A Quasi-Trefftz Method for the Iterative Solution of Time-Harmonic Wave Problems based on the Flux Reconstruction Method

Sébastien Pernet[1], Matthias Rivet[1,2], Sébastien Tordeux[2]

[1] DTIS, ONERA, Université de Toulouse, 31000, Toulouse, France
[2] EPI Makutu, INRIA, Université de Pau et des Pays de l'Adour, TotalEnergies, CNRS UMR 5142

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Context of the work

- Increasing need of 3D simulations of time-harmonic electromagnetic waves: **high-frequency**, **heterogeneous** environments, **wide** domains (in terms of wavelengths)...

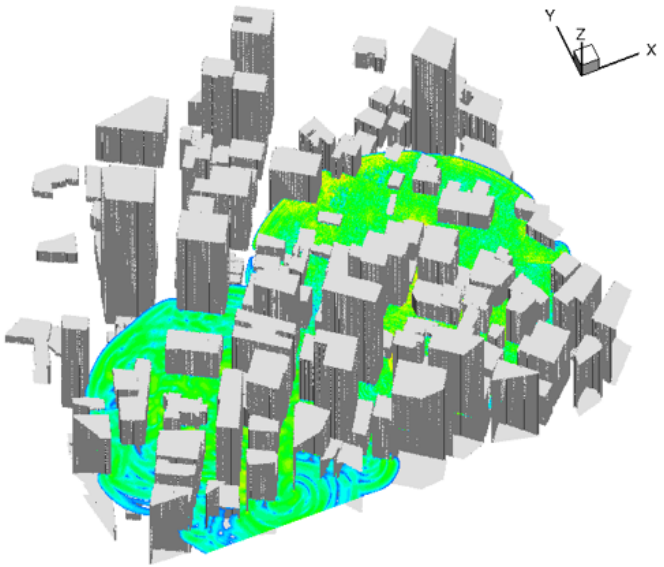

Figure: FDTD simulation on a Manhattan mesh of $700 \times 600 \times 300$m, Thibault Volpert (DEMR ONERA)

Example of computation challenge:
field radiated by two antennas in Manhattan

- **Large simulation domain**.
- **Wide range of frequencies** (1-18 GHz).

$\Longrightarrow$ Need of efficient methods to solve such ambitious cases!

Plan of the talk

1. 3D Maxwell: limitations and (quasi-)Trefftz method
2. Quasi-Trefftz numerical analysis
3. Information transfer optimisation

The 3D Maxwell problem in wide domains: current limitations and (quasi-)Trefftz method interests

Current issues: memory and iterative resolution

Time-harmonic Maxwell equations for the electromagnetic field $\mathbf{Y} := (\mathbf{E}, \mathbf{H}) \in [\mathrm{H}(\mathrm{curl}, \Omega)]^6$:

$$i\kappa \mathbf{M}\mathbf{Y} + \sum_{j=1}^{3} \frac{\partial \mathbf{F}^j \mathbf{Y}}{\partial x_j} = \mathbf{0} \text{ in } \Omega, \tag{1}$$

with impedance Boundary Conditions (BCs)

$$(\mathbf{n}_{\partial\Omega} \times \mathbf{E}) \times \mathbf{n}_{\partial\Omega} + Z_{\partial\Omega}\, \mathbf{n}_{\partial\Omega} \times \mathbf{H} = \mathbf{g} \text{ on } \partial\Omega. \tag{2}$$

Current issues: memory and iterative resolution

Many classic methods as Finite Differences [Yee 1966], Finite Elements [Nédélec 1980], Discontinuous Galerkin [Fezoui et al. 2005], but limitations for current applications:

- Wide domains lead to **very large linear systems**.
- Traditional direct solvers induce a **prohibitive memory consumption**.
- Usual numerical methods are **not well adapted to iterative resolution**.

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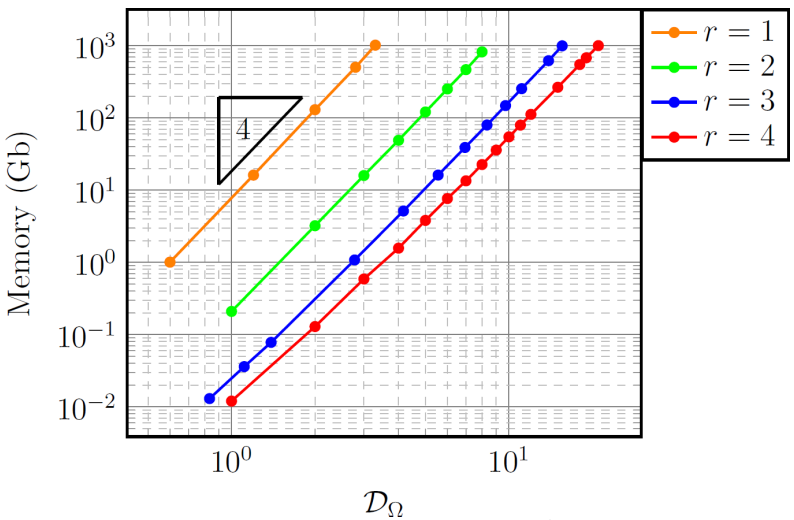

Figure: Memory consumption of a FEM w.r.t. the domain size (in wavelengths) [Sirdey 2022].

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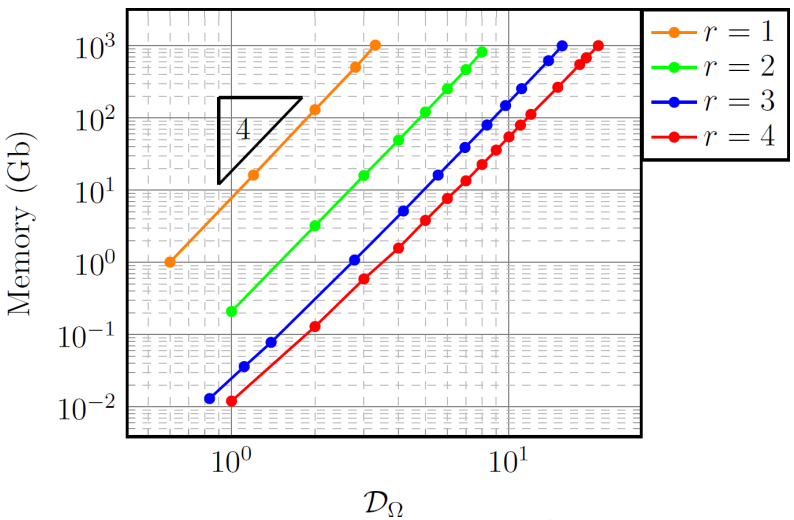

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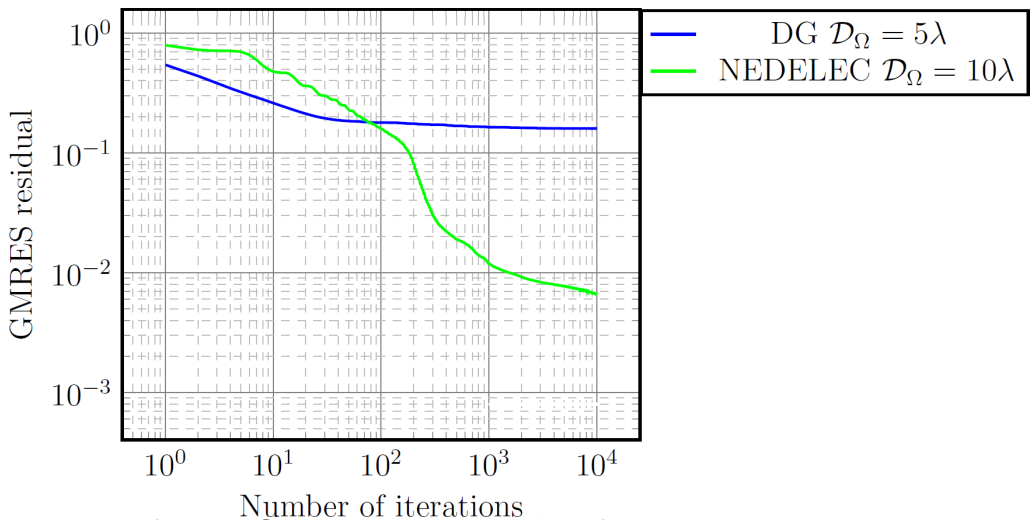

Figure: GMRES convergence of DG and FEM methods [Sirdey 2022].

Considered numerical method: the Trefftz approach

1. Belongs to the Discontinuous Galerkin methods, with the Galerkin space $\mathbb{X} := \prod_{T\in\mathcal{T}_h} \mathbb{X}_T$ made up of local solutions (*i.e.* in each mesh cell $T \in \mathcal{T}_h$) of the Maxwell equations.

2. Use of the reciprocity formula verified in each cell

$$\forall\, T \in \mathcal{T}_h,\ \int_{\partial T} \gamma_\times^T \mathbf{H}^T \cdot \overline{\gamma_t \mathbf{E}'^T} + \gamma_t \mathbf{E}^T \cdot \overline{\gamma_\times^T \mathbf{H}'^T} = 0, \tag{3}$$

in addition to the introduction of numerical traces [Sirdey 2022]:

Find $\mathbf{Y} = (\mathbf{E}, \mathbf{H}) \in \mathbb{X}$ such that $\forall\ \mathbf{Y}' = (\mathbf{E}', \mathbf{H}') \in \mathbb{X}$,

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \widehat{\gamma_\times^T \mathbf{H}^T} \cdot \overline{\gamma_t \mathbf{E}'^T} + \widehat{\gamma_t \mathbf{E}^T} \cdot \overline{\gamma_\times^T \mathbf{H}'^T} = \ell(\mathbb{E}'),$$

where $\widehat{\cdot}$ stands for *upwind* (or Riemann) numerical traces and with

$$\gamma_t \mathbf{u}^T = \mathbf{u}^T - \left(\mathbf{u}^T \cdot \mathbf{n}_T\right) \mathbf{n}_T \text{ and } \gamma_\times^T \mathbf{u}^T = \mathbf{n}_T \times \mathbf{u}^T.$$

The Trefftz approach: properties and classic limits

1. The formulation is posed on the mesh skeleton and adaptable to an iterative resolution (**contraction property**) [Cessenat, Després 1998].
2. A relevant choice of the numerical traces naturally implies formulation coercivity.
3. Basis functions have a **physical meaning**: leads to a reduced numerical pollution [Ihlenburg, Babuška 1995].

Limits of the classic Plane Waves (PWs) choice:

- Numerical dependence phenomena: ill-conditioned basis [Congreve et al. 2019].
- No adaptability of the basis to the local properties: expected singularities, complex interference phenomena...

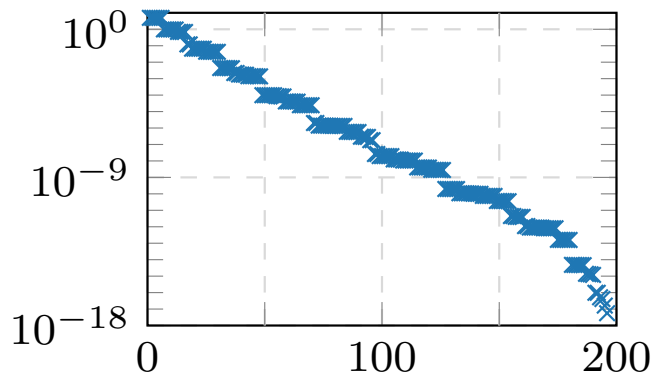

Figure: Eigenvalues of the mass matrix for 196 PWs.

$\Longrightarrow$ Possible algebraic adaptations [Barucq et al. 2021].

$\Longrightarrow$ Introduction of **well-conditioned basis** by parameterising the space of local solutions by polynomial boundary conditions.

A Quasi-Trefftz approach

Definition of the local problem

A local solution $\mathbf{Y}^T$ in T is parameterised by a tangential field $g^T \in \mathbb{V}^T$ such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T}\, \gamma_\times^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \tag{4}$$

$\Longrightarrow$ Consider a finite-dimensional subspace $\mathbb{V}_{\mathbf{h}}^T \subset \mathbb{V}^T$ and the associated Maxwell solutions as local basis [Fure et al. 2020].

