MIT 18.06 Advanced Standing Exam

Your name is: _____

August 2016

problem	score
1	
2	
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Problem 1:

Suppose A is a 10×10 real-symmetric matrix with eigenvalues 1,2,3,4,5,6,7,8,9,10, and corresponding (real) eigenvectors are $x_1, \ldots, x_{10}$. You are solving the equation $(A - \mu I)x = b$, for some number μ, and you notice that the solution x is blowing up ($\|x\| \to \infty$) as μ approaches some number μ_0.

(a) What are the possible values of μ_0 for which this could be true?

(b) If $\mu = 1.001$, give a good choice of approximate formula for x in terms of b and *one* of the eigenvectors of A, assuming b is chosen at random.

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Problem 2:

The complete solution to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + d \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

for any arbitrary constants c and d.

(a) If A is an $m \times n$ matrix with rank r, give as much true information as possible about the integers m, n, and r.

(b) Construct an explicit example of a possible matrix A and a possible right-hand side $\vec{b}$ with the solution $\vec{x}$ above. (There are many acceptable answers; you only have to provide one.)

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Problem 3:

A sequence of numbers $f_0, f_1, f_2, \ldots$ is defined by the recurrence

$$f_{k+2} = 4f_{k+1} + 3f_k,$$

with starting values $f_0 = 1$, $f_1 = 1$. (Thus, the first few terms in the sequence are $1, 1, 7, 31, 145, 673, 3127, \ldots$.)

(a) Defining $\vec{u}_k = \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix}$, re-express the above recurrence as $\vec{u}_{k+1} = A\vec{u}_k$, and give the matrix A.

(b) Find the eigenvalues of A, and use these to predict what the ratio f_{k+1}/f_k of successive terms in the sequence will approach for large k.

(c) The sequence above starts with $f_0 = f_1 = 1$, and $|f_k|$ grows rapidly with k. Keep $f_0 = 1$, but give a *different* value of f_1 that will make the sequence (with the *same recurrence* $f_{k+2} = 4f_{k+1} + 3f_k$) approach *zero* ($f_k \to 0$) as $k \to \infty$.

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Problem 4:

True or false. Give a counter-example if *false*. (You *don't* need to provide a reason or proof if true.)

(a) If Q is an orthogonal matrix, then $|\det Q| = 1$.

(b) If A is a Markov matrix, then $d\vec{u}/dt = A\vec{u}$ approaches some finite constant vector (a "steady state") for any initial condition $\vec{u}(0)$.

(c) If S and T are subspaces of $\mathbb{R}^3$, then their intersection (points in *both S and T*) is also a subspace.

(d) If S and T are subspaces of $\mathbb{R}^3$, then their union (points in *either S or T*) is also a subspace.

(e) The column space of AB contains the column space of A.

(f) The column space of AB is contained in the column space of A.

(blank page for your work if you need it)

Problem 5:

Recall that A and A^T have the same eigenvalues for a square matrix A.

(a) Let x be an eigenvector of A for an eigenvalue λ_x, and let y be an eigenvector of A^T for an eigenvalue λ_y. Show that x and y are orthogonal ($y^T x = 0$) if $\lambda_x \neq \lambda_y$. (Do *not* assume A is symmetric!)

(b) If $A = \begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix}$ for some number $\epsilon < 1$, then its eigenvalues are $\lambda_\pm = 1 \pm \sqrt{\epsilon}$ and the corresponding eigenvectors are $x_\pm = \begin{pmatrix} 1 \\ \pm\sqrt{\epsilon} \end{pmatrix}$. Find the eigenvectors $y_\pm$ of A^T, and check that $y_-^T x_+ = 0$, consistent with what you showed in the previous part.

(c) For $\epsilon = 0$, then $x_+ = x_-$ and the matrix A is _____ . In this case, $y_+^T x_+ = $ ____ .

Problem 6:

Your 18.06 TA has a matrix A and wants you to solve $A\vec{x} = \vec{b}$. Being a sadist like most TAs, however, she does not tell you what A is, and instead gives you a matrix $B = XA$. All you are told about X is that it is an invertible matrix.

(a) Why are you being asked this question? Specifically, what technique in the 18.06 is the transformation $A \to B$ most similar to: (i) Gaussian elimination (ii) Gram-Schmidt; (iii) diagonalization; (iv) similar matrices; or (v) least-squares? Why?

(b) Which of the four fundamental subspaces of A, if any, can you determine from B alone?

(c) What property/properties *must* B have in order for $A\vec{x} = \vec{b}$ to have a solution for *any* $\vec{b}$?

(d) Suppose $B = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 9 & -6 \\ 1 & 4 & -5 \end{pmatrix}$ and *one* solution is $\vec{x} = (1, 2, 3)^T$. What is $X\vec{b}$?

(e) Using the the information from part (d), find *all* solutions $\vec{x}$.

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