Sliding Mode Controllers Design based on Control Lyapunov functions for uncertain LTI systems

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Abstract: In this paper, a design method of sliding controllers based on control Lyapunov functions for uncertain LTI systems is proposed. Systems with matched non-vanishing disturbances, unmatched uncertainties and uncertain control coefficient matrix are considered. It is shown that whenever a robust nonlinear controller exists that renders the system quadratically stabilizable, in the absence of matched non vanishing disturbances, there is also an appropriate First-Order Sliding-Mode controller stabilizing the origin, that also renders the equilibrium point robustly stable in presence of perturbations. This shows that Sliding-Mode controllers are not more restrictive than other controllers, at least for the class of uncertain systems considered. Moreover, the design based on control Lyapunov functions provides a natural sliding surface, having an asymptotically stable sliding dynamics, so that with the corresponding sliding mode controller ensures global asymptotic stability of the uncertain system. The efficiency of the proposed SMC control is validated experimentally on a Furuta pendulum benchmark.

Keywords: robust control design, sliding mode control, Lyapunov methods, uncertain systems

1. INTRODUCTION

Nowadays, one of the main subjects of modern control theory is the problem of robustness against uncertainties and disturbances. Sliding Mode Controllers (SMC) have shown their efficiency in rejection of matched perturbations (Utkin, 1992; Edwards and Spurgeon, 1998; Ferrara et al., 2019; Shtessel et al., 2014).

Conventional sliding mode design consists in two steps. The first step is the selection of a sliding surface, leading to a *reduced* order system with the desired convergence properties. In the second step a discontinuous controller is designed, ensuring that the trajectories of the system are brought, in finite-time, to the sliding surface and are kept there despite of the uncertainties and disturbances.

Although there exists methodologies to design a sliding set for First-Order SMC, with some optimal regulator properties (Young et al., 1977; Edwards and Spurgeon, 1998), and to the case of Higher-Order Sliding-Mode controllers proposed in Castillo et al. (2016), such designs are unrelated to the (nominal) system. Ackermann and Utkin (1994) have designed the sliding dynamics without state transformations based on Ackermann's formula and for the MIMO case some sliding set designs are proposed in Peruničić-Draženović et al. (2016). An approach to sliding set design for systems with matched and unmatched perturbations based on H_∞ is presented by Choi (2003),Choi (1999) and for integral sliding mode in Alwi et al. (2011), Rubagotti et al. (2011) and Andrade-Da Silva et al. (2009).

In robust control, the *min-max* control (Gutman and Leitmann, 1976), later called *Lyapunov Redesign* by Khalil (2001), considers a nominal *stable* system, then, using the Lyapunov function of the nominal system, an extra discontinuous control term is added in order to compensate for the matched perturbations. With this robustifying control law, it is ensured that the nominal Lyapunov function remains a Lyapunov function for the system with the matched perturbations (Gutman, 1979; Leitmann, 1979). To attain this, the extra control term is designed such that the effect of the perturbation in the derivative of the Lyapunov function is compensated.

The concept of Control Lypunov function (CLF) was introduced by Artstein (1983) and Sontag (1983), giving a characterization of the stabilizability of dynamical systems. CLFs introduced a *universal* tool to stabilize a dynamical system, since if there exists a stabilizing control law, then there exists a CLF to construct it (Sontag, 1989). A good example on how CLFs have been exploited for control design is the so-called *Backstepping* technique (see Kokotović and Arcak (2001) and references therein). Moreover, the use of CLF is related to concepts of inverse optimality (Freeman and Kokotovic, 2008) and passivity (Byrnes et al., 1991). Therefore, it is convenient to have a characterization for SMC in terms of CLFs, an issue that seems not to have been done in the literature.