Construction of $\mathbb{V}_h$:

- Consider a cell $T \in \mathcal{T}_{\mathbf{h}}^T$.
- Define a mesh of $\mathcal{T}(\partial T)$.
- Consider a piecewise polynomial $\mathbf{g}^T$ of degree k_{QT} on $\mathcal{T}(\partial T)$.

Figure: 2D example of the mesh of ∂T for $k_{QT} = 1$.

$\Longrightarrow$ **Associated Maxwell solutions are unknown**: need of an auxiliary solver to compute approximations!

$\Longrightarrow$ **Other Quasi-Trefftz approaches**: generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

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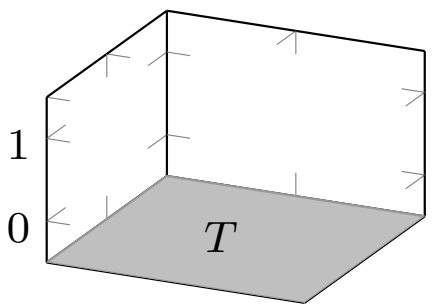

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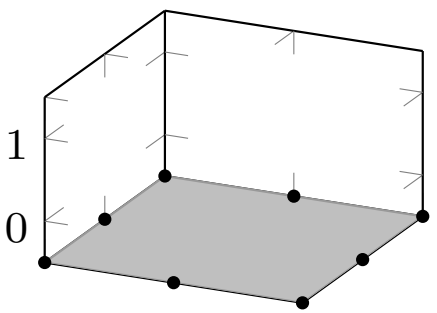

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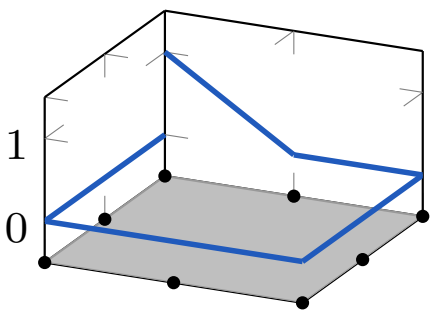

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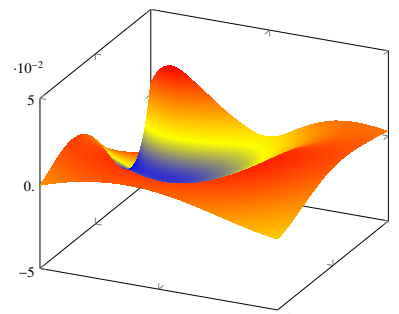

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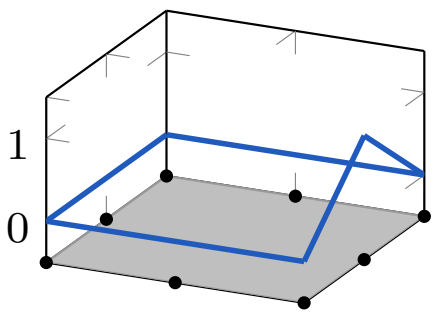

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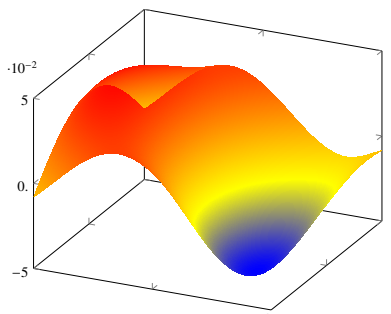

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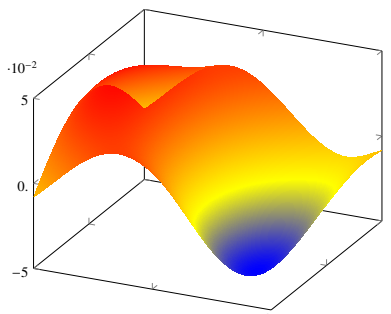

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Description of the FR local solver

Choice of the local solver: **the Flux Reconstruction method** [Huynh 2007]

- Based on the first-order strong formulation (solves both $\mathbf{E}$ and $\mathbf{H}$): no dependence on a quadrature rule.
- Idea: copy the **strong equation in each cell** of the mesh, which is solved on **piecewise polynomials**: the **flux is corrected** at interfaces thanks to numerical traces and correction polynomial functions.

$$i\kappa\mathbf{y} + \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}x} = \mathbf{0} \text{ with } \boldsymbol{\phi} = \mathcal{F}(\mathbf{y}) \implies \forall\, n \in [\![1, N]\!],\ i\kappa\mathbf{y}_h + \frac{\mathrm{d}\widetilde{\boldsymbol{\phi}}_h}{\mathrm{d}x} = \mathbf{0} \text{ in } [X_{n-1}, X_n]\,.$$

- Natural high-order method and adaptable to unstructured meshes.
- Choice of correction polynomials allows to retrieve usual methods: nodal DG (for Radau polynomials), Spectral Differences (for Lagrange polynomials)...
- Choice of the correction polynomials to the user: possibility to optimise them for wave propagation problems, especially in pre-asymptotic regime [Rivet, Pernet, Tordeux 2024].
- For identical cells, the **inversion has to be realised only once**!

$\implies$ **Any numerical method, solving the first-order system, may be used!**

A quasi-Trefftz approach based on a Flux Reconstruction auxiliary solver: numerical analysis

Summary of this Quasi-Trefftz approach

Variational formulation posed on $\mathbb{X} := \prod_{T \in \mathcal{T}_h} \mathbb{X}_T$ made up of local solutions.

Functions $(\mathbf{E}^T, \mathbf{H}^T)$ of $\mathbb{X}_T$ **parameterised** by $\mathbf{g}^T \in \mathbb{V}^T$ such that:

$$\gamma_t \mathbf{E}^T + Z_{\partial T}\, \gamma_\times^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \tag{5}$$

Polynomial approximation of the trace space: $\mathbb{V}_{\mathbf{h}}^T = \mathrm{span}(\mathbf{g}_i^T)$.
For all i, the associated solutions $(\mathbf{E}_i^T, \mathbf{H}_i^T)$ are taken as basis functions of $\mathbb{X}_T$.

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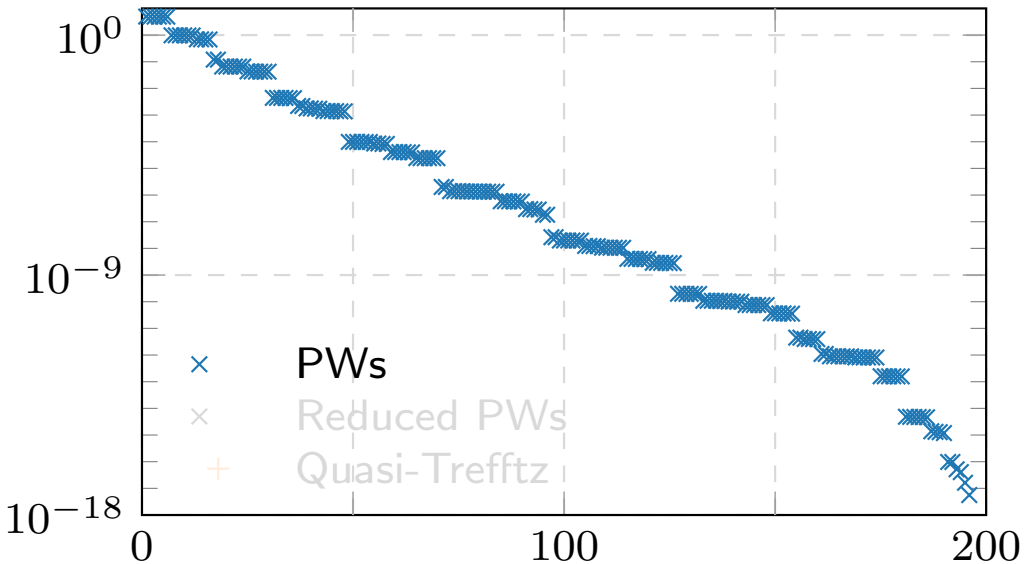

Figure: Eigenvalues of the mass matrix for 196 PWs (and reduction for $\epsilon = 10^{-6}$) and 192 Quasi-Trefftz basis functions.

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$\Longrightarrow$ Possible **reduction techniques** to restore the condition number.

$\Longrightarrow$ Avoids spurious numerical modes and ill-conditioning of the mass matrix!

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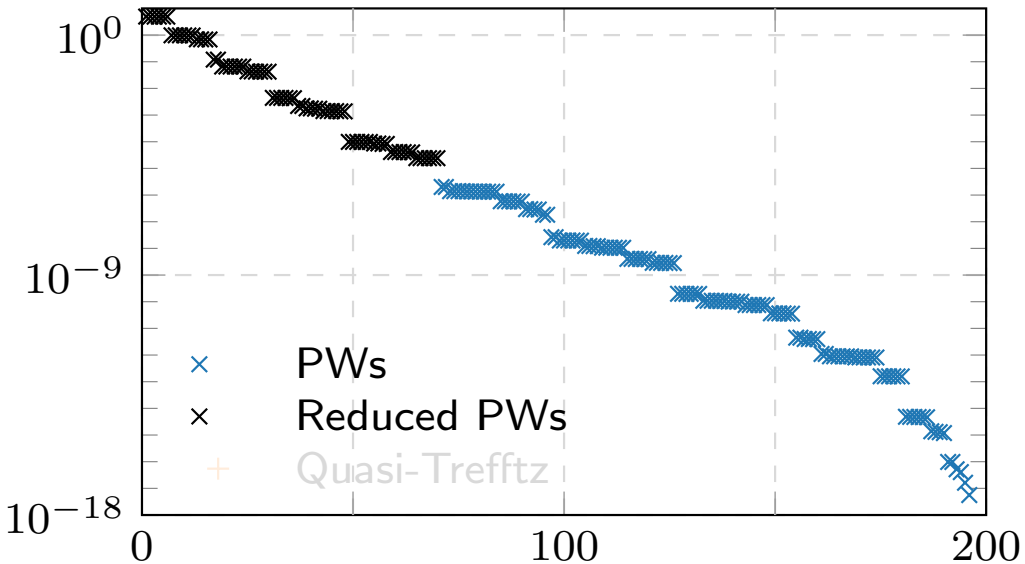

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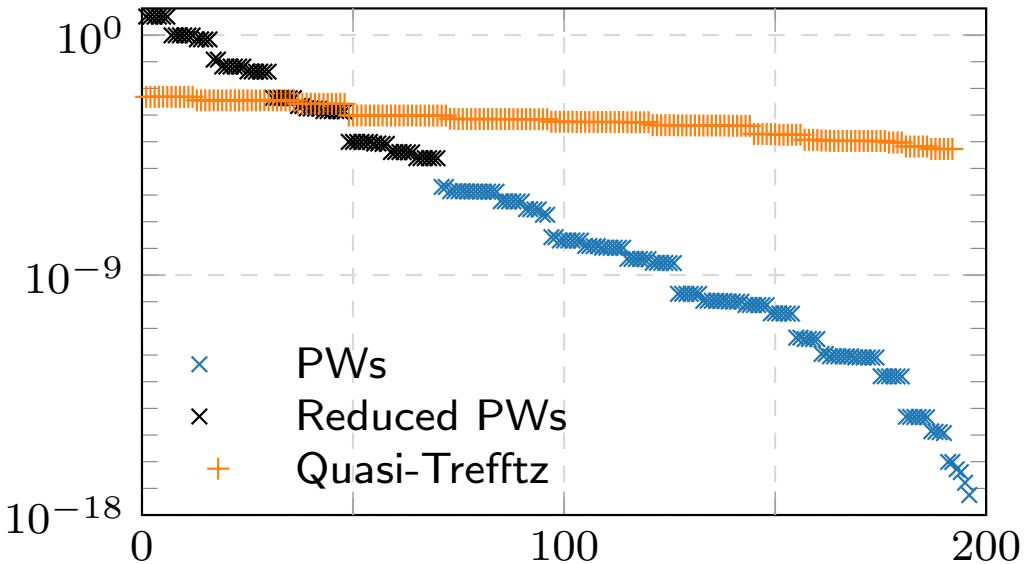

