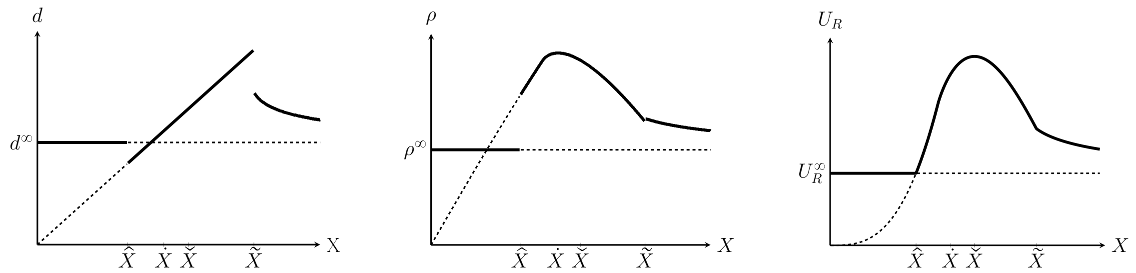
5.2 The Researcher's Decision

We now characterize the optimal choice of a researcher with access to knowledge $\mathcal{F}_k$. The researcher's problem can be formulated as follows:

$$\max_{X \in \{X_0, \dots, X_k\}} \underbrace{\max_{\substack{d \in [0, X/2], \\ \rho \in [0,1]}} \rho V(d; X) - \eta c(\rho, d; X)}_{=:U_R(X)}$$

If there is no cost ($\eta = 0$), we can use the insights from Section 3 to obtain the researcher's optimum. For any X, the researcher selects $\rho = 1$. For $X < \tilde{X}^0$ she selects $d = X/2$ and $d \in (3q, X/2)$ for $\tilde{X}^0 < X < \infty$. If $X = \infty$, the researcher

[24] A rationale for discarding nonfindings comes from moral hazard concerns: science is complex, and it is impossible to distinguish the absence of a finding from the absence of proper search. Our model can easily account for the possibility of publishing nonfindings; unsurprisingly, these increase the value of knowledge as well. The difficulty of publishing the absence of evidence, however, has been long recognized in the literature. See, for example, Sterling (1959). In principle, it is relatively straightforward to compute updated answer distributions based on null results in our setting. Including this in our researcher model, however, is beyond the scope of this paper.



(a) Distance d (b) Probability ρ (c) Payoff U_R

Figure 11: *Outcomes of the researcher's choices in areas of different length*
The graphs indicate optimal choices conditional on area length, X. We compare them with optimal choices on the expanding area $X = \infty$ (the horizontal line in each graph). On the x-axis we indicate the cutoffs $\hat{X}, \check{X}, \dot{X}$, and $\tilde{X}$ from Proposition 3.
The *solid graph* plots the optimal choice conditional on X being the best available area. For small areas ($X < \hat{X}$), the researcher prefers to expand knowledge. For areas of length $X > \hat{X}$, deepening knowledge is preferred to expanding knowledge. If the area has length $X < \tilde{X}$, the researcher selects the largest distance possible in area X—that is, $d = X/2$. If $X > \tilde{X}$, it is optimal to select a distance $d < X/2$ closer to one of the end points of the area. For small areas ($X < \dot{X}$), $\rho(X)$ increases with X. For large areas ($X > \dot{X}$), $\rho(X)$ decreases in X (apart from a discontinuous jump at $\tilde{X}$). The researcher's payoff increases in area length for $X < \check{X}$ and decreases for $X > \check{X}$. The order of the cutoffs is independent of the value of the cost parameter, η.

chooses $d = 3q$. She prefers to expand knowledge if and only if $X_i \leq \hat{X}^0$ for any area $X_i < \infty$ defined by $\mathcal{F}_k$ (see Corollary 2).

We now characterize the researcher's optimal decision when $\eta > 0$, that is, when considering the cost of research as well. Let $d(X)$ and $\rho(X)$ be the researcher's choices conditional on an area of length X, and let $U_R(X)$ be the associated payoff. Analogously, $d^{\infty}, \rho^{\infty}$, and U_R^{∞} are the respective objects for expanding research. Figure 11 illustrates the following proposition.

Proposition 3. *Fix $\eta \in (0, \infty)$. On the expanding area, we obtain optimal choices $d^\infty \in (0, 3q)$ and $\rho^\infty \in (0, 1)$, which deliver utility $U_R^\infty \in (0, \infty)$. There are cutoffs, $\hat{X} \leq \dot{X} \leq \check{X} \leq \tilde{X}$, such that the following holds:*

The optimal distance $d(X)$

- *increases in X if $X < \tilde{X}$ with $d(X) = X/2$ and*
- *declines on average on the interval of area lengths $(\tilde{X}, \infty)$.*[25]

The optimal probability $\rho(X)$

- *increases in X if $X < \dot{X}$,*
- *decreases in X if $X \in (\dot{X}, \tilde{X})$,*
- *discontinuously increases at $\tilde{X}$, and*
- *declines on average on the interval of area lengths $(\tilde{X}, \infty)$.*

The researcher's payoff, $U_R(X)$,

- *is smaller than U_R^∞ if and only if $X < \hat{X}$,*
- *increases in X if $X < \check{X}$, and*
- *decreases in X if $X > \check{X}$.*

Moreover, as $X \to \infty$, $d(X) \to d^\infty, \rho(X) \to \rho^\infty$, and $U_R(X) \to U_R^\infty$ from above.

Introducing cost, $\eta > 0$, distorts the researcher's decision from the choice that maximizes the benefits of a discovery. There are two additional effects: First, ceteris paribus the cost of research increase in the variance of the conjectures—a force that mitigates the researcher's incentives to aim for higher d. Second, cost increase faster in d when expanding knowledge because the variance increases faster—a force that mitigates the researcher's incentive to aim for expanding knowledge. Yet, in terms of the choice of question we obtain the same qualitative pattern.

Figure 11 sketches the researcher's expected value of conducting research in areas of different lengths and the corresponding probability of finding an answer. The optimal within-area distance is $d = X/2$ for all $X \in [0, \tilde{X}]$ and $d < X/2$ if $X > (\tilde{X}, \infty)$. The maximum payoff is achieved for research in area $\check{X}$. Expanding knowledge is the optimal strategy if all available interval lengths $X < \hat{X}$. As the area length $X \to \infty$, the value of conducting research on the area converges to that of expanding research, $X = \infty$. If several X are available, the researcher chooses the one with the largest value. We discuss the effect of η on the cutoff values below.

Yet, the cost of research introduce a second, interlinked dimension: the researcher's likelihood of finding an answer.

It is instructive to begin with the expanding-knowledge case, $X = \infty$. The researcher simultaneously chooses a probability of finding an answer, ρ, and a distance to current knowledge, d. Her expected benefit is $\rho V(d; \infty)$, and her cost is $\eta erf^{-1}(\rho)^2 d$. By setting $\rho = 0$—that is, by not looking for an answer at all—the researcher can guarantee herself a payoff of $u_R = 0$. The marginal cost of increasing the probability, starting at $\rho = 0$, are 0 and non-increasing. However, the marginal benefit of increasing ρ is $V(d; \infty)$ which is positive if $d > 0$. Thus, a higher utility than $u_R = 0$ is attainable by looking for the answer to a question with $(d, \rho) > 0$.

A low-effort search in a very narrow area is almost costless but provides strictly positive benefits. As $\rho \to 1$, the researcher has to look for an answer on the entire

[25]Numerically, $d(X)$ and $\rho(X)$ are decreasing throughout on $(\tilde{X}, \infty)$. However, we provide no formal proof.

real line for any $d > 0$—an infinitely expensive task. The optimal ρ is interior. By Proposition 1 we know that the benefit of discovery is increasing at first but decreasing once $d > 3q$. The cost of research, in turn, is monotone in d. The more distant the question, the more imprecise the conjecture and hence the more effort required to answer the question with any given probability ρ. The optimal distance is smaller than absent cost ($0 < d^\infty < 3q$). The probability of finding a solution is strictly positive but not 1 ($0 < \rho^\infty < 1$). The utility the researcher expects is positive and bounded: $0 < U_R^\infty < V(3q; \infty)$.

Now consider a researcher that deepens knowledge. The basic logic is identical. However, deepening knowledge is ceteris paribus less costly than expanding knowledge. For any given (ρ, d) the variance is lower the smaller X_i and, hence, the cheaper is the search. Regarding the researcher's choice of d the logic follows the discussion of Proposition 1.

The optimal ρ depends on the size of the area too. Consider a small area X_i. The scope for improvement is small as conjectures are already precise. Thus, investing into discovery has a small expected payoff. Albeit cost are small, the researcher does not invest much into the search for an answer—ρ is small. Now consider a large area. The benefits of deepening researcher are larger than in the small area. However—because conjectures are imprecise—the cost are larger too. The researcher does not invest much into a discovery—ρ is small. In an area of intermediate length, the benefits of a discovery are (relatively) high, yet conjectures are sufficiently precise and limit the cost of research. The return on investment is largest and ρ is high. We obtain a similar pattern for ρ as that for d: while ρ is low for small and large areas, it is higher for intermediate X_i.

Moreover, the researcher only trades off $d(X)$ against $\rho(X)$ if X is of intermediate length. If X is small, an increase in X increases the benefits of research. Cost are small, and the researcher has an incentive to increase $d(X)$ and $\rho(X)$. As X becomes larger, the marginal increase in the benefits of research declines, yet the marginal cost of research increase both for $d(X)$ and $\rho(X)$. Eventually the researcher faces a tradeoff: should she lower $\rho(X)$ to maintain $d(X) = X/2$? It turns out that this is optimal. While the researcher wants to remain at a boundary in her choice of $d(X)$ she mitigates the increased cost by lowering $\rho(X)$. As X increases further, the researcher eventually also lowers $d(X)$. After a discrete jump upwards at $\tilde{X}$ of $\rho(X)$ due to the jump downwards of $d(X)$, the two start to co-move again. Both decline.

The researcher's overall most preferred area length, $\check{X}$, is in a region in which a trade-off between $\rho(X)$ and $d(X)$ exists. The researcher would like to enter a larger X to increase the benefits of research, she would also prefer a smaller X to reduce the cost of finding an answer. Thus, $d(X)$ is increasing and $\rho(X)$ decreasing at the point at which $U_R(X)$ is maximal.

Note that we have only characterized the researcher's decision *conditional* on an area length X and compared these values so far. An explicit analytical characterization of the researcher's choice thus depends on the existing knowledge and thus all available X's. Solving for such an optimum is straightforward but not instructive.[26]

The extensive margin—that is, which X to choose—depends on the precise

[26]A computer program to numerically calculate the optima for different parameter values is available from the authors.

location of the cutoffs $\hat{X}, \dot{X}, \check{X}$, and $\tilde{X}$. We summarize the comparative statics of these cutoffs in the next corollary.[27]

Corollary 4. *All cutoff values are non-increasing in η. While $\dot{X}$ is constant for $\eta > 0$, the cutoffs $\hat{X}, \check{X}$, and $\tilde{X}$ are strictly decreasing and thus strictly smaller than their counterparts from Corollary 2.*

6 Application: Science Funding

In this section, we illustrate our model through an application. Consider a funding institution (the funder) with two instruments. Ex ante cost reduction for the researcher (by, for example, providing grants to reduce the researcher's cost) and ex post rewards (by, for example, handing out prizes for seminal contributions).[28] In addition, assume that the funder has to respect scientific freedom and can spend her budget on any combination of cost reductions and rewards. Which levels of novelty d and productivity ρ can the funder implement.

In the main text we present the stylized model to address this question in our framework and it's main result: a characterization of the feasible set in the (d, ρ)-space. In appendix C, we provide the full analysis of the mechanism combining our results from above with standard arguments from consumer theory.

Setup. Consider the following stylized model. Knowledge consists of a single question-answer pair, $\mathcal{F}_1 = (x_1, y(x_1))$. The researcher's cost parameter is η^0. The funder has a fixed budget K to invest in the researcher and has two funding technologies: ex ante cost reductions, h, and ex post rewards, ζ. Marginal cost of both are constant, and the cost ratio κ is such that the funder's budget constraint is given as follows:

$$K = \zeta + \kappa h$$

A cost reduction of h implies that the researcher faces cost parameter $\eta \equiv \eta^0 - h$. An ex post reward gives the researcher additional utility of ζ if she finds an answer. We assume rewards only come for "seminal contributions," that is, contributions that are sufficiently novel and thus difficult to obtain. We proxy that relation by a function $f(\sigma_{\mathcal{F}_k}) : \mathbb{R} \to [0, 1]$ that determines the probability of a reward. To keep the analysis simple, we assume a piecewise linear relationship

$$f(\sigma) = \begin{cases} \frac{\sigma^2}{s} & \text{if } \sigma^2 < s \\ 1 & \text{otherwise,} \end{cases}$$

for some $s > 0$. The parameter s determines the minimum level of difficulty that guarantees a reward.[29] To shorten notation define $\tilde{c}(\rho) := (erf^{-1}(\rho))^2$.

[27] Regarding the comparative statics for the intensive margin, $d(X)$ and $\rho(X)$, we conjecture that both are decreasing in η as well. However, so far we have been unable to provide a formal proof for the case $X > \tilde{X}$.

[28] There is a large literature debating different forms of science funding. For recent contributions, see Price (2019) and Azoulay and Li (2020) and references therein.

[29] The crucial assumption here is that f is a bounded function, which is true whenever f is indeed a probability. If f instead was unbounded, the researcher would naturally (for any $\zeta > 0$)

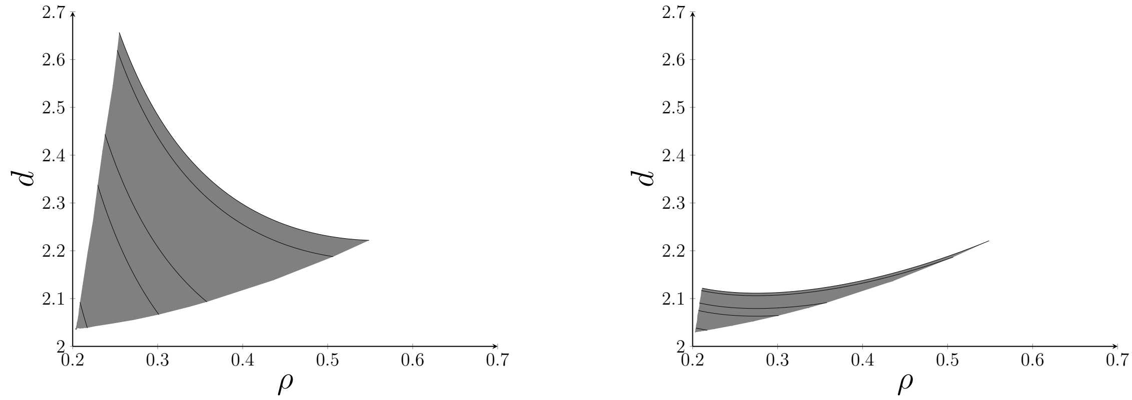


Figure 12: *Feasible set for different reward standards*
The shaded area shows the implementable (ρ, d) combinations of a funder for a given budget K. All points on a solid line require the same amount of funding, K. In both panels, the funder has a budget of $K = 10$, the price ratio is $\kappa = 6$, and the baseline cost factor is $\eta^0 = 2$. The status quo parameter has value $q = 1$. In the left panel, the reward technology is parameterized with $s = 40$; in the right panel, it is $s = 200$.

In our stylized model the researcher's problem becomes the following:

$$\max_{d,\rho} \rho\Big(V(d;\infty) + f(\sigma(d;\infty))\zeta\Big) - \eta\tilde{c}(\rho)\sigma^2(d;X).$$

The feasible set. In reality, funding may have many different objectives and can range from maximizing the externalities of research to answering particular questions. A demand-side discussion of research is beyond the scope of this paper. Instead, we characterize the feasible set of choices (d,ρ) that a funder can induce with a given budget. Computing this set provides a useful tool to analyze the optimal funding scheme given a particular preference relation over (d,ρ) bundles. As in a standard consumer problem, it can be readily applied, with this set being the analogue of a budget set. Let $\tilde{c}_\rho(\rho) := \partial\tilde{c}/\partial\rho(\rho)$.