The concept of quadratic Control Lyapunov Functions has been used in e.g. Rotea and Khargonekar (1989) and Khargonekar et al. (1990) to fully characterize the class of linear uncertain systems for which robust stabilization is achievable by means of continuous time-varying nonlinear feedback control laws. It is shown to be equivalent to the existence of a quadratic Control Lyapunov Function (see below Section 2). An important question in this context is if it is always possible to find a Sliding-Mode Controller to stabilize the system, whenever there is a continuous controller. The main objective of this paper is to provide a positive answer to this question. We consider the problem of sliding set design based on CLF for LTI, in the presence of unmatched uncertainties, uncertain control coefficient and matched non-vanishing disturbances. For the class of LTI uncertain systems which are quadratically stabilizable (without matched disturbances) via a continuous, time-varying nonlinear control law, it is shown that it is always possible to design a First-Order Sliding-Mode controller, with a linear sliding variable, achieving the stabilization even in presence of matched disturbances, based on the corresponding CLF. Moreover, the Sliding-Mode Controller provides insensitivity to matched non vanishing disturbances, that cannot be compensated by the continuous controller. The main advantages of such a design are:

- For the considered class of uncertain systems, necessary and sufficient conditions for the existence of a sliding set allowing to compensate for uncertain control coefficient, uncertain dynamic matrix and non vanishing matched perturbations are given.
- In contrast to the classical SMC design (Utkin, 1992; Edwards and Spurgeon, 1998; Shtessel et al., 2014), a transformation into the regular form is not needed.

The main message of this communication can be given as follows: *The knowledge of a CLF for an uncertain LTI system allows to generate a stable sliding set, and consequently design a stabilizing sliding mode controller, compensating for non-vanishing matched perturbations, matched and unmatched in the system matrix and control gain.*

The structure of the paper is as follows: In Section 2, the main result, the selection of a sliding surface based on a CLF, is presented. Section 3 presents a unit control law based on the proposed sliding variable. The advantages of the suggested control design are illustrated with experimental results for the Furuta pendulum in Section 4.

Notation Let $\mathbb{R}_+$ be the set of all positive real numbers. For any real value $y \in \mathbb{R}$, the function $\text{sign}(y) = y/|y|$ for $y \neq 0$ and $\text{sign}(0) \in [-1, 1]$. Let $\mathbb{I}$ be the identity matrix. $\|x\|$, $|w|$ are the euclidean norm and the absolute value of x and w, respectively. Solutions of differential equations with discontinuous right-hand side are understood in Filippov's sense (Filippov, 1988).

2. CLF BASED SLIDING SET DESIGN

For the design of a sliding-mode control, consider the uncertain LTI system (Rotea and Khargonekar, 1989):

$$\dot{x} =(A + D\Gamma(t)E)x + B\left[(\mathbb{I} + \Delta_B(t))u + \delta(t, x)\right], \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, A, B, D and E are known matrices of proper dimensions. The function $\Gamma(t) \in \mathbb{R}^{k\times q}$ is an uncertainty in the system matrix, Δ_B is the uncertainty in the control input and $\delta(t, x)$ are matched disturbances. Suppose that the following assumptions are fulfilled:

A1 The uncertainty in matrix A satisfies $\|\Gamma(t)\| \leq 1$.

A2 The uncertainty in the control matrix satisfies that for all $t \in \mathbb{R}_+$

$$\left\|B^T PB\, \Delta_B(t) \left(B^T PB\right)^{-1}\right\| \leq \epsilon < 1\,, \quad (2)$$

where P is a symmetric positive definite matrix that will be defined below.