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Approximation properties of the basis for a smooth solution: sum of random PWs

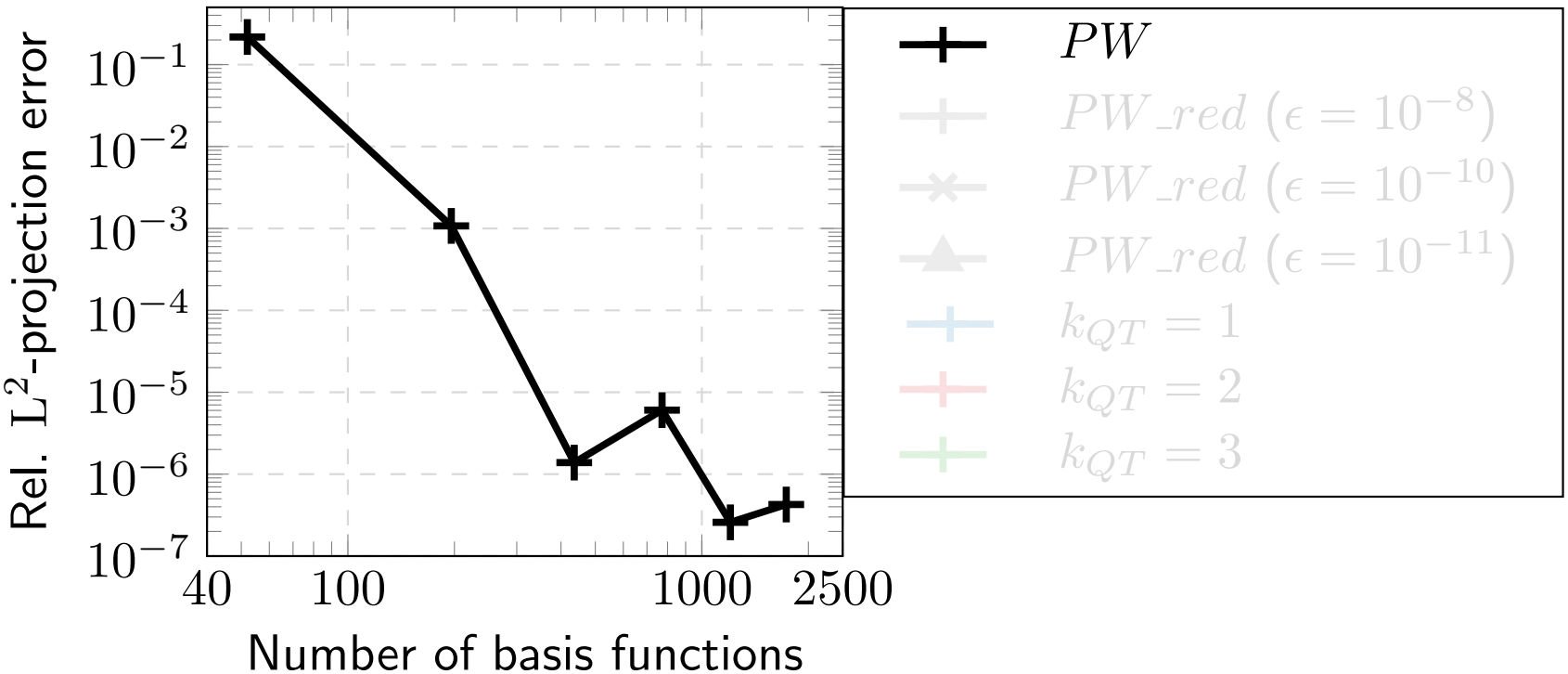

Figure: Projection error on the different basis for a sum of random PWs.

$\Longrightarrow$ Saturation phenomenon due to condition number.

$\Longrightarrow$ **Reduction techniques** avoid conditioning issues, but locked convergence because of rounding pollution error for an asked threshold.

$\Longrightarrow$ Robust approximation by FR basis.

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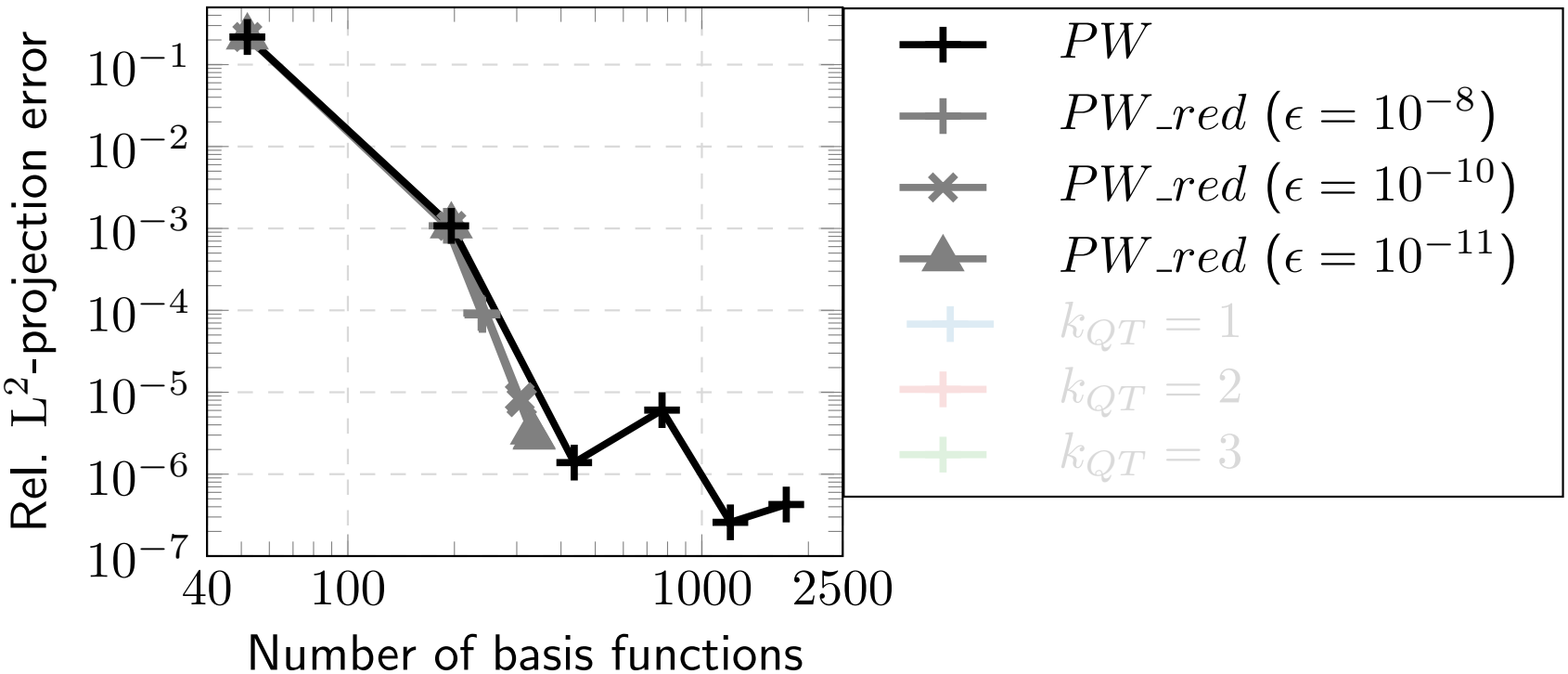

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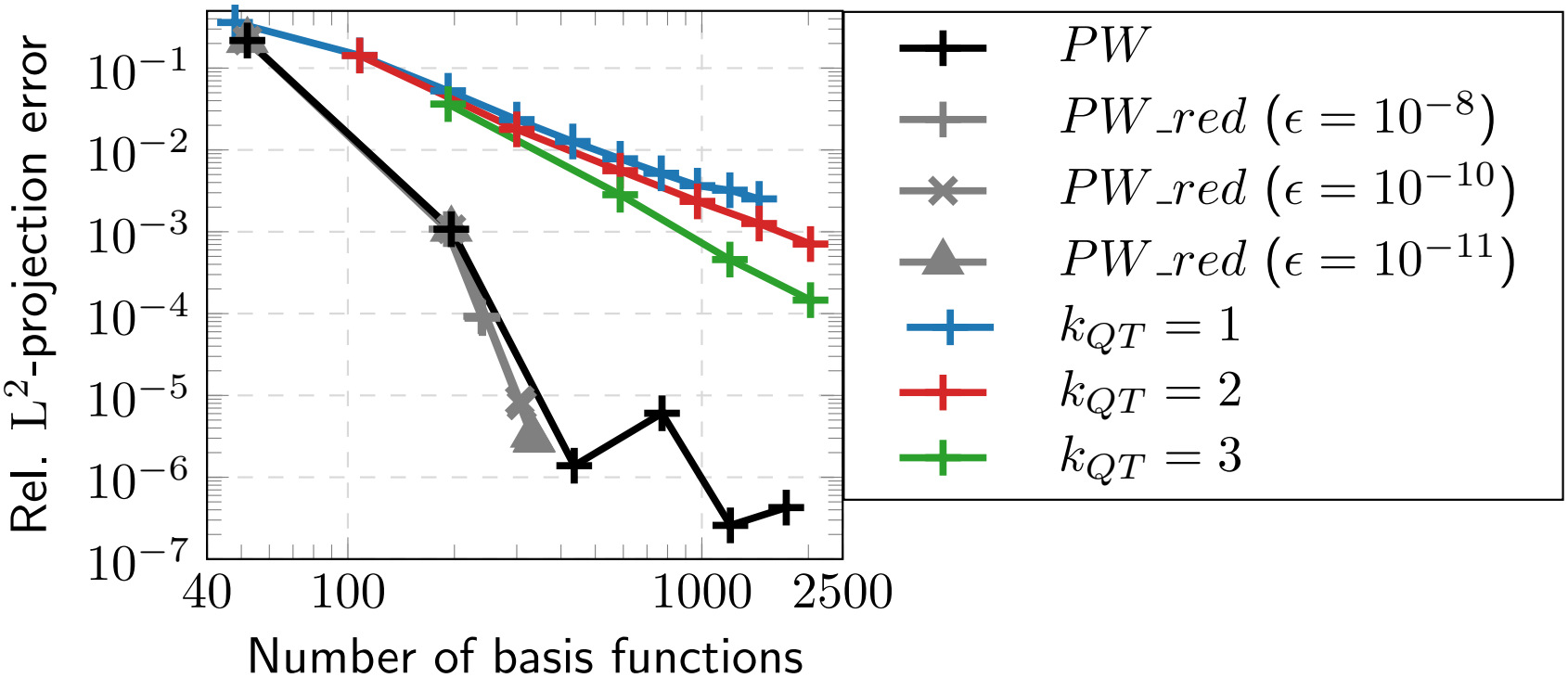

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Approximation properties of the basis for a 'non-smooth' solution: sum of random dipoles