Proposition 4. *The set of implementable (d,ρ)-combinations for a given cost ratio κ and a budget K is described by the (d,ρ) implementation frontier defined over $[\underline{\rho}, \overline{\rho}]$, which are the endogenous upper and lower bounds of ρ. These bounds are determined by the extreme funding schemes $(\zeta = 0, \eta = \eta^0 - K/\kappa)$ and $(\zeta = K, \eta = \eta^0)$. The research-possibility frontier between those polar points is as follows:*

$$d(\rho;K) = 6q(K + s - \kappa\eta^0)\frac{\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)}{2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) - \kappa\rho} \tag{1}$$

$d(\rho;K)$ can be increasing or decreasing, depending on whether d and ρ behave as substitutes or complements at the (ζ,η) mix inducing the current level of (d,ρ).

The proof of Proposition 4 is, together with the rest of the formal discussion of this application, in appendix C. The main takeaway from Proposition 4 is that the

choose to select $d = \infty$. The specific form, in turn, is chosen only for ease of computation.

slope of the research-possibility frontier can be positive or negative. If the slope is positive, changing the allocation of funds to increase output, ρ, implies an increase in novelty, d, as the two are complements from the funder's perspective. If the slope is negative, an increase in output implies a decrease in novelty, as the two are substitutes from the funder's perspective.

We illustrate the feasible set for two levels of parameter s in Figure 12. In the left panel ($s = 40$), novelty and output are substitutes as long as $\rho < \tilde{\rho} \approx 50\%$. As s increases to 200, that threshold decreases to $\tilde{\rho} \approx 29\%$. A funder that exclusively aims to maximize d (conditional on discovery) would choose to implement a risky choice by the researcher ($\rho \approx 25\%$) if $s = 40$. The same funder would choose a much safer strategy when $s = 200$ ($\rho \approx 65\%$).[30]

Novelty and output become complements because an increase in ρ has two effects: (i) it increases the marginal benefit of distance, $V_d(d, X) + \zeta/s\sigma_d^2(d, X)$, by increasing the probability that the researcher finds an answer, and (ii) it increases the marginal cost of distance, $\eta\tilde{c}(\rho)\sigma_d^2$. The uncertainty of the conjecture about questions, $\sigma^2(d, X)$, is increasing in distance. Moreover, for any distance, the interval that has to be covered to find an answer with probability ρ is increasing in this probability. For certain parameter constellations, it may be that, for example, an increase in ζ increases the weight placed on the marginal-benefit effect relatively more than the resulting increase in η increases the marginal-cost effect. Straightforwardly, the larger s, the smaller the effect on the marginal benefit.

7 Relation to Existing Models

In this section, we relate our model and our findings to the two closest models in the literature, those in Callander and Clark (2017) and Prendergast (2019).

Callander and Clark (2017) consider judicial decision making. Judges are interested in learning the realization of the entire path of a Brownian motion. As in our model, a decision maker (a lower court in their framework) adjusts decisions to the unknown state of the world. The decision maker's knowledge is the realization of the Brownian process at values that a higher court reveals. Aside from the application, the main differences between our model and that in Callander and Clark (2017) are the following. First, we assume that the researcher selects, in addition to the research question, her intensity of research. The latter affects both the cost of research and the probability of finding an answer. Second, we aim to learn where exactly the answer lies, in contrast to the "above or below the threshold" problem that Callander and Clark (2017) analyze. Unlike in their model, the value of a discovery in our framework does not lie in how the expectations relate to the threshold but in how they relate to other known points: as the conjectures become more precise and finding an answer comparatively easy, the benefits of research shrink too. Third, in Callander and Clark (2017), it is eventually optimal to stop discovery. In our model, science never stops. Although the cost of asking a specific question decreases when

[30] We assume in this exercise that s is exogenous to the funder. Under most funding schemes, the winners of a prize are determined by a jury of peers rather than by the funder itself. Thus, the the standard may not be under the funder's control.

similar questions have a known answer, there is always a question worth researching.

Prendergast (2019) considers a model of creative innovation with a different focus. He studies contracts that directly condition on an agent's choice. The benefit-of-discovery function Prendergast (2019) assumes is a special case of ours in which the area length of the best available area is larger than $\hat{X}$ (so that expanding knowledge is not beneficial) but smaller than $\tilde{X}$ (so that selecting the midpoint is always optimal). The novelty depending probability-of-success function he assumes in the extensions matches the features we derive for our endogenous probability of success. Thus, we can construct a special case in which model predictions coincide.

Our focus is on the microfoundation of the functions while his is on agency concerns in a reduced-form model. Yet, in terms of modeling approaches our model of knowledge differs at least in two crucial dimensions. First, we assume that it is possible to expand research beyond the frontier and show that it can also be optimal to do so. Prendergast (2019) instead assumes that research always takes place between two existing findings. Second, our decision maker has an outside option that limits her expected losses if conjectures are too imprecise. Existence of an outside option implies that there are bounds on the benefits of novelty. Once newly created areas become too large, conjectures in that area become too imprecise which mitigates the value of the area.[31] Therefore, and different from Prendergast (2019), we obtain a nonmonotonicity in the value of novelty.

Finally, Prendergast (2019) assumes an exogenous cost function. While our endogenous cost function has the basic properties he assumes (increasing and concave), we provide additional structure by microfounding the model as a search process. Moreover, if cost follow from the search for an answer, they endogenously depend on the area size: the smaller the area, the smaller the variance and hence the smaller the cost.

8 Conclusion

We propose a tractable model of knowledge and show how to embed it in a framework that allows us to study which questions researchers aim to answer and to what extent they invest in finding the answer. The main features of our model are that (i) finding the answer to one question spills over onto the conjectures about other questions, (ii) questions in close proximity to existing knowledge are easier to answer than questions that are far away from existing knowledge, (iii) the benefit of discovery depend on its effect on decision making, and (iv) researchers are motivated by the benefit their findings provide for decision making, but bear a cost of searching for an answer.

Using these four elements, we set up a model to derive an endogenous benefits-of-discovery function and an endogenous cost-of-research function and to characterize the researcher's optimal selection of research question and likelihood of finding an

[31] That result also formalizes the argument in Price (2020) that focusing on providing incentives that maximize novelty can backfire, as knowledge may be scattered to disconnected islands. In our model, research that is too far away from the current body of knowledge has little positive spillovers on society's conjectures about unanswered questions.

answer to it. The results crucially depend on the density of existing knowledge.

Recent empirical work—for example, Rzhetsky et al. (2015)—analyzes the impact of findings on future developments. That work suggests that scientists choose a dynamically suboptimal strategy when selecting their research questions. Rzhetsky et al. (2015) identify researcher myopia as one of the drivers of that effect. Our approach is static and focuses on the immediate effect of findings on decisions. However, the optimal choice of research questions has rich properties that depend on existing knowledge. A natural next step is to take dynamic considerations into account. One straightforward dynamic model would assume that researchers arrive sequentially each with a different cost factor.[32] From observed decisions, such as those documented in Rzhetsky et al. (2015), we could identify the distribution of these cost parameters. Our model would then allow us to compute the counterfactual outcomes of the various mitigation strategies proposed by Rzhetsky et al. (2015).

One such strategy is to incentivize the publication of null results, which we assume to be noncommunicable (an assumption motivated by observed reality). Within our model, knowing that the answer *is not* in a certain interval has a clear value that is straightforward to compute. In addition, because answers to questions are correlated, the knowledge provided by null results also generates a computable spillover onto the conjectures about other questions. Thus, if the researcher can credibly claim to have searched in an interval without finding the answer, publishing null results should be part of the benefit function and inform subsequent researchers' selection and search strategies.

Our paper starts by emphasizing the role of scientific freedom. Preserving that freedom remains a challenging task for science-funding institutions when designing a funding architecture that provides researchers the support to engage in research activities that might not be undertaken otherwise. The NSF emphasizes that it aims at funding high-risk/high-reward research to advance the knowledge frontier. While the question of optimal market design is beyond the scope of this paper, we hope that our modeling framework will serve as a stepping stone toward developing a structural model to evaluate funding incentives and to provide meaningful counterfactuals that can inform decision makers about how to provide incentives optimally.

Appendix

A Notation and Properties of erf^{-1}

Notation: We use argument subscripts to denote the *partial derivatives* with respect to the argument. We omit function argument whenever it is convenient. we use the notation $\frac{\mathrm{d}f(x,y)}{\mathrm{dx}}$ to indicate the total derivative $(f_x + f_y y_x)$.

Properties of erf^{-1}. From Appendix B.5 onwards we rely heavily on the proper-

[32] One could also have researchers arrive with an intuition or stroke of genius—that is, a private signal about the location of a specific answer. We thank Alex Frug for this suggestion.

ties of the inverse error function. It is instructive to keep in mind that

$$\tilde{c}(\rho) := erf^{-1}(\rho)^2$$

is convex and increasing on $[0, 1)$ with $\tilde{c}(0) = 0$ and $\lim_{\rho\to 1} \tilde{c}(\rho) = \infty$.[33] The derivative

$$\tilde{c}_\rho(\rho) = \sqrt{\pi} e^{\tilde{c}(\rho)} erf^{-1}(\rho)$$

is increasing and convex with the same limits.

Further, we make use of the fact that for $\rho \in (0, 1)$ $\tilde{c}(\rho)$ has a convex and increasing elasticity bounded below by 2 and unbounded above. Its derivative $\tilde{c}_\rho(\rho)$ has an increasing elasticity bounded below by 1 and unbounded above. We want to emphasize that these properties are not special to our quadratic cost assumption. To the contrary, $erf^{-1}(x)^k$ for any $k \geq 2$ admits similar properties with only the lower bounds changing. Formally, the following properties are invoked in the proofs:fun

$$\begin{aligned} \rho \frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)} &\in (1, \infty) \text{ and increasing,} \\ \rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho) &\in (0, \infty) \text{ and increasing,} \\ \frac{\tilde{c}(\rho)}{\rho\tilde{c}_\rho(\rho)} &\in (0, 0.5) \text{ and decreasing.} \end{aligned}$$

B Proofs

At various points we make use of inequality relations the proof of which we relegate to supplementary appendix D. In each of these cases proving the inequalities is done via algebra that produces little additional insight.

B.1 Proof of Proposition 1

Proof. The value of knowing $\mathcal{F}_k$ is

$$\int \max\left\{\frac{q - \sigma^2_{\mathcal{F}_k}(x)}{q}, 0\right\} \mathrm{d}x.$$

No matter which point of knowledge $(x, y(x))$ is added to $\mathcal{F}_k$, the value of knowledge outside the frontier is identical for both $\mathcal{F}_k$ and $\mathcal{F}_k \cup (x, y(x))$. Area lengths $X_1 = X_k = \infty$ do not depend on $\mathcal{F}_k$ and neither does the variance for a question $x < x_1$ or $x > x_k$ with a given distance d to $\mathcal{F}_k$. The conjectures about all questions outside $[x_1, x_k]$ deliver a total value of

$$2 \int_0^q \frac{q - x}{q} \mathrm{d}x = q,$$

[33] Due to this limit and the researcher's ability to choose $\rho = 1$, we augment the support of the cost function to include $\rho = 1$ with $\tilde{c}(1) = \infty$. However, the optimal ρ is always strictly interior unless the cost parameter η is chosen to be zero in which case we assume that $\eta\tilde{c}(\rho = 1) = 0$.

which is independent of $\mathcal{F}_k$. Moreover, answering question $\hat{x} \in [x_i, x_{i+1}]$ with $(x_i, y(x_i)), (x_{i+1}, y(x_{i+1}) \in \mathcal{F}_k$ only affects questions in the area $[x_i, x_{i+1}]$, i.e., $G(x|\mathcal{F}_k) = G(x|\mathcal{F}_k \cup (\hat{x}, y(\hat{x})) \;\forall\; x \notin (x_i, x_{i+1})$.

To simplify notation, let us consider the points in terms of distance to the lower bound of the area with X, $d \equiv x - x_i$.

The value of the area $[x_i, x_{i+}]$ is (with abuse of notation)

$$v(X) = \int_0^X \max\left\{\frac{q - \frac{d(X-d)}{X}}{q}, 0\right\} \mathrm{d}d.$$

Note that whenever $X \leq 4q$, $\frac{d(X-d)}{X} \leq q$. Hence, we can directly compute the value of any area with length $X \leq 4q$ as

$$v(X) = -\frac{X^2}{6q} + X.$$

Whenever $X > 4q$, value is only generated on a subset of points in the area. As the variance is a symmetric quadratic function with $X/2$ as midpoint, there is a symmetric area centered around $X/2$ which has a variance exceeding q. The points with variance equal to q are given by $\bar{d}_{1,2} = \frac{X}{2} \pm \frac{1}{2}\sqrt{X}\sqrt{X-4q}$. Hence, the value of an area with $X > 4q$ is (due to symmetry)

$$\begin{aligned} v(X) &= 2\int_0^{\bar{d}_1} \frac{q - \frac{d(X-d)}{X}}{q} \mathrm{d}d \\ &= -\frac{X^2}{6q} + X + \frac{X-4q}{6q}\sqrt{X}\sqrt{X-4q}. \end{aligned}$$

Hence, if a knowledge point on the boundary is added, a new area is created and no area is replaced. The value created is thus

$$V(d) = v(d) = -\frac{d^2}{6q} + d + \begin{cases} 0, & \text{if } d \leq 4q \\ \frac{d-4q}{6q}\sqrt{d}\sqrt{d-4q}, & \text{if } d > 4q. \end{cases}$$

If a knowledge point is added inside an area with length X with distance d to the closest existing knowledge, it generates two new areas with length d and $X - d$ that replace the old area with length X. The total value of the two intervals new is

$$\begin{aligned} v(d) + v(X-d) = -\frac{d^2}{6q} + d + &\begin{cases} 0, & \text{if } d \leq 4q \\ \frac{d-4q}{6q}\sqrt{d}\sqrt{d-4q}, & \text{if } d > 4q \end{cases} \\ +(-\frac{(X-d)^2}{6q}) + X - d + &\begin{cases} 0, & \text{if } X-d \leq 4q \\ \frac{X-d-4q}{6q}\sqrt{X-d}\sqrt{X-d-4q}, & \text{if } X-d > 4q \end{cases}. \end{aligned}$$

The benefit of discovery is then $V(d; X) = v(d) + v(X-d) - v(X)$ which is precisley the expression from the proposition.

□

B.2 Proof of Corollary 1

Proof. The optimality of $d = 3q$ follows directly from the first-order condition for $d \leq 4q$ which is

$$\frac{\partial V(d; \infty | d \geq 4q)}{\partial d} = -\frac{d}{3q} + 1 = 0$$

and the observation that the benefit is decreasing in d for $d > 4q$ which can be seen from the derivative with respect to d which is

$$\frac{\partial V(d; \infty | d > 4q)}{\partial d} = -\frac{d}{3q} + 1 + \sqrt{\frac{d-4q}{d}}\frac{d-q}{3q} < 0.$$

The inequality follows from Lemma 23 in appendix D. □

B.3 Proof of Corollary 2

We proceed in a series of lemmata. We outline the mapping to Corollary 2 first

- The first part of the first bullet point follows from Lemma 1 and Corollary 1. The second part follows from Lemma 2 to 6 and Lemma 8.
- The second bullet point follows from Lemma 2, 6 and 8
- The third bullet point follows from Lemma 2 to 7.
- The fourth bullet point follows from Lemma 5.
- Lemma 9 proves the order of the cutoffs.
- Lemma 10 provides an approximation to $\hat{X}$.