A3 For all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exists a known non-negative continuous function $\rho(t, x)$ such that

$$\|\delta(t, x)\| \leq \rho(t, x)\,. \quad (3)$$

2.1 Quadratic stabilizability and CLF

In (Rotea and Khargonekar, 1989; Khargonekar et al., 1990) system (1), without the matched perturbations, i.e.

$$\dot{x}(t) = (A + D\Gamma(t)E)x(t) + Bu(t) \quad (4)$$

is considered. System (4) is said to be *quadratically stabilizable via linear state feedback* if there exists a state feedback control $u = -Kx$ such that the closed-loop system is asymptotically stable for any admissible uncertainty $\Gamma(t)$, and it exists a quadratic Lyapunov function $V(x) = x^T Px$ assuring the stability, i.e. it exists $\alpha > 0$ such that

$$\begin{aligned}\frac{dV(x)}{dt} &= x^T \left[(A + D\Gamma(t)E)^T P + P(A + D\Gamma(t)E)\right] x \\ &\quad - 2x^T PBKx \leq -\alpha\|x(t)\|^2\,.\end{aligned} \quad (5)$$

In Rotea and Khargonekar (1989) it is shown that quadratically stabilizability via linear state feedback of system (4) is equivalent to the existence of a *quadratic control Lyapunov function (CLF)* $V(x) = x^T Px$, that is, for all $x \in \mathbb{R}^n$, $x \neq 0$, such that $B^T Px = 0$, then

$$x^T \left[(A + D\Gamma(t)E))^T P + P(A + D\Gamma(t)E))\right] x < 0. \quad (6)$$

This fact is already mentioned in Barmish (1985) while obtaining the necessary and sufficient conditions for quadratic stabilizability. Moreover, Rotea and Khargonekar (1989) have also shown that if system (4) is quadratically stabilizable via a nonlinear dynamic state feedback then it is also quadratic stabilizable via a linear nondynamic state feedback.

2.2 Characterization of all CLF's

The following Lemma characterizes all CLF for system (4).

Lemma 1. (Khargonekar et al. (1990)). The uncertain system (4) is quadratically stabilizable if and only if there exists $\mu > 0$ such that the algebraic Riccati equation

$$A^T P + PA + P\left(DD^T - \frac{1}{\mu}BB^T\right)P + E^T E + \mu\mathbb{I} = 0 \quad (7)$$

has a symmetric positive definite solution P.

This result is equivalent to the existence of $V(x) = x^T Px$ as a quadratic CLF (Rotea and Khargonekar, 1989). A suitable linear stabilizing controller is also given in Khargonekar et al. (1990).

2.3 Selection of a Sliding Set

The previous paragraphs recall that the necessary and sufficient conditions to quadratically stabilize system (4) using a continuous (time-varying nonlinear) controller is the existence of a quadratic CLF. Our main objective in this paper is to show that if system (4) is quadratically stabilizable then there is a *Sliding-Mode Controller*, with a *linear sliding variable* that robustly stabilizes system (1), with matched uncertainties and perturbations satisfying A2 and A3. Note that if V is a CLF for (4) it is also a CLF for (1).

Consider the definition of the sliding variable $w(x) : \mathbb{R}^n \mapsto \mathbb{R}^p$ of the form

$$w(x) = B^T P x. \tag{8}$$

In Proposition 2 below, it is shown that, if the trajectories of (1) are forced to stay on the set $B^T Px = 0$, i.e. the sliding mode $w(x(t)) \equiv 0$ is established, then the origin of the perturbed system (1) is globally exponentially stable.

Proposition 2. Assume that for system (1), a CLF $V = x^T Px$, that satisfies (6), is known. If the trajectories of (1) are restricted to stay on the set

$$W := \left\{ x(t) \in \mathbb{R}^n \mid B^T Px(t) = 0 \right\} \tag{9}$$

for all future times, then the origin of (1) is globally exponentially stable.