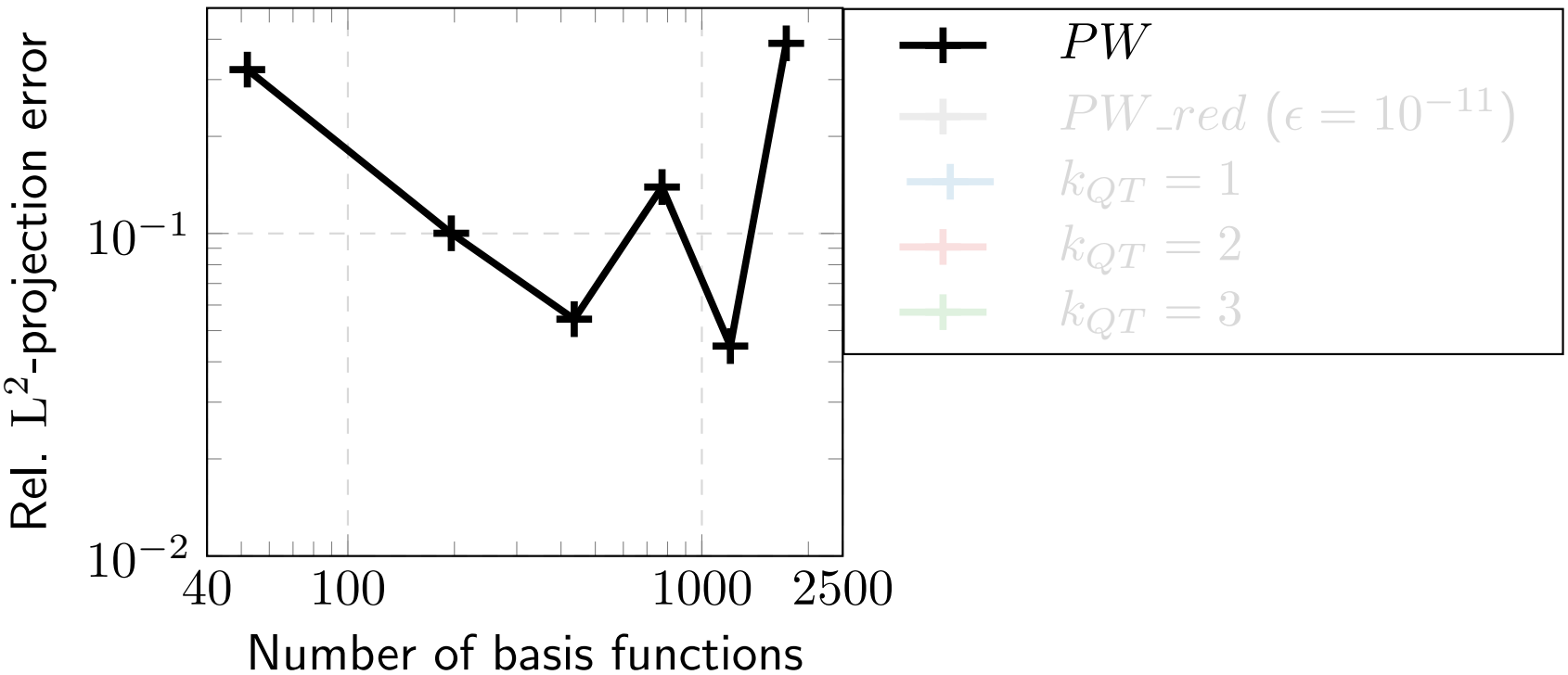

Figure: Projection error on the different basis for a sum of random dipoles.

$\Longrightarrow$ Saturation phenomenon due to condition number: **difficult representation of complex local solutions with PWs**.

$\Longrightarrow$ **Reduction techniques** avoid conditioning issues, but locked convergence because of rounding pollution error for an asked threshold.

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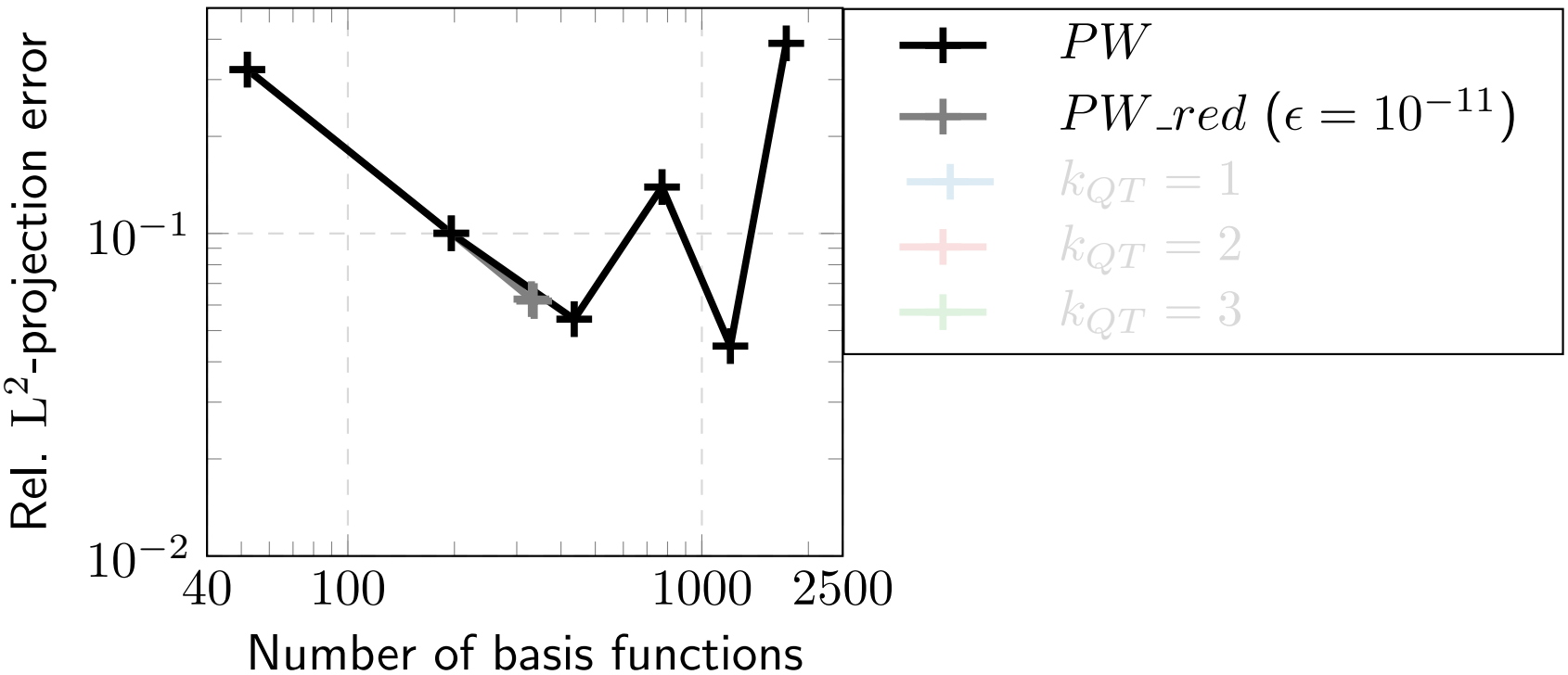
Approximation properties of the basis for a 'non-smooth' solution: sum of random dipoles



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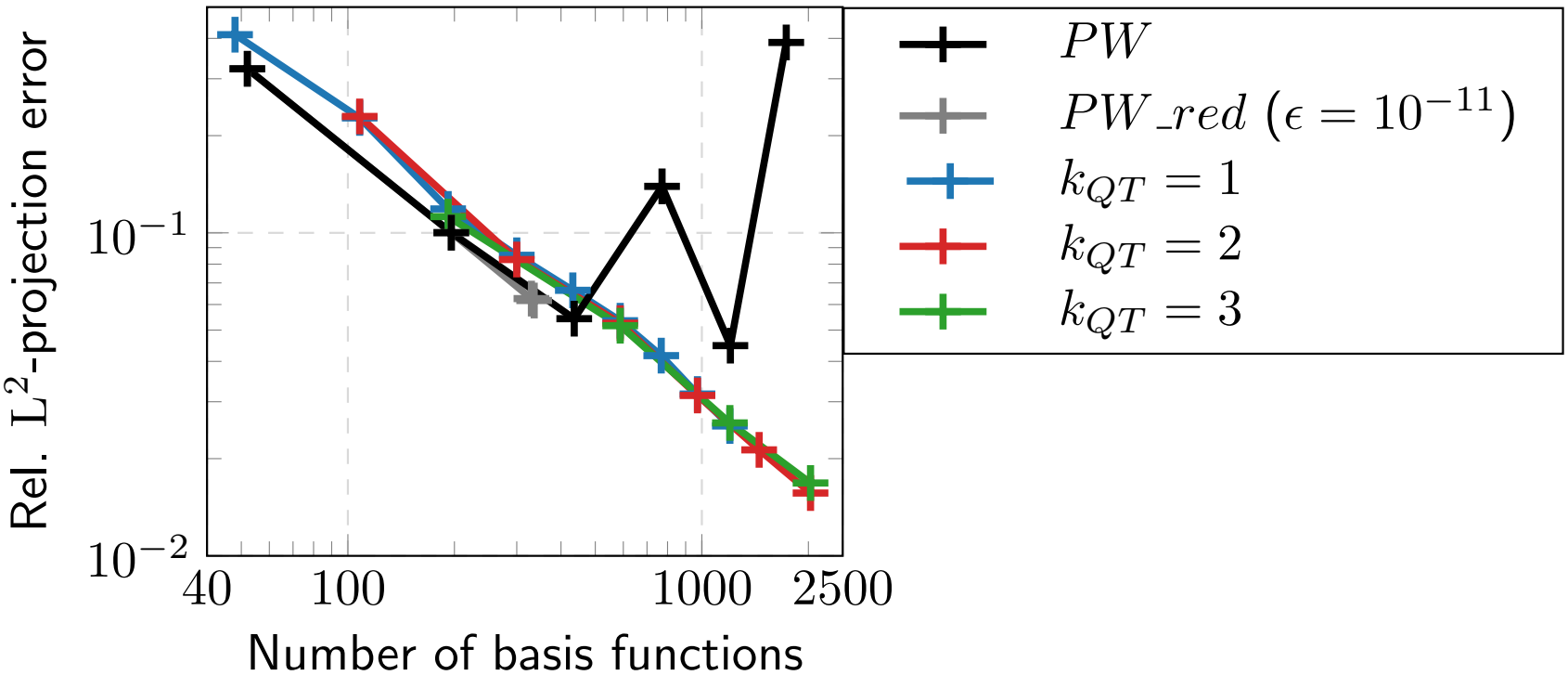

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Recovery of the Trefftz properties under local solver convergence

Trefftz approach rests on the use of exact solutions of the Maxwell equations:

Is using quasi-solutions enough?

Quasi-Trefftz properties at convergence

Hypothesis : the local solver verifies the convergence property for $\gamma_T \mathbf{E}^T + Z_{\partial T} \gamma_\times^T \mathbf{H}^T = g^T$ on ∂T:

$$\|\gamma_t(\mathbf{E}^T - \mathbf{E}_\mathbf{h}^T)\|_{\mathrm{L}^2(\partial T)} + \|\gamma_\times^T(\mathbf{H}^T - \mathbf{H}_\mathbf{h}^T)\|_{\mathrm{L}^2(\partial T)} \leq \varepsilon_{ls}^T \|\mathbf{g}^T\|_{\mathrm{L}^2(\partial T)} \text{ with } \varepsilon_{ls}^T \to 0,$$

when the local solver mesh is refined.

If the local solver mesh is sufficiently refined, the quasi-Trefftz formulation verifies the

- weak-coercivity,
- contraction properties,

of the original Trefftz formulation.

Calibration of the FR auxiliary solver

Trefftz approach rests on the use of exact solutions of the Maxwell equations:

Is using quasi-solutions enough?
How to adapt the FR order to the Trefftz one?

- Uniform Cartesian mesh $\mathcal{T}_H(\Omega)$ of the domain Ω: N **macro-cells** per direction.
- Uniform Cartesian mesh $\mathcal{T}_h(\partial T)$ of ∂T: M **micro-faces** per direction.
- Piecewise polynomial BCs of degree $k_{QT} = 1$.

Figure: Mesh $\mathcal{T}_H(\Omega)$ for $N = 3$.

Figure: Mesh $\mathcal{T}_h(\partial T)$ for $M = 2$ and $k_{QT} = 1$.

$\Longrightarrow$ No theoretical *a priori* error estimates in $\mathrm{L}^2(\Omega)$-norm: **numerical convergence in mesh and number of basis functions** [Fure et al. 2020].