Throughout, we refer to the distance d that maximizes $V(d; X)$ as $d^0(X)$. We relegate the algebra behind some inequalities to the supplementary appendix (see references below).

Proof.

Lemma 1. *$d^0(X) = X/2$ if $X \leq 6q$.*

Proof. **1. Assume** $X \leq 4q$.

The benefits of discovery are

$$V(d; X | X \leq 4q) = \frac{1}{3q}(Xd - d^2)$$

which is increasing in d and hence maximized at $d = X/2$. Moreover, $V(X/2; X) = X^2/(12q)$ which is increasing in X.

2. Assume $X \in (4q, 6q]$

(i) $d \geq X - 4q$ implies (since $d \leq 3q$)

$$V(d; X | d \geq X - 4q, X \in (4q, 6q])) = \frac{1}{6q}\left(2dX - 2d^2 - \sqrt{X}(X-4q)^{3/2}\right)$$

which is the same as in the first case up to constant $-\sqrt{X}(X-4q)^{3/2}$. Thus the optimal d conditional on $d \geq X - 4q$ is $d = X/2$.

(ii) For $d \leq X - 4q$ the benefit becomes

$$V(d; X|d \leq X - 4q, X \in (4q, 6q])) = \frac{1}{6q}\left(2dX - 2d^2 + \sqrt{X-d}(X-d-4q)^{3/2} - \sqrt{X}(X-4q)^{3/2}\right),$$

with derivative

$$V_d = \frac{1}{3q}\left(X - 2d - (X-d-q)\sqrt{\frac{X-d-4q}{X-d}}\right).$$

which is positive for $d \leq X - 4q, X \in (4q, 6q]$ by Lemma 24 from appendix D. Hence, $V_d(d; X|d \leq X - 4q, X \in [4q, 6q]) > 0$ for all d and X in the considered domain. Thus, $d = X - 4q$ maximizes $V(d; X|d \leq X - 4q, X \in (4q, 6q]))$ and hence $d = X/2$ maximizes $V(d; X|X \in (4q, 6q])$. □

Lemma 2. *If $X > 8q$ then $d^0(X) \neq X/2$.*

Proof. Take $\overline{d} = 4q < X/2$. That implies

$$V(\overline{d}; X|X > 8q) = \frac{1}{6}\left(8Xq - 32q^2 - \sqrt{X}(X-4q)^{3/2} + \sqrt{(X-4q)}(X-8q)^{3/2}\right).$$

By comparison

$$V(X/2; X|X > 8q) = \frac{1}{6}\left(\frac{X^2}{2} - \sqrt{X}(X-4q)^{3/2} + \frac{1}{2}\sqrt{X}(X-8q)^{3/2}\right)$$

The difference of the two is thus

$$\begin{aligned} V(\overline{d}; X|\cdot) - V(X/2; X|\cdot) &= \frac{1}{6q}\left(\sqrt{X-4q}(X-8q)^{3/2} - \frac{\sqrt{X}}{2}(X-8q)^{3/2} - \frac{(X-8q)^2}{2}\right) \\ &= \frac{1}{6}\frac{(X-8q)^{3/2}}{2}\left(2\sqrt{X-4q} - \sqrt{X} - \sqrt{(X-8q)}\right), \end{aligned}$$

which is positive if

$$4(X-4q) > 2X - 8q \Leftrightarrow X > 4q$$

and holds by assumption. □

Lemma 3. $d^0(X) < X/2 \Rightarrow \frac{dV(d^0(X);X)}{dX} < 0$.

Proof. By the envelope theorem,

$$\frac{dV(d^0(X); X)}{dX} = V_X(d^0(X); X).$$

This derivative is negative for $X \geq 4q$ and for all $d \in [0, X - 4q]$ by Lemma 25 in appendix D. If $X \geq 8q$ that claim is sufficient. By Lemma 1 we know that $X \geq 6q$ whenever $d^0(X) \neq X/2$. In 2.(i) in the proof of Lemma 25, page 59, we show that $V_d > 0$ for $d \in [X - 4q, X/2)$ if $X \leq 8q$. Hence if $d^0(X) \neq X/2$, then $d^0(X) \leq X - 4q$ and the inequality proved in Lemma 25 proves the lemma. $\square$

Lemma 4. *$d^0(X) < X/2$ for some $X \in [6q, 8q] \Rightarrow d^0(X) < X/2$ for all $X' > X$, $X' \in [6q, 8q]$.*

Proof. We prove the claim by showing that $V(d^0(X); X)$ for $d(X|X < 2) < X/2$ cuts $V(X/2; X)$ from below at *any* intersection point. Thus, there is at most one switch from $d^0(X) = X/2$ to $d^0(X) < X/2$ and no switch back. By continuity that implies the statement.

$V(d; X)$ is a continuously differentiable function in X and d. Thus any interior (local) optimum $d^0(X)$ is continuous as well and so are $V(d^0(X); X)$ and $V(X/2; X)$. We now show that if $V(d^0(X); X) = V(X/2; X)$ for some $d^0(X) < X/2$ and $X \in [6q, 8q]$, then $\mathrm{d}V(d^0(X); X)/\mathrm{d}X > \mathrm{d}V(X/2; X)/\mathrm{d}X$. Note that $\mathrm{d}V(d^0(X), X)/\mathrm{d}X < 0$ by Lemma 3. The first intersection therefore can occur only in a region when $V(X/2, X)$ is decreasing and must be such that $\mathrm{d}V(X/2, X)/\mathrm{d}X < \mathrm{d}V(d^0(X), X)/\mathrm{d}X$. We prove that this is the only potential intersection in Lemma 26 in appendix D where we show that $\mathrm{d}^2V(X/2, X)/(\mathrm{d}X)^2 < 0$ and $\mathrm{d}^2V(d^0(X), X)/(\mathrm{d}X)^2 > 0$. $\square$

Lemma 5. *As $X \to \infty$, $d^0(X) \to d^\infty$.*

Proof. Take any sequence of increasing X_n with $\lim_{n\to\infty} X_n = \infty$. For any $\delta(d), \exists n$ such that $V_n(d; X) - V(d; \infty) < \delta(d)$. Hence $V(d; X)$ converges uniformly to $V(d; \infty)$ and the result follows. $\square$

Lemma 6. *If $d^0(X) \leq X/2$ then $V(d^0(X); X) > V(d^\infty, \infty)$.*

Proof. By Lemma 1, we know that $d^0(X) < X \Rightarrow X > 6q$. But given any $X < \infty$ we obtain $V(3q; X) > V(3q; \infty)$. $\square$

Lemma 7. *$V(d^0(X); X)$ is continuous in X.*

Proof. $V(d^0(X); X) = \max_d V(d; X)$ with $V(d; X)$ continuous in both $d \in [0, X/2]$ and X. Hence, the max is continuous as well. $\square$

Lemma 8. *$V(d^0(X); X)$ is single peaked with an interior peak.* $\check{X} \approx 6.204q$

Proof. Follows from continuity of $V(X/2; X)$ (by Lemma 7) and Lemma 1 to 5.

The peak can be computed. It is the solution to

$$\frac{X}{X-q} = 2\frac{\sqrt{X-4q}}{\sqrt{X}}. \tag{2}$$

Replacing $X \equiv mq$ the above reduces to

$$\frac{m}{m-1} > 2\sqrt{\frac{(m-4)}{m}}$$

For $m > 4$, the LHS decreases in m while the RHS increases in m for $m > 4$. Moreover, both sides are equal at $m \approx 6.204$. $\square$

Lemma 9. $4q < \hat{X}^0 < 6q < \check{X}^0 < \tilde{X}^0$.

Proof. $4q < \hat{X}^0 < 6q$ by Lemma 1 and 6; $\check{X}^0 > 6q$ by Lemma 8; $8q > \tilde{X}^0 > \check{X}^0$ by claims Lemma 2, 4 and 8 $\square$

Lemma 10. $\hat{X} \approx 4.338q$.

Proof. For $X \in [4q, 6q]$ all we need to consider is $d^0(X) = X/2$ We compare

$$V(X/2; X) = \frac{X^2}{12q} - \frac{\sqrt{X}(X - 4q)^{3/2}}{6q}$$

with $V(3q; \infty) = \frac{3q}{2}$. Replacing $X \equiv \ell q$ and simplifying we the two intersect if

$$q\left(\frac{\ell^2}{12} - \frac{\sqrt{\ell}}{6}(\ell - 4)^2 - 3/2\right) = 0$$

which has a unique solution for $\ell < 6$ with $\ell \approx 4.338$. $\square$

$\square$

B.4 Proof of Proposition 2

Proof. As the normal distribution is symmetric around the mean with a density decreasing in both directions, it follows directly that the smallest interval that contains the realization with a particular likelihood is centered around the mean.

Take an interval of length $Z < \infty$ that is symmetric around the mean μ and assume a total mass of ρ is inside the interval. Then $(1 - \rho)/2$ lies to the left of the interval by the symmetry of the normal. Moreover, the left bound z_l of the interval has (by symmetry of the interval) distance $\mu - Z/2$ from the mean. From the properties of normal distributions

$$\Phi(z_l) = 1/2\left(1 + erf\left(\frac{z_l - \mu}{\sigma\sqrt{2}}\right)\right) = 1/2\left(1 + erf\left(\frac{-Z/2}{\sigma\sqrt{2}}\right)\right).$$

Solving (using symmetry of erf)

$$1/2\left(1 - erf\left(\frac{Z}{\sigma 2^{3/2}}\right)\right) = \frac{1 - \rho}{2}$$

or equivalently

$$erf\left(\frac{Z}{\sigma 2^{3/2}}\right) = \rho$$
$$\Leftrightarrow Z = 2^{3/2} erf^{-1}(\rho)\sigma. \square$$

B.5 Proof of Proposition 3

We proceed in a series of lemmata. We outline the our strategy below

- Lemma 11 proves existence of an interior optimum.
- Lemma 12 characterizes the choices hen expanding research.
- Lemma 13 and 14 show that the value of a boundary solution is strictly increasing on $[0, 4q]$ and has a unique maximum on $(4q, \check{X}^0]$.
- Lemma 15 proves existence and uniqueness of cutoff $\tilde{X}$.
- Lemma 16 proves the jump upwards in ρ at $\tilde{X}$.
- Lemma 17, shows that the researcher's value is single-peaked in X.
- Lemma 18 to 20 derived the order of the cutoffs.

We make frequent use of properties of $\tilde{c}(\rho) = erf^{-1}(\rho)^2$ as defined in appendix A.

Proof.

Lemma 11. *There is a non-trivial optimal choice with $\infty > d > 0, 1 > \rho > 0$ on any interval with positive length, $X \in (0, \infty)$.*

Proof. Recall that the researcher can always guarantee a non-negative payoff by choosing either $d = 0$ or $\rho = 0$. Hence, the researcher's value is bounded from below, $U_R(X) \equiv \max_{d,\rho} u_R(d, \rho; X) \geq 0$. Next, note that $u_R(\rho = 0, d > \varepsilon; X) = 0$ for some small $\varepsilon > 0$ and that $\frac{\partial u_R(\rho=0,d>\varepsilon;X)}{\partial\rho} = V(\varepsilon, X) > 0$ by Proposition 1. Hence, on any interval X there is a maximum with $d > 0, \rho > 0$.

Moreover, by Corollary 2 the value of knowledge is bounded $V(d, X) \leq M < \infty$ and $\lim_{\rho\to 1} \tilde{c}(\rho) = \infty$. Therefore the optimal $\rho < 1$. Finally, $V(d, \infty)$ is decreasing in d for d large enough while the cost $\eta\tilde{c}(\rho)\sigma^2(d, \infty)$ is increasing in d. Hence, the optimal distance is bounded $d \leq D < \infty$. □

Lemma 12. *On the expanding interval, $X = \infty$, the optimal choice is characterized by the first-order conditions (FOCs). The FOCs are sufficient and the optimal $d^\infty \in (2q, 3q)$. The researcher's value is strictly positive $U_R(X = \infty) > 0$.*

Proof. Fix any $\rho \geq 0$. Since $\sigma^2(d; \infty)$ is increasing it is immediate that the researcher's utility is non-increasing in d if $V(d; \infty)$ decreases in d. Thus, it is sufficient to restrict attention to $d \leq 3q$.

By Lemma 11, the researcher's optimal choice is interior and, hence, characterized by the first-order conditions. That the value is positive follows immediately from the choice being strictly interior and $X > 0$ by Lemma 11. To see sufficiency of the first-order conditions with a unique solution note first that the first principal minor of Hessian is $\rho V_{dd} - \eta c\sigma^2_{dd} = -\rho\frac{1}{3q} < 0$ as $\sigma^2_{dd} = 0$ and the second is given by the determinant of the Hessian which is positive

$$\begin{aligned}
&-\rho V_{dd}\eta\tilde{c}_{\rho\rho}(\rho)\sigma^2(d; \infty) - (V_d - \eta\tilde{c}_\rho(\rho)\sigma^2_d(d; \infty))^2 \\
=&\rho\frac{1}{3q}\eta\tilde{c}_{\rho\rho}(\rho)d - \left(-\frac{d}{3q} + 1 - \eta\tilde{c}_\rho(\rho)\right)^2 \qquad (3)\\
=&\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)}\frac{V(d; \infty)}{3q} - \left(-\frac{d}{3q} + 1 - \frac{V(d; \infty)}{\sigma^2(d; \infty)}\right)^2
\end{aligned}$$

where the last equality follows from substituting using the first-order conditions

$$\rho V_d(d;\infty) - \eta\tilde{c}(\rho)\sigma_d^2(d;\infty) = 0 \tag{4}$$

$$V(d;\infty) - \eta\tilde{c}_\rho(\rho)\sigma^2(d;\infty) = 0; \tag{5}$$

in particular, $\eta\sigma^2(d;\infty) = \frac{V(d;\infty)}{\tilde{c}_\rho(\rho)}$, $\eta\tilde{c}_\rho(\rho) = \frac{V}{\sigma^2}$. Rearranging the last term of equation (3) and substituting for V under the assumption that $d \leq 3q$.

$$\begin{aligned} \rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)} &> \left(-\frac{d}{3q} + 1 - \frac{\frac{-d^2}{6q} + d}{d}\right)^2 \frac{3q}{-\frac{d^2}{6q} + d} \\ \Leftrightarrow 2\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)} &> \frac{d}{(6q-d)}. \end{aligned}$$

where the inequality follows by the properties of $\tilde{c}(\rho)$ from appendix A implying $LHS \geq 2$ and the observation that $RHS \in [0,1]$ for $d \leq 3q$. Distance $3q$ is an upper bound for every critical d because the d-first-order condition yields

$$d^\infty = 3q\left(1 - \eta\frac{\tilde{c}(\rho)}{\rho}\right) < 3q. \tag{6}$$

Replacing η via equation (5) and solving for d we obtain

$$d^\infty = 3q\left(1 - \frac{\tilde{c}(\rho)}{2\tilde{c}_\rho(\rho)\rho - \tilde{c}(\rho)}\right) \in (2q, 3q)$$

where the lower bound follows from the properties of the error function. □

Lemma 13. *Fix $d = X/2$ and a assume that an interior optimum exists. Then $U_R(X|d = X/2)$ is maximal only if the total differential $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} \geq 0$.*

Proof. Under the assumption $d = X/2$, $U_R(X)$ is defined and continuously differentiable for all $X \in [0,\infty)$ despite the indicator functions. Because $X = 0$ implies $U_R(X = 0) = 0$ and Lemma 11 holds, there is an interior X at which $U_R(X)$.

Then, if $U_R(X)$ is maximal for some interior and differentiable X it needs to satisfy

$$\frac{\partial U_R}{\partial X} = 0$$

By assumption we have $d(\check{X}) = X/2$ and the first order condition with respect to ρ holds. At an interior and differentiable X we need that

$$\rho\frac{\mathrm{d}V(d = X/2; X)}{\mathrm{d}X} = \frac{\eta}{4}\tilde{c}(\rho).$$

The right hand side is non-negative, which implies the desired result.[34] □

[34]The inequality is weak as for $\eta = 0$, $\rho(X) = 1$ and $U_R(X) = V(X)$.