If $x \in W$, then $w = B^T Px = 0$. To keep the trajectory on W it is necessary to impose $\dot{w} = 0$, i.e.

$$\dot{w} = B^T P \left[(A + D\Gamma(t)E)x + B \left[(\mathbb{I} + \Delta_B(t))u + \delta(t, x) \right] \right] = 0,$$

what is achieved by the equivalent control

$$u = -\left(\mathbb{I} + \Delta_B\right)^{-1} \left[\left(B^T PB\right)^{-1} B^T P(A + D\Gamma E)x + \delta \right].$$

This leads to the dynamics on W

$$\dot{x} = \left[\mathbb{I} - B \left(B^T PB\right)^{-1} B^T P \right] (A + D\Gamma(t)E)\, x. \tag{10}$$

The restriction of $V(x)$ to the set W, $V_W(x)$, is continuous and positive definite. Its derivative along the trajectories on W can be calculated as

$$\dot{V}_W = 2x^T P \left[\mathbb{I} - B \left(B^T PB\right)^{-1} B^T P \right] (A + D\Gamma(t)E)\, x,$$

and since $w = B^T Px = 0$

$$\dot{V}_W = 2x^T P \left(A + D\Gamma(t)E\right) x < 0,$$

which is negative due to (6). Since V_W is quadratic and also its derivative, the stability is exponential. □

This Proposition shows that the existence of a quadratic CLF for system (1) implies that the *sliding set* (9) *has an asymptotically stable equilibrium at* $x = 0$.

Remark 3. This Proposition is remarkable. The set $B^T Px = 0$ is fundamental in the definition of a CLF, since on this set condition (6) has to be satisfied. Proposition 2 shows that this set also defines an appropriate sliding set! This shows a deep and unexpected relationship between a Control Lyapunov Function for a system and a suitable sliding manifold for stabilization using sliding-mode control.

It is important to note that this result is valid for every CLF $V(x)$, so that Lemma 1 characterizes the whole set of sliding surfaces for the uncertain system (1) obtained from the quadratic CLFs. Therefore, P can be obtained solving (7), and consequently a suitable sliding variable (with an stable sliding dynamics) is defined by (8).

3. SLIDING MODE CONTROLLERS DESIGN

From the previous section, we can select a CLF for any P that satisfies (7) and if the system dynamics is restricted to the set W, then the exponential stability of system's (1) origin is ensured. Therefore, the control u must be designed to enforce a sliding mode at (9). With this in mind, let us rewrite the dynamics of w as

$$\begin{aligned} \dot{w} &= B^T PA(x + D\Gamma(t)E)x(t) \\ &\quad + \left(B^T PB\right) \left[\left(\mathbb{I} + \Delta_B(t)\right) u + \delta(t, x) \right]. \end{aligned} \tag{11}$$

Proposition 4. Suppose that Assumptions A1, A2 and A3 are fulfilled. For the control law:

$$u = -(B^T PB)^{-1} \left(B^T PAx + \kappa(t, x) \frac{w(x)}{\|w(x)\|} \right), \tag{12}$$

where the gain κ is designed as

$$\begin{aligned} &\kappa(t, x) > \frac{\eta(t, x)}{(1 - \epsilon)} \\ &\eta(t, x) := \|B^T PB\| \rho(t, x) + \|B^T P(A + D\Gamma E)x\|, \end{aligned} \tag{13}$$

the trajectories of subsystem (11) converge to $w(x(t)) = 0$ in a finite time T_f. Moreover, there exists $\beta > 0$ such that

$$T_f \leq \frac{1}{\beta} \|w(x(t_0))\|. \tag{14}$$

Proof. The dynamics (11) in closed loop with control (12) has the form:

$$\dot{w} = -\kappa(t, x) \left(\mathbb{I} + \bar{\Delta}_B(t)\right) \frac{w(x)}{\|w(x)\|} + \bar{\delta}(t, x), \tag{15}$$

where $\bar{\Delta}_B = B^T PB \cdot \bar{\Delta} \cdot B^T PB$ and $\bar{\delta}(t, x) = B^T PB\delta(t, x) + \bar{\Delta}_B(t) B^T P(A + D\Gamma(t)E)x(t)$. Note that by A3, there always exists a known bound $\|\bar{\delta}(t, x)\| \leq \eta(t, x)$. For the closed loop system (15) consider the Lyapunov function candidate $V_u = \frac{1}{2}\|w\|^2$. Its time derivative has the form