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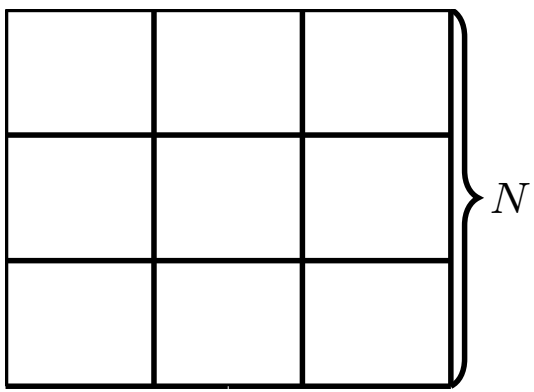

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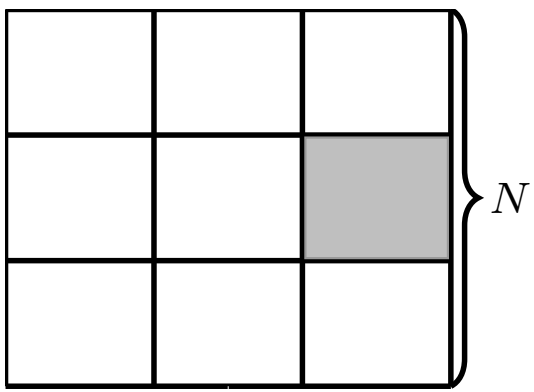

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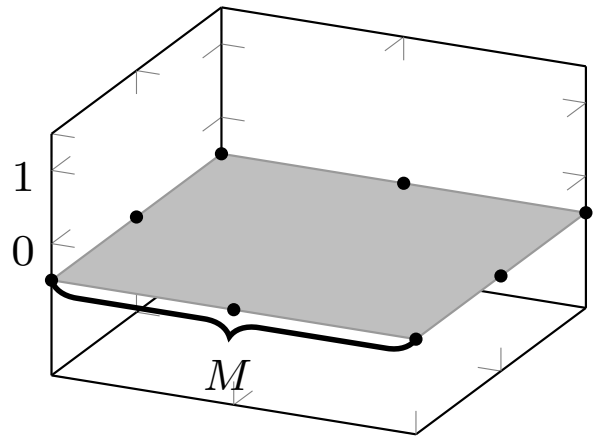

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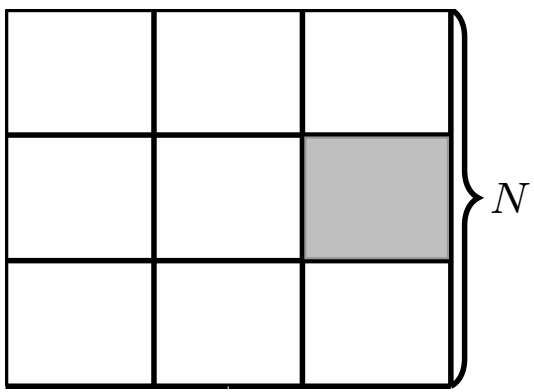

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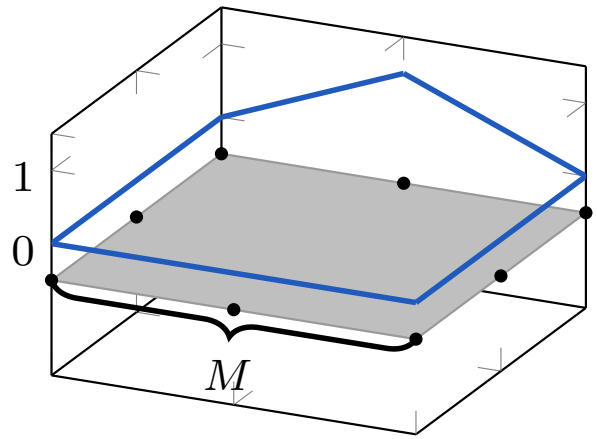

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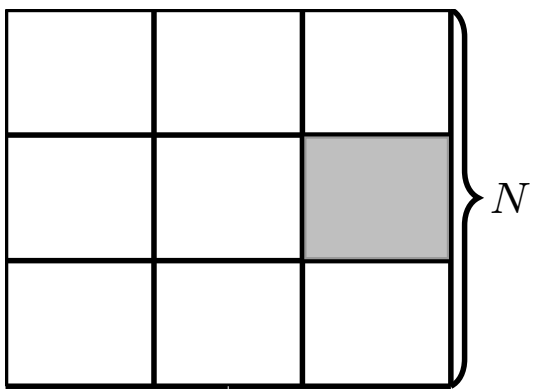

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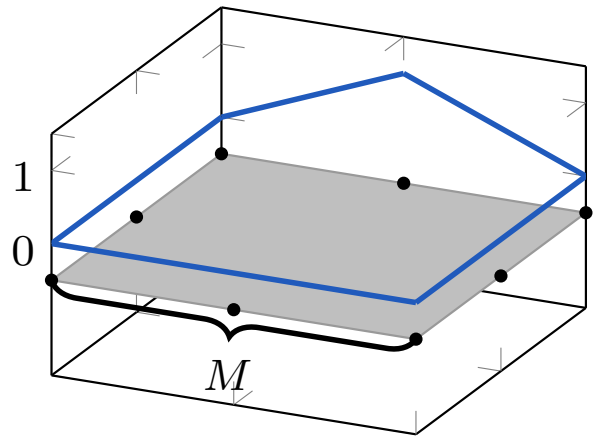

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Calibration of the FR auxiliary solver (2D Helmholtz)

Is using quasi-solutions enough?
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$\Longrightarrow$ Mesh-convergence: refinement of the macro-mesh $\mathcal{T}_H(\Omega)$ for $M = 2$.

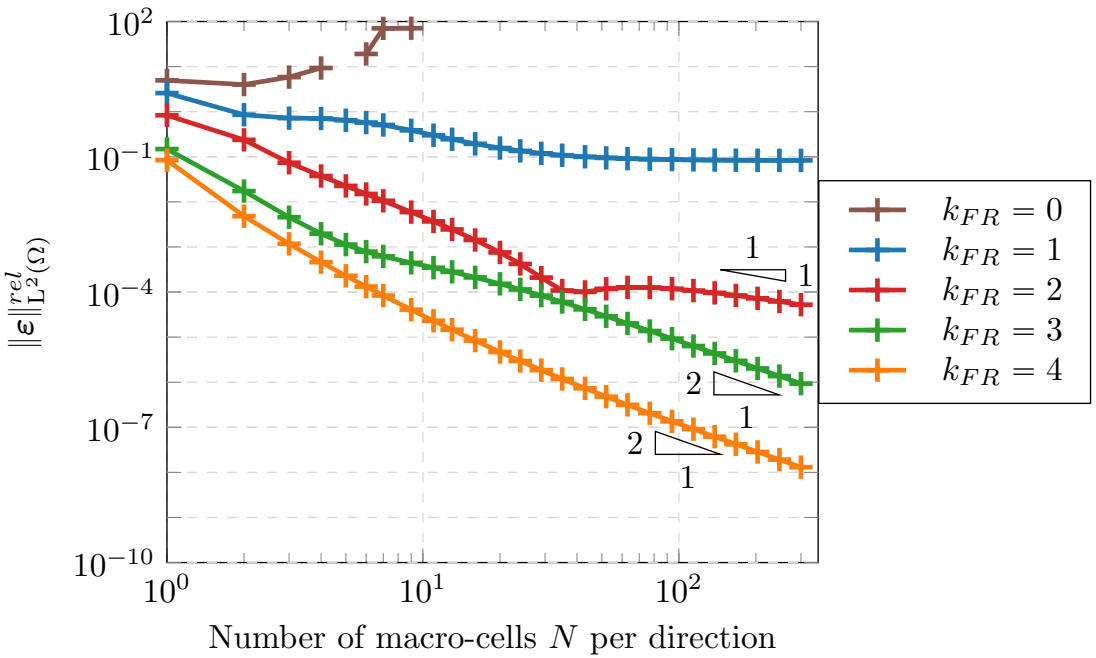

Figure: Evolution of the error for 1 nano-cell per micro-face.

Influence of the local solver precision (through the number of nano-cells):

- Regime 1 ('**imperfect**' local solver): quasi-optimal for $k_{FR} \geq k_{QT} + 2$ and the higher the order and the more refined the nano-mesh is, the later the transition.
- Regime 2 ('**perfect**' local solver): common super-convergence for $k_{FR} \geq k_{QT} + 1$.

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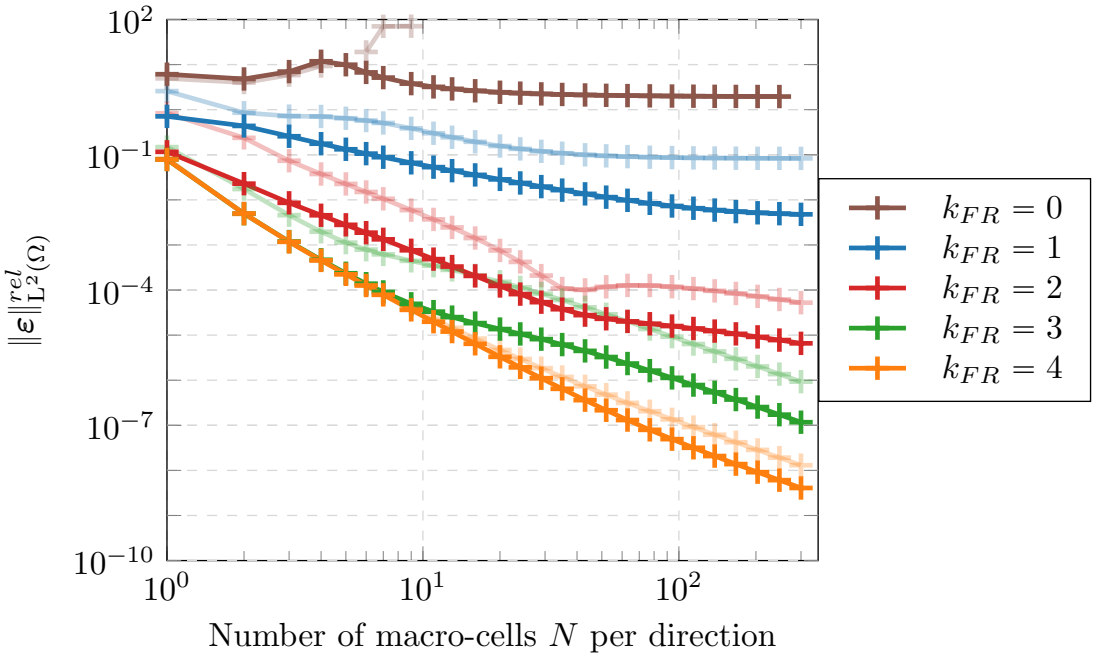

Figure: Evolution of the error for 2 nano-cells per micro-face.

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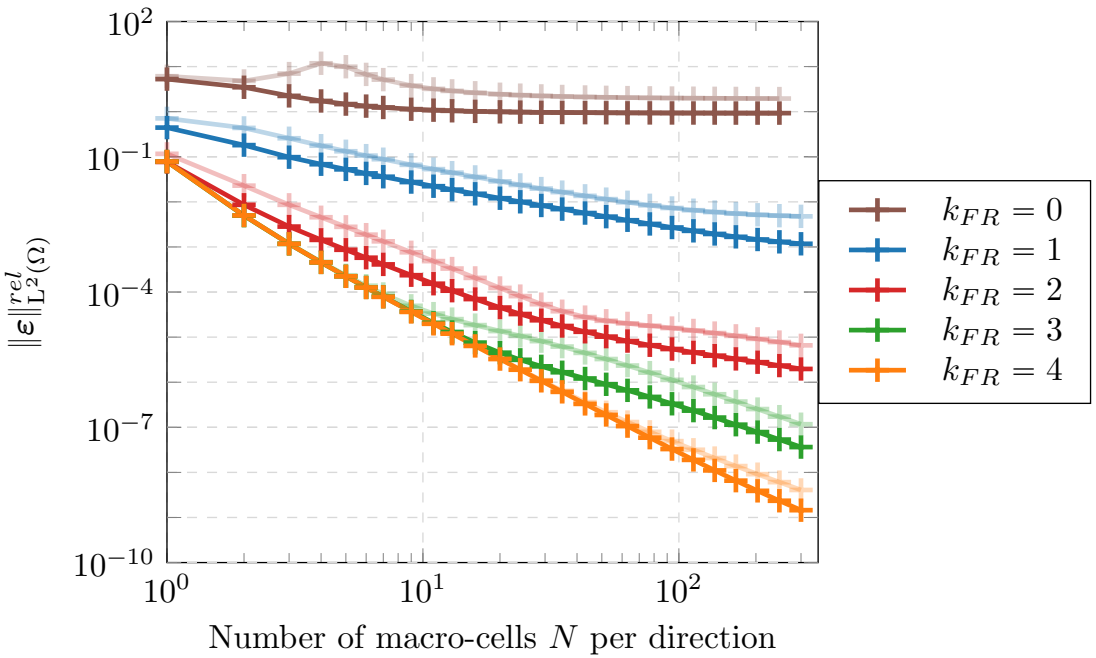

Figure: Evolution of the error for 3 nano-cells per micro-face.