Lemma 14. *The value of the deepening boundary solution $d = \frac{X}{2}$ peaks at $\check{X} \in (4q, \check{X}^0]$.*

Proof. Note that $U_R'(d \equiv X/2; X) > 0$ for $X \in [0, 4q]$ as in this case $U_R(d \equiv X/2, X) = \rho\frac{X^2}{12q} - \eta\tilde{c}(\rho)\frac{X}{4}$ and, hence, $U_R'(d \equiv X/2, X) = \rho\frac{X}{6q} - \eta\tilde{c}(\rho)\frac{1}{4}$. Using optimality of ρ via the FOC

$$\frac{X}{3q} = \eta\tilde{c}_\rho(\rho) \Rightarrow \frac{X}{6q} = \frac{\eta\tilde{c}_\rho(\rho)}{2}$$

which yields

$$\begin{aligned} U_R'(X) =& \rho\frac{\eta\tilde{c}_\rho(\rho)}{2} - \eta\tilde{c}(\rho)\frac{1}{4} \\ =& \frac{\tilde{c}_\rho(\rho)}{4}\rho\eta\left(2 - \frac{\tilde{c}(\rho)}{\rho\tilde{c}_\rho(\rho)}\right) > 0 \end{aligned}$$

where the inequality follows again from the properties of $\tilde{c}(\rho)$.

Moreover, $U_R(X)$ is strictly concave on $[4q, 8q]$ as $V(d = X/2, X)$ is concave on this interval (see Appendix B.3) and $\sigma^2_{XX}(d = X/2, X) = 0$ implying[35]

$$U_R''(X) = \rho\frac{\mathrm{d}\mathrm{d}V(d = X/2; X)}{\mathrm{d}X\mathrm{d}X} < 0.$$

For $X > \check{X}^0$, $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} < 0$ by the definition of $\check{X}^0$. By Lemma 13, it follows that the maximizing $X \in (4q, \check{X}^0]$ □

Lemma 15. *The researcher's optimal choice of distance is on the midpoint of the interval, $d = \frac{X}{2}$, for $X \leq \tilde{X}$ and interior, $d < \frac{X}{2}$, for $X > \tilde{X}$ with $\tilde{X} > \check{X}$. It converges from above to d^∞, $\lim_{X\to\infty} d(X) = d^\infty$. Any optimal distance choice satisfies $d \leq 4q$.*

Proof. Note first that the choice $d = \frac{X}{2}$ always constitutes a local maximum as the marginal cost of distance is zero at this point, $\frac{\partial\sigma^2(d,X)}{\partial d} = 1 - \frac{2d}{X}$, and, for any choice of d, there is a unique ρ that solves the first-order condition with respect to ρ because the first-order condition with respect to ρ for any d, $\frac{V(d,X)}{\sigma^2(d,X)} = \eta\tilde{c}_\rho(\rho)$, has a continuous, strictly increasing, unbounded right-hand side that starts at $\tilde{c}_\rho(0) = 0$ and a constant left-hand side. Hence, the boundary solution with $d = \frac{X}{2}$ is always a candidate solution.

We first show that for $X \leq 4q$, the optimal choice will always be the boundary solution with $d = X/2$

The first-order conditions for an interior solution are given by

$$\begin{aligned} \rho V_d(d, X) - \eta\tilde{c}(\rho)\sigma^2_d(d, X) = 0 \\ V(d, X) - \eta\tilde{c}_\rho(\rho)\sigma^2(d, X) = 0. \end{aligned}$$

[35]Note that we totally differentiate the value twice and all $\rho'(X)$ and $\rho''(X)$ terms drop out by optimality of ρ by applying the first-order condition directly and total differentiation of the first-order condition.

Replacing η from the second (ρ's) first-order condition in the first (d's) first-order condition, we obtain

$$\frac{\frac{V_d(d,X)}{\sigma_d^2(d,X)}}{\frac{V(d,X)}{\sigma^2(d,X)}} = \frac{\frac{\tilde{c}(\rho)}{\rho}}{\tilde{c}_\rho(\rho)}.$$

It follows from the properties of $\tilde{c}(\rho)$ that the $RHS \in [0, 1/2]$ and decreasing. Thus, whenever the $LHS > 1/2$ for all ρ, the boundary choice $d = \frac{X}{2}$ will be optimal. For $X \leq 4q$

$$\frac{\frac{V_d}{\sigma_d^2}}{\frac{V}{\sigma^2}} = \frac{\frac{2(X-2d)}{\frac{X-2d}{X}}}{\frac{2(dX-d^2)}{\frac{d(X-d)}{X}}} = 1.$$

Hence, for small intervals, the boundary choice is indeed optimal.

Next, we show that for $X > 8q$, the boundary solution is suboptimal to some interior solution. Note first that the variance of the question on the boundary is always larger than for any interior question as $\sigma^2 = \frac{d(X-d)}{X}$ is increasing in d. Hence, if the benefit of research V is larger for an interior question than for the boundary question, the researcher can obtain a higher payoff by choosing an interior question with the same ρ as for the boundary question: the cost will be lower, the success probability the same and the benefit upon success higher. The benefit of finding an answer on the boundary of an interval with $X > 8q$ is always smaller than for some interior distance by Lemma 2 from the proof of Corollary 2. Hence, an interior choice is optimal for $X > 8q$.

For $X \in (4q, 8q)$ and $X - d < 4q$,

$$\frac{\frac{V_d(d,X)}{\sigma_d^2(d,X)}}{\frac{V(d,X)}{\sigma^2(d,X)}} = \frac{2d(X-d)}{-2d^2 + 2dX - \sqrt{X}(X-4q)^{3/2}}$$

which is decreasing in d with limit

$$\lim_{d \to X/2} \frac{2d(X-d)}{-2d^2 + 2dX - \sqrt{X}(X-4q)^{3/2}} = \frac{X^2/2}{X^2/2 - \sqrt{X}(X-4q)^{3/2}}$$

which, in turn, is increasing in X and 1 for $X = 4q$. Hence, any interior solution must be such that $X - d > 4q$ as otherwise, the first-order condition with respect to d is always positive.

Thus, we know that (i) on intervals with $X < 4q$, the researcher's distance choice on the deepening interval will be a boundary solution, (ii) on intervals with $X > 8q$ the researcher's distance choice will be interior, (iii) on intervals with $X \in [4q, 8q]$ the researcher's distance choice may be interior or on the boundary, and (iv) any interior choice has to satisfy $X - d > 4q$ and $d < 4q$.[36]

It remains to show that the values, U_R of $d_1 = X/2$ and $d_2(X) < X/2$ with $d_2(X)$

[36]From Lemma 2, 4 and 6 any interior choice that maximizes V (ignoring cost) satisfies $X - d > 4q$ and $d < 4q$.

solving the first-order condition of $d_2(X)$ and $\rho_i(d_i, X)$ chosen optimally cross only once. We use three observations to show this.

1. First, at the area length X for which $U_R(d_1; X) = U_R(d_2; X)$, the payoff at the boundary must be decreasing faster than the payoff in the interior as the first switch is from the boundary solution to the interior solution by continuous differentiability of all terms and the observation from above that $d(X) = X/2$ for $X < 4q$.
2. Second, on the interval $[4q, 8q]$ the payoff of the boundary solution has a strictly lower second derivative with respect to X for all X than the interior solution. Hence, the two values can cross at most once on this interval.
3. Third, the value of the boundary solution is bounded from above by the value of the interior solution for all $X \geq 8q$.

The first observation is immediate.

For the second observation follows from totally differentiating U_R for the two types of local maxima. Using envelope conditions we obtain that the payoff is concave in the boundary solution and convex in the interior solution which implies the second observation. Define $\varphi(X) := \max_\rho u(d = X/2, \rho, X)$ for the boundary, we show in Lemma 27 in appendix D that $\varphi(X)$ is concave. In Lemma 28 in appendix D we in turn show that $\max_{\rho,d} u(d, \rho, X)$ is convex in X provided that the maximizer $d(X) < X/2$.

For the third observation, note that when $X \to \infty$, $V(d, X)$ converges to $V(d, \infty)$ and $\sigma^2(d, X)$ to $\sigma^2(d, \infty)$ and the researcher's optimization on the deepening interval converges to the optimization on the expanding interval which has a unique and interior maximum at (d^∞, ρ^∞). In particular, if such an interior optimum exists, the envelope condition implies that

$$U_R(X) = \rho V_X(d, X) - \eta \tilde{c}(\rho)\sigma^2_X(d, X) < 0$$

as $V_X(d, X) < 0$ according to Corollary 2 for $X > 4q$ and $X - d > 4q$ and $\sigma^2_X(d, X) > 0$.

Hence, the value of any optimal interior choice is decreasing in X. Because the payoff is continuous in X it follows that $\tilde{X} > \check{X}$ where $\tilde{X}$ denotes the first interval length such that the interior value with $d < X/2$ is equal to the boundary value with $d = X/2$. □

Lemma 16. $\lim_{X \searrow \tilde{X}} \rho(X) > \lim_{X \nearrow \tilde{X}} \rho(X)$.[37]

Proof. At $\tilde{X}$ there are two solutions to $d(X)$, a boundary solution $d^b = X/2$ and an interior solution $d^i < X/2$. Each comes with an associate ρ^d, ρ^i respectively. We show that $\rho^i(X) > \rho^d(X)$ which proves the claim as $\rho(X)$ is continuous in X if $d(X)$ is continuous in X which is true by the first-order conditions.

By the definition of $\tilde{X}$ and continuity of U_R we have that

$$\begin{aligned} U_R(d^b; \tilde{X}) &= U_R(d^i; \tilde{X}) \\ \rho^b V(d^b; \tilde{X}) - \eta \tilde{c}(\rho^b)\sigma^2(d^b; \tilde{X}) &= \rho^i V(d^i; \tilde{X}) - \eta \tilde{c}(\rho^i)\sigma^2(d^i; \tilde{X}). \end{aligned} \tag{7}$$

[37] We use $\searrow$ ($\nearrow$) to describe the one-sided limit "from above" ("from below").

Taking the first-order conditions w.r.t. ρ we get

$$V(d^b;\tilde{X}) = \eta\tilde{c}_\rho(\rho^b)\sigma^2(d^b;\tilde{X})$$
$$V(d^i;\tilde{X}) = \eta\tilde{c}_\rho(\rho^i)\sigma^2(d^i;\tilde{X})$$

Replacing $V(\cdot)$ accordingly in equation (7) and dividing by η we obtain

$$\sigma^2(d^b;\tilde{X})\rho^b\left(\tilde{c}_\rho(\rho^b) - \frac{\tilde{c}(\rho^b)}{\rho^b}\right) = \sigma^2(d^i;\tilde{X})\rho^i\left(\tilde{c}_\rho(\rho^i) - \frac{\tilde{c}(\rho^i)}{\rho^i}\right)$$

Or equivalently

$$\frac{\sigma^2(d^i;\tilde{X})}{\sigma^2(d^b;\tilde{X})} = \frac{\rho^b\left(\tilde{c}_\rho(\rho^b) - \frac{\tilde{c}(\rho^b)}{\rho^b}\right)}{\rho^i\left(\tilde{c}_\rho(\rho^i) - \frac{\tilde{c}(\rho^i)}{\rho^i}\right)}.$$

Since $d^b > d^i \Rightarrow \sigma^2(d^b;\tilde{X}) > \sigma^2(d^i;\tilde{X})$ and $\rho\left(\tilde{c}_\rho(\rho) - \frac{\tilde{c}(\rho)}{\rho}\right)$ increasing in ρ by the properties of the error function it follows that $\rho^i > \rho^b$ which proves the claim. □

Lemma 17. *The researcher's value $U_R(X)$ is single-peaked in X.*

Proof. Follows from the value being decreasing for the interior solution and the single-peakedness of the boundary value (increasing for $X < 4q$ and concave on $X \in [4q, 8q]$). □

Lemma 18. *Suppose $d = X/2$ is optimal for a range $[\underline{X}, \overline{X}]$. Then the optimal $\rho(X)$ is single peaked in that range. It is highest at $\dot{X} = \frac{8\cos\left(\frac{\pi}{18}\right)}{\sqrt{3}}$*

Proof. By Lemma 13 we know that $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} \geq 0$ and by Lemma 14 $\overline{X} > \hat{X}^0$. Moreover, recall $\sigma^2(d = X/2; X) = X/4$. The first-order condition with respect to ρ becomes

$$\frac{V(X/2;X)}{X} = \frac{\eta}{4}\tilde{c}_\rho(\rho),$$

With

$$\frac{V(X/2;X)}{X} = \frac{X}{12q} - \mathbf{1}_{X>4q}\frac{(X-4q)^{3/2}}{\sqrt{X}6q}.$$

The latter is continuous and concave. Since $\tilde{c}(\rho)$ is an increasing, twice continuously differentiable and convex function, ρ increases in X if and only if $V(X/2;X)/X$ increases in X. By concavity of $V(X/2;X)/X$ that implies single peakedness.

Thus, $\dot{X}$ is independent of η and given by $\dot{X} = \frac{8\cos\left(\frac{\pi}{18}\right)}{\sqrt{3}} \approx 4.548q$. □

Lemma 19. $\lim_{X \searrow \hat{X}} \rho(X) > \rho^\infty$ *and $\hat{X}$ decreases in η.*

Proof. We begin to show that the first claim holds if $\hat{X} < 6q$, then we show the second claim which together with the observation that $\hat{X}^0 < 6q$ is sufficient to prove the lemma.

At $\hat{X}$ we have

$$\begin{aligned} U_R(\hat{X}) &= U_R(\infty) \\ \rho(\hat{X})V(\hat{X}/2;\hat{X}) - \eta\tilde{c}(\rho(\hat{X}))\frac{\hat{X}}{4} &= \rho^\infty V(d^\infty;\infty) - \eta\tilde{c}(\rho^\infty)d^\infty. \end{aligned} \tag{8}$$

where the fact that $d(\hat{X}) = \hat{X}/2$ follows from Lemma 14, 15 and 17. Moreover, the following has to hold by optimality

$$V(d^\infty;\infty) = \eta\tilde{c}_\rho(\rho^\infty)d^\infty \tag{FOC ρ^∞}$$

$$V(\hat{X}/2;\hat{X}/2) = \eta\tilde{c}_\rho(\rho(\hat{X}))\frac{\hat{X}}{4} \tag{FOC $\rho^{\hat{X}}$}$$

Step 1: $\rho^\infty < \rho(\hat{X})$ if $\hat{X} < 6q$. Using (FOC ρ^∞) and (FOC $\rho^{\hat{X}}$) we obtain that by the properties of the error function $\rho(\hat{X}) > \rho^\infty$ if and only if

$$4\frac{V(\hat{X}/2;\hat{X}/2)}{X} > \frac{V(d^\infty;\infty)}{d^\infty}.$$

Case 1: $\hat{X} > 4q$. Substituting for the $V(\cdot)$'s the above becomes[38]

$$\begin{aligned} & \frac{\hat{X}}{3q} - \frac{2}{3q}\frac{(\hat{X}-4q)^{3/2}}{\sqrt{\hat{X}}} > 1 - \frac{d^\infty}{6q} \\ \Leftrightarrow \qquad & d^\infty + 2\hat{X} - 4\underbrace{\frac{(\hat{X}-4q)^{3/2}}{\sqrt{\hat{X}}}}_{<(\hat{X}-4q)} > 6q \end{aligned}$$

A sufficient condition for the above to hold is thus that

$$d^\infty - 2\hat{X} + 10q > 0$$

Using that $d^\infty > 2q$ by Lemma 12 we obtain that sufficient condition for $\rho(\hat{X}) > \rho^\infty$ is that $\hat{X} < 6q$.