$$\begin{aligned} \dot{V}_u &= -w^T \left[\kappa(t, x) \left(\mathbb{I} + \bar{\Delta}_B(t, x)\right) \frac{w}{\|w\|} + \bar{\delta}(t, x) \right] \\ &\leq -\|w\| \left[(1 - \epsilon)\kappa(t, x) - \eta(t, x) \right], \end{aligned} \tag{16}$$

which is negative if κ is selected as in (13). On the other hand, if for any $\beta > 0$ the gain is selected as

$$\kappa(t, x) = \frac{\eta(t, x) + \beta}{1 - \epsilon}, \tag{17}$$

then $\dot{V}_s \leq -\beta\|w\|$, which also implies that $\dot{V}_s \leq -\beta V_s^{1/2}$. Using the comparison Lemma, obtain inequality (12) as the estimation of convergence time to the set $w = 0$. □

4. EXPERIMENTAL RESULT

The mathematical model of pendulum produced by Quanser (2021) has the form:

$$\begin{aligned} &\left(m_p L_2^2 + \frac{1}{4} m_p L_p^2 \cos^2(\theta_p) + J_r \right) \ddot{\theta}_r - \left(\frac{1}{2} m_p L_p L_r \cos(\theta_p) \right) \ddot{\theta}_p \\ &+ \left(\frac{1}{2} m_p L_p^2 \sin(\theta_p) \cos(\theta_p) \right) \dot{\theta}_r \dot{\theta}_p + \left(\frac{1}{2} m_p L_p L_r \sin(\theta_p) \right) \dot{\theta}_p^2 = \tau \\ &-\frac{1}{2} m_p L_p L_r \cos(\theta_p) \ddot{\theta_r} + \left(J_p + \frac{1}{4} m_p L_p^2 \right) \ddot{\theta}_p \\ &-\frac{1}{4} m_p L_p^2 \cos(\theta_p) \sin(\theta_p) \dot{\theta}_r^2 - \frac{1}{2} m_p L_p g \sin(\theta_p) = 0, \end{aligned} \tag{18}$$

where θ_r and θ_p are the angles for the arm and the pendulum, respectively. The parameter m_p is the mass of

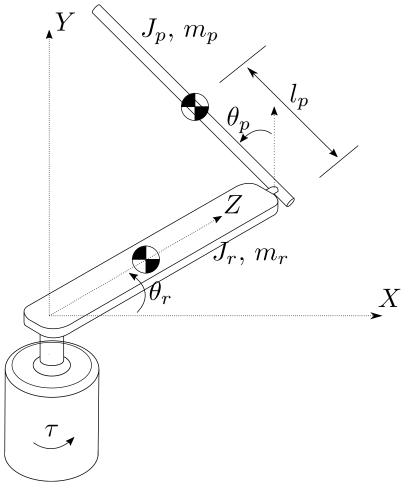

Fig. 1. Diagram of the furuta pendulum.

Fig. 2. Picture of the physical benchmark by *Quanser Inc* ®(Quanser, 2021).

the pendulum, L_p and L_r are the lengths of the pendulum and arm, and J_r is the inertia of the arm. One can compute the torque-voltage (input to the motor V_m) conversion as:

$$\tau = \frac{\eta_g K_g \eta_m k_t \left(V_m - K_g k_m \dot{\theta}_r\right)}{R_m}, \tag{19}$$

where $\eta_g, K_g, k_t, R_m, \eta_g$ are motor parameters. For $x_1 = \theta_r$, $x_2 = \theta_p$, $x_3 = \dot{\theta}_r$ and $x_4 = \dot{\theta}_p$, linearising around the point $x^T = [0\ 0\ 0\ 0]$ with x being a vector of the elements x_i, it yields to a form $\dot{x} = Ax + B\tau$, with matrices