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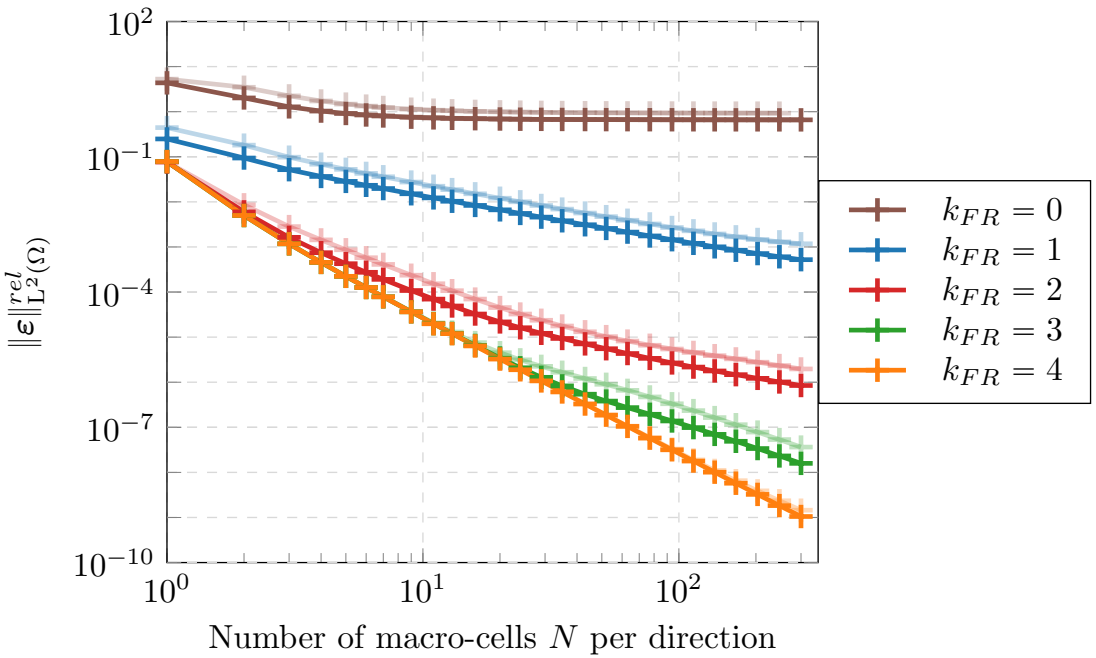

Figure: Evolution of the error for 4 nano-cells per micro-face.

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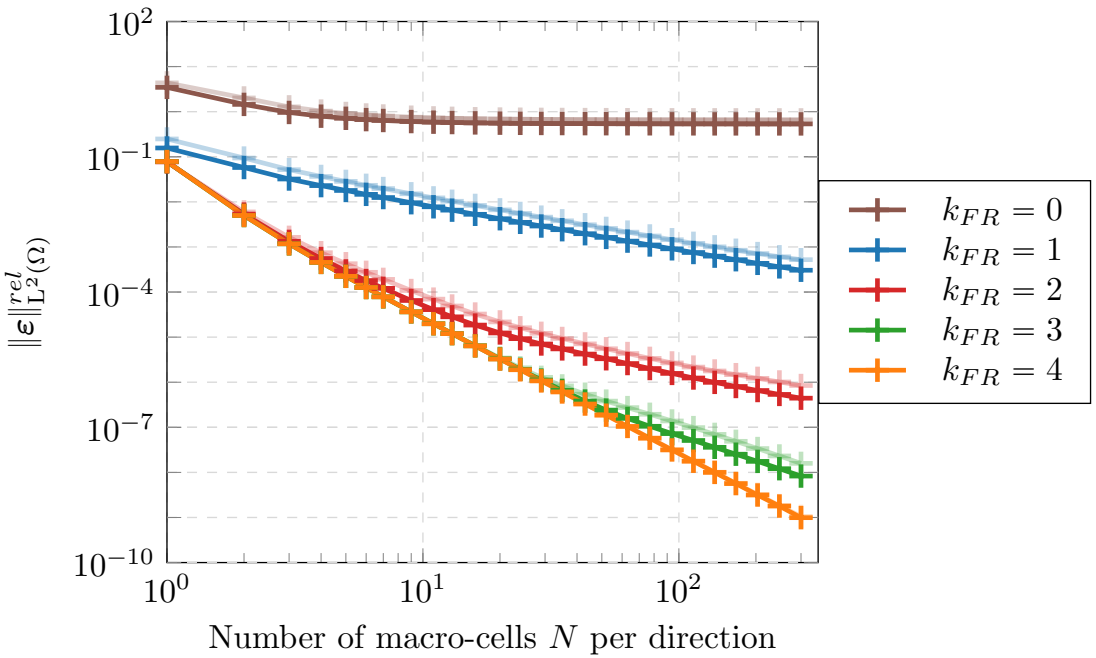

Figure: Evolution of the error for 5 nano-cells per micro-face.

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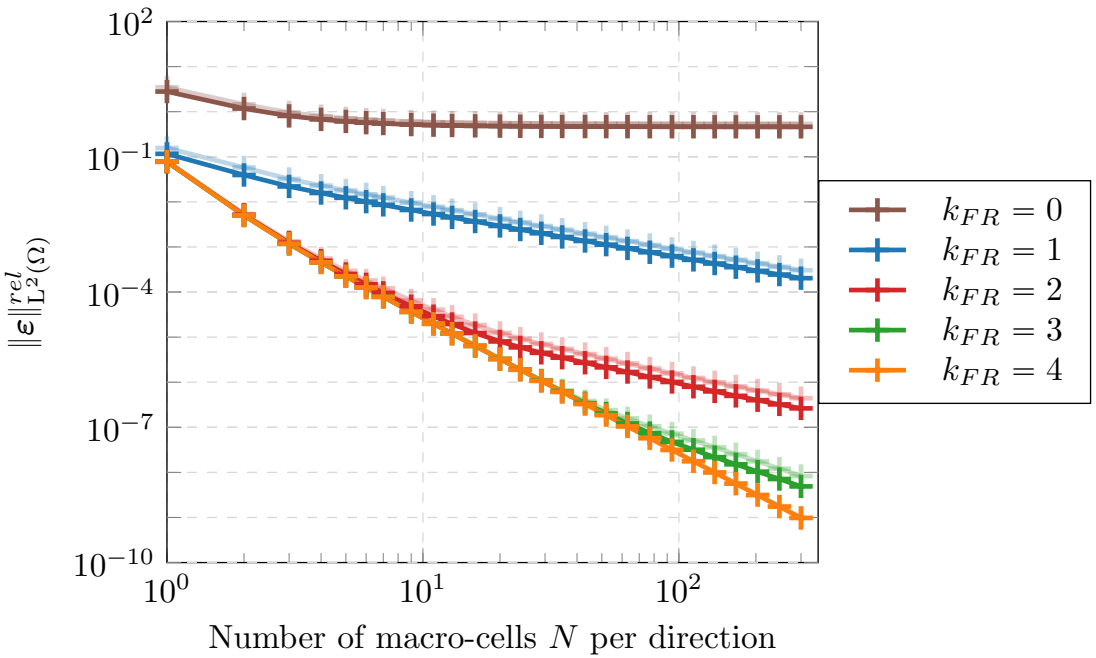

Figure: Evolution of the error for 6 nano-cells per micro-face.

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Calibration of the FR auxiliary solver (2D Helmholtz)

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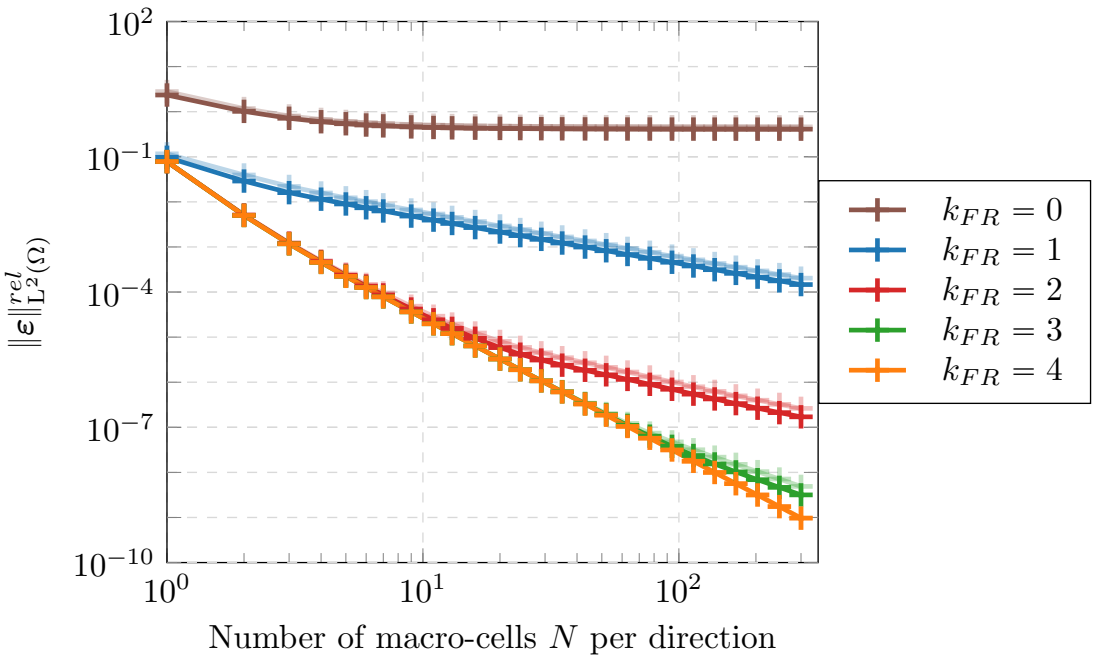

Figure: Evolution of the error for 7 nano-cells per micro-face.

Influence of the local solver precision (through the number of nano-cells):

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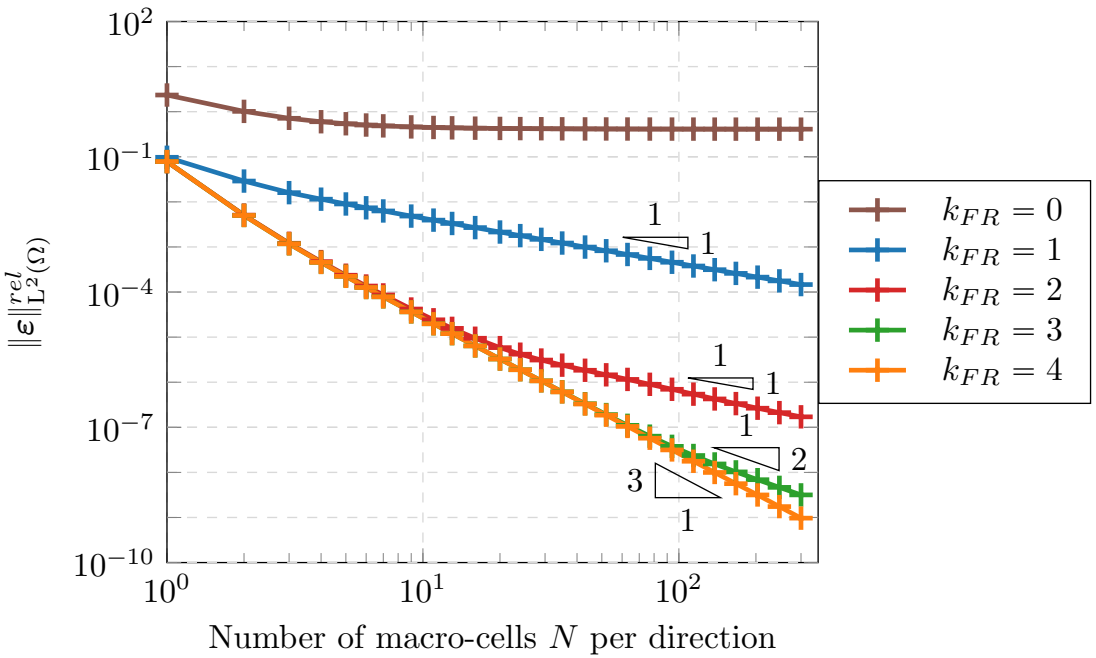