Case 2: $\hat{X} \in (2q, 4q]$. Performing the same steps only assuming that $\hat{X} \in [2q, 4q]$ we

$$\begin{aligned} \frac{\hat{X}}{3q} &> 1 - \frac{d^\infty}{6q} \\ \Leftrightarrow 2\hat{X} &> 6q - d^\infty > 4q \end{aligned}$$

which implies the desired result.

Case 3: $\hat{X} < 2q$ We show that case 3 never occurs, that is $\hat{X} > 2q$. To do so we compare $U_R(d = 2q; \infty)$ with $U_R(d = 1q; X = 2q)$ and show that the former is

[38] Since $\hat{X} \leq \check{X} \leq 8q$ that case is irrelevant.

always larger. Hence $X = 2q < \hat{X}$ for any η.

For $X = d = 2q$ we have that

$$\frac{\hat{X}}{3q} = 1 - \frac{d}{6q},$$

and thus $\rho(X = 2q) = \rho(d; \infty) = \rho$ (cf. case 2). Moreover we have that

$$V(1q; 2q) = q/3 \qquad\qquad V(2q; \infty) = 4/3q,$$

and (FOC ρ^X) implies

$$4V(1q; 2q)/2q = 2/3 = \eta\tilde{c}_\rho(\rho)$$

Since $\tilde{c}_\rho(\rho) > \tilde{c}(\rho)/\rho$ for any $\rho > 0$ that implies $\eta\tilde{c}(\rho)/\rho < 2/3$.

Now take

$$\begin{aligned} U_R(d = 2q; \infty) - U_R(X = 2q) \\ \rho\frac{4q}{3} - \eta\tilde{c}(\rho) - \rho\frac{q}{3} + \eta\tilde{c}(\rho)\frac{q}{2} \\ q\left(\rho - \frac{3}{2}\eta\tilde{c}(\rho)\right), \end{aligned}$$

which is positive whenever $\eta\tilde{c}(\rho)/\rho < 2/3$ which we know has to hold. Thus $U_R(d = 2q; \infty) > U_R(X = 2q)$ and therefore $\hat{X} < 2q$.

Step 2: If $\rho^\infty < \rho(\hat{X})$ then $\hat{X}$ decreases in η.

Using (FOC ρ^∞) and (FOC ρ^X) to replace the $V(\cdot)$'s in equation (8) and dividing by η we obtain

$$d^\infty\left(\rho^\infty\tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty)\right) = \tilde{X}/4\left(\rho(\hat{X})\tilde{c}_\rho(\rho(\hat{X})) - \tilde{c}(\rho(\hat{X}))\right)$$

from which we get

$$\hat{X}/4 = d^\infty\frac{\left(\rho^\infty\tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty)\right)}{\left(\rho(\hat{X})\tilde{c}_\rho(\rho(\hat{X})) - \tilde{c}(\rho(\hat{X}))\right)}.$$

Now we use the envelope theorem to calculate

$$\frac{\partial U_R(\hat{X}) - U_R(\infty))}{\partial\eta} = \tilde{c}(\rho(\hat{X}))\frac{\hat{X}}{4} - \tilde{c}(\rho^\infty)d^\infty.$$

Replacing for $\hat{X}$ implies that the RHS is positive if and only if

$$(\tilde{c}(\rho^\infty)) - \tilde{c}(\rho(\hat{X}))\frac{\rho^\infty\tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty)}{\rho(\hat{X})\tilde{c}_\rho(\rho(\hat{X})) - \tilde{c}(\rho(\hat{X}))} > 0.$$

Using that $\rho\tilde{c}_\rho(\rho) > \tilde{c}(\rho)$ by the properties of the error function and factoring out he denominator of the first term, the above holds if and only if

$$\tilde{c}(\rho^\infty)\rho^\infty\tilde{c}_\rho(\rho(\hat{X})) - \tilde{c}(\rho(\hat{X})\rho^\infty\tilde{c}_\rho(\rho^\infty) > 0$$
$$\frac{\rho(\hat{X})\tilde{c}_\rho(\rho(\hat{X}))}{\tilde{c}(\rho(\hat{X}))} > \frac{\rho^\infty\tilde{c}_\rho(\rho^\infty)}{\tilde{c}(\rho^\infty)}$$

which holds if and only if $\rho(\hat{X}) > \rho^\infty$ by the properties of the error function. Thus, $\hat{X}$ decreases if $\rho(\hat{X}) > \rho^\infty$.

Conclusion: Since $\hat{X}^0 \in [2q, 6q]$, $\Rightarrow \rho^\infty < \rho(\hat{X}) \Rightarrow \hat{X}$ is decreasing in η. □

Lemma 20. $\hat{X} \leq \dot{X} < \check{X} \leq \tilde{X}$

Proof. **Step 1: $\check{X} > \dot{X}$.** By the envelope theorem we need for $X = \check{X}$

$$\frac{\partial U_R(\check{X})}{\partial X} = \rho\frac{\mathrm{d}V(d = \check{X}/2; \check{X})}{\mathrm{d}X} - \frac{\eta}{4}\tilde{c}(\rho) = 0. \tag{9}$$

The FOC for ρ implies

$$\frac{V}{\check{X}} = \frac{\eta}{4}\tilde{c}_\rho(\rho)$$

Now assume for a contradiction that $\rho(\check{X})$ is increasing, then by Claim 11 $V(\cdot)/\check{X}$ must be increasing which holds if and only if

$$\frac{\mathrm{d}V(d = \check{X}/2; \check{X})}{\mathrm{d}X}\check{X} > V(d = \check{X}/2; \check{X}).$$

But then we obtain the following contradiction to $U_R(\check{X})$ being maximal

$$\frac{\mathrm{d}V(d = \check{X}/2; \check{X})}{\mathrm{d}X} > \frac{V(d = \check{X}/2; \check{X})}{\check{X}} = \frac{\eta}{4}\tilde{c}_\rho(\rho) > \frac{\eta}{4}\frac{\tilde{c}(\rho)}{\rho}.$$

The first inequality follow because $V(d = \check{X}/2; \check{X})/\check{X}$ must be increasing, the equality follows by equation (9). The last inequality is a consequence of the properties of erf. By Lemma 18, $\rho(X)$ is single peaked which proves the claim.

Step 2: Ordering. By Lemma 19 we know that $\hat{X} < \hat{X}^0$. Thus $\hat{X}^0 < \dot{X} \Rightarrow \hat{X} < \dot{X}$. Moreover, $\tilde{X} > \check{X}$ by Lemma 15 which concludes the proof. □

□

B.6 Proof of Corollary 4

Proof. The last claim follows from Lemma 18, and the first bullet point of the third claim follows from Lemma 19.

Lemma 21. $\partial\tilde{X}/\partial\eta < 0$.

Proof. To see hat $\tilde{X}$ is decreasing in η we use the same arguments as those in step 2 of the proof of equation (8). WE provide the steps again for completeness.

Recall our notations (ρ^b, d^b) for the boundary solution at $\tilde{X}$ and (ρ^i, d^i) for the interior solution at $\tilde{X}$. Using the first order conditions with respect to ρ we obtain

$$\eta\sigma^2(d^i;\tilde{X})\left(\rho^b\tilde{c}_\rho(\rho^b) - \tilde{c}(\rho^b)\right) = (\tilde{X}/4)\eta\left(\rho^i)\tilde{c}_\rho(\rho^i)) - \tilde{c}(\rho^i))\right)$$

from which we get

$$\tilde{X}/4 = \sigma^2(d^i;\tilde{X})\frac{\left(\rho^b\tilde{c}_\rho(\rho^b) - \tilde{c}(\rho^b)\right)}{(\rho^i)\tilde{c}_\rho(\rho^i)) - \tilde{c}(\rho^i)))}.$$

Now we use the envelope theorem to calculate

$$\frac{\partial U_R(d^b;\tilde{X}) - U_R(d^i;\tilde{X})}{\partial\eta} = \tilde{c}(\rho(\hat{X}))\frac{\tilde{X}}{4} - \tilde{c}(\rho^\infty)\sigma^2(d^i;\tilde{X}).$$

Replacing for $\tilde{X}/4$ implies that the RHS is positive if and only if

$$\left(\tilde{c}(\rho^i)\right) - \tilde{c}(\rho^b)\frac{\rho^i\tilde{c}_\rho(\rho^i) - \tilde{c}(\rho^i)}{\rho^b\tilde{c}_\rho(\rho^b) - \tilde{c}(\rho^b)} > 0.$$

Using that $\rho\tilde{c}_\rho(\rho) > \tilde{c}(\rho)$ by the properties of the error function and factoring out he denominator of the first term, the above holds if and only if

$$\begin{aligned}\tilde{c}(\rho^i)\rho^i\tilde{c}_\rho(\rho^b) - \tilde{c}(\rho^b\rho^i\tilde{c}_\rho(\rho^i) > 0\\ \frac{\rho^b\tilde{c}_\rho(\rho^b)}{\tilde{c}(\rho^b)} > \frac{\rho^i\tilde{c}_\rho(\rho^i)}{\tilde{c}(\rho^i)}\end{aligned}$$

which holds if and only if $\rho^b > \rho^\infty$ by the properties of the error function. Thus, $\hat{X}$ decreases if $\rho^b > \rho^i$. □

Lemma 22. $\partial\check{X}/\partial\eta < 0$.

Proof. Because $\check{X}$ is a maximizer of $U_R(X)$ we know that

$$\frac{\partial U_R(\check{X})}{\partial X} = \rho(\check{X})\frac{\mathrm{d}V(\check{X}/2;\check{X})}{\mathrm{d}X} - \eta\tilde{c}(\rho)\frac{\mathrm{d}\sigma^2(\check{X}/2;\check{X})}{\mathrm{d}X} = 0. \tag{10}$$

Taking derivatives with respect to η

$$\frac{\partial U_R(\check{X})}{\partial X\partial\eta} = \frac{\partial\check{X}}{\partial\eta}\frac{\partial U_R(\check{X})}{\partial X\partial X} + \frac{\partial\rho(X)}{\partial\eta}\frac{\partial U_R(\check{X})}{\partial X\partial\rho} - \tilde{c}(\rho(\check{X}))\frac{\mathrm{d}\sigma^2(\check{X}/2;\check{X})}{\mathrm{d}X} = 0,$$

or equivalently

$$\frac{\partial\check{X}}{\partial\eta}\frac{\partial U_R(\check{X})}{\partial X\partial X} = -\frac{\partial\rho(\check{X})}{\partial\eta}\frac{\partial U_R(\check{X})}{\partial X\partial\rho} + \tilde{c}(\rho(\check{X}))\frac{\mathrm{d}\sigma^2(\check{X}/2;\check{X})}{\mathrm{d}X}. \tag{11}$$

(FOC ρ^X) implies that for a given X

$$FOC^{\rho}(\eta,\rho) := V(\check{X}/2;\check{X})/\sigma^2(\check{X}/2;\check{X}) - \eta\tilde{c}_\rho(\rho(\check{X})) = 0$$

which in turn implies via the IFT that

$$\frac{\partial\rho(X)}{\partial\eta} = -\frac{\frac{\partial FOC^{\rho}}{\partial\eta}}{\frac{\partial FOC^{\rho}}{\partial\rho}} = -\frac{\tilde{c}_\rho(\rho(X))}{\tilde{c}_{\rho\rho}(\rho(X))\eta}.$$

Observe that

$$\frac{\partial U_R(\check{X})}{\partial X\partial\rho} = \frac{\mathrm{d}V(\check{X}/2;\check{X})}{\mathrm{d}\check{X}} - \eta\tilde{c}_\rho(\rho(\check{X})\frac{\mathrm{d}\sigma^2(\check{X}/2;\check{X})}{\mathrm{d}X},$$

and after replacing $\frac{\mathrm{d}V(\check{X}/2;\check{X})}{\mathrm{d}X}$ using equation (10)

$$\frac{\partial U_R(\check{X})}{\partial X\partial\rho} = \eta\left(\frac{\tilde{c}(\rho(\check{X}))}{\rho(\check{X})} - \tilde{c}_\rho(\rho(\check{X}))\right)\frac{\mathrm{d}\sigma^2(\check{X}/2;\check{X})}{\mathrm{d}X}.$$

That simplifies equation (11) to

$$\frac{\partial\check{X}}{\partial\eta}\frac{\partial U_R(\check{X})}{\partial X\partial X} = \left(\left(\frac{\tilde{c}(\rho(\check{X}))}{\rho(\check{X})} - \tilde{c}_\rho(\rho(\check{X}))\right)\frac{\tilde{c}_\rho(\rho(\check{X}))}{\tilde{c}_{\rho\rho}(\rho(\check{X}))} + \tilde{c}(\rho(\check{X}))\right)\frac{\mathrm{d}\sigma^2(\check{X}/2;\check{X})}{\mathrm{d}X}.$$

Recall from the proof of Lemma 3 in appendix B.3 that $\frac{\mathrm{d}^2V(X/2,X)}{\mathrm{d}X^2} < 0$ and $\frac{\mathrm{d}^2\sigma^2 X/2;X}{\mathrm{d}X\mathrm{d}X} = 0$. Hence $\frac{\partial U_R(\check{X})}{\partial X\partial X} < 0$. Moreover the term in brackets on the right hand side is positive by the properties of the error function and the variance $\sigma^2(\check{X}/2;\check{X})$ is increasing in X. Thus we can conclude that $\frac{\partial\check{X}}{\partial\eta} < 0$. □

□

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Supplementary Material

C Science Funding

In this part, we disccus the model of science funding from Section 6 in greater detail.

Setup. The setting is the same as in Section 6. For convenience we repeat the discussion here. We take the simplest possible setup within our modeling framework. Suppose there is only one known question-answer pair, $\mathcal{F}_1 = (x_1, y(x_1))$. In such a case, the researcher—absent any funding incentives—solves the following special case of the problem in Section 5:

$$\max_{d,\rho} \rho V(d;\infty) - \eta^0 \tilde{c}(\rho)\sigma^2(d;\infty),$$

with $\tilde{c}(\rho) = erf^{-1}(\rho)^2$.

Now assume that the funder has a fixed budget K to invest in the researcher and has two funding technologies: ex ante cost reductions, h, and ex post rewards, ζ. Marginal cost of both are constant, and the cost ratio κ is such that the funder's budget constraint is given as follows:[39]

$$K = \zeta + \kappa h$$

From the researcher's perspective, a cost reduction of h implies that she faces cost parameter $\eta \equiv \eta^0 - h$. An ex post reward gives the researcher additional utility of ζ if she finds an answer. Since research prizes are typically competitive, we assume that the researcher obtains the reward more likely if her contribution is seminal; otherwise the amount goes to some other (unmodeled) recipient. The probability of obtaining the prize positively correlates with the difficulty of solving the problem and is bounded. For the sake of concreteness, we assume the correlation is linear in variance:

$$f(\sigma) = \begin{cases} \frac{\sigma^2}{s} & \text{if } \sigma^2 < s \\ 1 & \text{otherwise,} \end{cases}$$

for some $s > 0$.[40]

We first consider two polar objectives of the funder: maximizing output and maximizing novelty. Then we determine the set of (ρ, d) combinations the funder can achieve. At the end, we consider what happens as prior knowledge expands.

[39]Economically, this assumption means that either the cost structure of the two instruments is linear everywhere or it can be interpreted as implicitly assuming that the funder's budget, K, is sufficiently small such that a linear cost structure is a decent approximation of the funding opportunities.

[40]The crucial assumption here is that f is a bounded function, which is true whenever f is indeed a probability. If f instead was unbounded, the researcher would naturally (for any $\zeta > 0$) choose to select $d = \infty$. The specific form, in turn, is chosen only for ease of computation.