$$\begin{aligned} A &= \frac{1}{J_T}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4}m_p L_p^2 L_r g & 0 & 0 \\ 0 & \frac{1}{2}m_p L_p g(J_r + m_p L_r^2) & 0 & 0 \end{bmatrix}, \\ B &= \frac{1}{J_T}\begin{bmatrix} 0 \\ 0 \\ J_p + \frac{1}{4}m_p L_p^2 \\ \frac{1}{2}m_p L_p L_r \end{bmatrix} \end{aligned} \tag{20}$$

and $J_T = J_p m_p L_r^2 + J_r J_p + \frac{1}{4}J_r m_p L_p^2$. Let us consider the following matrices $E = \mathbb{I}$ and

$$D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

for the case of an uncertainty in the mass m_p. As proposed in this work, one can design the surface solution of (7) with $Q = (\mathrm{diag}\,[1\ 5\ 20\ 50])$. The gain was chosen as $\rho = 10\|x(t)\| + 15$. For safety reasons of the experimental setup, let us choose a continuous approximation for SMC $u(t) = -(B^T PB)^{-1}\left(B^T PAx + v\right)$ with

$$v = \begin{cases} -\rho(x)\mathrm{sign}(w) & \text{if} \quad |w| > \varepsilon \\ -\rho(x)\frac{w}{\varepsilon} & \text{if} \quad |w| \le \varepsilon \end{cases} \tag{22}$$

and $\varepsilon = 1$. The sampling step provided is 1 ms and the following two scenarios were tested in the Furuta pendulum system by *Quanser Inc*®. Note that the input of the pendulum system is saturated from $\tau \in [-10, 10]$ volts to the motor, with the relation given by (19).

From Figure 3 it can be observed that the pendulum position remains near the origin, however, the angular position oscillates in some vicinity of zero but not exactly at zero due to the usage of the continuous approximation to the unit control for the scalar case. The angular velocities are presented in Figure 4. This velocities are estimated using

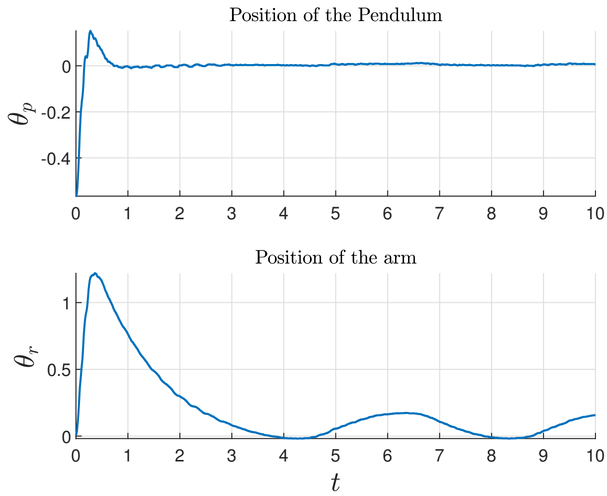

Fig. 3. Pendulum angle θ_p and arm position θ_r.

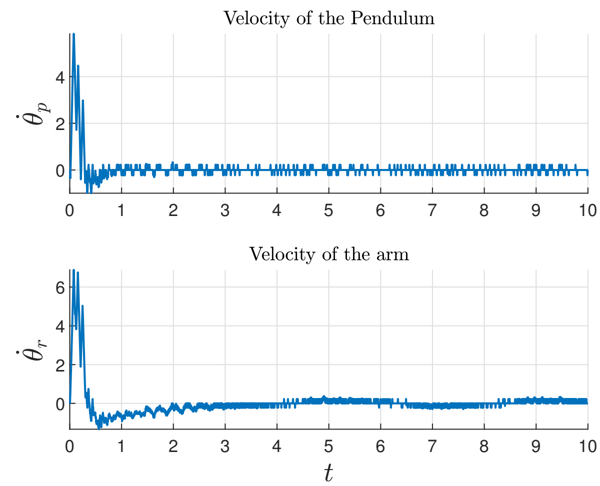

Fig. 4. Velocities of the pendulum $\dot{\theta}_p$ and arm $\dot{\theta}_r$.