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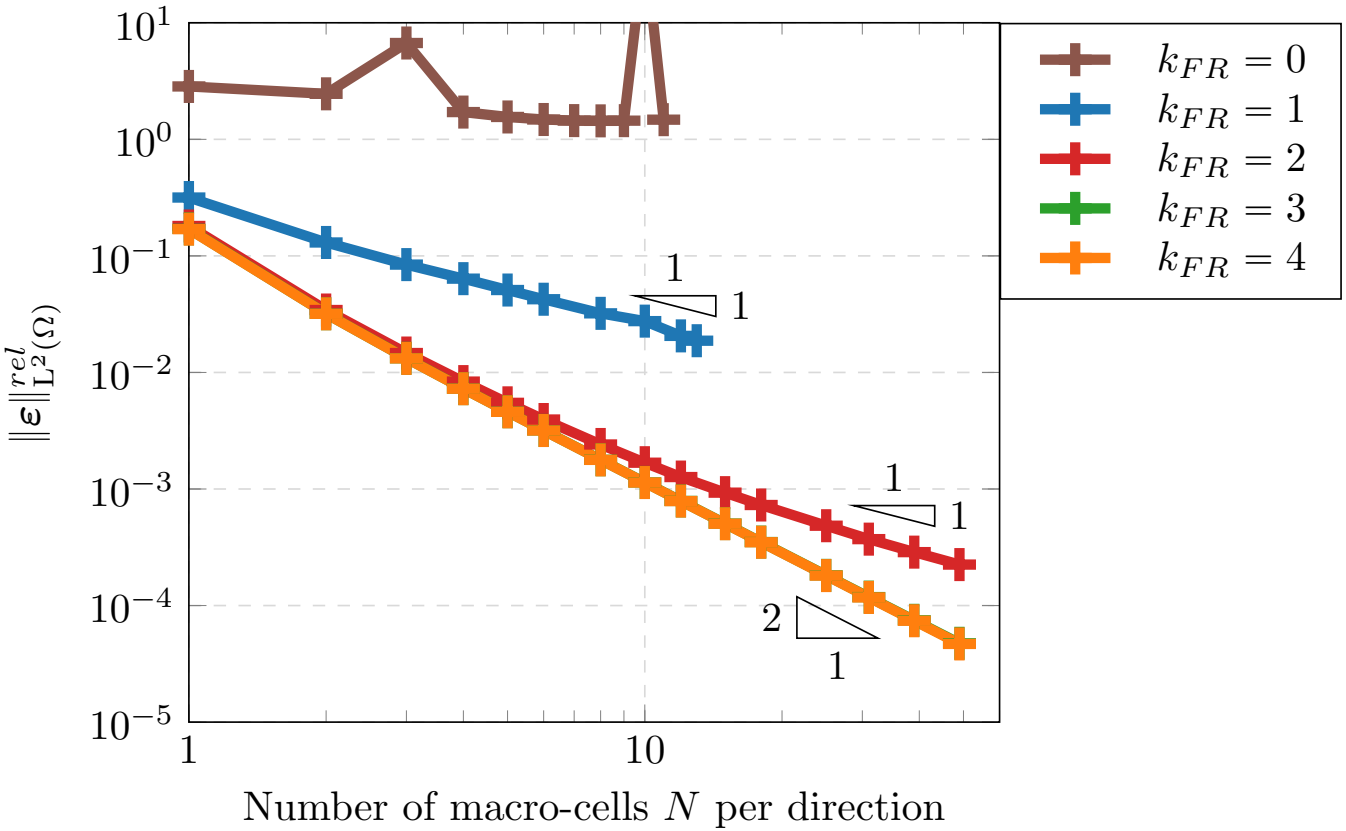
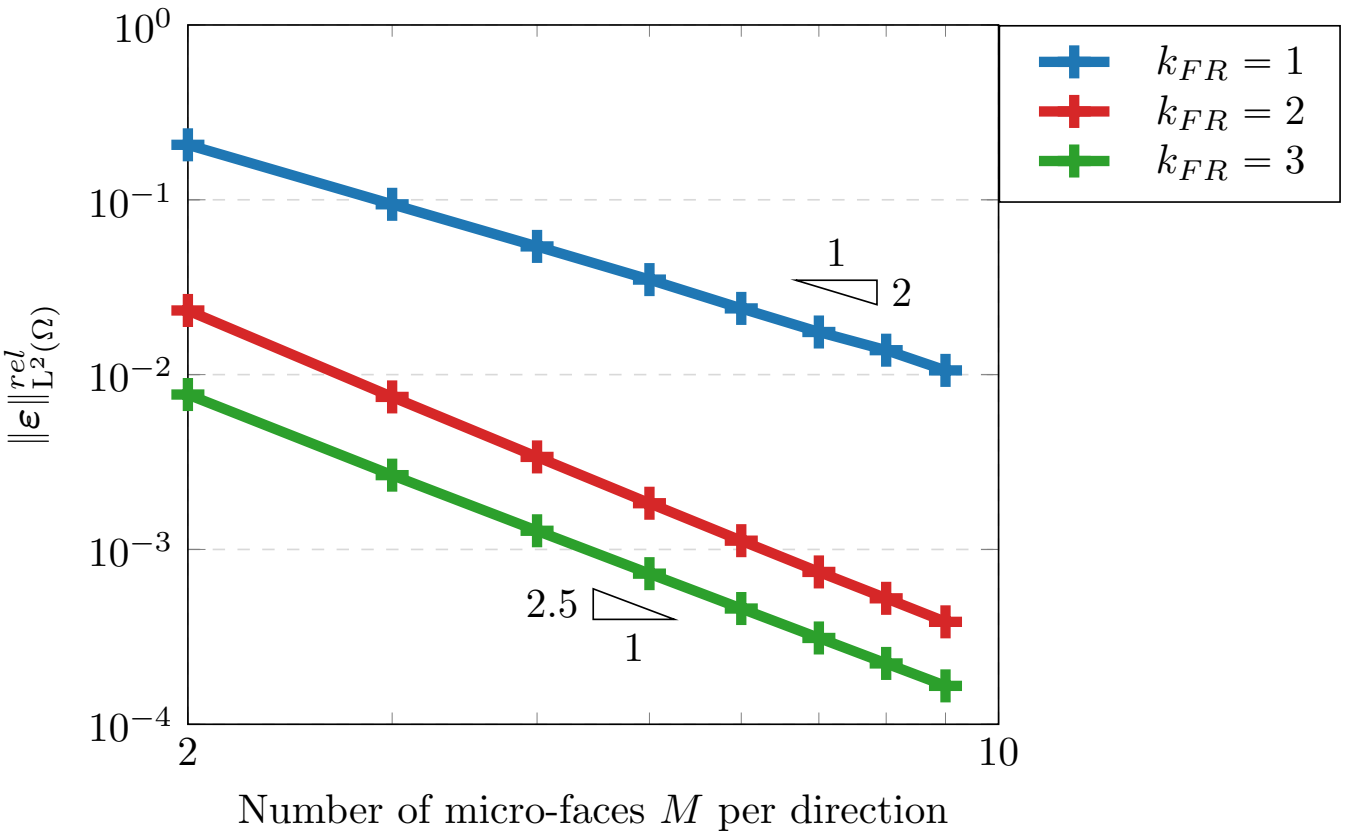


- Need of $k_{FR} \geq k_{QT} + 2$ for quasi-optimal orders, but **no interest in over-resolving**!

Numerical experiments: convergence in the number of basis functions

Is using quasi-solutions enough?
How to adapt the FR order to the Trefftz one?

$\Longrightarrow$ **Local basis enrichment** through refinement of the micro-mesh $\mathcal{T}_h(\partial T)$ for $N = 2$.



- Need of $k_{FR} \geq k_{QT}$ to be **sufficiently 'quasi-solution'**.
- Improvement for $k_{FR} \geq k_{QT} + 1$, but **limited interest in over-resolving**!

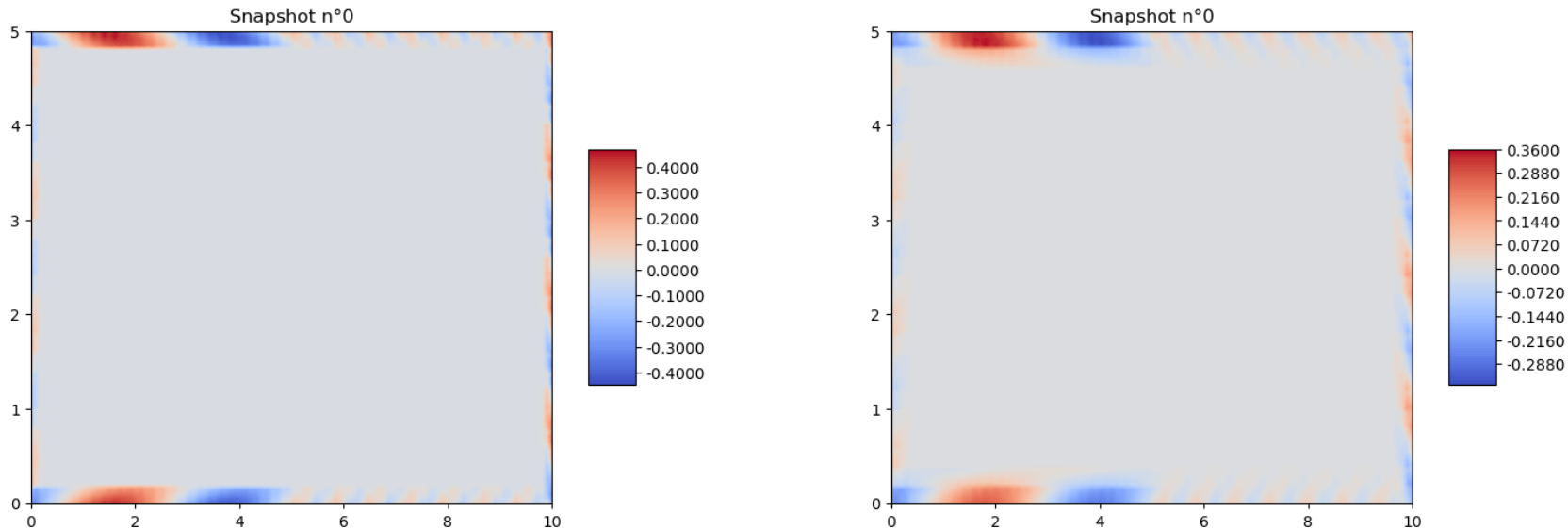
Conclusions on the quasi-Trefftz approach

- Classic numerical schemes are **not well adapted to wide domain simulations**, contrary to the Trefftz method.
- The classic choice of **Plane Waves** leads to limitations: we introduce a Quasi-Trefftz approach, in which a FR solver computes approximate Maxwell solutions associated to polynomial BCs in each cell.
- Good numerical independence properties of the basis functions.
- Robust approximation properties of the basis, even for complex local solutions.
- Numerical calibration of the local solver: $k_{FR} \geq k_{QT} + 2$ for asymptotic quasi-optimality, but **no need to over-resolve**!
- **2 convergence regimes** according to the local solver resolution: as refined as possible!

Improvement of the transfer of information: DtN approximation and local optimisation

Problematic: reduce the number of iterations to convergence

Iterative solution of the linear system thanks to a Krylov method, as GMRES: improving the information transfer would allow to **reduce the number of iterations to convergence**.



(i) Solution evolution (ii) Solution modification

Figure: Iterative convergence: information propagation through iterations.

Trefftz formulation generalisation

Principle of the proposed generalisation for the 2D Helmholtz equation:

- Introduction of the outgoing trace from a cell T ($\mathcal{O}^T = i\kappa Id$ classically) :

$$\gamma_{out}^T \mathbf{y}^T = -v^T + \mathcal{O}^T\left(u^T\right) \quad \text{avec} \quad v^T = \frac{\kappa}{\kappa_T}\partial_{\mathbf{n}_T} u^T.$$

- Consistent numerical traces $\widehat{u}$ and $\widehat{v^T}$ defined from $\gamma_{out}^T \mathbf{y}^T$ and $\gamma_{out}^K \mathbf{y}^K$.