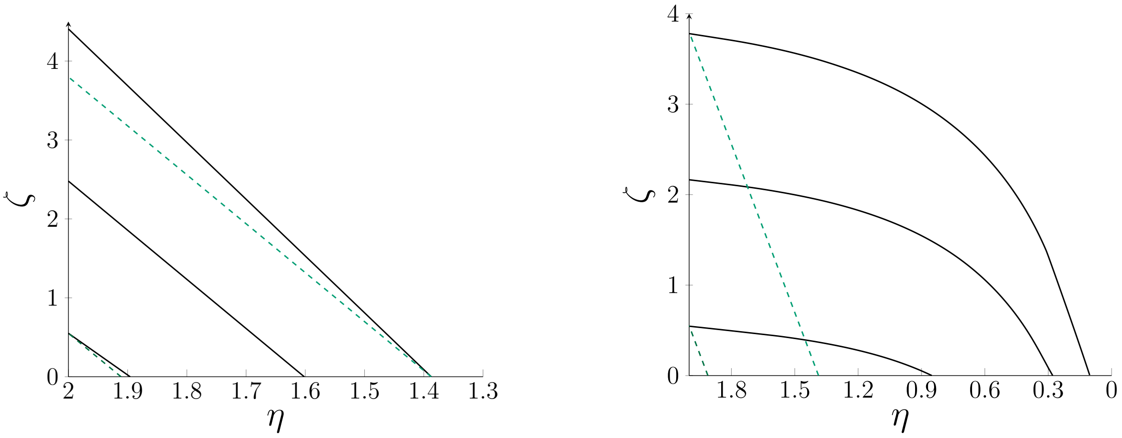

Figure 13: *Iso-ρ and iso-d curves with budget lines*
Dashed lines represent the budget lines for κ—$K < \overline{K}$ for the lower budget and $K > \overline{K}$ for the higher budget.
The *left panel* shows the iso-ρ curves. An *output-maximizing* funder follows the following investment rule: for $K < \overline{K}$, invest all of the budget in increasing ζ, $z = K, h = 0$; for the larger budget ($K > \overline{K}$), invest all of the budget in decreasing η, $z = 0, h = K/\kappa$.
The *right panel* shows the same budget lines but with iso-d curves for three levels of d. These curves are concave and flatter than the iso-ρ curves. Consequently, a *novelty-maximizing* funder can invest entirely in increasing ζ in situations in which an output-maximizing funder invests entirely in decreasing η.
Note: the x-axis moves in reverse order to plot *decreases* in the cost-function parameter compared to the base level $\eta^0 = 2$.

After observing the funding choices, the researcher's problem becomes the following:

$$\max_{d,\rho} \rho \left(V(d;\infty) + f(\sigma(d;\infty))\zeta\right) - \eta\tilde{c}(\rho)\sigma^2(d;X)$$

With our framework, we can straightforwardly apply basic consumer theory to study the funder's choices.

Example 1 (Maximizing Output). The funder's objective is to maximize the researcher's output:

$$\max_{h,\zeta} \rho$$

Proposition 5. *For an output-maximizing funder, it is optimal to focus exclusively on either of the two funding options. Moreover, there is a cutoff budget $\overline{K}$ such that the funder invests exclusively in ex post rewards if $K < \overline{K}$ and exclusively in ex ante cost reductions if $K > \overline{K}$.*

Proof. For the proof we characterise the "marginal rate of substitution" for ρ between the two funding options. That is, the (negative) of the ratio of the marginal effects of a an increase in η and ζ on ρ,

$$MRS^{\rho}_{\zeta\eta} := -\frac{\frac{\partial\rho}{\partial\eta}}{\frac{\partial\rho}{\partial\zeta}}.$$

Th MRS defines the slope of the funder's indifference curves. Thus we can use he simple economics from consumer theory to determine the funder's optimal decision.

Throughout, observe that since $X = \infty$ we have that $\sigma_d^2(d;\infty) = 1$ and $\sigma_{dd}^2(d;\infty) = 0$. As we restrict attention to small K (or large s) we implicitly assume that $d(\zeta,\eta) < 4q$ and hence $V_d = 1 - d/(3q)$ and $V_{dd} = -1/(3q)$.

In Lemma 29 in appendix D on page 62 we show that

$$MRS^{\rho}_{\zeta\eta} = 2\tilde{c}_{\rho}(\rho) - \tilde{c}(\rho)/\rho.$$

The marginal rate of substitution is a positive, increasing and convex function of ρ. The properties of the iso-ρ curves follow immediately. As it only depends on ρ, it is immediate that along an iso-ρ curve MRS^{ρ} is constant.

To maximize output, the funder chooses the highest iso-ρ curve on the budget line. Note that the steepest iso-ρ curve is the one with the highest ρ. Note that higher iso-ρ levels occur for higher ζ and lower η. Hence, it is sufficient to check whether the slope at one of the corner solutions is steeper or flatter than the cost ratio of η and ζ. Consider a corner solution in which all investment goes into ζ and the corresponding level of ζ. If $MRS^{\rho} < \kappa$, then this is the optimal funding scheme. If $MRS^{\rho} > \kappa$, then the optimal funding scheme invests only into ζ. Clearly, as the budget increases, the highest level of ρ that can be implemented increases. Thus, the MRS^{ρ} at the optimal funding scheme increases and a switch from a corner solution with only ζ to one with only η might occur. □

The intuition behind Proposition 5 builds on the properties of iso-ρ curves in (ζ,η) space as depicted in Figure 13. Like the consumer in a consumer problem, the funder chooses ζ and h (or equivalently $\eta \equiv \eta^0 - h$), which together imply a certain ρ. An iso-ρ curve depicts all choices (ζ,η) that induce the same ρ. The slope of the iso-ρ curves is described by the marginal rate of substitution (MRS) between ζ and η:[41]

$$MRS^{\rho}_{\zeta\eta}(\rho) = s(2\tilde{c}_{\rho}(\rho) - \tilde{c}(\rho)/\rho)$$

From the MRS and the properties of the inverse error function it is apparent that the slope of the iso-ρ curves is (i) linear with (ii) an increasing slope in ρ. For a given relative price (κ), the funder's solution is therefore generically a corner solution. Moreover, the slope determines whether it should focus on ex ante incentives or ex post incentives. As the slope changes with the output level ρ, so may the optimal funding scheme. If that ρ level is large, the iso-ρ curve is steep and ex ante cost reductions are optimal. If the ρ level is small, the iso-ρ curve is flat and ex post rewards are optimal. The highest level of ρ, and thus the type of the corner solution, depends on the available budget (K).

Intuitively, at low levels of ρ, the cost effect—which is convex—is not yet dominant and increasing the benefit of a discovery is more effective in incentivizing output. However, as ρ increases, the cost effect becomes more pronounced and weakening it with decreases in η becomes more effective instead.

Example 2 (Maximizing Novelty). The funder's objective is to maximize the novelty

[41] Note that with $\tilde{c}_{\rho}$, we denote the derivative of $\tilde{c}$ with respect to ρ to simplify notation.

of the researcher's question of choice:

$$\max_{h,\zeta} d$$

Proposition 6. *For a novelty-maximizing funder, it is optimal to focus on either of the two funding options. Moreover, there is a threshold $\overline{s} < 0.1$ such that whenever $s > \overline{s}$ and an output-maximizing funder finds it optimal to focus on ex post rewards, so does a novelty-maximizing funder. The reverse, however, does not hold.*

Proof. Using the implicit function theorem results from the proof of Lemma 29 above, we obtain for the marginal rate of substitution between ζ and η on the expanding interval by substituting first-oder conditions

$$MRS^d_{\zeta\eta} = \tilde{c}_\rho \frac{\tilde{c}/\rho - \tilde{c}_\rho + \frac{\tilde{c}}{\tilde{c}_\rho}\tilde{c}_{\rho\rho}}{\tilde{c}/\rho - \tilde{c}_\rho + \rho\tilde{c}_{\rho\rho}}.$$

This is a positive, increasing and convex function in ρ. Moreover, we obtain that $MRS^d_{\zeta\eta} > MRS^\rho_{\zeta\eta}$ as the difference reduces to

$$\tilde{c}_\rho - \tilde{c}/\rho + \tilde{c}_{\rho\rho}\frac{\rho\tilde{c}_\rho - \tilde{c}}{\tilde{c}/\rho - \tilde{c}_\rho + \rho\tilde{c}_{\rho\rho}} > 0$$

where the inequality follows as the cost is positive, increasing and convex and because $c_\rho > c/\rho$ and the denominator being positive.

Hence, moving along an iso-d curve ρ is decreasing and therefore the MRS^d is decreasing along the iso-d curve which is concave. Thus, a corner solution maximizes d on a given budget line.

Whenever $MRS^\rho < \kappa$, it follows from $MRS^\rho > MRS^d$ that $MRS^d < \kappa$. Hence, if the corner solution for ρ uses only ex post rewards so does the corner solution for d. The reverse is not true as Figure 13 illustrates. □

The iso-d curves are concave and flatter than the iso-ρ curve crossing at each point. Their slope is

$$MRS^d_{\zeta\eta}(\rho) = \tilde{c}_\rho(\rho)\frac{\tilde{c}(\rho)/\rho - \tilde{c}_\rho(\rho) + \frac{\tilde{c}(\rho)}{\tilde{c}_\rho(\rho)}\tilde{c}_{\rho\rho(\rho)}}{\tilde{c}(\rho)/\rho - \tilde{c}_\rho(\rho) + \rho\tilde{c}_{\rho\rho}(\rho)},$$

which can be shown to be smaller than $MRS^\rho_{\zeta\eta}$ for any $s > 0.1$.[42] Thus, if at any point it is optimal for the output-maximizing funder to invest more in increasing ζ, the novelty-maximizing funder would follow the same strategy. The reverse need not be true. The right panel of Figure 13 depicts the iso-d curves alongside the same budget lines as those in the left panel. Thus, the two funder types are observationally equivalent for small budgets, but it is possible to tell them apart by observing their behavior once the budget grows larger.

[42]Note that the pictures in Figure 13 are plotted on a reverse x-axis.

General funding objectives. In reality, funding may have many different objectives and can range from maximizing the externalities of research to answering particular questions. A demand-side discussion of research is beyond the scope of this paper. Instead, we characterize the feasible set of choices (d, ρ) that a funder can induce with a given budget. Computing this set provides a useful tool to analyze the optimal funding scheme given a particular preference relation over (d, ρ) bundles. As in a standard consumer problem, it can be readily applied, with this set being the analogue of a budget set. We restate (and prove) the Proposition 4 from the main text

Proposition (Identical to Proposition 4 from page 26)**.** *The set of implementable (d, ρ)-combinations for a given cost ratio κ and a budget K is described by the (d, ρ) implementation frontier defined over $[\underline{\rho}, \overline{\rho}]$, which are the endogenous upper and lower bounds of ρ. These bounds are determined by the extreme funding schemes $(\zeta = 0, \eta = \eta^0 - K/\kappa)$ and $(\zeta = K, \eta = \eta^0)$. The research-possibility frontier between those polar points is as follows:*

$$d(\rho; K) = 6q(K + s - \kappa\eta^0)\frac{\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)}{2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) - \kappa\rho} \tag{12}$$

$d(\rho; K)$ can be increasing or decreasing, depending on whether d and ρ behave as substitutes or complements at the (ζ, η) mix inducing the current level of (d, ρ).

Proof. **Deriving the Research Possibility Frontier:** Using the two first order conditions of the researcher and solving for ζ and η we obtain

$$\begin{aligned} \eta &= \frac{d}{6q}\frac{\rho}{\rho\tilde{c}_\rho - \tilde{c}} \\ \zeta &= \left(\frac{d}{3q} - 1 + \frac{d}{6q}\frac{\tilde{c}}{\rho\tilde{c}_\rho - \tilde{c}}\right)s. \end{aligned} \tag{13}$$

Using the calculated $MRS^{\rho}_{\zeta\eta}$ and $MRS^{d}_{\zeta\eta}$ we observe that any (ρ, d) can at most be implemented through one (ζ, η) combination because each iso-ρ curve crosses each iso-d curve at most once: both slopes are positive and the slope of the iso-ρ curves is steeper throughout than the slope of the iso-d curves.

Given budget K $\zeta \in [0, K]$, and $\eta = \eta^0 - h \in [\check{\eta}, \eta^0]$ where $\check{\eta} = \eta^0 - K/\kappa$. Moreover the budget line is $K = \kappa h + \zeta$.

The polar solutions induce $(\zeta = 0, \check{\eta})$ is a direct application of Proposition 3. More generally, we can plug conditions (13) into the the budget line and rearrange to obtain

$$d(\rho) = 6q(K + s - \kappa\eta^0)\frac{\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)}{2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) - \kappa\rho}. \tag{14}$$

our research possibility frontier.

Deriving the Boundary ρ's on the Research Possibility Frontier The first term in brackets, $K + s - \kappa\eta^0$ can be both positive or negative depending on the chosen parameters. Observe, however, that the minimum attainable cost factor

$\check{\eta} = \eta^0 - K/\kappa$. Thus we can rewrite

$$K + s - k\eta^0 = s(1 - \check{\eta}\kappa).$$

Replacing $\check{\eta} = 1/(2\tilde{c}_\rho(\check{\rho}) - \tilde{c}(\check{\rho})/\check{\rho}$ which is the solution of equation (5) from the proof of Lemma 12 performing the same steps as when solving for equation (6) (which is possible because $\zeta = 0$ in that case) that implies

$$K + s - \kappa\eta^0 = s\left(1 - \frac{\kappa}{\underbrace{s(2\tilde{c}_\rho(\check{\rho}) - \tilde{c}(\check{\rho})/\check{\rho})}_{=MRS^\rho_{\zeta\eta}(\check{\rho})}}\right) = s\left(\frac{MRS^\rho_{\zeta\eta}(\check{\rho}) - \kappa}{MRS^\rho_{\zeta\eta}(\check{\rho})}\right).$$

Thus,

$$K + s > \kappa\eta^0 \Leftrightarrow MRS^\rho_{\zeta\eta}(\check{\rho}) > \kappa.$$

Now consider the last term, $(\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho))/(2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) - \kappa\rho)$. It is positive if and only if

$$2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) > \kappa\rho$$
$$\kappa < \underbrace{s\left(\frac{2\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)}{\rho}\right)}_{=MRS^\rho_{\zeta\eta}}.$$

If $MRS^\rho_{\zeta\eta}(\check{\rho}) > \kappa$ then any iso-ρ curve that crosses the budgetline must satisfy $\rho > \check{\rho}$ and thus $MRS^\rho_{\zeta\eta}(rho) > \kappa$ by Proposition 5. Moreover, the largest implementable $\overline{\rho}$ on the research possibility frontier comes from the polar case $\zeta = K, \eta = \eta^0$. Similarily if $MRS^\rho_{\zeta\eta}(\check{\rho}) < \kappa$ then all iso-ρ curves that cross the budgetline must satisfy $\rho < \check{\rho}$ and the lowest implementable ρ on the research possibility frontier comes from the polar case $\zeta = K, \eta = \eta^0$

Deriving the slope of the Research Possibility Frontier: Let $n(\rho)$ be the numerator of the last term and $dn(\rho)$ the denominator. Then, the last term is increasing in ρ if and only if

$$n'(\rho)dn(\rho) > n(\rho)dn'(\rho)$$

or equivalently using that $n'(\rho) = \rho\tilde{c}_{\rho\rho}(\rho) > 0$, $dn'(\rho) = s\left(2\rho\tilde{c}_{\rho\rho} + \tilde{c}_\rho\right) - \kappa$ if and only if

$$\frac{\kappa}{s} < \underbrace{\frac{\tilde{c}_\rho(\rho)\tilde{c}(\rho) + \rho\tilde{c}(\rho)\tilde{c}_{\rho\rho}(\rho) - \rho\left(\tilde{c}_\rho(\rho)\right)^2}{\tilde{c}_{\rho\rho}(\rho)\rho^2 - \rho\tilde{c}_\rho(\rho) + \tilde{c}(\rho)}}_{=MRS^d_{\zeta\eta}(\rho)}.$$

Thus $d(\rho)$ is increasing if and only if $(MRS^\rho_{\zeta\eta}(\check{\rho}) - \kappa)(MRS^d_{\eta\zeta}(\rho) - \kappa/s) > 0$.