a second order Levant's differentiator toolbox provided by Reichhartinger et al. (2018). It can be observed in Figure 5 is continuous due to the usage of the saturation func-

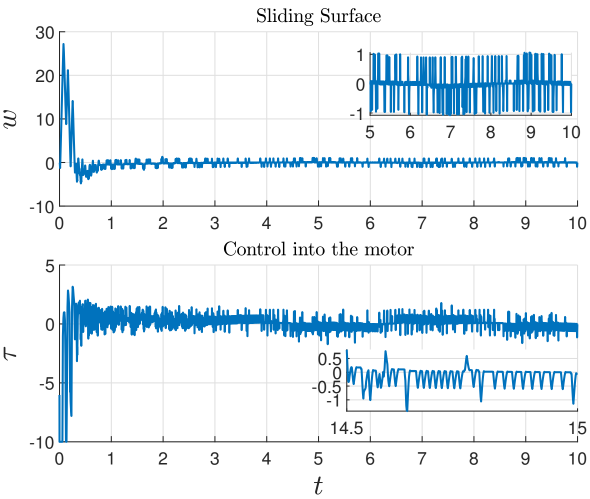

Fig. 5. Applied control τ and sliding surface w.

tion substituting discontinuous control signal. The sliding variable w from figure 5 appears to oscillate inside the ε-vicinity of $w = 0$. A video of the experimental result changing the nominal parameters is shown in the following video: https://youtu.be/Rl1ICutaHL4.

5. DISCUSSION

After the review process for the conference, the authors became aware of the works Fujisaki and Yasuda (1993)and Choi (1998, 1999), where the same problem of finding a linear sliding variable for uncertain linear time invariant systems was studied. In those papers, the idea of designing sliding sets using the quadratic stabilizability is presented, and the result is given by means of Riccati equations in Fujisaki and Yasuda (1993) and through LMI conditions in Choi (1998, 1999). The same class of systems as in this paper is considered, giving necessary and sufficient conditions for the existence of such sliding set.

Choi's papers offer an alternative using LMI's to the solution of the Algebraic Riccati Equation (7) proposed here and in Fujisaki and Yasuda (1993). However, it is not clear from Choi's papers that the conditions for the existence of an appropriate sliding set are exactly the same as those for the existence of a continuous nonlinear controller. This is clearly stated in Fujisaki and Yasuda (1993), and we rediscover it here.

In this sense, it is important to remark the key philosophical difference between the paper by Choi (1999) and ours. Here we emphasize the intrinsic connection between a CLF and a proper sliding manifold, unveiled in Proposition 2, and that reveals the universality of sliding mode control, since the existence of a CLF is equivalent to the existence of a stabilizing controller. This gives us a wider perspective towards generalizing the design of sliding sets of relative degree one for the nonlinear case, something that is not fully clear from the previous works (Fujisaki and Yasuda, 1993; Choi, 1998, 1999).

6. CONCLUSIONS

In this paper, a wider class of Perturbed LTI systems including: unmatched uncertainties in the open loop model, uncertainties in the control matrix and non-vanishing matched disturbances. It is shown that CLF method can be used to design a sliding set for first order sliding mode controllers, ensuring global asymptotically stabilization for the considered class of systems. We show that there exists a Sliding-Mode Controller always when there is a quadratically stabilizing continuous nonlinear controller for the system without the matched perturbations. This result shows the universality of the sliding mode control strategy for the class of systems considered.

The advantages of the proposed methodology are experimentally demonstrated in a Furuta Pendulum system.

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