$\Longrightarrow$ Need to have $\mathcal{O}^T$ in a **set of operators** $\mathcal{C}$ **which ensures the formulation conserves classic properties** (coercivity and contraction):

$$\mathcal{C} \subset \{\mathcal{O}, \text{ there exists a definite linear operator } \Lambda,\ \mathcal{O} = i\Lambda\Lambda^* \textbf{ for each face}\}.$$

Approximation of the Dirichlet-to-Neumann operator

Trefftz method can be seen as the variational formulation of

$$v^T + \mathcal{O}^K(u^T) = -v^K + \mathcal{O}^K(u^K) \quad \text{on} \quad \partial T \cap \partial K.$$

The theory of Domain Decomposition Methods (DDM) ensures the **optimal** operator $\mathcal{O}^T$ is the **exterior Dirichlet-to-Neumann** (DtN) one.

3 approximation types :

- Exact operator for classic outgoing waves with respect to the cell centre $\mathbf{x}_0$:

$$\mathcal{O}^T = i\kappa \mathbf{d} \cdot \mathbf{n}_{\partial\Omega} \text{ (directive PWs)} \quad \text{and} \quad \mathcal{O}^T = -\kappa \frac{\mathrm{H}_1^{(1)}(\kappa r)}{\mathrm{H}_0^{(1)}(\kappa r)} \text{ (Green kernel)},$$

  for $\mathbf{d} = (\mathbf{x} - \mathbf{x}_0)/r$ and $r = |\mathbf{x} - \mathbf{x}_0|$.
- Operator with same principal symbol as the classic approximation [El Bouajaji et al. 2014]

$$\mathcal{O}^T = i\kappa \sqrt{Id + \frac{\Delta_F}{\kappa^2}},$$

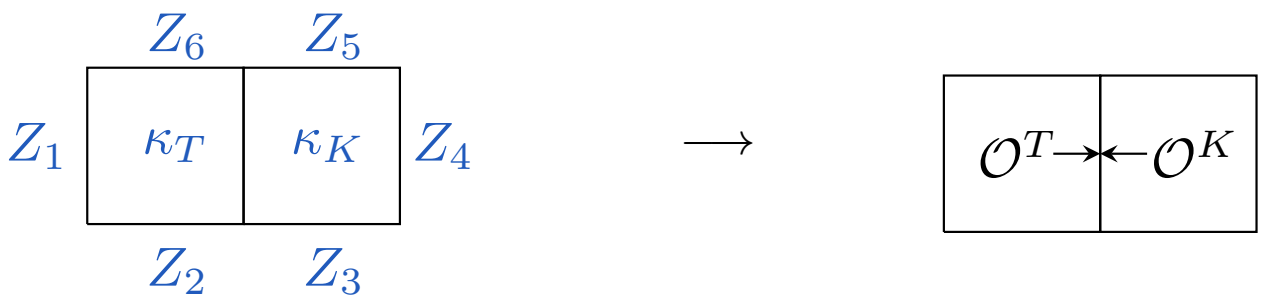
  as Padé-type approximations [Després et al. 2021]

$$\mathcal{O}^T = i\kappa \left(Id - \frac{\Delta_F}{2\kappa^2} \right)^{-1}.$$

  → Face point of view: BCs for the Laplace-Beltrami operator?
- Approximation thanks to neural networks (NN): allows to enforce $\mathcal{O}^T \in \mathcal{C}$.

Local optimisation

Being given **local parameters** P, we want to **optimise** the operators $\mathcal{O}^T$ and $\mathcal{O}^K$ with respect to an information transfer measure ρ_P.

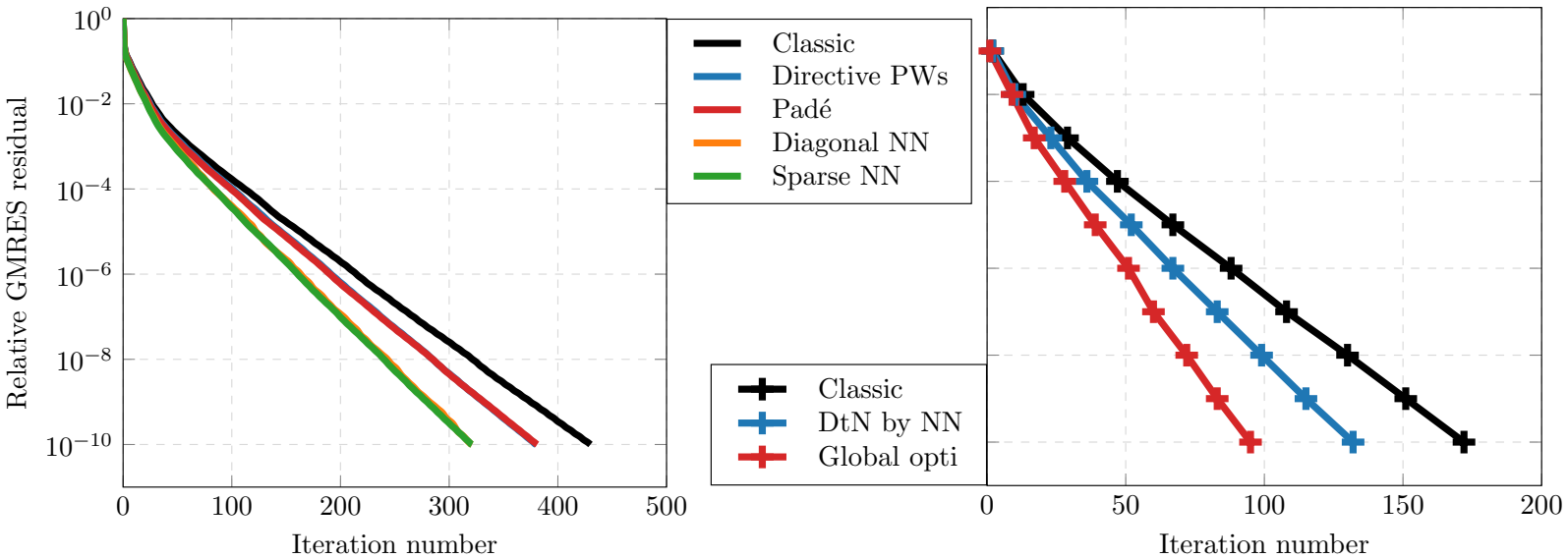


$$\mathcal{O}_P^{T,*}, \mathcal{O}_P^{K,*} = \underset{\mathcal{O}^T, \mathcal{O}^K \in \mathcal{C}}{\operatorname{argmin}} \rho_P(\mathcal{O}^T, \mathcal{O}^K).$$

Different problematics:

- Size of the local parameters P.
- Possibility to develop **interpolation approximations** of ρ_P for efficient optimisation.
- Which measure ρ_P ?
  - Reduce the distance of the eigenvalues to 1 (fixed point vision).
  - Increase the distance of the eigenvalues to 0 (GMRES point of view).
- **Does local optimisation lead to a similar behaviour at the global level?**

Preliminary results and conclusion



(i) DtN approximations. (ii) Optimisation and NN.

Figure: GMRES convergence for different generalisations: analytic, NN-based and optimisation-based.

- We proposed a generalisation of the Trefftz method, thanks to general numerical fluxes **keeping the original properties of the classic formulation**.
- Trefftz approach can be interpreted as a DDM one: we introduced approximations of the Dirichlet-to-Neumann operator, which is supposed to be optimal.
- Introduction of local optimisation problems for the operators, depending on the surrounding parameters.
- Promising preliminary results to reduce the number of iterations to convergence: **about 25% reduction by using NNs**.

Bibliography

[Sirdey 2022] M. Sirdey. 'Méthode itérative de Trefftz pour la simulation d'ondes électromagnétiques en trois dimensions', PHD, Université de Pau et des Pays de l'Adour, 2022.

[Yee 1966] Kane Yee. 'Numerical solution of initial boundary value problems involving maxwell's equations in isotropic media', IEEE Transactions on Antennas and Propagation, 14(3):302–307, 1966.

[Nédélec 1980] J. C. Nedelec, 'Mixed finite elements in $\mathbb{R}^3$', Numer. Math., vol. 35, no. 3, pp. 315–341, 1980.

[Fezoui et al. 2005] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno, 'Convergence and stability of a discontinuous Galerkin time-domain method for the 3D heterogeneous Maxwell equations on unstructured meshes', ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique, vol. 39, no. 6, pp. 1149–1176, 2005.

[Cessenat, Després 1998] O. Cessenat, B. Després. 'Application of an Ultra Weak Variational Formulation of Elliptic PDEs to the Two-Dimensional Helmholtz Problem', SIAM Journal on Numerical Analysis, vol. 35, pp. 255-299, 1998.

[Ihlenburg, Babuška 1995] F. Ihlenburg, I. Babuška. 'Finite element solution of the Helmholtz equation with high wave number Part I : The h-version of the FEM', Computers & Mathematics with Applications, vol. 30, pp. 9-37, 1995.

[Congreve et al. 2019] S. Congreve, J. Gedicke, I. Perugia. 'Numerical Investigation of the Conditioning for Plane Wave Discontinuous Galerkin Methods', in Numerical Mathematics and Advanced Applications ENUMATH 2017, pp. 493-500, 2019, Springer, Cham.

[Barucq et al. 2021] H. Barucq, A. Bendali, J. Diaz, and S. Tordeux, 'Local strategies for improving the conditioning of the plane-wave Ultra-Weak Variational Formulation', Journal of Computational Physics, vol. 441, p. 110449, 2021.

[Fure et al. 2020] H. Fure, S. Pernet, M. Sirdey, S. Tordeux, 'A discontinuous Galerkin Trefftz type method for solving the two dimensional Maxwell equations', SN Partial Differential Equations and Applications, vol. 1, pp. 1-25, 2020.

[Imbert-Gérard, Després 2013] L.-M. Imbert-Gérard, B. Després. 'A generalized plane wave numerical method for smooth non constant coefflcients'. IMA Journal of Numerical Analysis 34, 2024.

[Imbert-Gérard, Sylvand 2023] L.-M. Imbert-Gérard, G. Sylvand. 'Three types of quasi-Trefftz functions for the 3D convected Helmholtz equation: construction and approximation properties'. arXiv preprint, 2023.

[Lehrenfeld, Stocker 2023] C. Lehrenfeld and P. Stocker, 'Embedded Trefftz discontinuous Galerkin methods', International Journal for Numerical Methods in Engineering, vol. 124, no. 17, pp. 3637–3661, 2023.

[Barucq et al. 2017] H. Barucq, A. Bendali, M. Fares, V. Mattesi and S. Tordeux, 'A symmetric Trefftz-DG formulation based on a local boundary element method for the solution of the Helmholtz equation'. Journal of Computational Physics, Vol. 330, pp. 1069-92, 2017.

[Huynh 2007] H.T. Huynh, 'A Flux Reconstruction Approach to High-Order Schemes Including Discontinuous Galerkin Methods'. AIAA Paper 2007-4079, pp. 1-42, 2007.

[Rivet et al. 2023] M. Rivet, S. Pernet, S. Tordeux, 'Flux reconstruction method for time-harmonic linear propagation problems: 1D a priori error analysis'. HAL preprint, 2023.

[Rivet et al. 2024] M. Rivet, S. Pernet, S. Tordeux, 'Optimised Correction Polynomial Functions for the Flux Reconstruction Method in Time-Harmonic Electromagnetism'. Applied Mathematics Letters, 2024.

[El Bouajaji et al. 2014] M. El Bouajaji, X. Antoine, C. Geuzaine, 'Approximate local magnetic-to-electric surface operators for time-harmonic Maxwell's equationsm'. Journal of Computational Physics, 2014.

[Després et al. 2021] B. Després, A. Nicolopoulos, B. Thierry, 'Corners and stable optimized domain decomposition methods for the Helmholtz problem'. Numerische Mathematik, 2021.