□

While the expression in Equation (12) looks cumbersome at first sight, Proposition 5 and Proposition 6 give guidance on its interpretation. Any change in ρ can

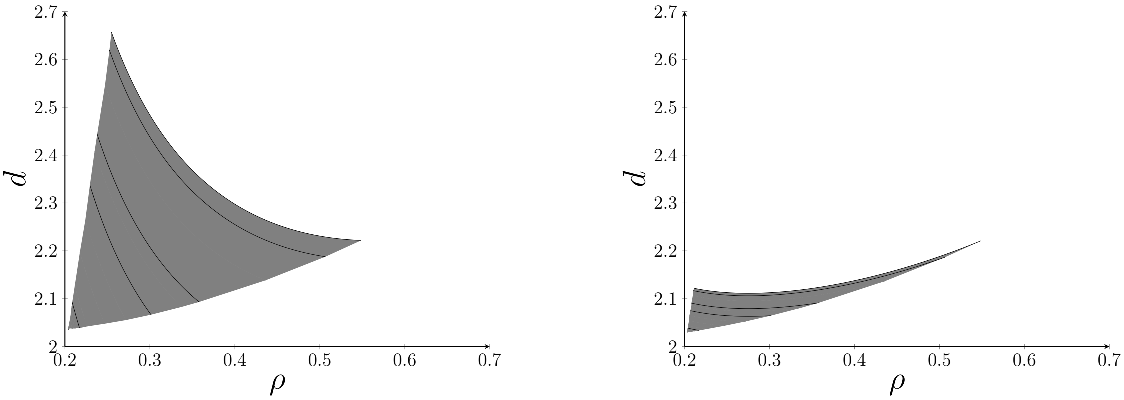

Figure 14: *Feasible set for different reward standards*
The shaded area shows the implementable (ρ, d) combinations of a funder for a given budget K. All points on a solid line require the same amount of funding, K. In both panels, the funder has a budget of $K = 10$, the price ratio is $\kappa = 6$, and the baseline cost factor is $\eta^0 = 2$. The status quo parameter has value $q = 1$. In the left panel, the reward technology is parameterized with $s = 40$; in the right panel, it is $s = 200$.

only be implemented through a movement along the (ζ, η) budget line. Whether the funding scheme has to increase rewards and reduce grants to increase the success probabilities is determined by the MRS of ρ between ζ and η relative to the cost ratio κ. In the proof of Proposition 4, we show that $K - \kappa\eta^0 + s$ has the same sign as

$$\underbrace{s\left(2\tilde{c}_\rho(\check{\rho}) - \frac{\tilde{c}(\check{\rho})}{\check{\rho}}\right)}_{=MRS^{\rho}_{\zeta\eta}(\check{\rho})} - \kappa,$$

where $\check{\rho}$ is the researcher's choice of ρ given the polar case $(\zeta = 0, \eta = \eta^0 - K/\kappa)$. Moreover, the denominator in Equation (12) is equivalent to $\rho(MRS^{\rho}_{\zeta\eta}(\rho) - \kappa)$. Thus, an equivalent formulation of equation (12) is

$$d(\rho) = \frac{MRS^{\rho}_{\zeta\eta}(\check{\rho}) - \kappa}{MRS^{\rho}_{\zeta\eta}(\check{\rho})} \frac{1}{MRS^{\rho}_{\zeta\eta}(\rho) - \kappa} \left(\tilde{c}_\rho(\rho) - \frac{\tilde{c}(\rho)}{\rho}\right) M,$$

where $M > 0$ is a constant.

By the properties of the inverse error function, the term in brackets is always positive. Because output maximization implies a corner solution (Proposition 5), the slopes of all iso-ρ curves that cross the budget line have to be in the same relation to the budget line's slope, κ. Thus, if $MRS^{\rho}_{\zeta\eta}(\check{\rho}) > \kappa$ then $MRS^{\rho}_{\zeta\eta}(\rho) > \kappa$ and vice versa. For the special case $MRS^{\rho}_{\zeta\eta}(\check{\rho}) = \kappa$, the research-possibility frontier is a vertical line because in that case, the slope of the budget line, κ, is precisely that of the $MRS^{\rho}_{\zeta\eta}(\rho)$, which implies that moving along the budget line is the same as moving along the iso-ρ curve for all (ζ, η) combinations. It is possible to implement various levels of distance, d, while ρ remains constant.

The slope of the research-possibility frontier can be positive or negative. If the slope is positive, changing the allocation of funds to increase output, ρ, implies an increase in novelty, d, as the two are complements from the funder's perspective. If the slope is negative, an increase in output implies a decrease in novelty, as the two are substitutes from the funder's perspective.

The slope of the research frontier is positive and finite if and only if

$$\underbrace{\left(K+s-\kappa\eta^0\right)}_{=s\frac{MRS^{\rho}_{\zeta\eta}(\tilde{\rho})-\kappa}{MRS^{\rho}_{\zeta\eta}(\tilde{\rho})}}\left(\underbrace{\frac{\tilde{c}_\rho(\rho)\tilde{c}(\rho)+\rho\tilde{c}(\rho)\tilde{c}_{\rho\rho}(\rho)-\rho\left(\tilde{c}_\rho(\rho)\right)^2}{\tilde{c}_{\rho\rho}(\rho)\rho^2-\rho\tilde{c}_\rho(\rho)+\tilde{c}(\rho)}}_{=MRS^{d}_{\zeta\eta}(\rho)}-\kappa/s\right)>0.$$

Note that both $MRS^{d}_{\zeta\eta}(\rho)$ and $MRS^{\rho}_{\zeta\eta}(\rho)$ are increasing in ρ by the properties of the inverse error function.

To illustrate, consider the two scenarios from Figure 14. Given the parameters, $K+s>\kappa\eta^0$ for both levels of s. In the left panel ($s=40$), novelty and output are substitutes as long as $\rho<\tilde{\rho}\approx 50\%$, the point at which $MRS^{d}_{\zeta\eta}$ crosses the threshold $\kappa/s=0.15$. As s increases to 200, that threshold decreases to $\kappa/s=0.03$ and $\tilde{\rho}\approx 29\%$. Moreover, while a novelty-maximizing funder would choose to implement a risky choice by the researcher ($\rho\approx 25\%$) if $s=40$, the same funder would choose a much safer strategy when $s=200$ ($\rho\approx 65\%$).[43]

Distance and success probabilities become complements because an increase in ρ has two effects: (i) it increases the marginal benefit of distance, $V_d(d,X)+\zeta/s\sigma^2_d(d,X)$, by increasing the probability that the researcher finds an answer, and (ii) it increases the marginal cost of distance, $\eta\tilde{c}(\rho)\sigma^2_d$. The uncertainty of the conjecture about questions, $\sigma^2(d,X)$, is increasing in distance. Moreover, for any distance, the interval that has to be covered to find an answer with probability ρ is increasing in this probability. For certain parameter constellations, it may be that, for example, an increase in ζ increases the weight placed on the marginal-benefit effect relatively more than the resulting increase in η increases the marginal-cost effect. Straightforwardly, the larger s, the smaller the effect on the marginal benefit.

Extensive margin. By assumption, the statements in Proposition 5, Proposition 6, and Proposition 4 refer to the intensive margin of ρ and d within the (unbounded) area outside the knowledge frontier. As we have seen in Section 5, such a focus covers only part of the researcher's decision. We conclude this application by illustrating how the funding architecture affects the extensive margin. To that end, we consider $\mathcal{F}_2$ such that $X_1=\hat{X}$. In that case, the researcher is indifferent between expanding knowledge and deepening knowledge. She chooses the midpoint $\hat{X}/2$, given the funding architecture ($\eta=\eta^0,\zeta=0$). We can straightforwardly compute the MRS between rewards and grants that keeps the researcher indifferent between the two intervals. Comparing this to the relative prices of rewards and grants, $1/\kappa$, determines how a funding institution can incentivize a shift toward expanding knowledge by shifting more funds either toward rewards or toward grants.

[43] We assume in this exercise that s is a parameter to the funder. Under most funding schemes, the winners of a prize are determined by a jury of peers rather than the funder itself. Thus, the standards may not be entirely under the funder's control.

This MRS is

$$MRS^{\hat{X}}_{\zeta\eta} = \frac{\tilde{c}(\rho^{\infty})/\rho^{\infty}}{s} \frac{\frac{\frac{\tilde{c}_{\rho}(\rho^{\infty})}{\tilde{c}(\rho^{\infty})/\rho^{\infty}}-1}{\frac{\tilde{c}_{\rho}(\rho^{\hat{X}})}{\tilde{c}(\rho^{\hat{X}})/\rho^{\hat{X}}}-1} - 1}{\frac{\tilde{c}_{\rho}(\rho^{\infty})-\tilde{c}(\rho^{\infty})/\rho^{\infty}}{\tilde{c}_{\rho}(\rho^{\hat{X}})-\tilde{c}(\rho^{\hat{X}})/\rho^{\hat{X}}} - 1} > 0.$$

Derivation of $MRS^{\hat{X}}_{\zeta\eta}$. Recall that at $\hat{X}$, the researcher is indifferent between expanding and deepening at $d = \hat{X}/2$

$$\rho^{\infty}(V(d^{\infty}) + \frac{\zeta}{s}d^{\infty}) - \eta\tilde{c}(\rho^{\infty})d^{\infty} = \rho^{\hat{X}}(V(\hat{X}/2, \hat{X}) + \frac{\zeta}{s}\frac{\hat{X}}{4}) - \eta\tilde{c}(\rho^{\hat{X}})\frac{\hat{X}}{4}$$

and replacing from the ρ-FOC, we obtain

$$\begin{aligned}\rho^{\infty}\left(\tilde{c}_{\rho}(\rho^{\infty}) - \tilde{c}(\rho^{\infty})/\rho^{\infty}\right)d^{\infty} &= \rho^{\hat{X}}\left(\tilde{c}_{\rho}(\rho^{\hat{X}}) - \tilde{c}(\rho^{\hat{X}})/\rho^{\hat{X}}\right)\frac{\hat{X}}{4} \\ \frac{\hat{X}}{4} &= d^{\infty}\frac{\rho^{\infty}\tilde{c}_{\rho}(\rho^{\infty}) - \tilde{c}(\rho^{\infty})}{\rho^{\hat{X}}\tilde{c}_{\rho}(\rho^{\hat{X}}) - \tilde{c}(\rho^{\hat{X}})}.\end{aligned}$$

From the envelope theorem it follows that

$$\begin{aligned}\frac{dU_R(\zeta, \eta; X)}{d\zeta} &= \rho^{X}\frac{\sigma^2(X)}{s} \\ \frac{dU_R(\zeta, \eta; X)}{d\eta} &= -\tilde{c}(\rho^{X})\sigma^2(X).\end{aligned}$$

Then, the marginal rate of substitution defined as

$$MRS^{\hat{X}}_{\zeta\eta} = -\frac{\frac{d}{d\eta}(U_R(\hat{X}) - U_R(\infty))}{\frac{d}{d\zeta}(U_R(\hat{X} - U_R(\infty))}$$

is

$$MRS^{\hat{X}}_{\zeta\eta} = \frac{\tilde{c}(\rho^{\hat{X}})\frac{\hat{X}}{4} - \tilde{c}(\rho^{\infty})d^{\infty}}{\rho^{\hat{X}}\frac{\hat{X}}{4s} - \rho^{\infty}\frac{d^{\infty}}{s}}$$

and the expression from the text follows by replacing $\frac{\hat{X}}{4}$ from above and factoring d^{∞} and s out.

D Omitted Proofs

Here we provide the steps that we have omitted in the proofs because they involve cumbersome algebraic manipulation with little economic or mathematical insight.

Lemma 23. $\frac{\partial V(d;\infty|d>4q)}{\partial d} < 0$.

Proof.
$$\frac{\partial V(d;\infty|d>4q)}{\partial d}=-\frac{d}{3q}+1+\sqrt{\frac{d-4q}{d}}\frac{d-q}{3q}$$
Letting $\tau := d/q$ the statement is negative if
$$\frac{3-\tau}{3}+\sqrt{\frac{\tau-4}{\tau}}\frac{\tau-1}{3}<0$$
The left-hand side is increasing in τ and converges to 0 as $\tau\to\infty$. □

Lemma 24. $\frac{1}{3q}\left(X-2d-(X-d-q)\sqrt{\frac{X-d-4q}{X-d}}\right)>0$ *if* $d\in[0,X-4q]$.

Proof. We show that the derivative V_d is a convex function which is positive at its minimum on $[0,X-4q]$ and hence throughout on that domain.

The relevant derivatives to consider are
$$V_d=\frac{1}{3q}\left(X-2d-(X-d-q)\sqrt{\frac{X-d-4q}{X-d}}\right).$$
$$V_{dd}=\frac{1}{3q}\left(-2+\frac{1}{\sqrt{X-d-4q}(X-d)^{3/2}}\left((X-d-4q)(X-d)+(X-d-q)2q\right)\right).$$
$$V_{ddd}=\frac{4q^2}{(X-d)^{5/2}(X-d-4q)^{3/2}}>0.$$
where $V_{ddd}>0$ follows immediately from $(X-d)>0$ and $(X-d-4q)>0$. It follows that, V_d is strictly convex over the relevant range. The maximal distance in this range, $d=X-4q$, $V_d|_{d=X-4q}=\frac{8q-X}{3q}>0$.

Hence, the minimum of the first derivative is either at $d=0$ or at some interior d such that $V_{dd}=0$. Suppose the minimum is at $d=0$, then $V_d|_{d=0}=\frac{1}{3q}\left(X-(X-q)\sqrt{\frac{X-4q}{X}}\right)>0$ because $\frac{X-4q}{X}<1$.

Hence, the only remaining case is when V_d attains an interior minimum. In this case, $V_{dd}=0$ must hold at the minimum and hence
$$\sqrt{X-d-4q}(X-d)^{3/2}=\frac{(X-d-4q)(X-d)+(X-d-q)2q}{2}.$$
The first derivative can be rewritten as
$$V_d=\frac{1}{3q}\left(X-2d-\frac{1}{\sqrt{X-d-4q}(X-d)^{3/2}}(X-d-q)(X-d-4q)(X-d)\right)$$
and plugging in for the minimum condition we obtain
$$\begin{aligned}&V_d|_{V_{dd}=0}\\&=\frac{1}{3q}\left(X-2d-\frac{2(X-d-q)(X-d-4q)(X-d)}{(X-d-4q)(X-d)+(X-d-q)2q}\right)\\&=\frac{1}{3q}\frac{(X-2d)((X-d-4q)(X-d)+(X-d-q)2q)-2(X-d-q)(X-d-4q)(X-d)}{(X-d-4q)(X-d)+(X-d-q)2q}.\end{aligned}$$

As the denominator and $\frac{1}{3q}$ are both positive, the sign of V_d at its minimum is determined by the sign of its numerator only. Note that the numerator is increasing in d because its derivative is $2(X-6q)(X-d-q) > 0$. Thus, the numerator of the derivative of V_d evaluated at the interior minimum d such that $V_{dd} = 0$ is greater than

$$-X(X^2 - 8qX + 10q^2) = -X((X-4q)^2 - 6q^2) > 0.$$

□

Lemma 25. *$V_X(d^0(X); X) < 0$ if $X \geq 4q$ and $d \in [0, X-4q]$.*

Proof. Observe that for any $X \geq 4q$ and $d \leq X - 4q$

$$V_{Xd} = \frac{1}{24q}\left(8 - 3\sqrt{\frac{X-d}{X-d-4q}} - (5(X-d)+4q)\frac{\sqrt{X-d-4q}}{(X-d)^{3/2}}\right).$$

Denote $a := X - d$, this is an increasing function in a as

$$\frac{dV_{Xd}}{da} = \frac{4q^2}{a^{5/2}(a-4q)^{3/2}} > 0.$$

Hence, the highest value of V_{Xd} is attained for $a \to \infty$ and

$$\lim_{a\to\infty} \frac{1}{24q}\left(8 - 3\underbrace{\sqrt{\frac{a}{a-4q}}}_{\to 1} - 5\underbrace{\frac{a\sqrt{a-4q}}{a^{3/2}}}_{\to 1} + 4q\underbrace{\frac{\sqrt{a-4q}}{a^{3/2}}}_{\to 0}\right) = 0.$$

It follows that the V_{Xd} converges to zero from below implying that $V_{Xd} < 0$. Thus, $V_X(d^0(X), X) < V_X(d = 0, X)$ and we obtain

$$\begin{aligned}
&V_X(d, X|d \leq 4q, X-d \geq 4q)\\
&\quad= \frac{1}{3q}\left(d + (X-d-q)\sqrt{\frac{X-d-4q}{X-d}} - (X-q)\sqrt{\frac{X-4q}{X}}\right)\\
&\quad< V(d = 0, X|d \leq 4q, X-d \geq 4q)\\
&\quad= \frac{1}{3q}\left((X-q)\sqrt{\frac{X-4q}{X}} - (X-q)\sqrt{\frac{X-4q}{X}}\right)\\
&\quad= 0
\end{aligned}$$

as desired. □

Lemma 26. $\mathrm{d}^2V(X/2, X)/(\mathrm{d}X)^2 < 0$ *and* $\mathrm{d}^2V(d^0(X), X)/(\mathrm{d}X)^2 > 0$.

Proof. Considering the boundary solution we obtain

$$
\begin{aligned}
\frac{d^2V(X/2,X)}{dX^2} &= -\frac{X^2-2qX-2q^2}{3qX^{3/2}\sqrt{X-4q}}+\frac{1}{6q}\\
\frac{d^3V(X/2,X)}{dX^3} &= \frac{4q^2}{X^{5/2}(X-4q)^{3/2}}>0
\end{aligned}
$$

implying that $\frac{d^2V(X/2,X)}{dX^2} \leq \frac{d^2V(4q,8q)}{dX^2}$ with

$$
\frac{d^2V(4q,8q)}{dX^2} = -\frac{64q^2-16q^2-2q^2}{3q8^{3/2}q^{3/2}2q^{1/2}}+\frac{1}{6q} = -\frac{46q^2}{96\sqrt{2}q^3}+\frac{1}{6q} = \frac{8-23/\sqrt{2}}{48q}<0.
$$

Next, consider the value of any interior solution and apply the envelope and implicit function theorem to obtain

$$
\begin{aligned}
\frac{dV(d^0(X),X)}{dX} &= V_X + d'(X)\underbrace{V_d}_{=0 \text{ by optimality of } d} = V_X\\
\frac{d^2V(d^0(X),X)}{dX^2} &= V_{XX} + d'(X)\underbrace{(V_{Xd}+V_{dd}d'(X))}_{=0 \text{ by IFT on FOC}} + d''(X)\underbrace{V_d}_{=0 \text{ by optimality}}\\
&= V_{XX}(d^0(X),X).
\end{aligned}
$$

Observing that

$$
V_{XXd}(d,X|d\leq 4q, X-d\geq 4q) = \frac{24q^3}{(X-d)^{5/2}(X-d-4q)^3/2}>0
$$

we can compute as lower bound for

$$
\begin{aligned}
V_{XX}(d^0(X),X) = \frac{1}{24q}\Bigg(3\left(\sqrt{\frac{X-d}{X-d-4q}}-\sqrt{\frac{X}{X-4q}}\right)+6\left(\sqrt{\frac{X-d-4q}{X-d}}-\sqrt{\frac{X-4q}{X}}\right)\\
+\left(\frac{X-4q}{X}\right)^{3/2}-\left(\frac{X-d-4q}{X-d}\right)^{3/2}\Bigg)\\
\geq V_{XX}(d=0,X)\\
=0
\end{aligned}
$$

implying that $\mathrm{d}^2V(d^0(X),X)/(\mathrm{d})^2 \geq 0$ and concluding the proof. □

Lemma 27. *Assume $X\in[4q,8q]$, then $\mathrm{d}^2U_R(d=X/2;X)/(\mathrm{d}X)^2<0$.*

Proof. Take the case of the boundary solution: we are analyzing a one-dimensional optimization problem with respect to ρ. Denote the objective $f(\rho;X)$ and the optimal value by $\varphi(X)=\max_\rho f(\rho;X)$. Then, the optimal ρ solves $f_\rho=0$. We obtain

$$
\begin{aligned}
\varphi'(X) &= \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho'(X) + f_X \\
\varphi''(X) &= \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho''(X) + \underbrace{(f_{\rho\rho}\rho'(X) + f_{X\rho})}_{=0 \text{ by total differentiation of FOC}} \rho'(X) + f_{XX} \\
&= f_{XX} \\
&= \rho(X) V_{XX}(X/2; X) < 0.
\end{aligned}
$$

Hence, the optimal value at the boundary solution is strictly concave as $\sigma^2_{XX}(X/2; X) = 0$ and $V_{XX} < 0$ in the region considered by Corollary 2 $\square$

Lemma 28. *Let $d^i < X/2$ be an local maximum of $u_r(rho, d, X)$. If $d^i(X)$ exists on $X \in [4q, 8q]$, then $\mathrm{d}^2 U_R(d = d^i(X); X)/(\mathrm{d}X)^2 > 0$.*

Proof. The implicit function theorem yields for $d'(X)$ and $\rho'(X)$

$$
\begin{pmatrix} d'(X) \\ \rho'(X) \end{pmatrix} = -\frac{1}{f_{dd} f_{\rho\rho} - f^2_{\rho d}} \begin{pmatrix} f_{dX} f_{\rho\rho} - f_{\rho X} f_{d\rho} \\ f_{\rho X} f_{dd} - f_{dX} f_{d\rho} \end{pmatrix}.
$$

Note that $-\frac{1}{f_{dd} f_{\rho\rho} - f^2_{\rho d}} < 0$ as this is $-\frac{1}{det(\mathcal{H})}$ and the determinant of the second principal minor being positive is a necessary second order condition for a local maximum given that the first ($f_{\rho\rho}$) is negative.

Denote the objective $f(\rho, d; X)$ and the optimal value by $\varphi(X) = \max_{\rho,d} f(d, \rho; X)$. Then, the optimal (d, ρ) solves $f_\rho = 0$ and $f_d = 0$. Differentiating the value of the researcher twice with respect to X yields

$$
\begin{aligned}
\varphi'(X) &= \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho'(X) + \underbrace{f_d}_{=0 \text{ by optimality}} d'(X) + f_X \\
\varphi''(X) &= \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho''(X) \\
&\quad + d'(X) \underbrace{(f_{dX} + f_{dd} d'(X) + f_{d\rho}\rho'(X))}_{=0 \text{ by total differentiation of foc wrt } d} \\
&\quad + \rho'(X) \underbrace{(f_{\rho X} + f_{\rho d} d'(X) + f_{\rho\rho}\rho'(X))}_{=0 \text{ by total differentiation of foc wrt } \rho} \\
&\quad + f_{dX} d'(X) + f_{\rho X}\rho'(X) + f_{XX} \\
&\quad = f_{dX} d'(X) + f_{\rho X}\rho'(X) + f_{XX}.
\end{aligned}
$$

Observe first that $f_{XX} > 0$ as $f_{XX} = \rho V_{XX}(d; X) - \eta\tilde{c}(\rho)\sigma^2_{XX}(d; X)$ and $V_{XX} > 0$ by proof of Corollary 2 and $\sigma^2_{XX}(d; X) = -\frac{2d^2}{X^3}$. Next, we show $f_{dX} d'(X) + f_{\rho X}\rho'(X) > 0$ using the implicit function theorem together with the property of the local maximum that $f_{\rho\rho} f_{dd} > f^2_{\rho d}$.

$$
f_{dX} d'(X) + f_{\rho X}\rho'(X) = -f_{dX}\left(\frac{f_{dX} f_{\rho\rho} - f_{\rho X} f_{d\rho}}{f_{dd} f_{\rho\rho} - f^2_{\rho d}}\right) - f_{\rho X}\left(\frac{f_{\rho X} f_{dd} - f_{dX} f_{d\rho}}{f_{dd} f_{\rho\rho} - f^2_{\rho d}}\right).
$$

As we only need the sign of this expression we can ignore the positive denominator to verify

$$\begin{aligned}
-f_{dX}(f_{dX}f_{\rho\rho} - f_{\rho X}f_{d\rho}) - f_{\rho X}(f_{\rho X}f_{dd} - f_{dX}f_{d\rho}) &> 0 \\
f_{dX}^2 f_{\rho\rho} + f_{\rho X}^2 f_{dd} - 2 f_{dX} f_{\rho X} f_{d\rho} &< 0 \\
\frac{f_{dX}}{f_{\rho X}}\frac{f_{\rho\rho}}{f_{d\rho}} + \frac{f_{\rho X}}{f_{dX}}\frac{f_{dd}}{f_{\rho d}} &> 2.
\end{aligned}$$

where we used the signs of the terms that follow because

$$\begin{aligned}
f_{\rho\rho} &= -\eta\tilde{c}_{\rho\rho}(\rho)\sigma^2(d;X) < 0 \\
f_{\rho X} &= V_X - \eta\tilde{c}_\rho(\rho)\sigma_X^2(d;X) \\
&< V_X - \eta\frac{\tilde{c}(\rho)}{\rho}\sigma_X^2(d;X) < 0 \\
f_{d\rho} &= V_d - \eta\tilde{c}_\rho(\rho)\sigma_d^2(d;X) \\
&< V_d - \eta\frac{\tilde{c}(\rho)}{\rho}\sigma_d^2(d;X) = 0 \\
f_{dX} &= \rho V_{dX} - \eta\tilde{c}(\rho)\sigma_{dX}^2 < 0
\end{aligned}$$

which in turn follow from the first-order conditions and Corollary 2.

Because $f_{\rho\rho}f_{dd} - f_{\rho d}^2 > 0$, we can replace $\frac{f_{\rho\rho}}{f_{d\rho}}$ with $\frac{f_{d\rho}}{f_{dd}}$ as $\frac{f_{\rho\rho}}{f_{d\rho}} > \frac{f_{d\rho}}{f_{dd}}$ yielding

$$2 < \frac{f_{dX}}{f_{\rho X}}\frac{f_{d\rho}}{f_{dd}} + \frac{f_{\rho X}}{f_{dX}}\frac{f_{dd}}{f_{\rho d}}$$

which is true as the right-hand side can be written as $g(a) = a + \frac{1}{a}$ with $a = \frac{f_{dX}}{f_{\rho X}}\frac{f_{d\rho}}{f_{dd}} > 0$. Note that $g(a)$ is a strictly convex function for $a > 0$ and minimized at $a = 1$ with $g(a = 1) = 2$. □

Lemma 29. $MRS_{\zeta\eta}^{\rho} = 2\tilde{c}_\rho(\rho) - \tilde{c}(\rho)/\rho$.

Proof. For any (η, ζ) the system of first-order conditions for a non-boundary choice is given by

$$\begin{aligned}
V_d(d,\infty) + \zeta\sigma_d^2(d,\infty)/s &= \eta\tilde{c}(\rho)/\rho \\
\frac{V(d,\infty) + \zeta\sigma_d^2(d,\infty)/s}{d} &= \eta\tilde{c}_\rho(\rho)
\end{aligned}$$

For an interior optimal choice of (d, ρ), we obtain using $\sigma^2(d, X) = d, \sigma_d^2(d, X) = 1$

and $\sigma^2_{dd}(d,X)=0$[44]

$$\begin{pmatrix} \frac{dd}{d\eta} \\ \frac{dd}{d\zeta} \\ \frac{d\rho}{d\eta} \\ \frac{d\rho}{d\zeta} \end{pmatrix} = -\frac{1}{det(\mathcal{H})}$$

$$\begin{pmatrix} d(\tilde{c}_\rho(V_d+\zeta/s-\eta\tilde{c}_\rho)+\eta\tilde{c}\tilde{c}_{\rho\rho}) \\ -d(V_d+\zeta/s-\eta\tilde{c}_\rho+\rho\eta\tilde{c}_{\rho\rho}) \\ -\rho\sigma^2\tilde{c}_\rho V_{dd}+\tilde{c}(V_d+\zeta/s-\eta\tilde{c}_\rho) \\ -\rho/s(V_d+\zeta/s-\eta\tilde{c}_\rho-dV_{dd}) \end{pmatrix}.$$

where $det(\mathcal{H})$ is the determinant of the Hessian matrix of the objective function which is given by

$$-\eta\sigma^2\tilde{c}_{\rho\rho}\rho V_{dd}-(V_d+\zeta/s-\eta\tilde{c}_\rho)^2>0.$$

Note that the determinant of the Hessian matrix for a local maximum is positive as the Hessian is negative semidefinite and the first principal minor $-\eta\tilde{c}_{\rho\rho}\sigma^2<0$ by convexity of the inverse error function.[45]

It follows that the sign of the derivatives are determined only by the negative of the sign of the respective terms in the matrix. Using the first-order conditions to rewrite these equations yields

$$\begin{aligned} \frac{dd}{d\eta} &= -\frac{d\eta}{det(\mathcal{H})}\left(\tilde{c}_\rho\left(\frac{\tilde{c}}{\rho}-\tilde{c}_\rho\right)+\tilde{c}\tilde{c}_{\rho\rho}\right)<0 \\ \frac{dd}{d\zeta} &= \frac{d\eta}{det(\mathcal{H})}\left(\frac{\tilde{c}}{\rho}-\tilde{c}_\rho+\rho\tilde{c}_{\rho\rho}\right)>0 \end{aligned}$$

where the inequalities hold due to the properties of the inverse error function.

$$\begin{aligned} \frac{d\rho}{d\eta} = &-\frac{\rho\eta}{det(\mathcal{H})} \\ &(2\tilde{c}_\rho-\tilde{c}/\rho)\,(\tilde{c}_\rho-\tilde{c}/\rho)<0 \end{aligned}$$

where we have used that $\sigma^2V_{dd}=-\frac{d}{3q}$ and from the first-order conditions we know that $\frac{d}{3q}=2\eta(\tilde{c}_\rho-\tilde{c}/\rho)$. The properties of $\tilde{c}$ imply that $\tilde{c}_\rho>\tilde{c}/\rho$. Finally,

$$\frac{d\rho}{d\zeta}=\frac{\rho\eta/s}{det(\mathcal{H})}\,(\tilde{c}_\rho-\tilde{c}/\rho)>0$$

[44] We suppress arguments of the functions for readability.

[45] In our case, one can actually show that this has to hold given that $d<\infty$. Plugging in from the first-order conditions yields

$$\eta^2(\tilde{c}_\rho-\tilde{c}/\rho)2(\rho\tilde{c}_{\rho\rho}-\tilde{c}_\rho+\tilde{c}/\rho)>0$$

where the inequality follows from the properties of $\tilde{c}$.

where the analogous reasoning as for the previous inequality applies. To conclude, we have

$$\begin{array}{ll} \frac{dd}{d\eta} < 0 & \frac{dd}{d\zeta} > 0 \\ \frac{d\rho}{d\eta} < 0 & \frac{d\rho}{d\zeta} > 0. \end{array}$$

Moreover, define we obtain for the marginal rate of substitution between ζ and η on the expanding interval

$$-\frac{\frac{d\rho}{d\eta}}{\frac{d\rho}{d\zeta}} = MRS^{\rho}_{\zeta\eta} = s(2\tilde{c}_{\rho} - \tilde{c}/\rho)$$

where we used the simplifications from above.

□